On the chromatic number of the $\text{AO}(2, k, k-1)$ graphs.

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On the Chromatic Number of the $AO(2, k, k - 1)$ Graphs

A thesis
presented to
the faculty of the Department of Mathematics
East Tennessee State University

In partial fulfillment of
the requirements for the degree
Master of Science in Mathematical Sciences

by
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May 2006

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ABSTRACT

On the Chromatic Number of the $AO(2,k,k-1)$ Graphs

by

Navya Arora

The alphabet overlap graph is a modification of the well known de Bruijn graph. De Bruijn graphs have been highly studied and hence many properties of these graphs have been determined. However, very little is known about alphabet overlap graphs. In this work we determine the chromatic number for a special case of these graphs.

We define the alphabet overlap graph by $G = AO(a,k,t)$, where $a$, $k$ and $t$ are positive integers such that $0 \leq t \leq k$. The vertex set of $G$ is the set of all $k$-letter sequences over an alphabet of size $a$. Also there is an edge between vertices $u$, $v$ if and only if the last $t$ letters in $u$ match the first $t$ letters in $v$ or the first $t$ letters in $u$ match the last $t$ letters in $v$. We consider the chromatic number for the $AO(a,k,t)$ graphs when $k > 2$, $t = k - 1$ and $a = 2$. 

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DEDICATION

I would like to dedicate this thesis to my parents, Rajpal Arora and Samidha Arora, for always being encouraging and supporting me with my decisions. I would also like to thank my sister, Ananya Arora, for always being there for me and last but not least my dog Leila.
I would like to thank my advisor, Dr. Debra Knisley for her continued guidance and support during the course of this project. Without her patience, encouragement and enthusiasm this thesis would have not been possible. I would also like to express my gratitude to Dr. Teresa Haynes for introducing me to the wonderful world of Graph Theory and for her encouragement during the past year. In addition, Dr. Anant Godbole has been an invaluable resource and has played a critical role during the course of my master’s degree and I am extremely grateful to him for his support.
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1 INTRODUCTION

1.1 Introduction to Alphabet Overlap Graphs

Alphabet overlap graphs were first introduced in [7], where properties such as hamiltonicity, planarity and domination were determined. We define the alphabet overlap graphs below.

Definition 1.1 Let $a$, $k$ and $t$ be positive integers with $0 \leq t \leq k$. We define alphabet overlap graph by $G = AO(a, k, t)$, where the vertex set of $G$ is the set of all $k$-letter sequences over an alphabet of size $a$. There is an edge between vertices $u$, $v$ if and only if the last $t$ letters in $u$ match the first $t$ letters in $v$ or the first $t$ letters in $u$ match the last $t$ letters in $v$.

In this project we focus on coloring the AO-graph and determining the chromatic number for $G = AO(2, k, k - 1)$. Figure 1 shows an example of the $AO(2, 3, 2)$ graph which is the alphabet overlap graph with alphabet size 2, sequence of length of 3, and tag size of 2. Note that 101 is adjacent to 110 since the first two digits in 101 are the same as the last two digits in 110. Also, 101 is adjacent to 011 since the last two digits in 101 are the same as the first two digits of 011. We will return to this figure to illustrate several examples that are discussed later in this paper.

1.2 Other Related Graphs

The de Bruijn graph $B(\alpha, k)$ is a directed graph with $\alpha^k$ vertices, where each vertex is labeled by words of length $k$ over an alphabet of size $\alpha$. There is an arc from vertex $u = u_1u_2\ldots u_k$ to vertex $v = v_1v_2\ldots v_k$ if and only if $u_i = v_{i-1}$ for $i = 2, \ldots, k$ [2].
Thus, the \( AO(2,k,k-1) \) graph is the underlying simple graph of the directed de Bruijn graph.

The de Bruijn graph and the hypercube, denoted by \( Q^k \), have applications in coding theory. We compare the \( AO(2,k,k-1) \) graph with the hypercube \( Q^k \). The hypercube is defined as follows: The hypercube is the graph \( K_2 \) if \( k = 1 \), while \( k > 1 \), \( Q^k \) is defined inductively as \( Q^{k-1} \times K_2 \). The hypercube can be thought of as having vertices labeled by binary \( k \)-tuples and two vertices are adjacent if and only if their \( k \)-tuples differ at exactly one coordinate. In coding theory, the Hamming distance
between two sequences of length \( k \) is the number of positions in which the sequences differ. For example, the Hamming distance between 10010 and 11110 is two. In the hypercube \( Q^5 \), the path between the two vertices labeled by these sequences is 10010 - 11010 - 11110, thus the graph theoretic distance is also 2. This illustrates the well-known property that relates the Hamming distance to the graph theoretic distance. Note that in the \( AO(2, k, k - 1) \), a path between two vertices is obtained by shifting \( k - 1 \) digits of the sequence, so the distance between these vertices is the number of shifts.

When comparing the \( AO(2, k, k - 1) \) graph with the hypercube \( Q^k \), there is a notable difference in the number of edges in \( G \) as \( k \) becomes very large. In the \( Q^k \) graph each vertex has degree \( k \). However for the \( AO(2, k, k - 1) \), the maximum degree is fixed at 4 as \( k \) increases. We note for all \( a \) that the \( AO(a, k, k - 1) \) graphs become sparse as \( k \) increases since the vertex degree is bounded by \( 2a \).

Finally, in [3] \( (\alpha, k) \) labeled graphs were studied. A graph \( H \) can be \((\alpha, k)\)-labeled if it is possible to assign a label of length \( k \) from an alphabet of size \( \alpha \) to each vertex \( x \) of \( H \), such that no two distinct vertices have the same label and if two vertices, \( u \) and \( v \) are adjacent, the last \( k - 1 \) digits of \( u \) are the same as the first \( k - 1 \) digits of \( v \). AO graphs can be thought of as \( complete(a, k) \) labeled graphs [7].
2 PROPERTIES OF AO-GRAPHS

2.1 Properties of \( AO(a, k, t) \)

Alphabet overlap graphs were introduced in [7] where several properties were studied, including hamiltonicity and planarity. In this section we give some definitions of the properties that were studied and state the results that were obtained.

A ***hamiltonian cycle*** of a graph is a cycle that contains all the vertices of the graph. A ***hamiltonian graph*** is a graph that has a hamiltonian cycle [8].

A graph is ***planar*** if it can be drawn in a plane with no edge crossings. A graph that is drawn with no edge crossing is called a ***plane graph***. Figure 2 shows a drawing of a planar graph that has edge crossings and also a drawing of the graph with no edge crossings.

![Figure 2: A planar graph with and without edge crossings](image-url)
The result below shows that $G$ is Hamiltonian for all non-trivial values of the parameters $a, k$ and $t$ [7].

**Theorem 2.1** Alphabet overlap graphs $G(a, k, t)$ are Hamiltonian for all $a, k \geq 2$ and $s \leq k - 1$, where $s = k - t$.

For example when for $AO(2, 3, 2)$, we have the following Hamiltonian cycle:

000 - 001 - 010 - 101 - 011 - 111 - 110 - 100.

The following theorem determines when an AO-graph is planar [7].

**Theorem 2.2** If $t \leq k/2$ the only non-trivial planar AO-graphs are when $d = 2, 3$, $t = 1$ and $k = 2$.

Because almost all of the AO-Graphs are non-planar we cannot apply known coloring results for planar graphs.

### 2.2 Properties of $AO(2, k, k - 1)$

In this project, we consider the special case of alphabet overlap graphs, namely the $G = AO(2, k, k - 1)$.

The overlap condition can be thought of as a left shift with an empty slot at the least significant digit, which can be filled with a 0 or a 1, or as a right shift with an empty slot at the most significant digit, which can be filled with a 0 or 1.

The neighbors of $v \in G$ can be obtained by left shifting and right shifting and filling the empty slot with a 0 or a 1. Since left shifting a binary sequence is equivalent to multiplication by 2 and right shifting is equivalent to division by 2, the following formulas can be used to generate the adjacencies for $v$. 

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Observation 2.3 For any vertex $v$ in $AO(2, k, k-1)$, if we use the base-10 representation of $v$’s binary labeling we can calculate the neighborhood of $v$ with the following formulas:

$$2v \mod 2^k$$ (Left shift, fill digit with 0)

$$(2v + 1) \mod 2^k$$ (Left shift, fill digit with 1)

$$\lfloor \frac{v}{2} \rfloor$$ (Right shift, fill digit with 0)

$$\lfloor \frac{v}{2} \rfloor + 2^{k-1}$$ (Right shift, fill digit with 1).

We illustrate this with an example. Consider vertex 1011 $\in AO(2, 4, 3)$, since 1011 in binary equals 11 in base-10, we have the following adjacencies:

$$2(11) \mod 16 = 6 = 0110_2$$

$$(2(11) + 1) \mod 16 = 7 = 0111_2$$

$$\lfloor \frac{11}{2} \rfloor = 5 = 0101_2$$

$$\lfloor \frac{11}{2} \rfloor + 2^3 = 13 = 1101_2.$$  

Thus, we have the neighborhood of 1011 = \{0110, 0111, 0101, 1101\}. Notice that for any vertex $0 = 00\ldots0 \in AO(2, k, k-1)$, we have $2(0) = \frac{0}{2} = 0$ so this vertex will only have degree 2.
Since each vertex in an $AO(2, k, k-1)$ graph can be labeled with a binary sequence of length $k$, and since all such sequences will exist in the graph, we see that $|V(G)| = 2^k$. We define a palindrome as a vertex with a sequence of length $k$, such that the $i^{th}$ digit and the $(k - i + 1)^{th}$ digit are the same. For example, when $k = 4$, 0110, 1001, 0000, and 1111 are the palindromes. The number of palindromes in a graph can be counted by: $2^k$ when $k$ is even and $2^{k+1}$ when $k$ is odd.

A geometric center vertex is defined as vertex with a sequence of length $k$, such that the $i^{th}$ digit and the $(k-i+1)^{th}$ digit are different. So when $k = 4$, 1010, 0101, 1100, and 0011 are the center vertices. It is easily seen that the number of geometric center vertices in a graph is counted using the formula as given above for palindromes.

In each graph in this family there will only be four vertices that do not have degree 4. The two palindromes 00...0 and 11...1 are each adjacent to themselves twice and have degree 2; that is, in a multigraph both of these vertices would have 2 loops. Also, the vertices 10...10 and 01...01 are adjacent to each other by two overlap conditions, and so have degree 3. So, in each of these graphs there will be two vertices of degree 2, two vertices of degree 3, and $2^k - 4$ vertices of degree 4. These results can be confirmed using the formulas given in Observation 2.3. When we consider the corresponding multigraphs, each vertex has degree 4 and hence each multigraph is 4-regular.

The graph $G$ can be drawn in such a way that it is symmetric about the geometric center vertices and palindromes. The reflection of vertex $v$ is defined as the mirror image of its sequence. Thus, when the $AO(a, k, t)$ graph is folded over the line formed by the palindromes, each vertex is folded onto its reflection. The binary complement is
obtained by flipping each bit in the sequence. For the center vertices, the complement and the reflection is the same. Thus, when the graph is folded along the line formed by the center vertices, each vertex will fold onto its binary complement. Note that since this graph is almost 4-regular and can be drawn symmetrically, it would be interesting to study if G could be made into a symmetric graph, as symmetric graphs are well-studied [6] [11][12].

Additionally, an interesting property we studied was odd and even cycles in the special case $AO(2, k, k - 1)$. Each $G \in AO(2, k, k - 1)$ graph contains both even and odd cycles. Note that an odd cycle is a combination of an even number of right shifts and an odd number of left shifts, or vice versa. An even cycle is a combination of an odd number of left and right shifts, or an even number of left and right shifts. The Hamiltonian cycle for $G$ (necessarily of length $2^k$) is the largest even cycle and the smallest cycle is a $C_3$. Also, $G$ will always contain an odd cycle of length $2^k - 1$, which can be obtained starting with the Hamiltonian cycle and removing the edges between vertices 000...0 and 000...1, 000...0 and 100...0 and adding the edge between 000...1 and 100...0. This removes vertex 000...0 from the cycle, leaving $2^k - 1$ vertices.

We studied methods for generating cycles in $G$ and discovered the following method for generating cycles of length $k$ and $k + 1$ for any vertex in a graph except 000...0 and 111...1, for $k > 2$.

1. Select a vertex $v$ from $G$.

2. Pick a vertex from the neighborhood of $v$ that is obtained by a left shift and has degree 4.
3. Apply a left shift and populate the last slot with the same digit as the 1<sup>st</sup> digit in $v$.

4. Repeat this procedure until the first $k - 1$ digits of $v$ have filled the empty $k - 1$ slots in the final vertex of the cycle.

To obtain a cycle of length $k$, simply omit step 2 above, which will remove one vertex from the cycle, leaving a cycle of length $k$. We illustrate this method with an example:

Consider $G = AO(2, 3, 2)$ and let $v = 010 \in G$. To generate an even cycle of length $k + 1$, select 100 which is adjacent to $v$ using a left shift. Now perform a left shift on this vertex and fill the empty slot with a 0, which is the 1<sup>st</sup> digit in $v$, to get 000. Finally, perform another left shift on 000 and fill the empty slot with a 1, which is the 2<sup>nd</sup> digit in $v$, to get 001 which is adjacent to $v$, completing the cycle.

The following even cycle of length 4 is generated:

- 010       Starting vertex
- 100       A vertex adjacent to $v$ by a left shift that has degree 4
- 000       Last digit is the 1<sup>st</sup> digit in $v$
- 001       Last digit is the 2<sup>nd</sup> digit in $v$
- 010       Back to the beginning to complete cycle of length 4.

To generate an odd cycle of length 3, we would omit the 2<sup>nd</sup> step and start populating the final slot immediately to generate: 010 – 100 – 001 – 010.

The following result is easily seen from the preceding discussion.

**Proposition 2.4** Each vertex in $G$ lies on both an odd and even cycle.
In general, cycles are an important topic in graph theory. Results on generalized de Bruijn cycles have been investigated in [5] and a study of the cycles in generalized AO graphs is an additional topic of interest.
One of the most important areas of study in graph theory is vertex coloring. A large part of its popularity is due to the Four-Color Conjecture, which was recently proven [1]. Vertex coloring also has a large number of useful applications which no doubt contribute to its popularity. Some examples of these applications are circuit board design, register allocation, and many forms of scheduling and assignment problems.

First, we define several important concepts. A coloring of a graph $G$ is an assignment of colors to the vertices of $G$, where each vertex gets exactly one color. A coloring is known as proper if no two adjacent vertices are assigned the same color. A set of vertices consisting of all vertices assigned the same color is known as a color class. If a graph can be colored using $k$ or fewer colors, then $k$ is said to be $k$-colorable. The minimum number $k$ for which a vertex is $k$-colorable is the chromatic number of $G$, denoted $\chi(G)$.

In general, vertex coloring is a very difficult problem. In fact, it was one of the original 21 problems listed by Richard Karp as being NP-complete [9]. It remains difficult to even find a provably good approximation to an optimum coloring. However there are some algorithms that deliver good results under certain conditions. Incremental coloring methods, where vertices are sequentially colored, are the most common choice in these algorithms. For example, the Greedy Coloring method is a common approach that takes an input of all vertices of the graph in some sorted order, and then assigns the lowest numbered color to each vertex such that the coloring remains proper. This method is called greedy because at each step it attempts to find the optimal solution up to that vertex.
The saturation degree of a vertex $v$, in a simple graph with a partial coloration, is defined as the number of different colored vertices to which $v$ is adjacent. The Dsatur algorithm [4] is a well-known vertex coloring method that uses the saturation degree of a vertex to color the graph. The Dsatur algorithm follows:

1. Arrange vertices by decreasing order of degrees.

2. Color a vertex of maximal degree color 1.

3. Choose a vertex with maximal saturation degree.

4. Color the chosen vertex with the lowest numbered possible color.

5. IF all vertices are colored STOP, ELSE return to 3.

One advantage of the Dsatur algorithm is that while it provides a good heuristic in the general case, it is known to be exact for coloring bipartite graphs. Because of this, it is a good method to determine if a graph is bipartite in polynomial time [4].

The following is known as the Five Color Theorem and was proven by Percy John Heawood in 1890.

**Theorem 3.1** Every planar graph is 5-colorable.

In 1976, the Four Color Theorem was proven by Appel, Haken, and Koch, using the aid of a computer to perform extensive calculations.

**Theorem 3.2** Every planar graph is 4-colorable.
It has already been established that AO-graphs are only planar for \(d = 2, 3, t = 1\) and \(k = 2\), thus we can not apply the Four Color Theorem to these graphs. We must use other results to attempt to find bounds for \(\chi(G)\).

The following inequality is due to Szekeres and Wilf [13].

**Theorem 3.3** For every graph \(G\),

\[
\chi(G) \leq 1 + \lambda(G)
\]

where \(\lambda(G)\) is the maximum of the minimal degrees of the subgraphs of \(G\).

From figure 1, we see that the minimum degree of any subgraph must be less than 4, which implies that \(\lambda(G) = 3\) for \(G = AO(2, k, k - 1)\) graph. So, by the previous result, 4 is an upper bound for the chromatic number of \(G\).

The following result, known as Brooks’ Theorem, gives us an upper bound for \(\chi(G)\) [8].

**Theorem 3.4** Let \(G\) be a non-complete, simple connected graph with \(\Delta(G) \geq 3\). Then \(\chi(G) \leq \Delta(G)\).

For each graph \(G\) in the family \(AO(2, k, k - 1)\), we have \(\Delta(G) = 4\); since each sequence can overlap in at most four different ways. Applying the previous result we find that \(\chi(G) \leq 4\), that is each graph \(G\) is 4-colorable. We now wish to find a lower bound for \(\chi(G)\).

A **clique** in a graph \(G\) is a maximal complete subgraph of \(G\). The **clique number** of a graph \(G\), denoted \(\omega(G)\), is equal to the number of vertices in a largest clique.
in $G$. The next theorem allows us to obtain a lower bound for $\chi(G)$ in terms of its clique number [8].

**Theorem 3.5** Let $G$ be a graph. Then $\chi(G) \geq \omega(G)$.

Again, looking at each $G$ in $AO(2, k, k - 1)$, we see that each $G$ contains a $K_3$ as an induced subgraph, therefore $\omega(G) \geq 3$. Using the previous theorem we see that the lower bound for the chromatic number of the $AO(2, k, k - 1)$ is 3.

**Theorem 3.6** Let $G$ be a bipartite graph, then $\chi(G) = 2$, unless $G$ is edgeless.

The cube graph $Q^k$ that we spoke of earlier is 2-colorable and hence, is bipartite [8]. However, since we have already established that $\chi(G) \geq 3$, $G$ cannot be bipartite. This is an important result, since there exist established algorithms which can color a bipartite graph in polynomial time [4].
For this project, we will be looking at the special case of alphabet overlap graphs, 
\( G = AO(2, k, k - 1) \) where \( a = 2 \) and \( t = k - 1 \).

We have previously shown that \( 3 \leq \chi(AO(2, k, k - 1)) \leq 4 \). Therefore, if we 
can demonstrate a proper 3-coloring for all such graphs, then we will have shown 
\( \chi(AO(2, k, k - 1)) = 3 \). In order to do this, we define a method of coloring the graphs 
by partitioning the vertices into 3 partite sets, and assigning each set a different color.

4.1 Method of Coloring

Our goal is to partition \( G \) into partite sets.

First we define 3 color classes \( A, B \) and \( C \). When \( k = 2 \), we let

\[
A_2 = \{00, 11\}
\]

\[
B_2 = \{01\}
\]

\[
C_2 = \{10\}
\]

which are clearly partite sets of \( AO(2, 2, 1) \).

Let \( A_k \) be the set of all possible combinations obtained by appending a 0 or 1 to 
the end of each vertex in \( A_{k-1} \), except for 000...0 and 111...1, and define \( B_k \) and \( C_k \) 
similarly. Now since 000...0 is adjacent to exactly 2 vertices, there must be some set 
not containing any vertices in its neighborhood, add 000...0 to this set. Repeat the 
procedure for 111...1.
The construction for $k=3$ using this method is shown below.

First, generate $A_3$ by appending a 0 or 1 to the end of each vertex in $A_2$:

$$A_3 = \{001, 110\}.$$  

Do not add 000 and 111 as they will be added in the last step. $B_3$ and $C_3$ are generated similarly:

$$B_3 = \{010, 011\}$$

$$C_3 = \{101, 100\}.$$  

Now add 000 to the set $B_3$ since it is not adjacent to any vertices in $B_3$. Similarly add 111 to $C_3$.

So now the three sets are:

$$A_3 = \{001, 110\}$$

$$B_3 = \{010, 011, 000\}$$

$$C_3 = \{101, 100, 111\}.$$  

A coloring is obtained by assigning the same color to all vertices in the same set. Figure 3 shows this coloring.

**Theorem 4.1** The sets $A_k$, $B_k$, and $C_k$ are partite sets of $G = AO(2, k, k - 1)$.

Proof: When $k = 2$, the sets $A_2$, $B_2$ and $C_2$ are clearly partite, so the statement is true for $k = 2$. We proceed by induction on $k$. Assume that $A_{k-1}$, $B_{k-1}$, and $C_{k-1}$ are partite sets. We show that $A_k$, $B_k$, and $C_k$ are partite.

Assume these sets are not disjoint, say $v \in A_k \cap B_k$, where $v$ is labeled as $v_1 v_2 \ldots v_k$. Then $v_1 v_2 \ldots v_{k-1}$ must exist in $A_{k-1} \cap B_{k-1}$, contradicting the induction hypothesis. Therefore, $A_k$, $B_k$ and $C_k$ are disjoint. Since by the method of construction, we know
that $|A_k| + |B_k| + |C_k| = 2^k$, and we have just shown the sets are disjoint, we must have $\forall v \in G, v \in A_k \cup B_k \cup C_k$.

All that remains to be shown is that these sets are independent. Assume, without loss of generality, that $A_k$ is not independent, that is, let $u, v \in A_k$ such that $u$ is adjacent to $v$. Label $u$ and $v$ by $u = u_i$ and $v = v_i$, where $1 \leq i \leq k$. Also, let $\overline{u} = \overline{u_i} = u_i$ and $\overline{v} = \overline{v_i} = v_i$, where $1 \leq i \leq k - 1$, that is $\overline{u}$ is labeled with the first $k - 1$ digits of $u$. We now have $\overline{u}, \overline{v} \in A_{k-1}$. We consider the two cases where $u$ is adjacent to $v$:
1. If $u$ is adjacent to $v$ by a right shift, then $u_i = v_{i+1}$ for all $1 \leq i \leq k - 1$. But then, $\overline{u_i} = \overline{v_{i+1}}$ for all $1 \leq i \leq k - 2$, which implies $\overline{u}$ is adjacent to $\overline{v}$ by a right shift.

2. If $u$ is adjacent to $v$ by a left shift, then $u_i = v_{i-1}$ for all $2 \leq i \leq k$. But then, $\overline{u_i} = \overline{v_{i-1}}$ for all $2 \leq i \leq k - 1$, which implies $\overline{u}$ is adjacent to $\overline{v}$ by a left shift.

In either case, we have $\overline{u}$ is adjacent to $\overline{v}$, contradicting the induction hypothesis. Hence, $A_k, B_k$, and $C_k$ are partite sets of $G$. □

**Corollary 4.2** All graphs in the family $AO(2, k, k - 1)$ are 3-colorable.

**Theorem 4.3** If $G$ is a graph in the family $AO(2, k, k - 1)$, then $\chi(G) = 3$.

Proof: Corollary 4.2 states that $G$ is 3-colorable and thus $\chi(G) \leq 3$. Also, since every graph $G$ contains a clique of size 3 we have $\chi(G) \geq 3$. Thus, $\chi(G) = 3$. □

### 4.2 Discussion

When $a = 3$, the $AO(3, 2, 1)$ graph can be properly colored with four colors. The method described in this thesis can be used to obtain a partition of the vertex set into 4 color classes. The $AO(3, 2, 1)$ graph does not contain a $K_4$. However, it does contain a subgraph that has chromatic number 4, see Figure 4a. Thus the construction method yields a minimum proper coloring for the $AO(3, 2, 1)$. Table 1 shows the partite sets constructed using the construction method and Figure 4b shows the graph with this 4-coloring.
Table 1: Partite sets of $AO(3, 2, 1)$

<table>
<thead>
<tr>
<th>$A_2$</th>
<th>$B_2$</th>
<th>$C_2$</th>
<th>$D_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>01</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>11</td>
<td>02</td>
<td>12</td>
<td>21</td>
</tr>
</tbody>
</table>

a) A drawing of $AO(3, 2, 1)$ with vertex tertiary labels.

b) A proper coloring of $AO(3, 2, 1)$

Figure 4: A proper 4-coloring of the $AO(3, 2, 1)$
5 CONCLUSION

In this paper, we studied a special case of alphabet overlap graphs. In particular, we considered all AO-graphs with an alphabet size 2, length \( k \), and tag size \( k - 1 \). We outlined a method for obtaining a proper 3-coloring for the \( AO(2, k, k - 1) \) that holds for all \( k \). We also showed that since these graphs contain a clique of size 3, we have \( \chi(AO(2, k, k - 1)) \geq 3 \). These two results combined give us our main result:

\[
\chi(AO(2, k, k - 1)) = 3.
\]

For future work, we plan to generalize this method for all alphabet sizes, that is \( AO(a, k, k - 1) \). When \( k = 2, a = 3 \), we demonstrated a proper 4-coloring. We conjecture that the chromatic number of \( AO(3, k, k - 1) \) is 4. We also conjecture that the \( \chi(AO(5, k, k - 1)) \) is 6. At this point, it might seem reasonable to conjecture that \( \chi(AO(a, k, k - 1)) = a + 1 \). However, a proper coloring of the \( \chi(AO(5, k, k - 1)) \) using only 5 colors has been given by Dr. Nigussie[10]. In general, we conjecture that the chromatic number of the \( AO(a, k, k - 1) \) graphs is less than or equal to \( a + 1 \).

The authors of this project are continuing to work with Dr. Nigussie on this open problem.
BIBLIOGRAPHY


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