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Strengthening the Precalculus Bridge: Enhancing the Precalculus Student's Understanding of Tangents Co conics, Biquadratic Equations, and Maxima and Minima.

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by

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ABSTRACT

Strengthening the Precalculus Bridge:

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Many students face tremendous difficulty in high school and/or college level calculus courses. The author hopes that by introducing students to the following nontraditional three topics prior to calculus, students' understanding of calculus will be enhanced. This thesis focuses on the following topics:

- Tangent Lines to Conics
- Maxima and Minima
- Biquadratic Equations

Because these topics are not generally covered in precalculus courses, there are several possible uses for them. An instructor could use the material as:

- An added classroom resource,
- Project assignments for outside classroom study,
- A student resource for precalculus advanced studies, or

- An independent study.

This thesis assumes that the student is well prepared for the precalculus course by having a good understanding of foundational algebra skills.

DEDICATION

This thesis is dedicated to my Lord, without his love and direction I would have never reached this level in my education.

This thesis is also dedicated to my children, Brad, and Emily, my daughter-in-law, Ashley; my grandchildren, Kaitlyn Faith, Chloe Elizabeth, Karlie Grace, and Parker Owen. My desire is that each one encounter God's hand in their lives as I have, and that they would live life believing that with GOD nothing is impossible. And finally, to my mother, Barbara Crusenberry, who in my early years taught me to search out God's best for my life and has been my number one cheerleader.

What a blessing it has been to again experience the pure joy and energy of the truth of my life verse, "He [Abraham] staggered not at the promise of God through unbelief, but was strong in faith, giving glory to God." Romans 4:20 (KJV) Because it was a challenge throughout this graduate experience to live by faith and not by fear.

ACKNOWLEDGMENTS

A special thank you to my committee chair, Dr. Michel Helfgott, whose unfailing patience and knowledge laid the foundation of success for this thesis; and to my thesis committee, Dr. Robert Gardner, the LaTeXmaster, and Spiderman and Batman's greatest fan, and to Dr. Daryl Stephens whose encouragement and direction always came at the right moment. And to all the many friends and family who took the time to pray for my success in this endeavor. God bless you.

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1 INTRODUCTION

In this thesis the author views precalculus in the same context as author Patrick Driscoll, in that precalculus can be viewed as a "bridge between secondary and post-secondary mathematics" [5]. Driscoll states that precalculus concepts are necessary because some of the concepts taught at the precalculus level may not have been covered in previous math courses [5]. Hence the importance of a "bridge" class referred to as precalculus. It is the author's desire to strengthen that bridge for students preparing for calculus. Author, physicist, and mathematician Freeman Dyson adds this perspective.

> The bottom line for mathematicians is that the architecture has to be right. In all the mathematics that I did, the essential point was to find the right architecture. It's like building a bridge. Once the main lines of the structure are right, then the details miraculously fit ... Freeman Dyson [1]

The beauty and strength of a bridge, as well as mathematics, is revealed with the right architecture [1]. (See Figure 1)

The topics addressed in this thesis are tangent lines involving conics, biquadratic equations, and maxima and minima. While in the classroom setting some attention is placed on finding the equation of the tangent line to circles, this thesis will focus primarily on tangent line equations to parabolas, ellipses, and hyperbolas. Biquadratic equations are a special type of quartic equation.

Figure 1: An Example Of Beauty And Strength When The Architecture Is Right

Its analysis will assist students in applying concepts normally covered in precalculus texts such as completing the square and complex numbers. Finally, the author will focus on maxima and minima using an alternative method to solve these problems without using calculus.

It is the hope of the author that by introducing precalculus students to these topics, not only will their understanding be enhanced, but it will also widen their mathematical horizon. This, in turn, should strengthen their precalculus bridge and improve their comprehension of calculus.

2 TANGENT LINES TO CONICS

2.1 Introduction

Menaechmus, (c. 375–325 BC), a Greek tutor to Alexander the Great, is credited with the discovery of conics [13]. Appollonius (c. 262–190 BC), the great geometer, compiled the information regarding conics into eight volumes [13]. “His work *Conics* was the first to show how all three curves, along with the circle, could be obtained by slicing the same right circular cone at continuously varying angles” [17]. Appollonius is also credited with naming the conic sections: ellipse, parabola and hyperbola [13].

A conic section (or conic) is defined as the intersection of a plane with a double-napped cone [12]. Depending on the angle of the plane to the vertex of the cone the resulting shape is described as a circle, ellipse, hyperbola, or a parabola as shown in Figure 2. The reader should note that with the basic conic sections the plane does not pass through the vertex of the cone [12].

While the Greeks are credited with the discovery of conics, it was not until the 17th century that practical applications were unveiled Johannes Kepler discovered that planetary motion follows an elliptical path [17].

Many other applications have been discovered since Kepler’s time. Here are just a few applications involving conic sections: “solar ovens use parabolic mirrors to converge light beams to use for heating…the parabola is used in the design of car headlights and in spotlights because it aids in concentrating the light beam…hyperbolas are used in a navigation system know as LORAN (long range navigation),” as well as “hyperbolics as parabolic mirrors and lenses used in systems of telescopes” [19].

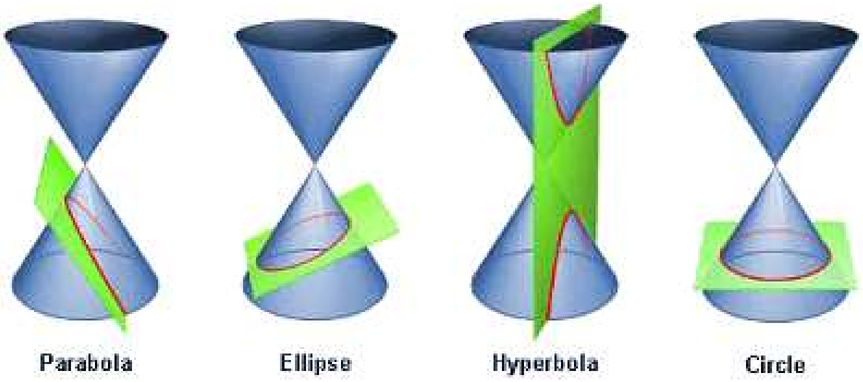

Figure 2: Basic Conics [17]

In this thesis, the author will focus the attention on tangent lines to conics. While the subject of tangent lines to circles is commonly discussed in the classroom setting, the author will focus specifically on tangent lines in relation to the parabola, ellipse, and hyperbola using elementary algebra. This approach should give precalculus students a good introduction to conics, and broaden students' mathematical horizons while enhancing their knowledge of conics prior to calculus.

The author will use a modification of a method first used by René Descartes (1596–1650) [2]. The modification focuses on lines instead of circles and is well suited to working with conics [2]. If the reader wants to further investigate Descartes original method, see the article by Baloglou and Helfgott [2]. Let us begin with the simplest conic, the parabola.

2.2 Parabolas

The standard form of the parabola equation with a vertical axis is

$$y - k = 4p(x - h)^2$$

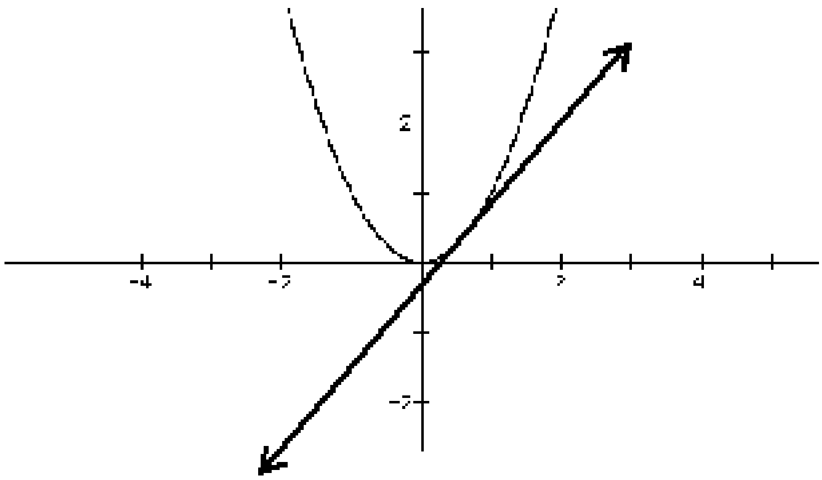

Figure 3: Tangent Line To The Parabola $y = x^2$

with a vertex (h, k) where the focus lies p units from the vertex; the basic parabola equation is $y = x^2$ [11].

Example 1: Find the equation of the tangent line $y = x^2$ at an arbitrary point (a, a^2) (See Figure 3):

We will begin with the standard equation of a line, $y - y_1 = m(x - x_1)$ our known point (a, a^2) and the unknown slope, m:

$$y - a^2 = m(x - a).$$

Since we know that $y = x^2$, we can substitute x^2 for y in the equation of a line:

$$\begin{aligned} x^2 - a^2 &= m(x - a) \\ x^2 - a^2 &= mx - ma \\ 0 &= x^2 - mx + (ma - a^2) \text{ Written in standard form.} \end{aligned}$$

The quadratic equation we just formed must have only one solution in order for the line to meet the parabola at just one point; therefore, the discriminant, Δ, must equal zero:

$$\Delta = b^2 - 4ac = 0 \text{ where: } a = 1, b = m, c = (ma - a^2)$$

$$\begin{aligned} \Delta &= m^2 - 4(1)(ma - a^2) \\ 0 &= m^2 - 4(ma - a^2) \\ 0 &= (m - 2a)^2 \\ 2a &= m. \end{aligned}$$

Now substitute $m = 2a$ into the standard equation of the line with our known point (a, a^2) and solve for the equation of the line $y = mx + b$:

$$\begin{aligned} y - a^2 &= 2a(x - a) \\ y &= a^2 + 2ax - 2a^2 \\ y &= 2ax - a^2. \end{aligned}$$

So the equation of the tangent line at the point (a, a^2) is: $y = 2ax - a^2$.

Now that we have found the equation of the tangent line $y = x^2$ at an arbitrary point, let us look at a different quadratic equation with a specific point.

Example 2: Find the equation of the tangent line to $y = 3x^2 + x + 1$ at the point $(0, 1)$. (See Figure 4)

Begin with the standard equation of a line that passes through the point $(0, 1)$:

$$\begin{aligned} y - 1 &= m(x - 0) \\ y &= mx + 1. \end{aligned}$$

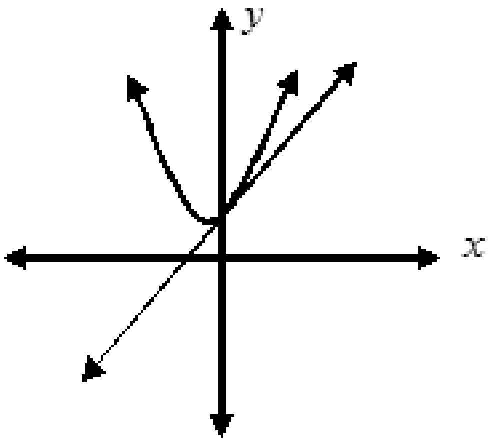

Figure 4: Tangent Line To The Parabola $y = 3x^2 + x + 1$

Since each equation is now solved for y we can set the equations equal to one another and solve:

$$\begin{aligned} 3x^2 + x + 1 &= mx + 1 \\ 3x^2 + x - mx &= 0 \\ 3x^2 + (1-m)x &= 0. \end{aligned}$$

We know that the newly formed quadratic equation must have only one solution so we will set the discriminant $\Delta = 0$ and solve for m. (Recall that the tangent line and the parabola can have only one point in common, something that can happen only if $\Delta = 0$.)

$$\Delta = b^2 - 4ac = 0 \text{ where: } a = 3, b = 1 - m, c = 0$$

$$\begin{aligned}\Delta &= (1-m)^2 - 4(3)(0) = 0\\ &= m^2 - 2m + 1\\ &= (m-1)^2\\ 1 &= m.\end{aligned}$$

Replace $m = 1$ in the equation of the line. Thus $y = x + 1$, which is the equation for the tangent line of the parabola $y = 3x^2 + x + 1$ at $(0, 1)$.

2.3 Ellipses

The standard form of an ellipse with a horizontal major axis, where $a > b$ and with the center (h, k) [11] is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Example 3: Find the equation of the tangent line of the ellipse $\dfrac{x^2}{2} + y^2 = 1$ at the point $\left(1, \dfrac{1}{\sqrt{2}}\right)$. (See Figure 5)

Begin with the standard equation of a line that passes through the point $\left(1, \dfrac{1}{\sqrt{2}}\right)$:

$$\begin{aligned}y - \frac{1}{\sqrt{2}} &= m(x-1)\\ y &= mx - m + \frac{1}{\sqrt{2}}\\ y &= mx + \left(\frac{1}{\sqrt{2}} - m\right).\end{aligned}$$

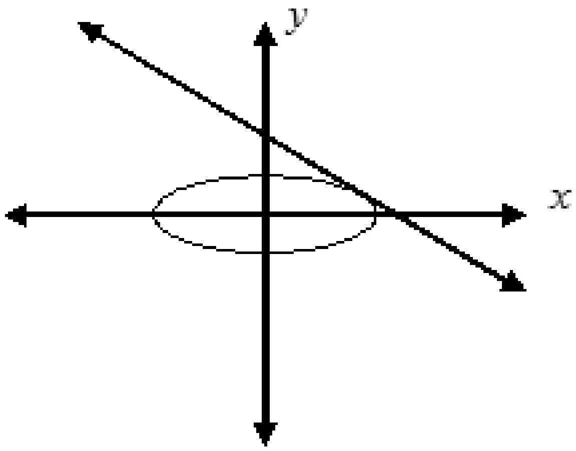

Figure 5: Tangent Line To The Ellipse $\dfrac{x^2}{2} + y^2 = 1$

This time we will insert the above equation into the ellipse equation substituting for y:

$$\begin{aligned} \frac{x^2}{2} + \Big[mx + \Big(\frac{1}{\sqrt{2}} - m\Big)\Big]^2 &= 1 \\ \Big(\frac{1}{2} + m^2\Big)x^2 + 2m\Big(\frac{1}{\sqrt{2}} - m\Big)x + \Big(\frac{1}{\sqrt{2}} - m\Big)^2 &= 1 \\ \Big(\frac{1}{2} + m^2\Big)x^2 + 2m\Big(\frac{1}{\sqrt{2}} - m\Big)x + \Big(m^2 - \frac{2}{\sqrt{2}}m - \frac{1}{2}\Big) &= 0. \end{aligned}$$

Since the quadratic equation can have at most one solution, the discriminant, Δ, must equal zero:

$$\Delta = b^2 - 4ac = 0 \text{ where: } a = \frac{1}{2} + m^2, b = 2m\Big(\frac{1}{\sqrt{2}} - m\Big), c = \Big(m^2 - \frac{2}{\sqrt{2}}\Big)m - \frac{1}{2}$$

$$\Delta = \Big(2m\Big(\frac{1}{\sqrt{2}} - m\Big)\Big)^2 - 4\Big(\frac{1}{2} + m^2\Big)\Big(m^2 - \frac{2}{\sqrt{2}}m - \frac{1}{2}\Big) = 0.$$

Which simplifies to:

$$2m^2 + \Big(\frac{4}{\sqrt{2}}\Big)m + 1 = 0$$

$$m = \frac{\frac{-4}{\sqrt{2}} \pm \sqrt{8-8}}{4}.$$

Thus,

$$m = -\frac{1}{\sqrt{2}}.$$

Finally, substitute $m = -\frac{1}{\sqrt{2}}$ into the standard line of the equation:

$$\begin{aligned}
y &= mx + \Big(\frac{1}{\sqrt{2}} - m\Big) \\
y &= \Big(-\frac{1}{\sqrt{2}}\Big)x + \frac{1}{\sqrt{2}} - \Big(-\frac{1}{\sqrt{2}}\Big) \\
y &= \Big(-\frac{1}{\sqrt{2}}\Big)x + \frac{2}{\sqrt{2}}.
\end{aligned}$$

So the equation of the tangent line at the point $\Big(1, \frac{1}{\sqrt{2}}\Big)$ is:

$$y = -\frac{1}{\sqrt{2}}x + \frac{2}{\sqrt{2}}.$$

If we were to apply calculus (implicit differentiation) to solve this problem, we would observe the following:

$$\frac{x^2}{2} + y^2 = 1 \text{ original equation}$$

$$x + 2y\frac{dy}{dx} = 0 \text{ implicit differentiation}$$

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

In particular we see

$$\frac{dy}{dx}\bigg|_{\left(1,\frac{1}{\sqrt{2}}\right)} = -\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)} = -\frac{1}{\sqrt{2}}.$$

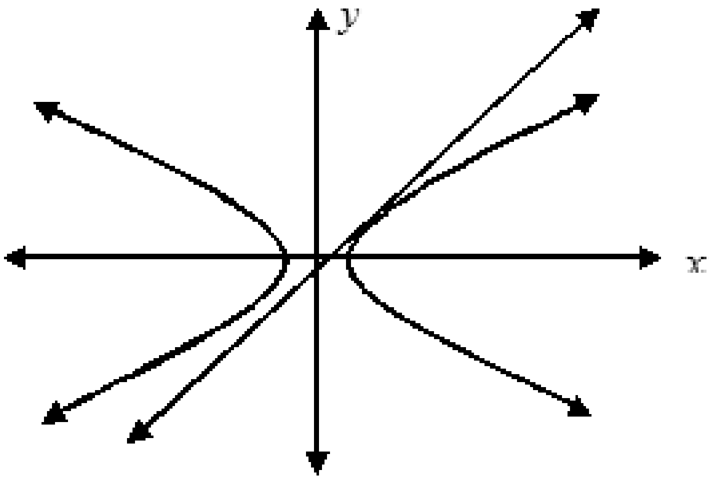

Figure 6: Tangent Line To The Hyperbola At $x^2 - y^2 = 1$

2.4 Hyperbolas

The reader will notice immediately that the standard equation of the hyperbola with a horizontal transverse axis is the same as the ellipse with the exception of the negative sign between the terms. The standard form of the hyperbola with the center at (h, k) is [11]:

$$\frac{(x-h)^2}{a^2} - \frac{(x-k)^2}{b^2} = 1.$$

Example 4: Find the equation of the tangent line to the hyperbola $x^2 - y^2 = 1$ at the point $(3, \sqrt{8})$. (See Figure 6)

Begin by substituting the point $(3, \sqrt{8})$ into the equation of the line:

$$y - \sqrt{8} = m(x-3)$$

$$y = mx + \left(\sqrt{8} - 3m\right).$$

Substitute the new equation for the y in the hyperbola equation:

$$\begin{aligned} x^2 - [mx + (\sqrt{8} - 3m)]^2 &= 1 \\ (1 - m^2)x^2 - 2m(\sqrt{8} - 3m)x - (\sqrt{8} - 3m)^2 &= 1 \\ (1 - m^2)x^2 - 2m(\sqrt{8} - 3m)x + (6\sqrt{8} - 9m^2 - 9) &= 0. \end{aligned}$$

Since the quadratic equation must have at most one solution, the discriminant, Δ, must equal zero:

$$\Delta = b^2 - 4ac = 0 \text{ where: } a = 1 - m^2, b = -2m(\sqrt{8} - 3m), c = (6\sqrt{8} - 9m^2 - 9)$$

$$\Delta = [2m(\sqrt{8} - 3m)]^2 - 4(1 - m^2)(6\sqrt{8} - 9m^2 - 9) = 0,$$

which simplifies to

$$32m^2 - 24\sqrt{8}m + 36 = 0$$

$$\text{i.e. } 8m^2 - 6\sqrt{8}m + 9 = 0.$$

Thus,

$$m = \frac{6\sqrt{8} \pm \sqrt{288 - 288}}{16}.$$

So,

$$m = \frac{3}{\sqrt{8}}.$$

Now substitute $m = \dfrac{3}{\sqrt{8}}$ into the equation:

$$\begin{aligned} y &= mx + (\sqrt{8} - 3m) \\ y &= \left(\frac{3}{\sqrt{8}}\right)x + \sqrt{8} - 3\left(\frac{3}{\sqrt{8}}\right) \\ y &= \frac{3}{\sqrt{8}}x - \frac{1}{\sqrt{8}}. \end{aligned}$$

So the equation of the tangent line at the point $(3, \sqrt{8})$ is: $y = \frac{3}{\sqrt{8}}x - \frac{1}{\sqrt{8}}$.

If we were to apply calculus (implicit differentiation) to solve this problem, we would observe the following:

$$\frac{x^2}{2} - y^2 = 1 \text{ original equation}$$
$$2x - 2y\frac{dy}{dx} = 0 \text{ implicit differentiation}$$
$$\frac{dy}{dx} = \frac{x}{y}.$$

In particular we see

$$\left.\frac{dy}{dx}\right|_{(3,\sqrt{8})} = \frac{3}{\sqrt{8}}.$$

2.5 Circles

The standard form of the equation of a circle with radius r and center (h, k) is [11]:

$$(x - h)^2 + (y - k)^2 = r^2.$$

Students are usually exposed to circles and tangent lines while at the precalculus level. There are various ways to explore tangent lines to circles; if interested, the reader is encouraged to review the article by G. Baloglou and M. Helfgott [2], wherein this and similar problems related to conics are discussed in detail.

Clearly, precalculus students can find the equation of the tangent line to conic sections using elementary algebra. Although the algebraic computations can become long and cumbersome, examining conics at the precalculus level will give the student additional experience in algebraic manipulations as well as applying the equation of a line formula. Knowing the difficulty many students face with algebra while learning calculus, this type of exercise is all the more important in solidifying their algebra skills. Although the precalculus student may not understand the two examples of calculus used in the ellipse and hyperbola sections to confirm the answer found algebraically, he/she can certainly appreciate the power generated by the calculus engine noticing the few steps required in solving the problem as compared to the algebraic solution.

3 BIQUADRATIC EQUATIONS

3.1 Introduction

Biquadratic equations are found within the family of quartic equations, which are fourth-degree polynomials. Quartic Equations were studied as early as 2nd century BC by Jaina mathematicians [15]. Gerolamo Cardano (1501–1576), an Italian renaissance mathematician, is famous for publishing the solutions to quartic equations in his 1545 book, *Ars Magna*, (The Great Art)[6]. While Cardano is famous for publishing the solutions, he gave credit to his disciple, Ludovico Ferrari (1522–1565), for solving quartic equations using the same rules Cardano used in solving cubic equations [3]. The general form of the quartic equation and the general form of the biquadratic equation are as follows:

Quartic: $x^4 + ax^3 + bx^2 + cx + d = 0$

Biquadratic: $x^4 + ax^2 + b = 0$

Note that in the biquadratic equation, both x^3 and x terms have been eliminated. The solutions for biquadratic equations, like its parent quartic equation, always results in four solutions. Some of the solutions may be complex numbers and some may have multiplicity greater than one. It is to be noted that we will deal only with equations whose coefficients are real numbers.

There are some special considerations to note regarding the solutions or roots of biquadratic equations: If the solution is a complex number, then the complex conjugate will also be a solution. If the solution or root is an integer, then it must be a divisor of the constant term within the biquadratic equation. Finally, if r is a root of a biquadratic, then $-r$ is a root as well. These properties, except the last one, are shared by all polynomials of n^{th} degree with real coefficients. In this section, the author will use complex numbers as well as the method of completing the square to solve biquadratic equations. It is assumed that the precalculus student has covered this information prior to attempting to understand how to manipulate biquadratic equations.

We will investigate several ways of solving biquadratic equations.

- Factorization method
- Transformation method
- Transformation method and Euler's formula
- Perfect Squares method.

3.2 Solving Biquadratics

3.2.1 Factorization Method

Factorization can be used for very simple biquadratics.

Example 1: Solve by factorization:

$$x^4 + 2x^2 = 24$$

Well,

$$x^4 + 2x^2 - 24 = 0$$

$$(x^2 - 4)(x^2 + 6) = 0$$

$$(x-2)(x+2)(x^2+6)=0$$

Example 2: Solve by factorization:

$$x^4+11x^2=0$$

So,

$$x^2(x^2+11)=0$$

$$x^2=0 \text{ and } x^2+11=0$$

3.2.2 Transformation Method

Example 3: Solve by transformation:

Using the transformation method, we will convert the biquadratic to a quadratic equation. Let us begin by looking at one of the simplest quartic equations, namely $x^4 = 1$, using the transformation defined as $y = x^2$:

$$x^4 = 1 \text{ transformed becomes } y^2 = 1$$

$$\begin{aligned} y^2 &= 1 \\ y &= \pm\sqrt{1} \\ y &= \pm 1. \end{aligned}$$

Thus the problem redefined ($y = x^2$) is reduced to the following two equations:

$$x^2 = 1 \text{ and } x^2 = -1.$$

By taking the square root of both sides the resultant roots are 1, -1, $i, -i$, which gives us the four solutions to the original problem. Referencing the special considerations stated previously, note that the integer solutions, 1 and -1 are divisors of the constant 1 in the original equation.

Example 4: Solve the biquadratic equation by using the transformation $y = x^2$:

$$x^4 + 42 = 13x^2.$$

Well,

$$\begin{aligned} x^4 - 13x^2 + 42 &= 0 \\ y^2 - 13y + 42 &= 0 \qquad \text{transformed} \\ (y-7)(y-6) &= 0. \end{aligned}$$

By factoring, we obtain $y = 7$ and $y = 6$. Now redefine the solution as in example 3 and the problem is reduced to the following two equations:

$$x^2 = 7 \text{ and } x^2 = 6.$$

The resultant roots are $\sqrt{7}, -\sqrt{7}, \sqrt{6}, -\sqrt{6}$.

Example 5: Solve the biquadratic equation by using the transformation $y = x^2$:

$$x^4 - 6x^2 - 3 = 0.$$

Therefore,

$$y^2 - 6y - 3 = 0 \qquad \text{transformed.}$$

Since this polynomial is not factorable, we must use the quadratic formula:

$$\begin{aligned} y &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-3)}}{2(1)} \\ &= \frac{6 \pm \sqrt{48}}{2} \\ &= \frac{6 \pm 4\sqrt{3}}{2} \\ &= 3 \pm 2\sqrt{3}. \end{aligned}$$

Now redefine the problem in terms of x^2:

$$\begin{aligned} x^2 &= 3 + 2\sqrt{3} \\ \sqrt{x^2} &= \sqrt{3 + 2\sqrt{3}} \\ x &= \pm\sqrt{3 + 2\sqrt{3}} \end{aligned}$$

and

$$\begin{aligned} x^2 &= 3 - 2\sqrt{3} \\ x^2 &= (2\sqrt{3} - 3)(-1) \\ \sqrt{x^2} &= \sqrt{(2\sqrt{3} - 3)(-1)} \\ x &= \pm i\sqrt{2\sqrt{3} - 3}. \end{aligned}$$

With the resultant roots:

$$x = \sqrt{3 + 2\sqrt{3}}, -\sqrt{3 + 2\sqrt{3}}, i\sqrt{3 - 2\sqrt{3}}, -i\sqrt{3 - 2\sqrt{3}}.$$

3.2.3 Transformation Method and Euler's Formula

Example 6: Solve the biquadratic equation by using the transformation $y = x^2$:

$$x^4 + x^2 + 1 = 0.$$

Well,

$$y^2 + y + 1 = 0 \quad \text{transformed.}$$

Since this polynomial is not factorable, we must use the quadratic formula:

$$\begin{aligned} y &= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1-4}}{2} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \\ &= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \end{aligned}$$

Redefine $(y = x^2)$:

$$x^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad \text{and} \qquad x^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Solving for x we find the resultant roots, which involve the square root of a complex number. This situation requires us to use Euler's formula. This formula was developed in the 18th century, by the Swiss mathematician Leonhard Euler (1707–1783) [16]: The n roots of $x^n = d$ are given by

$$x_k = |d|^{\frac{1}{n}} \left(\cos\left(\frac{360^\circ + \theta}{n}\right) + i \sin\left(\frac{360^\circ + \theta}{n}\right)\right) k = 0, 1, \ldots, n-1,$$

where θ is the argument of the complex number $d = a + ib$. We choose to work with the complex number $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$. We could have easily chosen the other solution of $y^2 + y + 1 = 0$ and come up with the same results. Before applying Euler's formula, we must rewrite $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ in polar form, (i.e., find $|d|$ and θ). The symbol $|d|$ in Euler's formula represents the distance from the origin $(0,0)$ to the point (a,b). Thus $|d| = \sqrt{a^2+b^2}$, also called the absolute value of the complex number $d = a + ib$. When $d = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ we get

$$\begin{aligned} |d| &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \sqrt{\frac{1}{4} + \frac{3}{4}} \\ &= 1 \end{aligned}$$

As preliminary work let us write:

$$\tan \mu = \frac{b}{a}$$

Now solve for μ:

$$\begin{aligned} \tan \mu &= \left(\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) \\ \mu &= -\tan^{-1}(\sqrt{3}) \\ \mu &= 60^\circ \end{aligned}$$

The value μ shows up in quadrant I of the unit circle, but our point $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ lies in quadrant III. Thus we must add 180° to μ, resulting in $\theta = 180^\circ + 60^\circ = 240^\circ$. Since we are working with a quadratic equation, we are only interested in the case where $n = 2$. Now we are ready to use Euler's formula when $n = 2$ and $k = 0, 1$:

$$\begin{aligned} x_0 &= |1|^{\frac{1}{2}} \cos\left(\frac{360^\circ(0) + 240^\circ}{2}\right) + i \sin\left(\frac{360^\circ(0) + 240^\circ}{2}\right) \\ &= \cos\left(\frac{240^\circ}{2}\right) + i \sin\left(\frac{240^\circ}{2}\right) \\ &= \cos(120^\circ) + i \sin(120^\circ) \end{aligned}$$

Based on $\cos(120°) + i\sin(120°)$, the solution will be $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$\begin{aligned} x_1 &= |1|^{\frac{1}{2}} \cos\left(\frac{360°(1) + 240°}{2}\right) + i\sin\left(\frac{360°(1) + 240°}{2}\right) \\ &= \cos\left(\frac{600°}{2}\right) + i\sin\left(\frac{600°}{2}\right) \\ &= \cos(300°) + i\sin(300°) \end{aligned}$$

Based on $\cos(300°)+i\sin(300°)$, the solution this time will be $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Referencing the special considerations previously stated regarding complex conjugates, we have the following four solutions:

$$x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Now let's look at another way to solve biquadratics:

3.2.4 Perfect Squares Method

In order to understand the process, let us analyze the general equation for biquadratics. Suppose $x^4 + ax^2 + b = 0$. We assume that $b \neq 0$ because $x^4 + ax^2 = 0$ can be solved by the factoring process, namely: $x^2(x^2 + a) = 0$. We have two cases to consider:

CASE 1:

$\Delta = a^2 - 4b \geq 0$ Δ is the discriminant of the quadratic formula

$x^4 + ax^2 + b = 0$ General biquadratic equation

$x^4 + ax^2 = -b$ Rewrite equation

$$\left(x^2+\frac{a}{2}\right)^2=\frac{a^2}{4}-b \qquad \text{Complete the square}$$

$$\left(x^2+\frac{a}{2}\right)^2=\frac{a^2-4b}{4} \qquad \text{Find LCD on right sign of equal sign}$$

$$\left(x^2+\frac{a}{2}\right)^2=\left(\frac{\sqrt{a^2-4b}}{2}\right)^2 \qquad \text{Rewrite the right side as a squared term}$$

$$x^2+\frac{a}{2}=\pm\frac{\sqrt{a^2-4b}}{2} \qquad \text{Basic algebra fact }^1$$

CASE 2:

$\Delta = a^2 - 4b < 0$ Δ is the discriminant of the quadratic formula

First of all we must note that $a^2 < 4b$, so $a \leq |a| < 2\sqrt{b}$. Thus $2\sqrt{b} - a > 0$.

$$x^4+ax^2+b=0 \qquad \text{General biquadratic equation}$$

$$x^4+b=-ax^2 \qquad \text{Rewrite equation}$$

$$(x^2+\sqrt{b})^2=2\sqrt{b}x^2-ax^2 \qquad \text{Use rule } (a+b)^2=a^2+2ab+b^2$$

$$(x^2+\sqrt{b})^2=x^2(2\sqrt{b}-a) \qquad \text{Factor out the } x^2 \text{ term}$$

$$(x^2+\sqrt{b})^2=x^2(\sqrt{2\sqrt{b}-a})^2 \qquad \text{Rewrite } (2\sqrt{b}-a) \text{ as a squared term}$$

$$(x^2+\sqrt{b})^2=(x\sqrt{2\sqrt{b}-a})^2 \qquad \text{Combine two squared terms on right side}$$

$$x^2+\sqrt{b}=\pm x\sqrt{2\sqrt{b}-a} \qquad \text{Basic algebra fact}^1$$

The key to success in this procedure is to manipulate the equation such that you have a perfect square term that appears on both sides of the equation.

[1] Basic Algebra Fact: $u^2 = v^2$ implies $u = v$. Why? $u^2 = v^2$ is equivalent to $u^2 - v^2 = 0$, (i.e.(u + v)(u - v)= 0). Then $u + v = 0$ or $u - v = 0$. That is to say $u = -v$ or $u = v$

Example 7: Solve the biquadratic equation $x^4+4x^2+2=0$ using the Perfect Squares method. We will follow case 1 for this example since $\Delta = 4^2-4(2)>0$:

$$\begin{aligned} x^4+4x^2+2 &= 0 \\ x^4+4x^2 &= -2 \\ x^4+4x^2+4 &= 4-2 \\ (x^2+2)^2 &= 2 \\ (x^2+2)^2 &= (\sqrt{2})^2 \\ x^2+2 &= \pm\sqrt{2}. \end{aligned}$$

Now you are ready to set the two equations equal to zero and solve for x. The resultant roots are:

$$x = i\sqrt{2-\sqrt{2}}, \quad -i\sqrt{2-\sqrt{2}}, \quad i\sqrt{2+\sqrt{2}}, \quad -i\sqrt{2+\sqrt{2}}.$$

Example 8: Let's revisit the same biquadratic equation used in example 6 by means of the perfect squares method. We will follow case 2 for this example since $\Delta = 1^2-4<0$

$$\begin{aligned} x^4+x^2+1 &= 0 \\ x^4+1 &= -x^2 \\ (x^2+\sqrt{1})^2 &= 2\sqrt{1}x^2-x^2 \\ (x^2+1)^2 &= x^2(2-1) \\ (x^2+1)^2 &= x^2 \\ x^2+1 &= \pm x. \end{aligned}$$

Now set the two equations equal to zero and solve for x. Using the quadratic formula we have the same results as in example 6 with much less work involved:

$$x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

I think that the reader will agree that the Perfect Squares method is much simpler than using Euler's formula.

REMARK: The discriminant, Δ, explored in the previous two cases of the general equation, gives some very interesting insights into which method to use to solve biquadratic equations. If the discriminant, $\Delta = a^2 - 4b \geq 0$, then use the transformation method or perfect squares method case 1. Often the transformation method will be the simplest. If the discriminant, $\Delta = a^2 - 4b < 0$, then use the perfect squares method case 2, since using the transformation would involve applying Euler's formula.

Example 9: Solve the biquadratic equation $x^4 + 3x^2 + 1 = 0$. Begin by solving for Δ.

$$x^4 + 3x^2 + 1 = 0$$

Well,

$$\begin{aligned} \Delta &= a^2 - 4b \\ \Delta &= 9 - 4 \\ \Delta &= 5 \\ \Delta &> 0 \end{aligned}$$

Since $\Delta > 0$, we will compare the transformation method with the perfect squares method using case 1.

Transformation method:

$$\begin{aligned} x^4 + 3x^2 + 1 &= 0 \\ y^2 + 3y + 1 &= 0 \qquad \text{transformed} \end{aligned}$$

Since the transformed equation will not factor we must use the quadratic formula:

$$\begin{aligned} y &= \frac{\sqrt{(3)2 - 4(1)(1)}}{2(1)} \\ y &= \frac{-3 \pm \sqrt{5}}{2} \end{aligned}$$

It follows that

$$x^2 = \frac{-3+\sqrt{5}}{2} \qquad \text{or} \qquad x^2 = \frac{-3-\sqrt{5}}{2}$$

Hence,

$$x = \pm\sqrt{\frac{-3+\sqrt{5}}{2}} \qquad\qquad x = \pm\sqrt{\frac{(3+\sqrt{5})(-1)}{2}} = \pm i\sqrt{\frac{3+\sqrt{5}}{2}}$$

And we have the resultant four roots:

$$x = \sqrt{\frac{-3+\sqrt{5}}{2}}, \quad -\sqrt{\frac{-3+\sqrt{5}}{2}}, \quad i\sqrt{\frac{3+\sqrt{5}}{2}}, \quad -i\sqrt{\frac{3+\sqrt{5}}{2}}.$$

Now let's compare the same equation using the Perfect Squares method (case 1):

Well,

$$\begin{aligned} x^4 + 3x^2 + 1 &= 0 \\ x^4 + 3x^2 &= -1 \end{aligned}$$

$$\left(x^2 + \frac{3}{2}\right)^2 = \frac{9}{4} - 1$$
$$\left(x^2 + \frac{3}{2}\right)^2 = \frac{5}{4}$$
$$\left(x^2 + \frac{3}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2$$
$$x^2 + \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$$

Application of Perfect Squares method results in the following two equations:

$$x^2 = \frac{-3 + \sqrt{5}}{2} \qquad \text{or} \qquad x^2 = \frac{-3 - \sqrt{5}}{2}$$

We have the same resultant roots as with the transformation method:

$$x = \sqrt{\frac{-3 + \sqrt{5}}{2}},\ -\sqrt{\frac{-3 + \sqrt{5}}{2}},\ i\sqrt{\frac{3 + \sqrt{5}}{2}},\ -i\sqrt{\frac{3 + \sqrt{5}}{2}}$$

Which method do you prefer? It seems to be a matter of whether you prefer working with the quadratic formula as in the transformation method or completing the square as in the perfect squares method. When $\Delta > 0$, the choice is up to you; the author prefers the transformation method.

Example 10: Solve the biquadratic equation $x^4 + 2x^2 + 4 = 0$. Begin by solving for Δ.

$$x^4 + 2x^2 + 4 = 0$$

Well,

$$\Delta = a^2 - 4b$$

$$\Delta = 4 - 16$$

$$\Delta = -12$$

$$\Delta < 0$$

Since $\Delta < 0$, we will compare the transformation method and Euler's formula with the perfect squares method (case 2).

Transformation method with Euler's formula:

$$\begin{aligned} x^4 + 2x^2 + 4 &= 0 \\ y^2 + 2y + 4 &= 0 \qquad \text{transformed} \end{aligned}$$

Since the transformed equation will not factor we must use the quadratic formula:

$$\begin{aligned} y &= \frac{-2 \pm \sqrt{(2) - 4(1)(4)}}{2(1)} \\ y &= \frac{-2 \pm \sqrt{-12}}{2} \\ y &= -1 \pm i\sqrt{3} \end{aligned}$$

So,

$$x^2 = -1 + i\sqrt{3} \quad \text{or} \quad x^2 = -1 - i\sqrt{3}.$$

At this point it is obvious that the solution will result in a square root of a complex number. This means that we will need to apply Euler's formula, but first we must choose one root to manipulate into polar form. The author chooses $-1 - i\sqrt{3}$:

Modulus:

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{(-1)^2+(-\sqrt{3})^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Argument:

$$\begin{aligned} \tan\mu &= \left(\frac{b}{a}\right) \\ \tan\mu &= \left(\frac{-\sqrt{3}}{-1}\right) \\ \tan\mu &= \sqrt{3} \\ \mu &= \tan^{-1}(\sqrt{3}) \\ \mu &= 60^\circ \end{aligned}$$

Referencing the unit circle, 60° is located in quadrant I, but our point $-1-i\sqrt{3}$, is located in quadrant III. So we will add 180° to μ; hence, $\theta = 180^\circ + 60^\circ = 240^\circ$. The polar form becomes $2(\cos 240^\circ + i\sin 240^\circ)$. Now we are ready to apply Euler's formula:

$$x_k = 2^{\frac{1}{2}}\left(\cos\frac{360^\circ k+240^\circ}{2} + i\sin\frac{360^\circ k+240^\circ}{2}\right) \text{ when } n=2, k=0,1,$$

$$
\begin{aligned}
x_0 &= \sqrt{2}\left(\cos\frac{240^\circ}{2} + i\sin\frac{240^\circ}{2}\right) \\
&= \sqrt{2}(\cos 120^\circ + i\sin 120^\circ) \\
&= \sqrt{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2}
\end{aligned}
$$

$$
x = \frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}, \quad -\frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}, \quad \frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2}, \quad -\frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2}
$$

Now let's use the Perfect Squares method (case 2) and compare the results with the Transformation method and Euler's formula:

$$x^4 + 2x^2 + 4 = 0$$

So,

$$
\begin{aligned}
(x^2+2)^2 &= 4x^2 - 2x^2 \\
(x^2+2)^2 &= x^2(4-2) \\
(x^2+2)^2 &= x^2(\sqrt{2})^2 \\
(x^2+2)^2 &= (x\sqrt{2})^2 \\
x^2 + 2 &= \pm x\sqrt{2}
\end{aligned}
$$

Now set the equations equal to zero and solve for x. Using the quadratic formula you have the same four resultant roots as when you applied Euler's formula, again with much less work involved. It is good for the reader to experience several ways to solve a problem.

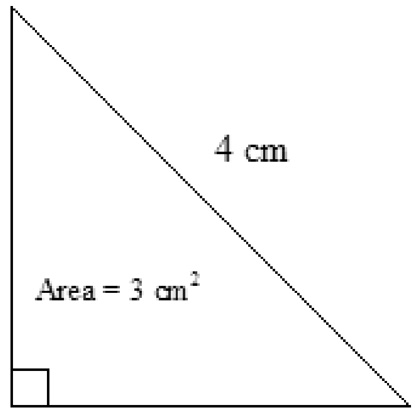

Figure 7: Right Triangle With Hypotenuse 4 cm And Area 3 cm^2

3.2.5 A Problem From Geometry

We will finish this section by solving a simple problem: Given a right triangle with hypotenuse 4 cm and area 3 cm^2, find the legs. (See Figure 7)

Solution:

We have $x^2+y^2=16$ and $xy=6$. Thus $x^2+\frac{36}{x^2}=16$, i.e. $x^4-16x^2+36=0$. We note that $\Delta = 16^2-4(36)=112$. Let $z=x^2$. The equation $z^2-16z+36=0$ has two positive roots, namely $\dfrac{16\pm 4\sqrt{7}}{2}$. Let us choose $8-2\sqrt{7}$. Next we have to solve $x^2=8-2\sqrt{7}$. Obviously, $x=\sqrt{8-2\sqrt{7}}$ while $y^2=16-(8-2\sqrt{7})=8+2\sqrt{7}$. Thus $y=\sqrt{8+2\sqrt{7}}$. If we had chosen $8+2\sqrt{7}$ then $x^2=8+2\sqrt{7}$ implies $x=\sqrt{8+2\sqrt{7}}$, while $y^2=16-(8+2\sqrt{7})=8-2\sqrt{7}$ implies $y=\sqrt{8-2\sqrt{7}}$. This is no surprise since the given system of equations is symmetric in x and y. In summary, the length of the legs are $\sqrt{8+2\sqrt{7}}$ and $\sqrt{8-2\sqrt{7}}$.

3.3 Practice Exercises

Solve by factorization method:

1. $x^4 - 9x^2 = 36$
2. $x^4 - 125 = -20$
3. $x^4 + 4x^2 = 0$
4. $x^4 = -15x^2$

Solve using the transformation method $y = x^2$:

5. $x^4 = 16$
6. $x^4 = 49$
7. $x^4 + 6 = 5x^2$
8. $x^4 - 10x^2 = -11$
9. $x^4 - 5x^2 - 5 = 0$
10. $x^4 + 8x^2 + 3 = 0$

Solve using the appropriate perfect square method and the transformation method:

11. $x^4 + 2x^2 + 2 = 0$
12. $x^4 + 3x^2 - 2 = 0$
13. $x^4 + 6x^2 + 7 = 0$

4 MAXIMA AND MINIMA

4.1 Introduction

Maxima and minima problems involve finding the values in the domain of a function that represent a largest (maxima) or smallest (minima) value within a predetermined boundary. Maxima and minima are also referred to as extrema. If you are looking for the largest or smallest value within the entire domain, the maxima (maximum) is called the absolute maximum, and similarly the smallest the value within the entire domain is called the absolute minimum [12].

Maximum and minimum problems are very applicable in today's world; for example, when searching for the maximum/minimum value for volume, area, and profit. It is important for businesses to know the least amount of tin needed to construct a can for a fixed volume of product. They can also determine the greatest or least amount of profit from particular sales promotions. Extrema problems can help decide whether one needs to limit the amount of product sold to a customer for maximum profit. What if you have a limited amount of fencing that needs to be installed and you must get the maximum amount of area enclosed, or perhaps it is your job to design a box given a fixed amount of cardboard? How do you determine the size of the box that will give you the maximum volume? All these examples and more come under the umbrella of maxima and minima problems.

There has been some discussion among mathematics educators as to whether or not maxima and minima problems can be understood prior to learning calculus [7]. The purpose of this section is to show that there are ways to solve maxima and minima problems before calculus; that, in fact, it is good for students to learn a variety of ways to solve mathematics problems at each level they might be encountered.

We will discuss, in detail, ten problems from geometry, several of them based on the work of I.P. Natanson [18]. Whenever appropriate, we will compare the non-calculus approach with the usual calculus approach. The arithmetic-geometric mean inequality is going to be one of our main tools.

4.2 Quadratic Functions

Let us begin by exploring quadratic functions. The general form of a quadratic function is:

$$y = ax^2 + bx + c, \text{ where } a \neq 0.$$

If $a = 0$, we would lose the second-degree term and be left with a linear function. The quadratic function produces a parabola for its graph, which only changes direction once; thus, parabolas only have one maximum or one minimum.

Example 1: Let's begin with a very simple example, where $a > 0$ and $b = 0$:

$$y = x^2 + 2.$$

Today's technology makes it very simple to get a quick visual image of the equation and easily identify maxima and minima.

As the reader can see, the quadratic function gives a parabola as shown in Figure 8. When determining the maxima or minima you are looking for the greatest (maximum) or smallest (minimum) value of y determined by the given quadratic, in this case $y = x^2 + 2$.

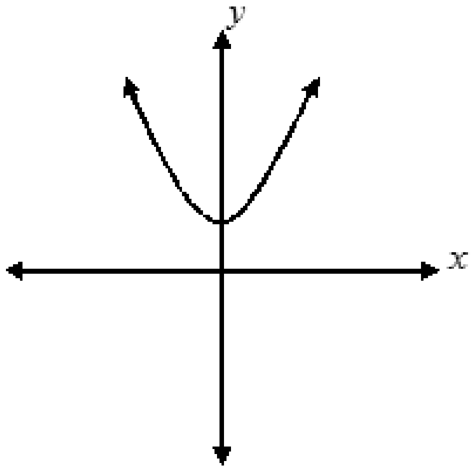

Figure 8: Quadratic Function, $y = x^2 + 2$

When you examine the graph from Figure 7, it is obvious that the function $y = x^2 + 2$, does not have a greatest value for y. It follows then, that since the greatest value of y does not exist, there is no maximum. However, there is a least value of y; which means the minimum does exist. In this parabola, the minimum is located at the lowest point in the bowl-shaped curve with the open end of the parabola facing upward. Let us take an algebraic look at the equation $y = x^2 + 2$ and solve for the unknown minimum.

$$y = x^2 + 2$$

The expression is written as the sum of two terms. The second term, the constant, is independent of the value of x. The first term, x^2, being a second-degree term can never be a negative number, even when x is equal to a negative number

$$[\text{i.e. } (-1)^2 = 1, (-2)^2 = 4, (-3)^2 = 9, \text{ etc.}];$$

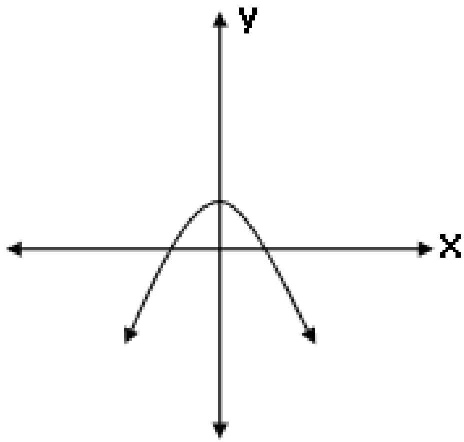

Figure 9: Quadratic Function, $y = -x^2 + 2$

it can however be equal to zero. In fact, this is the key to finding the minimum. When $x = 0$, we are able to find the smallest value of y and solve for the minimum. When $x = 0$

$$y = (0)^2 + 2$$
$$y = 2$$

So, the answer is $(0, 2)$, i.e. the minimum, namely 2, is adopted when $x = 0$. What happens when $a < 0$? Let's explore that condition in example 2.

Example 2: Solve for maxima or minima when $a < 0$ and $b = 0$:

$$y = -x^2 + 2$$

Once again, let's take a look at the graph, this time, in Figure 9. This graph has taken on a slightly different appearance than example 1. This time we have a mountain or hill-shaped curve with the open end of the parabola facing downward. The graph simply flipped directions.

Upon examination of the graph from Figure 8, it is obvious that the function $y = -x^2 + 2$ does not have a smallest value for y. It follows that since the smallest value of y does not exist, there is no minimum. However, there is a largest value of y; which means in this graph a maximum does exist. In this parabola, the maximum is located at the highest point on the hill-shaped curve. We note that when $x = 0$

$$\begin{aligned} y &= -(0)^2 + 2 \\ y &= 2 \end{aligned}$$

Thus the answer is $(0, 2)$, i.e. the maximum, namely 2, is adopted at $x = 0$.

REMARK: Note that the change in the direction of the parabola was directly related to the sign of the coefficient a. When $a > 0$ (positive), the parabola is concave up and has a minimum. When $a < 0$ (negative) the parabola is concave down and has a maximum.

What happens when $b \neq 0$? In the quadratic function, when $b < 0$ or $b > 0$ it brings a new component into the problem. Let us take a look at the following example:

Example 3: The function below is a quadratic trinomial with $a > 0$ and $b < 0$:

$$y = 2x^2 - 20x + 45.$$

In our previous examples y was equal to two terms, one second-degree term and the other a constant term. If we are to solve the problem using the same process, we must manipulate the expression until it has two terms: one second-degree term and one constant. In order to do that we will break up the given expression, completing the square for the x^2 and x terms:

$$\begin{aligned} y &= 2x^2 - 20x + 45 \\ &= 2(x^2 - 10x) + 45 \\ &= 2(x^2 - 10x + 25 - 25) + 45 \\ &= 2(x^2 - 10x + 25) + 45 - 50 \\ &= 2(x-5)^2 - 5. \end{aligned}$$

Now our function is as before (examples 1 and 2). We have two terms, one second-degree and one constant, and we are ready to solve for the maxima or minima. Since $a > 0$ we know that the parabola is concave upward. If you need additional convincing, graph the equation either by hand or with a graphing calculator. We know that a maximum does not exist, but we can find the minimum when $2(x - 5) = 0$. With this example it is easy to see that when $x = 5$ the resultant second-degree term will equal zero. In fact:

$$\begin{aligned} y &= 2(5-5) - 5 \\ y &= 2(0) - 5 \\ y &= -5. \end{aligned}$$

When $x = 5$, y adopts its minimum value (namely, $y = -5$).

Example 4: The function below is a quadratic trinomial with $a < 0$ and $b > 0$:

$$y = -6x^2 + 12x - 4.$$

We will follow the same procedure for this example as we did in example 3:

$$
\begin{aligned}
y &= -6x^2 + 12x - 4 \\
&= -6(x^2 - 2x) - 4 \\
&= -6(x^2 - 2x + 1 - 1) - 4 \\
&= -6(x^2 - 2x + 1) - 4 + 6 \\
&= -6(x - 1)^2 + 2.
\end{aligned}
$$

When $x = 1$, y adopts its maximum value (namely, $y = 2$).

We have based much of the discussion of finding maxima or minima on whether $a > 0$ or $a < 0$. It is clear that the value of a changes the behavior of the graph and determines whether the parabola has a maximum or minimum value. Let us investigate this more conclusively analyzing the general quadratic function:

$$
\begin{aligned}
y &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + c
\end{aligned}
$$

It is important to recognize that the term $c - \dfrac{b^2}{4a^2}$ is the constant we have been working with through out this section, and gives the value of y for the maxima or minima when $x = -\dfrac{b}{2a}$. Consequently, when $a > 0$ and $x = -\dfrac{b}{2a}$, we have a

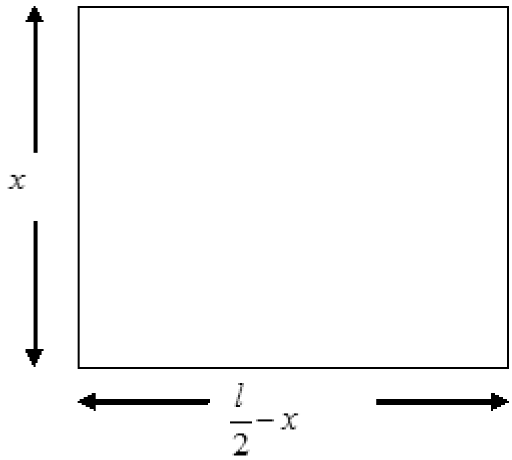

Figure 10: Problem 1 — Maximizing The Area Of A Rectangle

minimum value for y and a maximum does not exist. On the other hand, when $a < 0$ and $x = -\frac{b}{2a}$, we have a maximum value for y and a minimum does not exist.

4.3 Ten Problems from Geometry

Problem 1:

Max has been given a roll of chain link fence of length l, to protect his garden. He wants the largest possible area enclosed in a rectangle. Find the dimensions of the rectangle. We know that to find the area of a rectangle we multiply length times width (i.e. $A = lw$). (See Figure 10) We will designate the sides of the rectangle as x and y. Hence we will have $y = \left(\frac{l}{2} - x\right)$. Then, $A = x\left(\frac{l}{2} - x\right)$ or $A = -x^2 + \frac{l}{2}x$

We see that $a = -1, b = \frac{l}{2}$, which results in our maximum value being located at

$$x = -\frac{\frac{l}{2}}{2(-1)} = \frac{l}{4}.$$

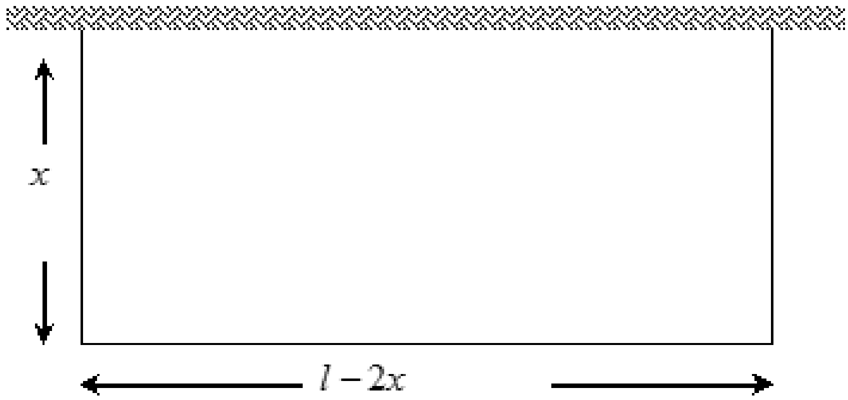

Figure 11: Problem 2 — Maximizing The Area Of A Playground

Thus,

$$y = \frac{l}{2} - \frac{l}{4} = \frac{l}{4}.$$

Consequently the maximum area will be attained when the rectangle becomes a square of side $\frac{l}{4}$.

Problem 2:

A local day care has received a donation of materials to create a rectangular fence around a play ground for the children. There is enough fencing to construct a l foot fence. For safety reasons, the director of the day care wants to use the back side of the daycare building as part of the enclosed play ground and wants the play ground to be as large as possible. Find the dimensions of the fence.

Let us review what we know: we have a fixed amount of fencing and we need

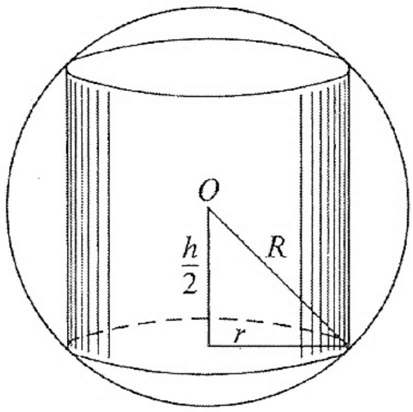

Figure 12: Problem 3 — The Cylinder With Greatest Lateral Area

fencing for only three sides of the play ground. Let the width of the fence be x, thus the length of the fence will be $(l - 2x)$ (See Figure 11). [2]

Then,

$$A = x(l - 2x) \quad \text{or} \quad A = -2x^2 + lx.$$

We see that $a = -2, b = l$, which results in our maximum value to be located at

$$x = -\frac{l}{2(-2)} = \frac{l}{4}.$$

The width of the fence will be $\frac{l}{4}$ feet and the length $\frac{l}{2}$ feet.

[2] Several of the figures in this section have been adapted from the work of I.P. Natanson [18]

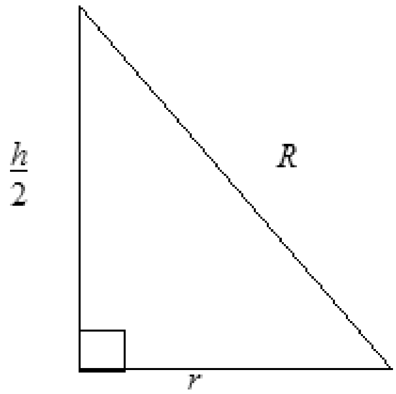

Figure 13: Problem 3 — A Right Triangle Within The Cylinder

Problem 3:

Suppose you have a cylinder inscribed within a sphere, and you wish the cylinder to have the greatest lateral area. (See Figure 12) Find the dimensions of the cylinder.

The formula for the lateral area of the cylinder is: $S = 2\pi rh$, where the variables are defined as follows:

$$r = \text{radius of the cylinder}$$

$$h = \text{altitude of the cylinder}$$

$$(R = \text{is the fixed radius of the sphere}).$$

Let us investigate the relationship between $R, r,$ and $\frac{h}{2}$. (See Figure 13) The variables $\frac{h}{2}$ and r are equal to the legs of the triangle, and R is the hypotenuse. Let us use the Pythagorean Theorem to solve for h:

$$\begin{aligned}
\left(\frac{h}{2}\right)^2 + r^2 &= R^2 \\
\frac{h^2}{4} + r^2 &= R^2 \\
\frac{h^2}{4} &= R^2 - r^2 \\
h^2 &= 4(R^2 - r^2) \\
h &= 2\sqrt{R^2 - r^2}.
\end{aligned}$$

Now insert the value of h into the lateral area formula:

$$\begin{aligned}
S &= 2\pi r h \\
S &= 2\pi r h(2\sqrt{R^2 - r^2}) \\
S &= 4\pi r h\sqrt{R^2 - r^2}.
\end{aligned}$$

And set $y = S^2$ to eliminate the radical:

(Note that when you square a function or multiply it by a constant, the point where the maximum or minimum is attained does not change)

Next define $x = r^2$:

$$\begin{aligned}
y &= 16\pi^2 x(R^2 - x) \\
y &= 16\pi^2 x R^2 - 16\pi^2 x^2 \\
y &= -16\pi^2 x^2 + 16\pi^2 x R^2.
\end{aligned}$$

The maximum is attained at:

$$x = \frac{-b}{2a} = \frac{-16\pi^2 R^2}{2(-16)\pi^2} = \frac{R^2}{2}.$$

Thus, the greatest lateral area is obtained when:

$$\begin{aligned} r^2 &= \frac{R^2}{2} \\ \text{i.e.} \qquad r &= \sqrt{\frac{R^2}{2}} = \frac{R}{\sqrt{2}} \\ r &= \frac{R}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ r &= \frac{R\sqrt{2}}{2}. \end{aligned}$$

Now that we know the optimum value for r we can solve for h and we are done.

Since

$$\begin{aligned} h &= 2\sqrt{R^2 - r^2}, \text{ then} \\ h &= 2\sqrt{R^2 - \frac{R^2}{2}} \\ h &= \sqrt{4R^2 - 2R^2} = \sqrt{2R^2} \\ h &= R\sqrt{2}. \end{aligned}$$

Thus the dimensions of the inscribed cylinder with the greatest lateral area are such that the radius equals $\frac{R\sqrt{2}}{2}$ while the height equals $R\sqrt{2}$.

REMARK: In the preceding application we had to deal with the function

$$y(r) = -16\pi^2 r^4 + 16\pi^2 R^2 r^2.$$

This is a biquadratic function. We provided a plausible path to study where the maximum is attained: analyzing the quadratic function

$$g(x) = -16\pi^2 x^2 + 16\pi^2 R^2 x$$

we find that its maximum is attained at

$$x = \frac{-16\pi^2 R^2}{2(-16\pi^2)} = \frac{R^2}{2},$$

then the maximum of $y(r)$ is attained at

$$\sqrt{\frac{R^2}{2}} = \frac{R}{2}.$$

However, a plausible argument is not a proof. We need to provide the latter to be convinced that the above-mentioned argument is valid. Indeed, let us consider an arbitrary biquadratic function

$$f(x) = ax^4 + bx^2 + c, \text{ where } a < 0 \text{ and } b > 0.$$

Define the function

$$g(z) = az^2 + bz + c.$$

Since g is a quadratic function we can conclude that its maximum is attained at $-\dfrac{b}{2a}$,

$$\text{i. e.} \quad g(z) \leq g\left(-\frac{b}{2a}\right) \quad \forall z.$$

Thus

$$g(x^2) \leq g\left(-\frac{b}{2a}\right) \quad \forall x,$$

so

$$ax^4 + bx^2 + c \leq a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c \quad \forall x.$$

But

$$f\left(\sqrt{-\frac{b}{2a}}\right) = a\left(\sqrt{-\frac{b}{2a}}\right)^4 + b\left(\sqrt{-\frac{b}{2a}}\right)^2 + c = a\left(-\frac{b}{2a}\right)^2 - \frac{b^2}{2a} + c.$$

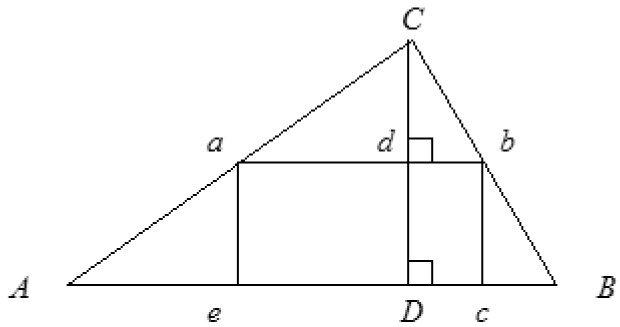

Figure 14: Problem 4 — Rectangle Inscribed Within An Acute Triangle

Therefore

$$f(x) \leq f\left(\sqrt{-\frac{b}{2a}}\right) \forall x,$$

as we wished to prove. It should be noted that

$$f(x) \leq f\left(-\sqrt{\frac{-b}{2a}}\right) \forall x,$$

too. Nonetheless, in the context of geometry we are only interested in positive quantities. A thorough analysis of maxima and minima of biquadratic functions can be found in [8].

Problem 4:

Given an acute triangle ABC with a rectangle $abce$ inscribed, where ab is parallel to AB. Where should the line segment ab be constructed so as to give the rectangle $abce$ maximum area? (See figure 14)

The line segments are defined as follows.

$$\overline{AB} = L \qquad \overline{CD} = H \qquad \overline{ab} = l \qquad \overline{bc} = h$$

($\overline{CD}$ is the height of the big triangle).

We notice similar triangles, which are triangles that have the same shape, but different sizes; for instance $\triangle ABC$ is similar to $\triangle abC$. Since both triangles are similar, their corresponding sides and heights are proportional.

Thus

$$\frac{l}{L} = \frac{H-h}{H} \qquad \text{Next we solve for } l: \qquad l = \frac{L}{H}(H-h)$$

The area of the rectangle *abce* equals length times width. So,

$$A(h) = h\left(\frac{L}{H}(H-h)\right) = -\frac{L}{H}h^2 + Lh$$

The maximum will be attained at

$$h = \frac{-L}{2\left(-\frac{L}{H}\right)} = \frac{H}{2}.$$

4.3.1 An Important Inequality

Let us expand our investigation of maxima and minima to include some further geometric applications. With this purpose in mind, we will explore the relationship between the arithmetic and geometric means. Let us recall that the arithmetic mean is $\dfrac{x+y}{2}$ and the geometric mean is $\sqrt{xy}$ (x, y any two positive numbers).

<u>Arithmetic-Geometric Mean Inequality.</u>

We have the following chain of equivalent inequalities:.

$$\begin{aligned} \sqrt{xy} &\le \frac{x+y}{2} \qquad\qquad x, y > 0 \qquad\qquad (1) \\ xy &\le \left(\frac{x+y}{2}\right)^2 \\ xy &\le \frac{x^2+2xy+y^2}{4} \\ 4xy &\le x^2+2xy+y^2 \\ 0 &\le x^2+2xy-4xy+y^2 \\ 0 &\le x^2-2xy+y^2 \\ 0 &\le (x-y)^2 \end{aligned}$$

Since $(x-y)^2 \ge 0$ is always true we can conclude that (1) is true. Now, let us take a look at the equality $\sqrt{xy} = \dfrac{x+y}{2}$ where $x, y > 0$. We have the following chain of equalities:

$$\begin{aligned}
\sqrt{xy} &= \frac{x+y}{2} \\
2\sqrt{xy} &= x+y \\
(2\sqrt{xy})^2 &= (x+y)^2 \\
4xy &= x^2+2xy+y^2 \\
0 &= x^2+2xy-4xy+y^2 \\
0 &= x^2-2xy+y^2 \\
0 &= (x-y)^2
\end{aligned}$$

We then note that

$$\sqrt{xy} = \frac{x+y}{2} \quad \text{if and only if } x = y.$$

Theorem. Assume $P > 0$. Define the function $z(x) \leq x + \dfrac{P}{x}$ for any $x > 0$. The minimum of this function is attained at $x\sqrt{P}$.

Proof. Let $x > 0$. By the Arithmetic-Geometric Mean (AGM) inequality we have

$$\sqrt{x \times \frac{P}{x}} \leq \frac{x + \frac{P}{x}}{2}, \quad \text{so} \quad 2\sqrt{P} \leq x + \frac{P}{x}.$$

Hence $2\sqrt{P} \leq z(x)$ for any $x > 0$. Moreover, $2\sqrt{P} = x + \dfrac{P}{x}$ if and only if $x = \dfrac{P}{x}$. Thus $2\sqrt{P} = x + \dfrac{P}{x} = z(x)$ if and only if $x = \sqrt{P}$. In other words, $z(x)$ adopts its minimum at $x = 2\sqrt{P}$

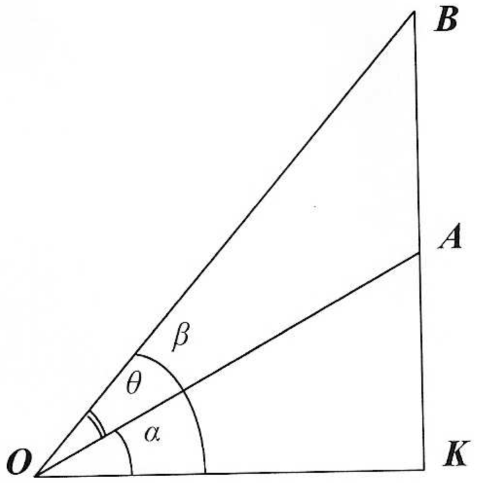

Figure 15: Problem 5 — The Flat-Screen TV Problem

4.3.2 Additional Geometric Applications

Problem 5:

A flat-screen TV AB hangs on the wall BK above the eye level of the observer, O sitting on a sofa. At what distance from the wall must the sofa be positioned in order for the angle θ, created by the TV screen, to be greatest?

First we must establish some definitions. We chose K to equal the point of intersection between the wall BK, and the horizontal line of sight of the observer O sitting on the sofa. This is the distance we are looking for.

Let $x = OK, KA = a$, and $KB = b$. We will let the angle $KOA = \alpha$ and the angle $KOB = \beta$ so that $\theta = \beta - \alpha$.

We begin by using a basic trigonometric identity:

$$\tan\beta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

From Figure 15, we see that

$$\tan\alpha = \frac{a}{x} \text{ and } \tan\beta = \frac{b}{x}, \quad \text{therefore:}$$

$$\tan\theta = \frac{\frac{b}{x} - \frac{a}{x}}{1 + \left(\frac{a}{x}\right)\left(\frac{b}{x}\right)}$$

We need to manipulate the denominator of $\tan\theta$ to look like $x + \dfrac{P}{x}$ shown in the previous theorem. So, we will multiply the entire fraction by $\dfrac{x}{x}$, which gives the desired result:

$$\tan\theta = \frac{b-a}{x + \frac{ab}{x}}.$$

$$\text{Thus } \theta(x) = \arctan\left(\frac{b-a}{x + \frac{ab}{x}}\right)$$

Since arctan is a strictly increasing function, we need to maximize the fraction

$$\frac{(b-a)}{\left(x + \frac{ab}{x}\right)}$$

That is to say, we need to locate the value for x that will give the smallest possible denominator possible; which will make the fraction

$$\frac{b-a}{x + \frac{ab}{x}}$$

the largest possible number, hence the maximum. According to the theorem, that value of x will be attained when $x = \sqrt{ab}$, which is the distance between the wall where the flat-screen TV hangs and the observer O sitting on the sofa.

4.3.3 Arithmetic - Geometric Mean Inequality for $(n = 3)$

In the following applications, we will include an additional component to the Arithmetic - Geometric Mean Inequality: it also works for three positive numbers $(n = 3)$. Let $u, v, w > 0$.

Then:

(i) $\quad \sqrt[3]{uvw} \leq \frac{u+v+w}{3}$, or its equivalent $uvw \leq \frac{(u+v+w)^3}{27}$

(ii) $\quad \sqrt[3]{uvw} = \frac{u+v+w}{3} \Leftrightarrow u = v = w.$

Proof. Although it may seem strange at first sight, we need to prove the case $n = 4$ before attempting to prove the case $n = 3$. Let a, b, c, d be any positive numbers. Our goal is to show that

$(\ i)$ $\quad \frac{a+b+c+d}{4} \geq (abcd)^{\frac{1}{4}}$

$(\ ii)$ $\quad \frac{a+b+c+d}{4} \geq (abcd)^{\frac{1}{4}} \Leftrightarrow a = b = c = d.$

Indeed $a + b \geq 2\sqrt{ab}, c + d \geq 2\sqrt{cd}$ thanks to AGM inequality $(n = 2)$. Then $\frac{a+b+c+d}{4} \geq \sqrt{ab} + \sqrt{cd}$. Once more we use AGM, this time to the numbers $\sqrt{ab}$ and $\sqrt{cd}$, obtaining

$$\sqrt{ab} + \sqrt{cd} \geq 2\sqrt{\sqrt{ab} \times \sqrt{cd}} = 2(abcd)^{\frac{1}{4}}.$$

Therefore $a + b + c + d \geq 4(abcd)^{\frac{1}{4}}$.

Next we have to prove $(\ ii)$,

in particular $\frac{a+b+c+d}{4} = (abcd)^{\frac{1}{4}} \Rightarrow a = b = c = d,$ or its logical contrapositive,

namely:

$$\neg(a = b = c = d) \Rightarrow \frac{a+b+c+d}{4} \neq (abcd)^{\frac{1}{4}}.$$

Assume that

$$\neg(a = b = c = d), \text{ say } a \neq b.$$

Then

$$a + b > 2\sqrt{ab} \text{ and } c + d \geq 2\sqrt{cd}.$$

Therefore

$$\frac{a+b+c+d}{2} > \sqrt{ab} + \sqrt{cd}.$$

But let us recall that

$$\sqrt{ab} + \sqrt{cd} \geq 2(abcd)^{\frac{1}{4}}.$$

Thus

$$\frac{a+b+c+d}{2} > 2(abcd)^{\frac{1}{4}},$$

which implies that

$$\frac{a+b+c+d}{2} \neq 2(abcd)^{\frac{1}{4}}.$$

On the other hand, the implication

$$a = b = c = d \Rightarrow \frac{a+b+c+d}{4} = (abcd)^{\frac{1}{4}}$$

is obviously true because $a = b = c = d$ implies that

$$\frac{a+b+c+d}{4} = \frac{4a}{a} = a$$

while $(abcd)^{\frac{1}{4}} = (a^4)^{\frac{1}{4}} = a$. We have finished proving AGM $(n = 4)$.

We will now provide a proof of AGM $(n = 3)$ for three arbitrary positive numbers u, v, w : First of all we have

$$\frac{u+v+w+(uvw)^{\frac{1}{3}}}{4} \geq (uvw(uvw)^{\frac{1}{3}})^{\frac{1}{4}} = ((uvw)^{\frac{4}{3}})^{\frac{1}{4}} = (uvw)^{\frac{1}{3}}.$$

(Note that we have just used AGM ($n = 4$), to the four positive numbers u, v, w, and $(uvw)^{\frac{1}{3}}$.)

Then

$$u + v + w \geq 3(uvw)^{\frac{1}{3}}, \text{ i.e. } \frac{u+v+w}{3} \geq \sqrt[3]{uvw},$$

thus proving part (i) of the inequality. Next we have to prove that

$$\frac{u+v+w}{3} \geq \sqrt[3]{uvw}$$

implies $u = v = w$, or its logical contrapositive, namely

$$\neg(u = v = w) \Rightarrow \frac{u+v+w}{3} \neq \sqrt[3]{uvw}.$$

Assume that $\neg(u = v = w)$, say $u \neq v$. The AGM inequality ($n = 4$, second part) leads to

$$\frac{u + v + w + (uvw)^{\frac{1}{3}}}{4} > (uvw(uvw)^{\frac{1}{3}})^{\frac{1}{4}} = (uvw)^{\frac{1}{3}}.$$

Thus

$$u + v + w > +(uvw)^{\frac{1}{3}} > 4(uvw)^{\frac{1}{3}}, \text{ i.e. } u + v + w > 3(uvw)^{\frac{1}{3}},$$

which implies

$$\frac{u+v+w}{3} \neq \sqrt[3]{uvw}.$$

Finally, it only remains to prove that

$$u = v = w \Rightarrow \frac{u+v+w}{3} = \sqrt[3]{uvw}.$$

This task is pretty easy because $(u = v = w)$ implies

$$\frac{u+v+w}{3} = \frac{3u}{3} = u$$

while

$$\sqrt[3]{uvw} = \sqrt[3]{u^3} = u.$$

Problem 6:

This application is similar to problem three (3) in that we are dealing with a cylinder inscribed within a sphere. (See Figure 16) This time we will try to find the cylinder of greatest volume instead of area. We can use the same figure and definitions as in problem 3; however, we will use the formula for the volume of the cylinder; namely

$$V = \pi r^2 h.$$

Recall that:

$$R = \text{ fixed radius of the sphere}$$

$$r = \text{ radius of the cylinder}$$

$$h = \text{ altitude of the cylinder}$$

Again we have the Pythagorean Theorem relationship between $R, r,$ and $\frac{h}{2}$ where we will solve for h. To avoid redundancy please see problem 3 for the step by step solution for $h = 2\sqrt{R^2 - r^2}$. Now insert the value of h into the volume formula:

$$\begin{aligned}
V &= \pi r^2 h \\
V &= \pi r^2 (2\sqrt{R^2 - r^2}) \\
\frac{V}{2\pi} &= r^2 \sqrt{R^2 - r^2} \\
\left(\frac{V}{2\pi}\right)^2 &= \left(r^2 \sqrt{R^2 - r^2}\right)^2 \\
\frac{1}{4\pi^2} V^2 &= r^4 (R^2 - r^2)
\end{aligned}$$

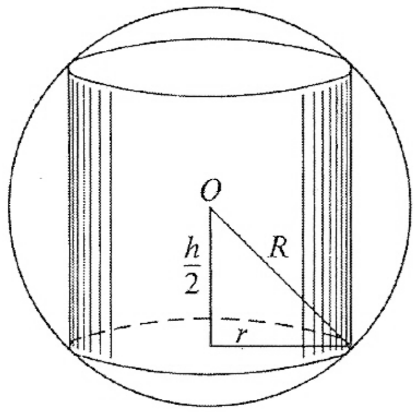

Figure 16: Problem 6 — The Cylinder With Greatest Volume

Notice that we squared both sides of the function, an operation that does not change the point where the maximum is attained. To simplify the notation let

$$z = \frac{1}{4\pi^2}V^2, \text{ thus } z = r^4(R^2 - r^2).$$

The function z is a sixth degree polynomial. Rather than analyzing it directly we will apply AGM for three positive numbers.

By the AGM inequality we have

$$\frac{r^2}{2} \times \frac{r^2}{2} \times (R^2 - r^2) \leq \frac{(R^2)^3}{27} = \frac{R^6}{27}.$$

Moreover

$$\frac{r^2}{4}(R^2 - r^2) = \frac{R^6}{27} \Leftrightarrow \frac{r^2}{2} = R^2 - r^2$$

i.e

$$\frac{r^4}{4}(R^2 - r^2) = \frac{R^6}{27} \Leftrightarrow r = R\sqrt{\frac{2}{3}}.$$

Thus $\frac{1}{4}z$ attains its maximum at $r = R\sqrt{\frac{2}{3}}$ which is the same point where V adopts its maximum since

$$\frac{1}{4}z = \frac{1}{16\pi^2}V^2 \left(V, V^2, \text{ and } \frac{1}{16\pi^2}V^2 \text{ attain their maximum at the same point} \right).$$

In conclusion to this problem, we have found the height of the inscribed cylinder as $2\sqrt{R^2 - r^2}$ with a radius of $r = R\sqrt{\frac{2}{3}}$.

REMARK: In the previous five problems, the non-calculus approach compares favorably with the calculus approach. Problem 6 is rather different because the use of calculus leads to a fast conclusion. Indeed, the derivative of

$$f(r) = r^4R^2 - r^6 \text{ is } 4R^2r^3 - 6r^5,$$

which we make equal to zero. Then $r = \sqrt{\frac{2}{3}}R$ is the only candidate for extrema. Since

$$f"(r) = 12R^2r^2 - 30r^4$$

we can conclude that

$$f"\left(\sqrt{\frac{2}{3}}R\right) = 8R^4 - \frac{120}{9} < 0,$$

thus $f(r)$ and consequently V, attain its maximum at $r = \frac{2}{3}R$.

Problem 7:

Find the dimensions of the cylinder of maximum volume inscribed within a right circular cone. (See Figure 17) Recall the volume formula of a cylinder is $V = \pi r^2 h$.

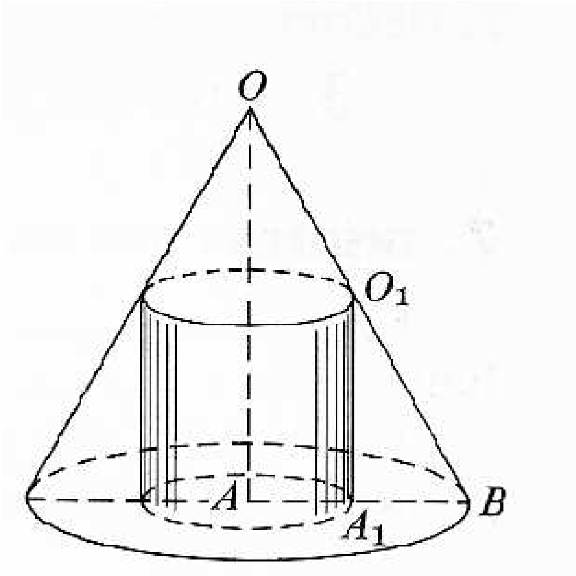

Figure 17: Problem 7 — Cylinder Inscribed Within A Right Circular Cone.

Definitions based on Figure 16 are as follows:

$$AB = R = \text{ fixed radius of cone base}$$

$$OA = H = \text{ altitude of the cone}$$

$$AA_1 = r = \text{ radius of cylinder}$$

$$O_1A_1 = h = \text{ altitude of cylinder}$$

$$A_1B = R - r = \text{ base of triangle } O_1A_1B$$

As in problem five (5), we will explore similar triangles, OAB and O_1A_1B, (See Figure 18).

In comparing similar triangles we have the following proportion:

$$\frac{h}{H} = \frac{R - r}{R}.$$

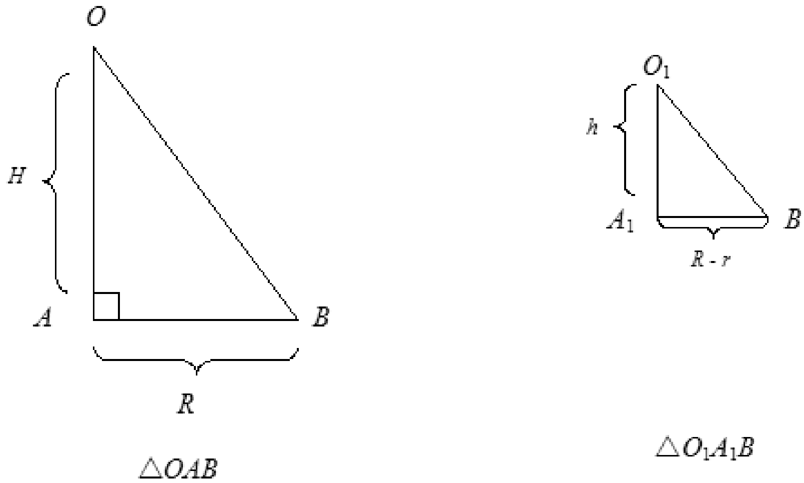

Figure 18: Problem 7 — Similar Triangles

So

$$h = \frac{H}{R}(R - r).$$

Now replace the value for h into the inscribed cylinder volume formula:

$$\begin{aligned} V &= \pi r^2 \left[\frac{H}{R}(R - r) \right] \\ \frac{VR}{\pi H} &= r^2(R - r). \end{aligned}$$

Again, to simplify the notation let $z = \dfrac{VR}{\pi H}$, so it follows that $z = r^2(R - r)$. The functions z and V will attain its maximum at the same point as r because they only differ by a factor $\left(\text{namely } \dfrac{R}{rH} \right)$. Now, apply AGM for three positive numbers by breaking up the expression on the right side of the equal sign into three factors.

$$z = r^2(R - r)$$

$$\frac{1}{4}z = \frac{r}{2} \times \frac{r}{2} \times (R - r)$$

We note that

$$\frac{r}{2} \times \frac{r}{2} \times (R - r) \leq \frac{R^3}{27}.$$

Moreover $\frac{1}{4}z = \frac{R^3}{27}$ if and only if

$$\begin{aligned} \frac{r}{2} &= R - r \\ \frac{2r}{2} + \frac{r}{2} &= R \\ \frac{3}{2}r &= R \\ r &= \frac{2}{3}R. \end{aligned}$$

The height or altitude of the cylinder is $\frac{H}{R}(R-r)$ where $r = \frac{2}{3}R$, which completes this problem. We have found the dimensions of the cylinder of maximum volume. It should be noted that a knowledge of calculus provides an alternative solution.

The function

$$g(r) = r^2(R - r) = Rr^2 - r^3,$$

a cubic polynomial, can be easily derived. Indeed,

$$g'(r) = -3r^2 + 2Rr.$$

Then $-3r^2 + 2Rr = 0$ leads to $r = \frac{2}{3}R$. In turn $g''\left(\frac{2}{3}R\right) = -2R < 0$, thus confirming that g, and hence V, attains its maximum at $r = \frac{2}{3}R$.

Problem 8:

We have a right circular cone inscribed in a sphere. (See Figure 19) We are looking for the dimensions of the inscribed cone of greatest volume. The formula for the volume of a right circular cone is $v = \frac{1}{3}\pi r^2 h$:

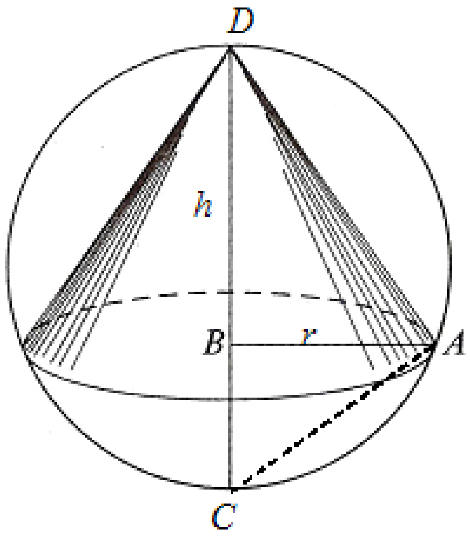

Figure 19: Problem 8 — The Cone With Greatest Volume

$$R = \text{ fixed radius of the sphere}$$

$$r = \text{ radius of the cone base}$$

$$h = \text{ altitude of cone}$$

So,

$$\begin{aligned} AB &= r \\ BD &= h \\ DC &= 2R \\ BC &= 2R - h. \end{aligned}$$

We note that $\triangle DAC$ has a right angle at A since DC is a diameter of the circle that passes through $A, B, C,$ and D. Thus $r^2 = h(2R - h)$.

Now insert the value of r^2 into the volume formula:

$$\begin{aligned} V &= \frac{1}{3}\pi[h(2R-h)]h \\ V &= \frac{\pi}{3}(2R-h)h^2 \\ \frac{3V}{\pi} &= h^2(2R-h). \end{aligned}$$

We simplify the notation by defining:

$$z = \frac{3V}{\pi}.$$

It follows that:

$$z = h^2(2R-h).$$

So,

$$\frac{1}{4}z = \frac{h}{2} \times \frac{h}{2} \times (2R-h).$$

When we apply the AGM theorem we get:

$$\frac{h}{2} \times \frac{h}{2} \times (2R-h) \leq \frac{(2R)^3}{27}.$$

Furthermore, $\frac{1}{4}z = \frac{(2R)^3}{27}$ if and only if $\frac{h}{2} = 2R-h$. Thus,

$$\begin{aligned} h &= 2(2R-h) \\ h &= 4R-2h \\ 3h &= 4R \\ h &= \frac{4}{3}R. \end{aligned}$$

Let us recall that $r^2 = h(2R - h)$. So for the optimal value $h = \frac{4}{3}R$ we have:

$$\begin{aligned} r^2 &= \frac{4}{3}R\left(2R - \frac{4}{3}R\right) \\ &= \frac{8}{3}R^2 - \frac{16}{9}R^2 \\ &= \frac{8}{3}R^2. \end{aligned}$$

Therefore,

$$\begin{aligned} r &= \sqrt{\frac{8}{3}R^2} \\ r &= \frac{2\sqrt{2}}{3}R. \end{aligned}$$

The inscribed cone of greatest volume has altitude $\frac{4}{3}R$ and radius $\frac{2\sqrt{2}}{3}R$. As in the two previous problems, the problem of the cone of maximum volume inscribed in a sphere can be solved using calculus with great advantage. We need only to analyze $f(h) = -h^3 + 2Rh^2$.

4.4 Final Considerations

Let us take a look at two distinct problems. Problem 9 involves a cylinder with a fixed volume, commonly used in calculus texts, while problem 10 investigates triangles with fixed perimeter. The latter is not seen in calculus texts primarily because of the difficulty of the calculus machinery required to solve the problem.

Problem 9:

When Dr. John T. Dorrance was hired by the Joseph Campbell Preserve Company in 1897, no one knew the impact he would have on the company. Dr. Dorrance, a

European trained chemist, developed condensed soups [4]. This new development gave Campbell's an edge on the market by selling a 10-ounce can of condensed soup for ten cents over the larger 32-ounce can of soup selling for thirty-four cents [4].

How could Campbell's build a cylinder (soup can) in such a way that it will hold the new 10-ounce product, yet minimize the surface area of the can? In other words, the company needed to minimize the size of the can and cost of the product used to make the new smaller-sized soup can. With a fixed volume of soup, how should Campbell's build a new soup can (cylinder) in such a way as to minimize the surface area? The "soup can" problem is a common problem used in most calculus texts.

Let the fixed volume be denoted V, the surface area S, and let x and y represent the radius and the height of the cylinder, respectively. We will use the standard formulas for volume and surface area of a cylinder:

$$V = \pi x^2 y \qquad \text{and} \qquad S = 2\pi x^2 + 2\pi xy.$$

Now, take the formula for surface area and break it up into three positive numbers $2\pi x^2, \pi xy, \pi xy$, and apply the first part of the AGM inequality. So,

$$(i) \qquad \sqrt[3]{2\pi^3 x^4 y^2} \leq \frac{2\pi x^2 + \pi xy + \pi xy}{3} = \frac{S}{3}.$$

$$\text{Since, } V^2 = \pi^2 x^4 y^2, \qquad \text{we will have} \qquad 2\pi V^2 = 2\pi^3 x^4 y^2.$$

Consequently $\sqrt[3]{2\pi V^2} \leq \frac{S}{3}$ and therefore $3\sqrt[3]{2\pi V^2} \leq S$, so that S will always be bigger than or equal to the constant value $\sqrt[3]{2\pi V^2}$. Its minimum is $\sqrt[3]{2\pi V^2} = S$. However, according to the second part of the AGM inequality:

$$(ii) \quad \sqrt[3]{2\pi^3 x^4 y^2} = \frac{S}{3} \qquad \text{if and only if} \qquad 2\pi x^2 = \pi xy; \text{ in fact,}$$

$$\sqrt[3]{2\pi V^2} = \frac{S}{3} \quad \text{if and only if} \quad 2x = y.$$

Recall that x represents the radius and y the height of the cylinder. Thus, S will adopt the minimum value if the cylinder is built in such a way that the height (y) of the cylinder is equal to twice the radius ($2x$). So, since $V = \pi x^2 y$, this proportion will happen when $V = \pi x^2(2x)$ or when

$$x = \sqrt[3]{\frac{V}{2\pi}} \quad \text{and} \quad y = 2\sqrt[3]{\frac{V}{2\pi}}.$$

Did Dr. Dorrance use this to reconfigure the Campbell's new soup can size? My research did not reveal the answer; however, it does seem possible and makes for interesting reading, and the application is useful for the reader in real world situations.

Problem 10:

In our final application, let us take a look at all possible triangles with fixed perimeter p. Is there one particular triangle that encloses the largest area? In order to evaluate this question, we will need to use Heron of Alexandria's formula for the area of a triangle [9]. Heron was a famous 1st century geometer who worked in mechanics and is most famous for the formula we are about to use [14].

Let a, b, c be the length of the sides of any triangle with perimeter p, hence $p = a + b + c$. Let

$$s = \frac{a+b+c}{2} = \frac{p}{2} \text{ (semi perimeter).}$$

Heron's formula states:

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Let us use the three positive numbers, $s-a, s-b, s-c$ and apply the AGM Inequality, which implies the following:

$$(i) \qquad \sqrt[3]{s(s-a)(s-b)(s-c)} \leq \frac{(s-a)+(s-b)+(s-c)}{3}$$
$$\leq \frac{3s-a-b-c}{3}$$
$$\leq \frac{3s-(a+b+c)}{3}$$
$$\leq \frac{3s-2s}{3}$$
$$\leq \frac{s}{3}.$$

$$(ii) \qquad \sqrt[3]{s(s-a)(s-b)(s-c)} = \frac{s}{3} \text{ if and only if } s-a=s-b=s-c,$$

in other words $a=b=c$.

By using the inequality (i) and cubing both sides we get:

$$(s-a)(s-b)(s-c) \leq \frac{s^3}{3^3}$$

So, by Heron's formula we see:

$$\sqrt{s(s-a)(s-b)(s-c)} \leq \sqrt{\frac{s(s^3)}{3^3}}$$
$$\leq \sqrt{\frac{s^4}{3^3}} \quad \text{and we conclude that area } \leq \frac{s^2}{3\sqrt{3}}.$$

So, we have shown that the area of any triangle with perimeter p is less than or equal to the fixed quantity $\frac{s^2}{3\sqrt{3}}$. The best scenario would be the equality: area $= \frac{s^2}{3\sqrt{3}}$. Then

$$\sqrt{s(s-a)(s-b)(s-c)} = \frac{s^2}{3\sqrt{3}}.$$

Thus (squaring both sides)

$$s(s-a)(s-b)(s-c) = \frac{s^4}{3^3} \qquad \text{i.e.} \quad (s-a)(s-b)(s-c) = \frac{s^3}{3^3}.$$

So

$$\sqrt[3]{(s-a)(s-b)(s-c)} = \frac{s}{3}.$$

And from (ii) it follows that $a = b = c$. Of course, one can check if $a = b = c$, then the area $\dfrac{s^2}{3\sqrt{3}}$. The triangle with perimeter p, which encloses the largest area, is the equilateral triangle.

Clearly, there are many applications available to the motivated precalculus student that desires a broader understanding of maxima and minima, without the use of calculus. This thesis has covered only a few applications using algebra and a few formulas. After vigorously working through these applications, the precalculus student should have an enhanced appreciation of maxima and minima prior to taking their first calculus course. It might also benefit calculus students to review these applications to deepen their mathematical insight.

5 CONCLUSION

After reviewing tangent lines to conics, biquadratics, and maxima and minima, it is clear that each of these topics can be addressed and understood with algebra. As students engage and thoroughly examine the topics in this thesis, not only will their computational skills in algebra be strengthened, but perhaps a general enhancement of mathematical skills will also be developed. It is the hope of the author that this thesis can be used in conjunction with any precalculus course to strengthen the precalculus bridge, and promote a more positive calculus experience for the student.

As stated in the introduction:

> The bottom line for mathematicians is that the architecture has to be right. In all the mathematics that I did, the essential point was to find the right architecture. It's like building a bridge. Once the main lines of the structure are right, then the details miraculously fit ... Freeman Dyson [1]

May the student find the right architecture within the pages of this thesis.

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APPENDICES

A Precalculus Textbook Review

Twenty-two standard precalculus textbooks were reviewed in regard to the three non-traditional topics to see whether the non-traditional topics were addressed within the text. The textbook numbers with an asterick(*) represent textbooks adopted by the state of Tennessee for education.

Table 1: Precalculus Textbook Chart.

Text	Biquadratics	Tangent Lines	Maxima and Minima
1*	No	No	No
2*	No	No	No
3*	No	Parabola	No
4*	No	No	No
5*	No	No	No
6*	No	No	Yes
7*	No	Yes	No
8*	No	No	No
9	No	No	No
10	No	No	No
11	No	No	No
12	No	No	No
13	No	No	No
14	No	No	No
15	No	No	No
16*	No	Yes	No
17*	No	No	No
18*	No	Parabola	No
19	No	No	Yes
20	No	No	No
21	No	Parabola	Yes
22	No	No	No

B Biquadratic Equations Practice Exercise Solutions

1. $x = \pm 2\sqrt{3} \qquad x = \pm i\sqrt{3}$

2. $5i, -5i, \sqrt{5}, -\sqrt{5}$

3. 0 (multiplicity two)$, 2i, -2i$

4. 0 (multiplicity two)$, i\sqrt{15}, -i\sqrt{15}i$

5. $2, -2i, 2i, -2i$

6. $\sqrt{7}, -\sqrt{7}, i\sqrt{7}, -i\sqrt{7}$

7. $\sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}$

8. $i, -i, \sqrt{11}, -\sqrt{11}$

9. $\sqrt{\frac{5+3\sqrt{3}}{2}}, \quad -\sqrt{\frac{5+3\sqrt{3}}{2}}, \quad \sqrt{\frac{5-3\sqrt{3}}{2}}, \quad -\sqrt{\frac{5-3\sqrt{3}}{2}},$

10. $\sqrt{\sqrt{13}-4}, \quad -\sqrt{\sqrt{13}-4}, \quad i\sqrt{\sqrt{13}+4}, \quad -i\sqrt{\sqrt{13}+4}$

11. $\frac{\sqrt{2\sqrt{2}-2}+i\sqrt{2+2\sqrt{2}}}{2}, \qquad \frac{\sqrt{2\sqrt{2}-2}-i\sqrt{2+2\sqrt{2}}}{2},$

    $-\frac{\sqrt{2\sqrt{2}-2}+i\sqrt{2+2\sqrt{2}}}{2}, \qquad -\frac{\sqrt{2\sqrt{2}-2}-i\sqrt{2+2\sqrt{2}}}{2},$

    or

    $0.455 + 1.09868i, 0.455 - 1.09868i, -0.455 + 1.09868i, -0.455 - 1.09868i$

12. $\sqrt{\frac{\sqrt{17}-3}{2}}, \quad -\sqrt{\frac{\sqrt{17}-3}{2}}, \quad i\sqrt{\frac{\sqrt{17}+3}{2}}, \quad -i\sqrt{\frac{\sqrt{17}+3}{2}}$

13. $\sqrt{\sqrt{12}-3}, \quad -\sqrt{\sqrt{12}-3}, \quad i\sqrt{\sqrt{12}+3}, \quad -i\sqrt{\sqrt{12}+3},$

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