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Double Domination Edge Critical Graphs.

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by
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ABSTRACT

Double Domination Edge Critical Graphs

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In a graph $G = (V, E)$, a subset $S \subseteq V$ is a double dominating set if every vertex in V is dominated at least twice. The minimum cardinality of a double dominating set of G is the double domination number. A graph G is double domination edge critical if for any edge $uv \in E(\overline{G})$, the double domination number of $G + uv$ is less than the double domination number of G. We investigate properties of double domination edge critical graphs. In particular, we characterize the double domination edge critical trees and cycles, graphs with double domination numbers of 3, and graphs with double domination numbers of 4 with maximum diameter.

DEDICATION

I dedicate this thesis to my parents, Charles and Vivian Thacker, who always encouraged and supported me in my educational endeavors, and instilled in me a desire to always do my best. I would also like to thank my wife Jessica Thacker for her love, understanding, and support. And mostly I thank God, the source of all wisdom, for being my strength, lighting my paths, and making my life worthwhile.

ACKNOWLEDGEMENTS

I would like to give my sincere heartfelt thanks to my advisor Dr. Teresa Haynes. I can honestly say this thesis would not have been possible without your encouragement, influence, and support. Much thanks and may God bless.

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1 INTRODUCTION

The branch of mathematics known as graph theory emerged from various questions and ideas from the study of games and other recreational mathematics. With the prominence in recent years of computer science, operations research, and other engineering areas, the field of graph theory has flourished as its application in these areas is quite natural. Motivated by numerous applications, much work has been done on the graph theory topic of domination. We start with the basic terminology in order to discuss domination and related parameters. A graph G consists of an ordered pair (V, E) where V is a finite nonempty set of objects called vertices and E is a set of unordered pairs of distinct vertices of G called edges. An edge is denoted by the two vertices it joins, and two vertices are called *adjacent* if there is an edge between them. We also consider the definition of the *complement* of a graph G, denoted $\overline{G}$. The *complement* $\overline{G}$ of a graph G is that graph with vertex set V such that two vertices are adjacent in $\overline{G}$ if and only if the vertices are not adjacent in G. An example of a graph G and $\overline{G}$ is given in Figure 1.

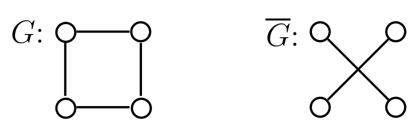


Figure 1: The Complement $\overline{G}$ of a Graph G

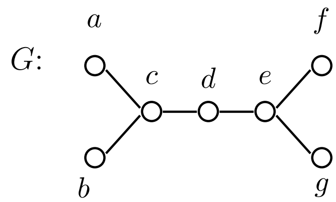


Figure 2: Example of a Graph

Next consider the graph $G = (V, E)$ displayed in Figure 2. Notice $V = \{a, b, c, d, e, f, g\}$ and $E = \{ac, bc, cd, de, ef, eg\}$. For an example of adjacency, we can see that c is adjacent to a, b, and d, thus we would say the *open neighborhood* of c is $\{a, b, d\}$. In general, for any vertex $v \in V$, the *open neighborhood* of v is $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Hence $N[c] = \{a, b, c, d\}$. The open neighborhood of a subset S of the vertex set V is the union of all the open neighborhoods of the vertices in S, or $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is denoted $N[S] = N(S) \cup S$. For example, in the graph G in Figure 2, if we consider the set $S = \{c, e\}$, then $N(S) = \{a, b, d, f, g\}$ and $N[S] = \{a, b, c, d, e, f, g\} = V$. When $N[S] = V$, we say S is a *dominating set* of G, since by definition any vertex v of a graph G *dominates* $N[v]$. Therefore as shown in Figure 3, $S = \{c, e\}$ is a *dominating set*.

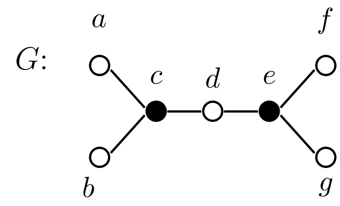


Figure 3: Domination Example

Furthermore the minimum cardinality of any dominating set of G is the *domination number* $\gamma(G)$. Again considering the graph in Figure 2, with $S = \{c, e\}$, we know that $\gamma(G) \leq 2$. Since no single vertex can dominate all of the remaining vertices, we have $\gamma(G) \geq 2$. Thus it follows that $\gamma(G) = 2$.

Another type of dominating set that has been widely studied is a *total dominating* set. A set S is a *total dominating* set if $N(S) = V$, in other words every vertex in V is adjacent to some vertex in S. The minimum cardinality of any total dominating set of G is the *total domination number* $\gamma_t(G)$. Consider again the graph shown in Figure 2. The set $S = \{c, d, e\}$ forms a total dominating set of G and we note that $\gamma_t(G) = 3$ which can be seen in Figure 4 below.

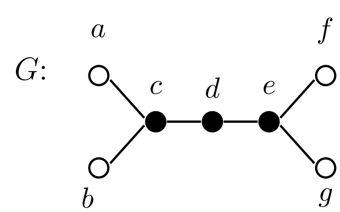


Figure 4: Total Domination Example

Now we will consider a domination parameter more central to this thesis, namely *double domination.* A subset S of V is a *double dominating* set of G, or DDS, if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$, that is, v is in S and has at least one neighbor in S or v is in $V - S$ and has at least two neighbors in S(see [7]). So a double dominating set dominates every vertex in G at least twice, and the *double domination number*, denoted $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. Note that a double dominating set of minimum cardinality is called a $\gamma_{\times 2}$-set. Again consider the graph from Figure 2. The set $S = \{a, b, c, e, f, g\}$ forms a double dominating set of G and we note that $\gamma_{\times 2}(G) = 6$ which can be seen in Figure 5.

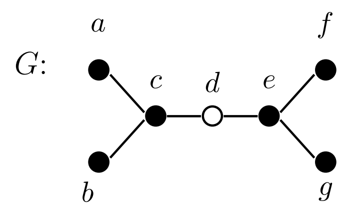


Figure 5: Double Domination Example

We note that the total and double domination numbers are only defined for graphs with no isolated vertices. Also for any graph without isolated vertices, every double dominating set is a total dominating set, so $\gamma(G) \leq \gamma_t(G) \leq \gamma_{\times 2}(G)$. For a more detailed treatment of domination related parameters and for terminology not defined here, the reader is referred to [2, 8].

Adding an edge cannot increase any of the three aforementioned domination parameters. Hence the addition of an edge either decreases each of these parameters or leaves it unchanged. If the addition of an edge decreases the parameter under consideration, it is called a *critical edge*. For domination number, Sumner and Blitch [15] studied graphs where the addition of any edge changed the domination number. They called graphs with this property *domination edge critical*. The *total domination edge critical* graphs, that is, graphs where the addition of any edge decreased the total domination number were studied by Haynes, Mynhardt, and van der Merwe in [9]-[12]. Consider the domination edge critical graph G and the total domination edge critical graph H in Figure 6.

Figure 6: Domination and Total Domination Edge Critical Graphs

Although much work has been done on the domination and total domination edge critical graphs, neither class of graphs has been characterized (not even for domination and total domination numbers as small as three). The work done on domination edge critical and total domination edge critical graphs has motivated the study of the same concept for double domination in this thesis. Formally, we define a graph G as *double domination edge critical*, or just $\gamma_{\times 2}$-critical, if $\gamma_{\times 2}(G + uv) < \gamma_{\times 2}(G)$ for any edge $uv \in E(\overline{G})$. Specifically we characterize the $\gamma_{\times 2}(G)$-critical trees, cycles, graphs G having $\gamma_{\times 2} = 3$, and graphs G having $\gamma_{\times 2} = 4$ with maximum diameter.

2 Properties of Double Domination Edge Critical Graphs

We begin this section with a trivial, but useful observation about double domination edge critical graphs.

Observation 2.1 *If a graph G is $\gamma_{\times 2}$-critical and $uv \in E(\overline{G})$, then every $\gamma_{\times 2}(G+uv)$-set contains at least one of u and v.*

Note that adding an edge can decrease the domination number by at most one. However, it was shown in [9] that adding an edge can decrease the total domination number by as much as two.

We show the same result holds for double domination.

Proposition 2.2 *For any graph G without isolates and edge $uv \in E(\overline{G})$,*

$$\gamma_{\times 2}(G) - 2 \leq \gamma_{\times 2}(G + uv) \leq \gamma_{\times 2}(G).$$

Proof. Obviously, adding an edge to a graph cannot increase the double domination number, so the upper bound holds. Let $G' = G + uv$ for some $uv \in E(\overline{G})$, and assume that $\gamma_{\times 2}(G') < \gamma_{\times 2}(G)$. Let S' be a $\gamma_{\times 2}(G')$-set. From Observation 2.1, we know that at least one of u and v is in S'. If, without loss of generality, $u \in S'$ and $v \notin S'$, then $w \in N_G(u)$ is in S to double dominate u. Moreover, v has at least two neighbors in S' in G', that is, v has at least one neighbor in $S' - \{u\}$. Thus, $S' \cup \{v\}$ double dominates G, and so

$$\gamma_{\times 2}(G) \leq |S'| + 1 = \gamma_{\times 2}(G') + 1.$$

Thus assume that both u and v are in S'. Then S' double dominates $V - S'$ in G. If each of u and v has a neighbor in $S' - \{u, v\}$, then S' is a DDS of G with cardinality less than $\gamma_{\times 2}(G)$, a contradiction. Hence assume, without loss of generality, that v is the only neighbor of u in S'. Since u is not an isolate in G, u has a neighbor, say w, in $V - S'$. If v has a neighbor in $S' - \{u\}$, then $S' \cup \{w\}$ is a double dominating set of G, and

$$\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G') + 1.$$

If neither u nor v has a neighbor in $S' - \{u, v\}$, then again since G has no isolates, both u and v have a neighbor in $V - S'$. Hence, $S \cup \{w, y\}$, where y is a neighbor of v, is a DDS of G, and we have $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G') + 2$. □

Based on this proposition we have our first corollary that describes the double dominating set for the case where adding an edge to a graph decreases the double domination number by two.

Corollary 2.3 *If $\gamma_{\times 2}(G + uv) = \gamma_{\times 2}(G) - 2$, then every $\gamma_{\times 2}(G + uv)$-set contains both u and v.*

Thus adding an edge can decrease the double domination by 0, 1, or 2. If adding an edge from $E(\overline{G})$ decreases the double domination number by i, we call such an edge an i-edge. It is possible for a single graph G to have i-edges for all i, $0 \leq i \leq 2$. Consider the following example. A *corona* $G \circ K_1$ is the graph formed from G by adding a new vertex v' for each $v \in V(G)$ and the edge vv'. Let G be the *corona* $C_4 \circ K_1$, as shown in Figure 7, where the darkened vertices represent a double dominating set.

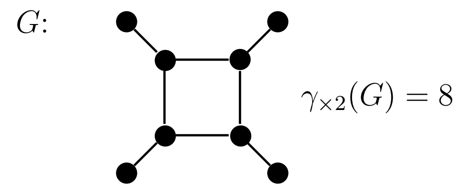


Figure 7: The Corona $C_4 \circ K_1$

Then an edge between nonadjacent vertices of the C_4 is a 0-edge. Letting G' be the corona $C_4 \circ K_1$ plus an edge between nonadjacent vertices of the C_4, we can see in Figure 8 that the double domination number does not decrease.

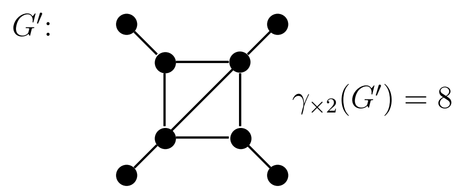


Figure 8: Example of a 0-edge

An edge from a leaf to a vertex of the C_4 is a 1-edge. Letting G'' be the corona $C_4 \circ K_1$ plus an edge from a leaf to a vertex of the C_4, we can see in Figure 9 that the double domination number decreases by 1.

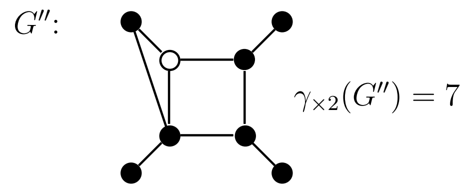


Figure 9: Example of a 1-edge

An edge between two leaves in $C_4 \circ K_1$ is a 2-edge, and we will denote the graph with such an edge added as G'''. Figure 10 shows that such an edge decreases the double domination number by 2.

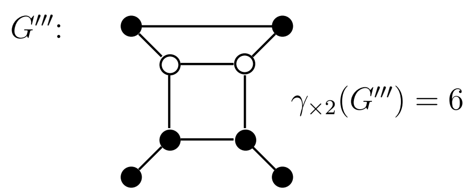


Figure 10: Example of a 2-edge

In [9], a graph is called *total domination supercritical* if the total domination number is decreased by two for any edge added, and these graphs were characterized as follows.

Theorem 2.4 [9] *A graph G is total domination supercritical if and only if G is the union of two or more nontrivial complete graphs.*

For example, consider the total domination supercritical graph G in Figure 11, and the graph $G' = G + e$ where $e \in E(\overline{G})$. Obviously the addition of any edge in

$E(\overline{G})$ will give similar results

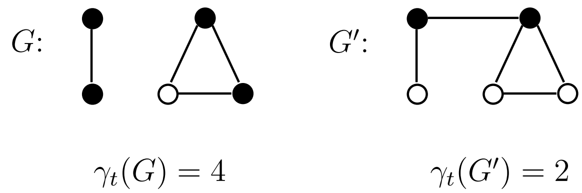


Figure 11: Total Domination Supercritical Graph Example

Next we show that double domination supercritical graphs do not exist.

Theorem 2.5 *There are no double domination supercritical graphs.*

Proof. Suppose G is a double domination supercritical graph, and consider $G' = G + uv$ for some $uv \in E(\overline{G})$. Let S be a $\gamma_{\times 2}(G')$-set. By Corollary 2.3, we know $\{u, v\} \subset S$. By the proof to Proposition 2.2, neither u nor v has another neighbor in S. Since G has no isolates, each of u and v has a neighbor in $V - S$. If x is a common neighbor of u and v, then $S \cup \{x\}$ double dominates G, contradicting that G is supercritical. Hence, we may assume that $N(u) \cap N(v) = \emptyset$ for every pair of nonadjacent vertices in G. This implies that every component of G is complete. But then adding an edge does not change the double domination number, contradicting our assumption that G is supercritical. □

We conclude this section with some useful observations.

Observation 2.6 *Every double dominating set of a graph G contains the leaves and support vertices of G.*

An example of this observation can be seen in Figure 5.

Since adding an edge between support vertices does not decrease the double domination number, we have the following result.

Observation 2.7 *If G is $\gamma_{\times 2}$-critical, then the set of support vertices of G induces a complete graph.*

3 Double domination Critical Trees and Cycles

Often when it is difficult to characterize graphs with particular parameters, it is helpful to restrict one's attention to trees. It has been found that no tree is domination or total domination edge critical (see [15, 10]). However there are double domination edge critical trees. First we must define a *star* and *double star*. A *star* is a tree with exactly one vertex that is not a leaf. Consider for example the star $K_{1,4}$ in Figure 12, where $\gamma_{\times 2}(K_{1,4}) = 5$ while $\gamma_{\times 2}(K_{1,4} + e) = 4$ for any edge $e \notin E(K_{1,4})$. In general, for the star $K_{1,m}$, where $\gamma_{\times 2}(K_{1,m}) = m + 1$, then $\gamma_{\times 2}(K_{1,m} + e) = m$ for any edge $e \notin E(K_{1,m})$. A *double star* is a tree with exactly two vertices that are not leaves, as shown if Figure 13. Using this information we characterize the double domination edge critical trees.

Proposition 3.1 *A tree T is double domination edge critical if and only if T is a star or a double star.*

Proof. Clearly, stars and double stars of order n have $\gamma_{\times 2}(G) = n$ and are $\gamma_{\times 2}$-critical. For the converse, let T be a $\gamma_{\times 2}$-critical tree. By Observation 2.7, every pair

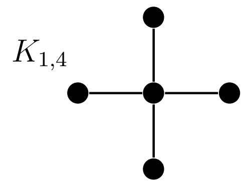


Figure 12: The Star $K_{1,4}$

of support vertices of T must be adjacent. Since T is a tree, it follows that T has at most two support vertices implying our result. $\square$

To illustrate, consider the double star shown below where the darkened vertices represent a double dominating set.

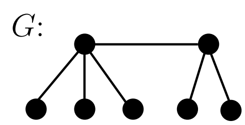


Figure 13: Double Star Example

Let $G' = G + uv$ for any $uv \in E(\overline{G})$ and again the darkened vertices represent a double dominating set. Note that any edge added will decrease $\gamma_{\times 2}(G)$ by one, as seen in Figure 14.

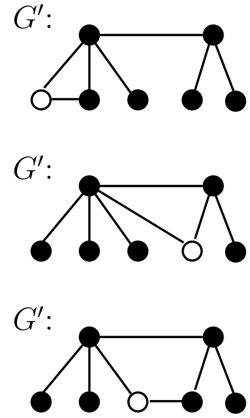


Figure 14: Examples of a Critical Edge Added to a Double Star

Now we will characterize double domination edge critical cycles in the next Proposition.

Proposition 3.2 *A cycle C_n is $\gamma_{\times 2}$-critical iff $n \in \{3, 4, 5\}$.*

Proof. The cycle C_3 is vacuously $\gamma_{\times 2}$-critical. It is a simple exercise to show that C_4 and C_5 are $\gamma_{\times 2}$-critical, while C_6 is not. Let C_n be a cycle of order $n \geq 7$, and label the vertex set $V(C_n) = \{v_1, v_2, ..., v_n\}$, and assume C_n is $\gamma_{\times 2}$-critical. Consider $G = C_n + v_1v_5$, and let S be an arbitrary $\gamma_{\times 2}(G)$-set. By Observation 2.1, $S \cap \{v_1, v_5\} \neq \emptyset$. Assume without loss of generality that $v_1 \in S$. If $v_5 \in S$ and both v_1 and v_5 have neighbors in $S - \{v_1, v_5\}$, then S doubly dominates C_n implying that $|S| \geq \gamma_{\times 2}(C_n)$. Hence we may assume that at most one of v_1 and v_5 has a neighbor in $S - \{v_1, v_5\}$. Moreover to doubly dominate v_3, $|S \cap \{v_2, v_3, v_4\}| \geq 2$. Thus we may assume that $\{v_1, v_2, v_3, v_5\} \subseteq S$.

It follows that $(S - \{v_3\}) \cup \{v_4\}$ is a double dominating set of C_n, and so $|S| \geq \gamma_{\times 2}(C_n)$. If $v_5 \notin S$, then $\{v_3, v_4, v_6, v_7\} \subseteq S$ to doubly dominate v_4 and v_6. Also at least one of v_2 and v_n is in S. Hence S is a double dominating set of C_n and again $|S| \geq \gamma_{\times 2}(C_n)$. Thus v_1v_5 is not a critical edge and hence, C_n is not $\gamma_{\times 2}$-critical for $n \geq 6$. □

4 Critical Graphs with Small Double Domination Numbers

Our aim in this section is to characterize the $\gamma_{\times 2}$-critical graphs G with $\gamma_{\times 2}(G) = 3$ and graphs with $\gamma_{\times 2}(G) = 4$ with maximum diameter.

4.1 Characterization for Graphs G having$\gamma_{\times 2}(G) = 3$

We begin with a lemma determining the diameter of a 3-critical graph G.

Lemma 4.1 *If G is a 3-critical graph, then* diam(G) *=2.*

Proof. Obviously, diam$(G) \geq 2$. Assume to the contrary that diam$(G) \geq 3$. Let $u_0, u_1, u_2, \ldots, u_d$ be a diametrical path where $d = \text{diam}(G) \geq 3$. Partition the vertices, $V(G)$, into sets $V_0, V_1, V_2, \ldots, V_d$ where $u_0 \in V_0$ and $x \in V_i$ if $dist(u_0, x) = i$.

By Observation 2.1, we know that at least one of u_0 and u_2 are in any $\gamma_{\times 2}(G + u_0u_2)$-set S. If $S = \{u_0, u_2\}$, then V_3 is not double dominated by S. If $S = \{u_0, x\}$ and $x \neq u_2$, then $x \in V_1$ and again the vertices of V_3 are not double dominated. If $S = \{u_2, x\}$ and $x \neq u_0$, then $x \in V_2$ or $x \in V_3$ to double dominate V_3. But then u_0 is only dominated once, a contradiction. Therefore, diam$(G) \leq 2$. □

Since we know that diam$(G) = 2$ for any 3-critical graph G, we can choose a vertex, say u_0, such that u_0 is at distance 2 from some vertex u_2. Let u_0, u_1, u_2 be a diametrical path of the graph G. Then starting at u_0 we can partition the other vertices of G with respect to u_0. We will let $N(u_0) = V_1$ and $V - N[u_0] = V_2$. This partitioning of G is illustrated in Figure 15 below.

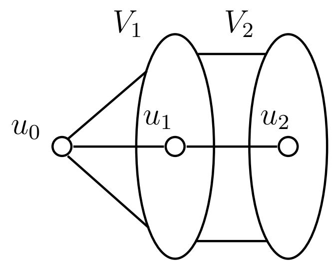


Figure 15: Partition of Graph G with $\text{diam}(G) = 2$

To denote that all edges are present between the vertices in a set A and the vertices in a set B, we say that $[A, B]$ is full.

Theorem 4.2 *A graph G is 3-critical if and only if one of the following holds*

(a) $\overline{G}$ is a set of independent edges or

(b) $\overline{G}$ is a nontrivial galaxy with exactly 1 isolated vertex

Proof. It is a straightforward exercise to show that a graph G as described in the theorem has $\gamma_{\times 2}(G) = 3$ and adding any edge reduces the double domination number to two. For the converse, assume that G is 3-critical. By Lemma 4.1, $\text{diam}\, G = 2$. Let u_0,u_1,u_2 be a diametrical path of G and partition the vertices of G as described previously and illustrated in Figure 15.

We first show that $\langle V_2 \rangle$ is complete. Suppose $\{u, v\} \subset V_2$ and $uv \in E(\overline{G})$. Let S be a $\gamma_{\times 2}(G + uv)$-set. Since by Observation 2.1, at least one of u and v is in S, it follows that S cannot double dominate u_0, a contradiction. Thus, $\langle V_2 \rangle$ is complete. If $|V_1| = 1$, then $\overline{G}$ is a galaxy with exactly one isolate and one nontrivial star, and condition (b) holds. Hence assume that $|V_1| \geq 2$. We consider two cases.

Case 1: $[V_1, V_2]$ is full.

We note that since $\gamma_{\times 2}(G) = 3$, $\langle V_1 \rangle$ is not complete. Let u and v be nonadjacent vertices in V_1, and let S be a $\gamma_{\times 2}(G + uv)$-set. Then $|S| = 2$, and at least one of u and v is in S. Without loss of generality, if $S = \{u, x\}$ where $x \neq v$, then $x \in V_1$ and x dominates G. Moreover, u dominates $G - v$ implying that u has degree one in $\overline{G}$. Since uv is an arbitrary missing edge from $\langle V_1 \rangle$, it follows that at least one endvertex of every edge in the subgraph induced by V_1 in $\overline{G}$ has degree one. In other words the subgraph induced by V_1 in $\overline{G}$ is a collection of stars. Since no adjacent pair double dominates G it follows that x is the only isolate in $\overline{G}$. Since $\{u_0\} \cup V_2$ induces a $K_{1,|V_2|}$ in $\overline{G}$, we conclude that $\overline{G}$ is a galaxy with exactly one isolate, and again condition (b) holds.

If $S = \{u, v\}$, then u dominates $G - v$ and v dominates $G - u$. If there is a vertex x in V_1 such that x dominates G, then $\{u, x\}$ or $\{v, x\}$ double dominates $G + uv$ and we have the previous case. Hence assume that no vertex of V_1 dominates G. We show in this case that $|V_2| = 1$. Suppose to the contrary that $|V_2| \geq 2$, and let S' be a $\gamma_{\times 2}(G + u_0 u_2)$-set. Then $|S'| = 2$ and by Observation 2.1 at least one of u_0 and u_2 is in S'. If $u_0 \in S'$, then $V_2 - \{u_2\}$ is not double dominated by S'. Thus $S' = \{u_2, z\}$, $z \in V_1$ and z dominates G, contradicting our assumption that no vertex dominates

G. Hence, $|V_2| = 1$. Thus in $\overline{G}$, V_1 induces a collection of independent edges. This implies that $\overline{G}$ is a set of independent edges, m K_2s, where $m \geq 2$, and u_0u_2 is an edge, and condition (a) holds.

Case 2: $[V_1, V_2]$ is not full.

We first show that the size of $|V_2| = 1$. Suppose to the contrary that $|V_2| \geq 2$. Let $uv \in E(\overline{G})$ where $u \in V_1$ and $v \in V_2$, and let S be a $\gamma_{\times 2}(G + u_0v)$-set. Then $|S| = 2$ and at least one of u_0 and v is in S. If $u_0 \in S$, then $V_2 - \{v\}$ is not double dominated, and if $v \in S$, then u is not double dominated. In either case, we have a contradiction. Hence, $|V_2| = 1$, that is, $V_2 = \{u_2\}$.

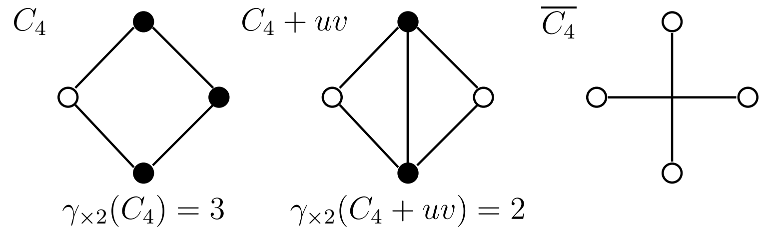
Partition the vertices of V_1 into sets A and B where $N(u_2) = A$. Observe that since $[V_1, V_2]$ is not full $B \neq \emptyset$. Moreover, $\langle B \rangle$ is complete for otherwise adding an edge between two vertices in B implies that at least one of these vertices is in any double dominating set of the resulting graph, and hence, u_2 is not double dominated.

Case 2(a): $[A, B]$ is full. If $|A| = 1$, then G is a complete graph with a pendant edge, that is, u_2 is a leaf. This implies that $\overline{G}$ is the union of a star and exactly one isolate satisfying condition (b). Hence we may assume $|A| \geq 2$. Since $\gamma_{\times 2}(G) = 3$, no adjacent pair of vertices in A double dominate G. In other words, at most one vertex in A dominates A. Let $b \in B$ and S be a $\gamma_{\times 2}(G + u_2b)$-set. By Observation 2.1, $|S \cap \{u_2, b\}| \geq 1$. If $u_2 \in S$, then u_0 is not double dominated, so $S = \{b, x\}$ and $x \in A$ (to double dominate u_2 and u_0). Hence there is exactly one vertex $x \in A$ that dominates G. Let u and v be a nonadjacent pair of vertices in A, and S' be a $\gamma_{\times 2}(G + uv)$-set. Then without loss of generality $S' = \{u, z\}$. Then u dominates $G - v$ and if $z = v$, v dominates $G - u$. In any case at least one of u and v dominates

$G - u$ or $G - v$ for each $uv \in E(\overline{G})$. Thus $\overline{G}$ is a galaxy with exactly one isolated vertex satisfying condition (b).

Case 2(b): $[A, B]$ is not full. Partition A into two sets A_1 and A_2 where $[A_2, B]$ is full. We show first that $A_2 \neq \emptyset$. Let S be a $\gamma_{\times 2}(G+u_0u_2)$-set. Then $|S \cap \{u_0, u_2\}| \geq 1$ from Observation 2.1. Note that $u_2 \notin S$ for otherwise B is not double dominated by S. Hence, $S = \{u_0, x\}$ and $x \in A_2$ because x dominates G. Moreover, as before, x is the only vertex in G that dominates G since $\gamma_{\times 2}(G) = 3$. Let $a \in A_1$ and $b \in B$ (note that a is not adjacent to b). Let S' be a $\gamma_{\times 2}(G+u_2b)$-set. Then Observation 2.1 implies at least one of u_2 and b is in S'. If $u_2 \in S'$ then u_0 is not double dominated. If $b \in S'$ then a is not double dominated, contradicting our assumption that G is 3-critical, so $[A, B]$ must be full. Thus the theorem is proven. □

Now we present examples of the type of graph described in Theorem 4.2. The simplest example of a $\gamma_{\times 2}$-critical graph with $\gamma_{\times 2} = 3$ such that $\overline{G}$ is a set of independent edges, condition (a) in Theorem 4.2, is C_4, shown in Figure 16. We also present an example of adding an edge $uv \in E(\overline{G})$ and show the complement of C_4 in Figure 16. We also show an example of a $\gamma_{\times 2}$-critical graph with $\gamma_{\times 2} = 3$ that meets condition (b) in Theorem 4.2. Again we let the darkened vertices represent a double dominating set.



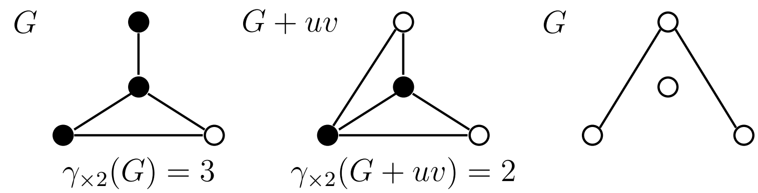


Figure 16: Example of a Double Domination Edge Critical Graph with $\gamma_{\times 2} = 3$

4.2 Characterization for Graphs G having $\gamma_{\times 2} = 4$ with Maximum Diameter

As before, let $u_0, u_1, u_2, ..., u_d$ be a diametrical path where $d = \text{diam}(G)$. Partition the vertices, $V(G)$, into sets $V_0, V_1, V_2, ..., V_d$ where $u_0 \in V_0$ and $x \in V_i$ if $\text{dist}(u_0, x) = i$.

Lemma 4.3 *If $G \neq K_n$ is a $\gamma_{\times 2}$-critical graph with $\gamma_{\times 2}(G) = 4$, then* $\text{diam}(G) \in \{2, 3\}$.

Proof. Obviously $\text{diam}(G) \geq 2$. Assume to the contrary that $\text{diam}(G) \geq 4$.

By Observation 2.1, we know that at least one of u_0 and u_4 are in any $\gamma_{\times 2}(G + u_0u_4)$-set S. First assume that $S = \{u_0, u_4, x\}$. Then $x \in V_1$ or $x \in V_3 \cup V_4$. In both cases, V_2 is not double dominated by S. Thus, if $S = \{u_0, x, y\}$, then $\{x, y\} \subseteq V_1 \cup V_2$, so V_4 is not double dominated by S. Finally, if $S = \{u_4, x, y\}$, then $\{x, y\} \subseteq V_2 \cup V_3$ and in this case u_0 is not double dominated. Thus, $\text{diam}(G) \leq 3$. □The *sequential join*, as defined by Akiyama and Harary, for three or more disjoint graphs $G_1, G_2, ..., G_n$, denoted as $G_1 + G_2 + ... + G_n$, is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup ... \cup (G_{(n-1)} + G_n)$. We use the definition of a sequential join in order to characterize the $\gamma_{\times 2}$-critical graphs with $\gamma_{\times 2}(G) = 4$ with maximum diameter, that is $\text{diam}(G) = 3$.

Theorem 4.4 *A graph G with $diam(G) = 3$ and $\gamma_{\times 2}(G) = 4$ is $\gamma_{\times 2}$-critical if and only if G is the sequential join $K_1 + K_s + K_t + K_1$ for positive integers s and t.*

Proof. It is a straightforward exercise to show that a graph G as described in the theorem has $\gamma_{\times 2}(G) = 4$ and adding any edge reduces the double domination number. For the converse, assume that $\gamma_{\times 2}(G) = 4$, $\text{diam}(G) = 3$, and that G is $\gamma_{\times 2}$-critical. Let u_0, u_1, u_2, u_3 be a diametrical path of G and partition the vertices of G as follows: $\{\{u_0\}, V_1, V_2, V_3\}$ where the vertices of V_i are at distance i from u_0.

We first show that $|V_3| = 1$.

Suppose to the contrary that $|V_3| \geq 2$. Let S be a $\gamma_{\times 2}(G + u_1u_3)$-set. Then $|S| \leq 3$ and by Observation 2.1, at least one of u_1 and u_3 is in S. Moreover, at least two vertices in S are in $u_0 \cup V_1$ to double dominate u_0. But then $V_3 - \{u_3\}$ is not double dominated by S, a contradiction. Hence, $|V_3| = 1$.

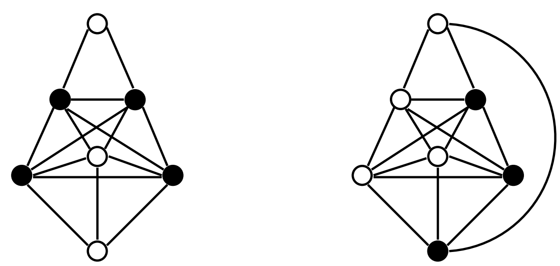
We now show that $N(u_3) = V_2$.

Suppose to the contrary that $v \in V_2$ and $vu_3 \in E(\bar{G})$, and let S be a $\gamma_{\times 2}(G + vu_3)$-set. To double dominate u_0, $|S \cap (\{u_0\} \cup V_1)| \geq 2$ and to double dominate u_3, $|S \cap (\{u_3\} \cup V_2)| \geq 2$, contradicting the fact that $|S| \leq 3$. Hence $N(u_3) = V_2$.

Next we show that the only missing edges in G are incident to u_0 or u_3.

Suppose $uv \in E(\overline{G})$ and neither u nor v is in $\{u_0, u_3\}$. Consider a $\gamma_{\times 2}(G+uv)$-set S. Then to double dominate u_0 and u_3, $|S \cap (\{u_0\} \cup V_1)| \geq 2$ and $|S \cap (\{u_3\} \cup V_2)| \geq 2$, so $|S| \geq 4$ contradicting that G is $\gamma_{\times 2}$-critical. Hence, $\langle V_1 \rangle$ is complete, $\langle V_2 \rangle$ is complete, and $[V_1, V_2]$ is full.□

Consider the graph of $K_1 + K_2 + K_3 + K_1$ in Figure 17 as an example of a $\gamma_{\times 2}$-critical graph with $\gamma_{\times 2}(G) = 4$ and a diameter of 3. Again let the darkened vertices represent a $\gamma_{\times 2}$ set.



$\gamma_{\times 2}(G) = 4$ $\gamma_{\times 2}(G+uv) = 3$

Figure 17: Example of a Double Domination Edge Critical Graph with $\gamma_{\times 2} = 4$ with diam$(G) = 3$

We conclude this thesis with a comment on a direction for future work. To complete the characterization of $\gamma_{\times 2}$-critical graphs with $\gamma_{\times 2}(G) = 4$, we are working on characterizing such graphs G with minimum diameter (i.e., diam$(G) = 2$). The $\gamma_{\times 2}$-critical graphs G with $\gamma_{\times 2}(G) \geq 5$ have not been studied.

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