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Decomposition, Packings and Coverings of Complete Digraphs with a Transitive-Triple and a Pendant Arc.

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Decomposition, Packings and Coverings of Complete Digraphs with a
Transitive-Triple and a Pendant Arc

A thesis
presented to
the faculty of the Department of Mathematics
East Tennessee State University
In partial fulfillment
of the requirements for the degree
Master of Science in Mathematical Sciences

by
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Keywords: graph theory, design theory, decompositions, packings, coverings
In the study of design theory, there are eight orientations of the complete graph on three vertices with a pendant edge, $K_3 \cup \{e\}$. Two of these are the 3-circuit with a pendant arc and the other six are transitive triples with a pendant arc. Necessary and sufficient conditions are given for decompositions, packings and coverings of the complete digraph with each of the six transitive triples with a pendant arc.
DEDICATION

This thesis is dedicated to my father, Frank Hierholzer, for all those science kits and educational toys that he bought for me and for the many hours he spent helping me with my homework. His own academic example (M.S. in E.E and over 50 graduate credits beyond that) and love of learning set the stage for me to excel.
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1 INTRODUCTION

Design theory, an intriguing branch of combinatorial mathematics, has applications in many fields. Some applications in the area of computer science and electronics are “the theory of parallel algorithms, the design of file organization schemes, the design of hardware switches, and the analysis of algorithms[17].” In design theory, using graphs to represent structures, we study decompositions, packings and coverings with a smaller structure in order to better understand the whole structure.

Graphs provide a visible link between theory and application that makes them ideal for design theory. A graph $G$ consists of a set of elements together with a binary relation defined on the set. In a graph, the elements are represented by points (vertices) and the binary relation is represented by lines (edges) joining pairs of points. A directed graph (digraph) $D$ is simply a graph where the edges (arcs) have been assigned a direction. If two vertices have an edge between them, we say that they are adjacent. A complete graph on $v$ vertices, $K_v$, is a graph where every vertex is adjacent to every other vertex in the graph. The complete digraph on $v$ vertices, $D_v$, is formed by replacing each edge in $K_v$ with two arcs of opposite orientation. As an example, see Figure 1 which shows $K_3$, the complete graph on three vertices and $D_3$, the complete digraph on three vertices.

The degree of a vertex $u$ in a graph, $G$, is defined as the number of edges that are adjacent to $u$. Directed graphs, however, have out degrees and in degrees for each vertex. The out degree, $od(u)$, of vertex $u$ in $D$ is defined as the number of vertices of $D$ that are adjacent from $u$. The in degree, $id(u)$ of vertex $u$ in $D$ refers to the number of vertices of $D$ that are adjacent to $u$. The total degree of vertex $u$ in $D$ is
$od(u) + id(u)$.

A decomposition of a digraph with isomorphic copies of digraph $d$ is a set $\{d_1, d_2, ..., d_n\}$ where $d_i \cong d$ and $V(d_i) \subset V(D)$ for all $i$ and $A(d_i) \cap A(d_j) = \emptyset$ for $i \neq j$ and the union over all $d_i$’s gives the digraph $D$. The $d_i$’s are called the blocks of the decomposition while $V(D)$ is the vertex set of $D$ and $A(D)$ is the arc set.

Many types of decompositions have been studied that led to this research. A triple system is a graph (or digraph) decomposition into isomorphic copies of a graph (or digraph) on three vertices. Steiner Triple systems, denoted $STS(v)$, are decompositions of $K_v$ into $K_3$’s. Figure 2 shows the two orientations of $D_3$ (called directed triples) which are known as the 3-circuit and the transitive triple. The 3-circuit
and the *transitive triple* decompositions of $D_v$ are called *Mendelsohn triple systems* ($MTS(v)$) and *directed triple systems* ($DTS(v)$), respectively. In addition, $L$ (from Figure 2) decompositions of $D_v$ have been studied.

The concentration of this thesis is the three *transitive triple* orientations, applied to the $K_3$ subgraph in the $L$ and the two different orientations on the pendent arc. We will decompose, pack and cover complete digraphs with these six orientations. These orientations are labeled $d_1, d_2, d_3, d_4, d_5$, and $d_6$ as shown in Figure 3.

![Figure 3: Six Orientations of $K_3 \cup \{e\}$](image)

Figure 3: Six Orientations of $K_3 \cup \{e\}$.
These six orientations are denoted as \((a, b, c) - (d)_{d1}, (a, b, c) - (d)_{d2}, \ldots, (a, b, c) - (d)_{d6}\), respectively.
2 DECOMPOSITIONS

2.1 INTRODUCTION

Many mathematicians have studied decompositions of a complete graph. These are some of the results of their studies that motivated the research for this thesis:

- A $STS(v)$ exists if and only if $v \equiv 1 \text{ or } 3 \pmod{6}$ [13],
- A $MTS(v)$ exists if and only if $v \equiv 0 \text{ or } 1 \pmod{3}$, $v \neq 6$ [12],
- A $DTS(v)$ exists if and only if $v \equiv 0 \text{ or } 1 \pmod{3}$ [8], and
- A decomposition of $K_v$ into copies of $K_3$ with a pendant edge (the graph $L$ of Figure 2) exists if and only if $v \equiv 0 \text{ or } 1 \pmod{8}$ [1].

Figure 4 shows the decomposition of $D_3$ into two isomorphic copies of transitive triples.

![Figure 4: $D_3$ Decomposition with Transitive Triples.](image)

Figure 4: $D_3$ Decomposition with Transitive Triples.
2.2 DIFFERENCE METHOD

The *difference method* is a method of decomposing a complete graph using the distances (differences) between vertices. We define *differences* as follows. Suppose we have a complete graph with \( v \) vertices. We start by labelling the vertices 0 through \( v - 1 \). The *difference* of the arc from vertex \( a \) to vertex \( b \), is defined as \( b - a \mod (v - 1) \). The complete graph, \( D_v \), has \( N = v - 1 \) differences.

We define *orbit* as follows. The *orbit* of a block \( b \) under permutation \( \pi \) is the set \( \{ \pi^n(b) | n = 0, 1, 2, \ldots \} \). A set \( \{b_1, b_2, \ldots, b_n\} \) is a set of *base blocks under permutation* \( \pi \) of a \( b \)-decomposition of graph \( G \) if each \( b_i \) is isomorphic to \( b \) and the orbits of the blocks of the \( b_i \) generate a \( b \)-decomposition of \( G \) and the orbit of \( b_i \) is disjoint from the orbit of \( b_j \) when \( i \neq j \).

To illustrate the difference method and the use of fixed points, we will decompose the complete digraph, \( D_v \), with \( v \equiv 0 \mod 4 \). Suppose we have a complete digraph \( D_v \) of 24 vertices (see Figure 5). This implies that \( v \equiv 0 \mod 4 \) or \( v = 4k \). Since \( v = 24 \), we have 23 differences. We are decomposing with \( d_1 \) which has four arcs so the number of differences has to be divisible by 4. We should then use one fixed point because that will bring the number of differences down to 22. Then, we use two differences with our base block containing the fixed point, making the number of differences 20. So, we use fixed points when 4 does not divide \( v \). Since each fixed point uses up three differences, we keep adding fixed points until the number of differences left (after we have subtracted three differences per fixed point) is a multiple of 4.

Now we let vertex \( c \) in \( d_1 \) equal \( \infty \) for the base block containing the fixed point. Next we write all the differences down (1 through 22 in this case). We use the last
two differences in the base block containing the fixed point. We write our base blocks as \((a, b, c)-(d)_{d_1}\). The base block containing the fixed point is \((0, 2, \infty)-(1)\) which takes care of the arcs \((\infty, 0), (2, \infty), (1,0),\) and \((2,0)\). Then we permute this base block as follows \(\{(0 + j, 2 + j, \infty) - (1 + j)_{d_1}|j = 0, 1, \ldots, 22\}\) in order to cover all the arcs to and from \(\infty\) and the differences of 21 and 22.

Next, we write the blocks for the differences of 10, 15, 16, and 1. We chose these using the difference method for transitive triples which is: if \(a, b\) and \(c\) are vertices of the base block, then we choose our differences such that \((c-b)+(a-c)=(a-b)\). So our base block for the differences of 10, 15, 16, and 1 is \((0, 7, 8)-(13)_{d_1}\). We then take this base block and permute it around the circle shown in Figure 5 in order to use up all these differences. In mathematical notation, this is \(\{(0 + j, 7 + j, 8 + j) - (13 + j)_{d_1}|j =\)
0, 1, . . . , 22}. Similarly, the rest of our blocks are \{(j, 6 + j, 9 + j) − (15 + j)_{d1}, (j, 5 + j, 10 + j) − (17 + j)_{d1}, (j, 4 + j, 11 + j) − (19 + j)_{d1}, (j, 3 + j, 12 + j) − (21 + j)_{d1} | j = 1, 2, . . . , 22\}.

Now, we check to see if we have covered all the arcs in $D_{24}$. In our base block with the fixed point, we have used 23 times 4 arcs which equals 92. In the other five blocks, we’ve used 23 times 4 times 5 arcs which equals 460. When we add these together, we get 552 arcs which is 24 times 23 which equals the number of arcs in $D_{24}$.

This process is then extended to all the complete digraphs, $D_v$, with $v \equiv 0 \pmod{4}$, i.e. $v = 4k$. Figure 6 shows the differences marked for the general case where $v = 4k$. Next we write the blocks which are annotated in Theorem 2.1, case 4.

Figure 6: A $d_1$ Decomposition of $D_v$ with $v = 4k$ Vertices Using the Difference Method.
2.3 RESULTS

Theorem 2.1 A $d_1$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$.

Proof. The necessary condition for a $d_1$-decomposition of $D_v$ to exist is $v \equiv 0$ or $1 \pmod{4}$ since 4 divides $v$ if and only if $v \equiv 0$ or $1 \pmod{4}$. We show sufficiency in four cases.

Case 1. Suppose $v \equiv 1 \pmod{12}$, say $v = 12k + 1$. Consider the blocks:

$$\{(j, 6k - i + j, 12k - 2i + j) - (3k + 1 + i + j)_{d_1},$$

$$(j, 5k - i + j, 10k - 2i + j) - (8k + 1 + 2i + j)_{d_1}$$

$$| i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 12k\}$$

$$\bigcup\{(j, k - 1 - i + j, 12k - 3 - 2i + j) - (2k + 2 + i + j)_{d_1}$$

$$| i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 12k\}$$

$$\bigcup\{(j, k + j, 12k - 1 + j) - (k + 1 + j)_{d_1} | j = 0, 1, \ldots, 12k\}.$$ 

Case 2. Suppose $v \equiv 5 \pmod{12}$, say $v = 12k + 5$. Consider the blocks:

$$\{(j, 6k + 2 - i + j, 12k + 4 - 2i + j) - (3k + 1 + i + j)_{d_1},$$

$$(j, 5k + 1 - i + j, 10k + 2 - 2i + j) - (8k + 5 + 2i + j)_{d_1}$$

$$| i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 12k + 4\}$$

$$\bigcup\{(j, k - 1 - i + j, 12k + 1 - 2i + j) - (2k + 2 + i + j)_{d_1}$$

$$| i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 12k + 4\}.$$
Case 3. Suppose \( v \equiv 9 \pmod{12} \), say \( v = 12k + 9 \). Consider the blocks:

\[
\bigcup \{(j, 5k + 2 + j, 10k + 4 + j) - (4k + 1 + j)_{d_1}, (j, k + j, 12k + 3 + j) - (k + 1 + j)_{d_1} \mid j = 0, 1, \ldots, 12k + 4 \}\]

In each of Cases 1–3, the given set of blocks forms a decomposition of \( D_v \) where \( V(D_v) = \{0, 1, \ldots, v - 1\} \) and vertex labels in the blocks are reduced modulo \( v \).

Case 4. Suppose \( v \equiv 0 \pmod{4} \), say \( v = 4k \). Consider the blocks:

\[
\bigcup \{(j, 6k + 4 - i + j, 12k + 8 - 2i + j) - (3k + 4 + i + j)_{d_1}, (j, 5k + 3 - i + j, 10k + 6 - 2i + j) - (8k + 7 + 2i + j)_{d_1}, (j, k - i + j, 12k + 5 - 2i + j) - (2k + 4 + i + j)_{d_1} \mid i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 12k + 8 \}
\]

\[
\bigcup \{(j, 5k + 4 + j, 10k + 8 + j) - (8k + 6 + j)_{d_1}, (j, k + 1 + j, 12k + 7 + j) - (k + 2 + j)_{d_1} \mid j = 0, 1, \ldots, 12k + 8 \}\]

In Case 4, the given set of blocks forms a decomposition of \( D_v \) where \( V(D_v) = \{\infty, 0, 1, \ldots, v - 2\} \) and numerical vertex labels in the blocks are reduced modulo \( v - 1 \). □

Corollary 2.2 A \( d_2 \)-decomposition of \( D_v \) exists if and only if \( v \equiv 0 \) or 1 \( (\text{mod} \ 4) \).

Proof. The necessary condition follows as in Theorem 2.1. Since the converse of \( d_1 \) is \( d_2 \) and the \( D_v \) is self converse, the result follows trivially from Theorem 2.1. □
Corollary 2.3 A $d_3$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$.

Proof. The necessary condition follows as in Theorem 2.1. In the case $v \equiv 1 \pmod{4}$, blocks for such a system can be constructed from the $d_1$ system of Theorem 2.1 by replacing every block of the form $(j, a+j, b+j) - (c+j)d_1$ with a block of the form $(a-b+j, a+j, j) - (a-b+c+j)d_3$. In the case $v \equiv 0 \pmod{4}$, blocks for such a system can be constructed from the $d_1$ system of Theorem 2.1 by replacing every block of the form $(j, a+j, b+j) - (c+j)d_1$ with a block of the form $(a-b+j, a+j, j) - (a-b+c+j)d_3$ and by replacing every block of the form $(j, a+j, \infty) - (c+j)d_1$ with a block of the form $(a+j, \infty, j) - (a+c+j)d_3$. □

Corollary 2.4 A $d_4$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$.

Proof. The necessary condition follows as in Theorem 2.1. Since the converse of $d_4$ is $d_3$ and the $D_v$ is self-converse, the result follows trivially from Corollary 2.3. □

Corollary 2.5 A $d_5$-decomposition of $D_v$ exists if and only if $v \equiv 1 \pmod{4}$.

Proof. As in Theorem 2.1, one necessary condition is that $v \equiv 0$ or $1 \pmod{4}$. Notice that the vertices of $d_5$ are of in-degree $0, 0, 2, 2$. Therefore another necessary condition for a $d_5$-design on $D_v$ is that each vertex of $D_v$ is of in-degree even — that is, $v$ must be odd. Therefore $v \equiv 1 \pmod{4}$ is necessary.

Blocks for such a system can be constructed from the $d_1$ system of Theorem 2.1 by replacing every block of the form $(j, a+j, b+j) - (c+j)d_1$ with a base block of the form $(b+j, j, a+j) - (b+c+j)d_5$. □
Corollary 2.6 A $d_6$-decomposition of $D_v$ exists if and only if $v \equiv 1 \pmod{4}$.

Proof. The necessary condition follows as in Corollary 2.5. Since the converse of $d_5$ is $d_4$ and the $D_v$ is self-converse, the result follows trivially from Corollary 2.5. □
3 PACKINGS

3.1 INTRODUCTION

When a decomposition does not exist, we ask, “How close to a decomposition can we get?” In this section, we will consider one way to answer this question which is with packings. In a \( g \)-packing of \( D_v \), we remove isomorphic copies of \( g \) without repeating any arcs until we cannot remove more copies of \( g \). The packing is said to be maximal if the number of arcs left over or not used (the leave) in \( D_v \) is minimal. The formal definition for a maximal packing follows.

A maximal packing of a directed graph \( G \) with isomorphic copies of a graph \( g \) is a set \( \{ g_1, g_2, \ldots, g_n \} \) where \( g_i \cong g \) and \( V(g_i) \subset V(G) \) for all \( i \) and \( A(g_i) \cap A(g_j) = \phi \) for \( i \neq j \) and \( \bigcup^n_i g_i \subset G \) and

\[
|A(L)| = |A(G)/\bigcup^n_i g_i|
\]

is minimal, where \( V(G) \) is the vertex set, \( A(G) \) is the arc set of the graph \( G \) and \( L \) represents the leave of the packing.

The following lemma is an example of a maximal \( d_1 \) packing of \( D_6 \). Figure 7 illustrates the maximal \( d_1 \) packing of \( D_6 \) which consists of six \( d_1 \)'s.
Lemma 3.1  A $d_1$ packing of $D_6$ with minimal leave $L$ exists and consists of exactly six $d_1$'s with $|A(L)| = 6$.

Proof.
Since $|A(D_6)| = 30$ and $|A(d_1)| = 4$, seven $d_1$'s should fit in $D_6$. However, since each vertex in $D_6$ has an in-degree of five, and vertex $a$ of $d_1$ has an in-degree of three, only one vertex $a$ of $d_1$ will fit per vertex of $D_6$. Therefore the maximum packing is six $d_1$'s and not seven. Here are the blocks and the leave of a maximal packing:

\[
\{ (0,2,4) - (3)_{d_1}, (1,2,3) - (0)_{d_1}, (2,4,3) - (1)_{d_1}, \\
(3,5,1) - (0)_{d_1}, (4,5,6) - (3)_{d_1}, (5,1,0) - (4)_{d_1} \}
\]

\[A(L) = \{ (3,5), (0,2), (2,5), (5,2), (1,4), (4,1) \} \]

□

![Diagram](image)

Figure 7: The Packing and Leave of a Maximal $d_1$-Packing of $D_6$. 

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3.2 RESULTS

Theorem 3.1 A maximal $d_1$-packing of $D_v$ with leave $L$ satisfies

(i) $|A(L)| = 0$ if $v \equiv 0$ or 1 (mod 4), and

(ii) $|A(L)| = 2$ if $v \equiv 2$ or 3 (mod 4), $v \neq 6$, and

(iii) $|A(L)| = 6$ if $v = 6$.

Proof. Clearly it is necessary that $|A(L)| \equiv |A(D_v)|(\text{mod } 4)$. We show that, with the exception of $v = 6$, $|A(L)| = |A(D_v)|(\text{mod } 4)$.

Case 1. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. If $v \equiv 2$ (mod 4), say $v = 4k + 2$, where $k \neq 1, 2, 3, 4, 5$ then consider the blocks in $A \cup B$ where

$$A = \{j, k + 9 - i + j, k + 10 + i + j) - (2k + 5 + 2i + j)d_1$$

$$| i = 0, 1, \ldots, k - 6, j = 0, 1, \ldots, 4k - 6\}$$

$$\cup \{(j, 2i + 2 + j, \infty_i) - (2i + 1 + j)d_1 | i = 0, 1, \ldots, 6, j = 0, 1, \ldots, 4k - 6\}$$

and

$$B = \{(\infty_0, \infty_1, \infty_3) - (\infty_6)d_1, (\infty_1, \infty_2, \infty_4) - (\infty_6)d_1, (\infty_2, \infty_5, \infty_3) - (\infty_1)d_1,$$

$$ (\infty_3, \infty_6, \infty_4) - (\infty_2)d_1, (\infty_4, \infty_5, \infty_0) - (\infty_3)d_1, (\infty_5, \infty_1, \infty_6) - (\infty_4)d_1,$$

$$ (\infty_6, \infty_2, \infty_0) - (\infty_5)d_1, (\infty_2, \infty_4, \infty_0) - (\infty_6)d_1,$$

$$ (\infty_1, \infty_3, \infty_6) - (\infty_5)d_1, (\infty_5, \infty_0, \infty_3) - (\infty_2)d_1\}.\]
Then $A \cup B$ is a maximal $d_1$-packing of $D_v$ with leave $L$ where $A(L) = \{(\infty_1, \infty_4), (\infty_4, \infty_6)\}$, and so the packing is maximal. The given set of blocks forms a packing of $D_v$ where $V(D_v) = \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, 0, 1, \ldots, v-8\}$ and vertex labels in the blocks are reduced modulo $v - 7$.

Case 3. If $v \equiv 3 \pmod{4}$, say $v = 4k + 3$, $k \neq 1$ then consider the blocks in $A \cup B$

where

$A = \{(j, k+3-i+j, 4k-2i+j)-(2k+3+2i+j)_{d_1} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 4k\}$

and

$B = \{(j, 1+2i+j, \infty_{i}) – (2+2i+j)_{d_1} \mid i = 0, 1, j = 0, 1, \ldots, 4k\}$

Then $A \cup B$ is a maximal $d_1$-packing of $D_v$ with leave $L$ where $A(L) = \{(\infty_0, \infty_1), (\infty_1, \infty_0)\}$, and so the packing is maximal. The given set of blocks forms a packing of $D_v$ where $V(D_v) = \{\infty_0, \infty_1, 0, 1, \ldots, v-3\}$ and vertex labels in the blocks are reduced modulo $v - 2$.

Case 4. If $v = 6$ then Lemma 3.1 applies.

Case 5. If $v = 7$ then consider the blocks:

$\{(0, 1, 3) – (6)_{d_1}, (1, 2, 4) – (0)_{d_1}, (2, 5, 3) – (1)_{d_1}, (3, 6, 4) – (2)_{d_1}, (4, 5, 0) – (3)_{d_1}, (5, 1, 6) – (4)_{d_1}, (6, 2, 0) – (5)_{d_1}, (2, 4, 0) – (6)_{d_1}, (1, 3, 6) – (5)_{d_1}, (5, 0, 3) – (2)_{d_1}\}$

The leave $L$ is $A(L) = \{(1, 4), (4, 6)\}$ so the packing is maximal. □
Corollary 3.2 A maximal $d_2$-packing of $D_v$ with leave $L$ satisfies

(i) $|A(L)| = 0$ if $v \equiv 0$ or $1 \pmod{4}$, and

(ii) $|A(L)| = 2$ if $v \equiv 2$ or $3 \pmod{4}$, $v \neq 6$, and

(iii) $|A(L)| = 6$ if $v = 6$.

Proof. The necessary condition follows as in Theorem 3.1. Since the converse of $d_1$ is $d_2$ and the $D_v$ is self-converse, the result follows trivially from Theorem 3.1. □

Theorem 3.3 A maximal $d_3$-packing of $D_v$ with leave $L$ satisfies

(i) $|A(L)| = 0$ if $v \equiv 0$ or $1 \pmod{4}$, and

(ii) $|A(L)| = 2$ if $v \equiv 2$ or $3 \pmod{4}$.

Proof. The necessary condition follows as in Theorem 3.1. We consider sufficiency in five cases.

Case 1. If $v \equiv 0$ or $1 \pmod{4}$, then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. Suppose $v \equiv 2 \pmod{4}$, blocks for such a system can be constructed from the $d_1$ system of Theorem 3.1 by replacing every block of the form $(j, a+j, b+j)-(c+j)_{d_1}$ with a block of the form $(a-b+j, a+j, j)-(a-b+c+j)_{d_3}$.

Case 3. Suppose $v \equiv 3 \pmod{4}$, blocks for such a system can be constructed from the $d_1$ system of Theorem 3.1 by replacing every block of the form $(j, a+j, b+j)-(c+j)_{d_1}$ with a block of the form $(a-b+j, a+j, j)-(a-b+c+j)_{d_3}$ and by replacing every block
of the form \((j, a+j, \infty) - (c+j)_{d1}\) with a block of the form \((a+j, \infty, j) - (a+c+j)_{d3}\).

Case 4. Suppose \(v = 6\), consider the following blocks:

\[
\{(4,1,3)-(0)_{d3}, (4,5,2)-(3)_{d3}, (5,3,0)-(1)_{d3}, (3,1,2)-(0)_{d3},
\]

\[
(0,5,1)-(2)_{d3}, (1,4,0)-(2)_{d3}, (2,5,4)-(0)_{d3}\}.
\]

The leave \(L\) is \(A(L) = \{(2,3), (3,5)\}\) so the packing is maximal.

Case 5. Suppose \(v = 7\), consider the following blocks:

\[
\{(0,5,1)-(3)_{d3}, (1,5,2)-(4)_{d3}, (2,5,3)-(0)_{d3}, (3,5,4)-(1)_{d3},
\]

\[
(4,5,0)-(2)_{d3}, (0,6,3)-(1)_{d3}, (1,6,4)-(2)_{d3}(2,6,0)-(3)_{d3},
\]

\[
(3,6,1)-(4)_{d3}(4,6,2)-(0)_{d3}\}.
\]

The leave \(L\) is \(A(L) = \{(5,6), (6,5)\}\) so the packing is maximal. □

Corollary 3.4 A maximal \(d_4\)-packing of \(D_v\) with leave \(L\) satisfies

(i) \(|A(L)| = 0\) if \(v \equiv 0\) or 1 \((\text{mod } 4)\), and

(ii) \(|A(L)| = 2\) if \(v \equiv 2\) or 3 \((\text{mod } 4)\).

Proof. The necessary condition follows as in Theorem 3.1. Since the converse of \(d_3\)
is \(d_4\) and the \(D_v\) is self converse, the result follows trivially from Theorem 3.3. □

Lemma 3.5 A maximal \(d_5\)-packing of \(D_v\) with leave \(L\) has \(|A(L)| \geq v\) if \(v \equiv 0\) or 2 \((\text{mod } 4)\).
Proof. Each vertex of $D_v$ is of in-degree $v - 1$ (which is odd) and each vertex of $d_5$ is of in-degree even. Therefore, in a maximal packing, we are left with each vertex of $D_v$ of in-degree at least 1. Thus, $|A(L)| \geq v$. □

Theorem 3.5 A maximal $d_5$-packing of $D_v$ with leave $L$ satisfies

(i) $|A(L)| = 0$ if $v \equiv 1 \pmod{4}$,

(ii) $|A(L)| = v$ if $v \equiv 0$ or $2 \pmod{4}$, and

(iii) $|A(L)| = 2$ if $v \equiv 3 \pmod{4}$.

Proof. The necessary conditions follow as in Theorem 3.1 when $v \equiv 1$ or $3 \pmod{4}$ and follow from Lemma 3.5 when $v \equiv 0$ or $2 \pmod{4}$. We consider sufficiency in four cases.

Case 1. If $v \equiv 1 \pmod{4}$, then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. If $v \equiv 0 \pmod{4}$, then by Lemma 3.5, $|A(L)| \geq v$. Consider the following blocks in $A \cup B$ where:

$$A = \{(2i, 4k - 1 + 2i, 1 + 2i) - (4k - 2 + 2i)_{d_5} | i = 0, 1, \ldots, 2k - 1\}$$

and

$$B = \{(j, 3k - 3 + j, 4k - 2 + j) - (3k - 2 + j)_{d_5} | j = 0, 1, \ldots, 4k - 1\}$$

$$\cup\{(j, 2k - 1 + i + j, 2k + 2 + 2i + j) - (2k - 3 - 2i + j)_{d_5} | i = 0, 1, \ldots, k - 3, j = 0, 1, \ldots, 4k - 1\}.$$ 

Then $A \cup B$ is a maximal $d_5$-packing of $D_v$ with leave $L$ where

$$A(L) = \{(j - 1, j) | j = 0, 1, \ldots, 4k - 1\}.$$
The given set of blocks forms a packing of $D_v$ where $V(D_v) = \{0, 1, \ldots, v - 1\}$ and vertex labels in the blocks are reduced modulo $v$.

Case 3. If $v \equiv 2 \pmod{4}$, then by Corollary 3.5, $|A(L)| \geq v$. Consider the following blocks:

$$A = \{(j, k + 2 + i + j, 1 + 2i + j) - (2k + 2 + 2i)d_5 | i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 4k + 1\}$$

Then $A$ is a maximal $d_5$-packing of $D_v$ with leave $L$ where

$$A(L) = \{(j - 1, j) | j = 0, 1, \ldots, 4k - 1\}.$$

The given set of blocks forms a packing of $D_v$ where $V(D_v) = \{0, 1, \ldots, v - 1\}$ and vertex labels in the blocks are reduced modulo $v$.

Case 4. If $v \equiv 3 \pmod{4}$ Consider the following blocks in $A \cup B$ where:

$$A = \{(2i, 4k + 2 + 2i, 1 + 2i) - (4k + 1 + 2i)d_5 | i = 0, 1, \ldots, 2k\}$$

and

$$B = \{(j, 3k - 1 + j, 4k + 2 + j) - (4k + j)d_5 | j = 0, 1, \ldots, 4k + 2\}$$

$$\cup \{(j, 2k + i + j, 2k + 4 + 2i + j) - (2 + 2i + j)d_5 | i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 4k - 1\}.$$  

Then $A \cup B$ is a maximal $d_5$-packing of $D_v$ with leave $L$ where

$$A(L) = \{(4k + 1, 4k + 2), (4k, 4k + 2)\}.$$

The given set of blocks forms a packing of $D_v$ where $V(D_v) = \{0, 1, \ldots, v - 1\}$ and vertex labels in the blocks are reduced modulo $v$. □
Corollary 3.6 A maximal $d_6$-packing of $D_v$ with leave $L$ satisfies

(i) $|A(L)| = 0$ if $v \equiv 1 \pmod{4}$,

(ii) $|A(L)| = v$ if $v \equiv 0$ or $2 \pmod{4}$, and

(iii) $|A(L)| = 2$ if $v \equiv 3 \pmod{4}$.

Proof. The necessary condition follows as in Theorem 3.5. Since the converse of $d_5$ is $d_6$ and the $D_v$ is self converse, the result follows trivially from Theorem 3.5. □
4 COVERINGS

4.1 INTRODUCTION

In answering the question, “How close to a decomposition can we get?”, our second response is to consider coverings. In a \textit{g-covering} of \(D_v\), we cover \(D_v\) with isomorphic copies of \(g\) until every arc of \(D_v\) is covered. The covering is said to be \textit{minimal} if the number of arcs repeated (\textit{the padding}) in \(D_v\) is minimal. The formal definition for a minimal covering follows.

A \textit{minimal covering} of a simple graph \(G\) with isomorphic copies of a graph \(g\) is a set \(\{g_1, g_2, \ldots, g_n\}\) where \(g_i \cong g\) and \(V(g_i) \subset V(G)\) for all \(i\), \(G \subset \bigcup_{i=1}^{n} g_i\), and

\[
|A(P)| = |\bigcup_{i=1}^{n} A(g_i) \setminus A(G)|
\]

is minimal (the graph \(\bigcup_{i=1}^{n} g_i\) may not be simple and \(\bigcup_{i=1}^{n} E(g_i)\) may be a multiset). The graph \(P\) is called the \textit{padding} of the covering.

The following lemma is an example of finding the minimal covering of \(D_7\) with isomorphic copies of \(d_5\). Figure 8 shows \(d_5\) and blocks \(A\) and \(B\) from the proof of Lemma 4.1. Block \(A\) is permuted by adding 2 to each vertex three times. It produces all arcs with differences of 1 and 2 and it produces the padding. Block \(B\) is permuted by adding 1 to each vertex (six times) and produces all arcs with differences of 4, 5 and 6. Block \(A\) is another example of the difference method in that a difference of 1 plus a difference of 1 equals a difference of 2. Block \(B\) also demonstrates the difference method because a difference of 5 plus a difference of 6 equals a difference of 4 (mod 7).
Lemma 4.1 A $d_5$ covering of $D_7$ with minimal padding $P$ exists and consists of exactly eleven $d_5$’s with $|A(L)| = 2$.

Proof.

If $v = 7$, then consider the blocks in $A \cup B$ where (See Figure 8):

\[ A = \{(1 + 2i, 0 + 2i, 2 + 2i) - (6 + 2i)_{d_5} | i = 0, 1, 2, 3\} \]

and

\[ B = \{(1 + j, 3 + j, 0 + j) - (5 + j)_{d_5} | j = 0, 1, ..., 6\} \].

Then $A \cup B$ is a minimal $d_5$-covering of $D_7$ with padding $P$ where

\[ A(P) = \{(0, 1), (6, 1)\}. \]

\[ \square \]

Figure 8: Blocks for a Minimal $d_5$-Covering of $D_7$. 
4.2 RESULTS

Theorem 4.1 A minimal \( d_1 \)-covering of \( D_v \) with padding \( P \) satisfies

(i) \( |A(P)| = 0 \) if \( v \equiv 0 \) or \( 1 \) (mod 4), and

(ii) \( |A(P)| = 2 \) if \( v \equiv 2 \) or \( 3 \) (mod 4).

Proof. Clearly it is necessary that \( |A(D_v)| - |A(P)| \equiv 0 \) (mod 4). We show that, \( |A(P)| \) is the smallest value possible, namely \( |A(P)| = |A(D_v)| \) (mod 4).

Case 1. If \( v \equiv 0 \) or \( 1 \) (mod 4), then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. If \( v \equiv 2 \) (mod 4), say \( v = 4k + 2 \), where \( k \neq 1, 2, 3, 4, 5 \) then consider the blocks in \( A \cup B \cup \{ (\infty_6, \infty_1, \infty_4) - (\infty_3, d_1) \} \) where sets \( A \) and \( B \) are defined in Theorem 3.1. This is a minimal covering of \( D_v \) with padding \( P \) where \( A(P) = \{ (\infty_1, \infty_6), (\infty_3, \infty_6) \} \).

Case 3. If \( v = 6 \), then consider the blocks

\[
\{(5, 0, 1) - (4)_{d_1}, (1, 5, 4) - (2)_{d_1}, (3, 1, 0) - (5)_{d_1}, (2, 4, 3) - (1)_{d_1},
(4, 3, 1) - (0)_{d_1}, (0, 2, 4) - (3)_{d_1}, (5, 2, 3) - (4)_{d_1}, (2, 5, 0) - (3)_{d_1}\}.
\]

The padding \( P \) is \( A(P) = \{ (3, 2), (4, 5) \} \) so the covering is minimal.

Case 4. If \( v \equiv 3 \) (mod 4), say \( v = 4k + 3 \), where \( k \neq 1 \) then consider the blocks in \( A \cup B \cup \{ (\infty_1, 3, 0) - (\infty_0)_{d_1} \cup (0, \infty_1, \infty_0) - (4)_{d_1} \} \) where sets \( A \) and \( B \) are defined in Theorem 3.1. This is a minimal covering of \( D_v \) with padding \( P \).
where $A(P) = \{(\infty_0, 0), (\infty_1, 0)\}$.

Case 5. If $v = 7$, then consider the blocks

$$
\{ (0,1,3) - (6)_{d1}, (1,2,4) - (0)_{d1}, (2,5,3) - (1)_{d1}, (3,6,4) - (2)_{d1},

(4,5,0) - (3)_{d1}, (5,1,6) - (4)_{d1}, (6,2,0) - (5)_{d1}, (2,4,0) - (6)_{d1},

(1,3,6) - (5)_{d1}, (5,0,3) - (2)_{d1}, (6,1,4) - (3)_{d1} \}.
$$

The padding $P$ is $A(P) = \{(1,6), (3,6)\}$ so the covering is minimal. □

Corollary 4.2 A minimal $d_2$-covering of $D_v$ with padding $P$ satisfies

(i) $|A(P)| = 0$ if $v \equiv 0$ or $1 \pmod{4}$, and

(ii) $|A(P)| = 2$ if $v \equiv 2$ or $3 \pmod{4}$.

Proof. The necessary condition follows as in Theorem 4.1. Since the converse of $d_1$ is $d_2$ and the $D_v$ is self converse, the result follows trivially from Theorem 4.1. □

Theorem 4.3 A minimal $d_3$-covering of $D_v$ with padding $P$ satisfies

(i) $|A(P)| = 0$ if $v \equiv 0$ or $1 \pmod{4}$, and

(ii) $|A(P)| = 2$ if $v \equiv 2$ or $3 \pmod{4}$.

Proof. The necessary condition follows as in Theorem 4.1. We consider sufficiency in five cases.

Case 1. If $v \equiv 0$ or $1 \pmod{4}$, then there is a decomposition by Theorem 2.1 and the result follows.
Case 2. Suppose $v \equiv 2 \pmod{4}$, blocks for such a system can be constructed from the $d_1$ system of Corollary 4.1 by replacing every block of the form $(j, a+j, b+j) - (c+j)_{d_1}$ with a block of the form $(a-b+j, a+j, j) - (a-b+c+j)_{d_3}$.

Case 3. Suppose $v \equiv 3 \pmod{4}$, blocks for such a system can be constructed from the $d_1$ system of Corollary 4.1 by replacing every block of the form $(j, a+j, b+j) - (c+j)_{d_1}$ with a block of the form $(a-b+j, a+j, j) - (a-b+c+j)_{d_3}$ and by replacing every block of the form $(j, a+j, \infty) - (c+j)_{d_1}$ with a block of the form $(a+j, \infty, j) - (a+c+j)_{d_3}$.

Case 4. Suppose $v = 6$, consider the following blocks:

\[
\{(4,1,3) - (0)_{d_3}, (4,5,2) - (3)_{d_3}, (5,3,0) - (1)_{d_3}, (3,1,2) - (0)_{d_3},
\]

\[
(0,5,1) - (2)_{d_3}, (1,4,0) - (2)_{d_3}, (2,5,4) - (0)_{d_3}, (2,3,5) - (1)_{d_3}\}.
\]

The padding $P$ is $A(P) = \{(1,2), (2,5)\}$ so the covering is minimal.

Case 5. Suppose $v = 7$, consider the following blocks:

\[
\{(0,5,1) - (3)_{d_3}, (1,5,2) - (4)_{d_3}, (2,5,3) - (0)_{d_3}, (3,5,4) - (1)_{d_3},
\]

\[
(4,5,0) - (2)_{d_3}, (1,6,4) - (2)_{d_3}(2,6,0) - (3)_{d_3},
\]

\[
(3,6,1) - (4)_{d_3}, (4,6,2) - (0)_{d_3}, (6,0,3) - (5)_{d_3}, (0,6,5) - (1)_{d_3}\}.
\]

The padding $P$ is $A(P) = \{(6,0), (0,5)\}$ so the covering is minimal. □

Corollary 4.4 A minimal $d_4$-covering of $D_v$ with padding $P$ satisfies

(i) $|A(P)| = 0$ if $v \equiv 0$ or $1 \pmod{4}$, and

(ii) $|A(P)| = 2$ if $v \equiv 2$ or $3 \pmod{4}$.
Proof. The necessary condition follows as in Theorem 4.1. Since the converse of $d_3$ is $d_4$ and the $D_v$ is self converse, the result follows trivially from Theorem 4.3. □

Lemma 4.5 A minimal $d_5$-covering of $D_v$ with padding $P$ has $|A(P)| \geq v$ if $v \equiv 0$ or $2 \pmod{4}$.

Proof. Each vertex of $D_v$ is of in-degree $v - 1$ (which is odd) and each vertex of $d_5$ is of in-degree even. Therefore, in a minimal covering, each vertex of the covering is of in-degree at least 1. Thus, $|A(P)| \geq v$. □

Theorem 4.5 A minimal $d_5$-covering of $D_v$ with padding $P$ satisfies

(i) $|A(P)| = 0$ if $v \equiv 1 \pmod{4}$, and

(ii) $|A(P)| = v$ if $v \equiv 0$ or $2 \pmod{4}$, and

(iii) $|A(P)| = 2$ if $v \equiv 3 \pmod{4}$.

Proof. The necessary conditions follow as in Theorem 4.1 when $v \equiv 1$ or $3 \pmod{4}$ and follow from Lemma 4.5 when $v \equiv 0$ or $2 \pmod{4}$. We consider sufficiency in four cases.

Case 1. If $v \equiv 1 \pmod{4}$, then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. If $v \equiv 0 \pmod{4}$, then by Lemma 4.5, $|A(P)| \geq v$. Consider the following blocks in $A \cup B$ where:

$$A = \{(j, 2k + j, 2k - 1 + j) - (4k - 1 + j)d_5 | j = 0, 1, \ldots, 4k - 1\}$$
and

\[ B = \{(j, k + 1 + i + j, 1 + 2i + j) - (4k - 2 + j)_{d_5} | i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 4k - 1\}. \]

Then \( A \cup B \) is a minimal \( d_5 \)-covering of \( D_v \) with padding \( P \) where

\[ A(P) = \{(j, j + 1) | j = 0, 1, \ldots, 4k - 1\}. \]

The given set of blocks forms a covering of \( D_v \) where \( V(D_v) = \{0, 1, \ldots, v - 1\} \) and vertex labels in the blocks are reduced modulo \( v \).

Case 3. If \( v \equiv 2 \ (\text{mod} \ 4) \), then by Lemma 4.5, \( |A(P)| \geq v \). Consider the following blocks in \( A \cup B \) where:

\[ A = \{(2i, 4k + 1 + 2i, 1 + 2i) - (4k + 2i)_{d_5} | i = 0, 1, \ldots, 2k\} \]

and

\[ B = \{(j, k + 1 + j, 1 + j) - (2k + 1 + j)_{d_5} | j = 0, 1, \ldots, 4k + 1\} \]

\[ \cup \{(j, k + 2 + i + j, 3 + 2i + j) - (4k - 2 - 2i + j)_{d_5} | i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 4k + 1\}. \]

Then \( A \cup B \) is a minimal \( d_5 \)-covering of \( D_v \) with padding \( P \) where

\[ A(P) = \{(j, j + 1) | j = 0, 1, \ldots, 4k + 1\}. \]

The given set of blocks forms a packing of \( D_v \) where \( V(D_v) = \{0, 1, \ldots, v - 1\} \) and vertex labels in the blocks are reduced modulo \( v \).

Case 4. If \( v \equiv 3 \ (\text{mod} \ 4) \) Consider the following blocks in \( A \cup B \) where:

\[ A = \{(2i, 4k + 2 + 2i, 1 + 2i) - (4k + 1 + 2i)_{d_5} | i = 0, 1, \ldots, 2k + 1\} \]
and

\[ B = \{(j, 3k - 1 + j, 4k + 2 + j) - (4k + j)d_5 | j = 0, 1, \ldots, 4k + 2\} \]

\[ \bigcup \{ (j, 2k + i + j, 2k + 4 + 2i + j) - (2 + 2i + j)d_5 | i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 4k - 1\}. \]

Then \( A \bigcup B \) is a minimal \( d_5 \)-packing of \( D_v \) with padding \( P \) where

\[ A(P) = \{(4k + 1, 0), (4k + 2, 0)\}. \]

The given set of blocks forms a covering of \( D_v \) where \( V(D_v) = \{0, 1, \ldots, v - 1\} \) and vertex labels in the blocks are reduced modulo \( v \). \( \square \)

Corollary 4.6 A minimal \( d_6 \)-covering of \( D_v \) with padding \( P \) satisfies

(i) \( |A(P)| = 0 \) if \( v \equiv 1 \pmod{4} \), and

(ii) \( |A(P)| = v \) if \( v \equiv 0 \) or \( 2 \pmod{4} \), and

(iii) \( |A(P)| = 2 \) if \( v \equiv 3 \pmod{4} \).

Proof. The necessary conditions follow as in Theorem 4.5. Since the converse of \( d_5 \) is \( d_6 \) and the \( D_v \) is self converse, the result follows trivially from Theorem 4.5. \( \square \)
5 CONCLUSION

Here is a summary of our decomposition results:

(i) \(d_1, d_2, d_3, \text{ or } d_4\) decompositions of \(D_v\) exist if and only if \(v \equiv 0 \text{ or } 1 \pmod{4}\), and

(ii) \(d_4 \text{ or } d_5\) decompositions of \(D_v\) exist if and only if \(v \equiv 1 \pmod{4}\).

The most similar decomposition result to compare this to is the transitive triple decompositions of \(D_v\) as studied by Hung and Mendelsohn [8]. That is, a \(DTS(v)\) exists if and only if \(v \equiv 0 \text{ or } 1 \pmod{3}\) [8]. We have similar results for \(d_1 \text{ through } d_4\) decompositions — these decompositions exist if and only if \(v\) is equivalent to 0 or 1 modulo the number of arcs in the figure with which we are decomposing. This makes sense because the number of arcs in \(D_v\) is divisible by 3 if and only if \(v \equiv 0 \text{ or } 1 \pmod{3}\), i.e. \(v = 3k \text{ or } v = 3k + 1\). (Recall that \(|A(D_v)| = v(v - 1)|\). So if \(v = 3k\), then \(|A(D_v)| = 3k(3k - 1)\) and if \(v = 3k + 1\), then \(|A(D_v)| = 3k + 1(3k)\). Similarly, the number of arcs in \(D_v\) is divisible by 4 if and only if \(v \equiv 0 \text{ or } 1 \pmod{4}\).

What was surprising though, was that \(d_5 \text{ or } d_6\) decompositions of \(D_v\) did not exist if \(v \equiv 0 \pmod{4}\). A closer look at the structure of \(d_5 \text{ and } d_6\) revealed the reason for this. The vertices of \(d_5\) were of in-degree even and \(D_v\) had an even number of vertices which meant that the in-degree of each vertex in \(D_v\) was odd. Thus a decomposition was not possible because the sum of any number of even numbers can never equal an odd number.
Here is a summary of our packing results:

(i) Maximal $d_1$ or $d_2$ packings of $D_v$ with leave $L$ has $|A(L)| = 0$ if $v \equiv 0$ or 1 (mod 4) and $|A(L)| = 2$ if $v \equiv 2$ or 3 (mod 4), $v \neq 6$, and $|A(L)| = 6$ if $v = 6$.

(ii) Maximal $d_3$ or $d_4$ packings of $D_v$ with leave $L$ has $|A(L)| = 0$ if $v \equiv 0$ or 1 (mod 4) and $|A(L)| = 2$ if $v \equiv 2$ or 3 (mod 4).

(iii) Maximal $d_5$ or $d_6$ packings of $D_v$ with leave $L$ has $|A(L)| = 0$ if $v \equiv 1$ (mod 4), and $|A(L)| = v$ if $v \equiv 0$ or 2 (mod 4), and $|A(L)| = 2$ if $v \equiv 3$ (mod 4).

Again, we want to compare this to the transitive triple packing results of $D_v$. Here are the results from R. Gardner’s research in [5]:

Theorem 5.1 [5] A maximal packing of $D_v$, where $v \neq 6$, with copies of the transitive triple, $T$, and a leave $L$ satisfies:

1) $|A(L)| = 0$ if $v \equiv 0$ or 1 (mod 3), or

2) $|A(L)| = 2$ and $L=C_2$ if $v \equiv 2$ (mod 3).

Our results were very similar to R. Gardner’s but were dependent on which graph configuration with which we were packing. In packing with $d_1$ or $d_2$, we found that $|A(L)| = 6$ when $v = 6$. The reason for this was that one of the vertices in $d_1$ had an in-degree of three (See Lemma 3.1). Dr. Gardner also had an exception when $v = 6$, but this would not be for the same reason since there are no vertices of in-degree three in a transitive triple.

Another unusual result was that in packing with $d_5$ or $d_6$, we found $|A(L)| = v$ if $v \equiv 0$ or 2 (mod 4). Again the reason for this was the structure of $d_5$ and $d_6$ as discussed above in the decomposition section (See Lemma 3.5).
Finally, here is the summary of our covering results:

(i) Minimal \(d_1\), \(d_2\), \(d_3\) or \(d_4\) coverings of \(D_v\) with padding \(P\) has \(|A(P)| = 0\) if \(v \equiv 0\) or 1 (mod 4) and \(|A(P)| = 2\) if \(v \equiv 2\) or 3 (mod 4).

(ii) Minimal \(d_5\) or \(d_6\) coverings of \(D_v\) with padding \(P\) has \(|A(P)| = 0\) if \(v \equiv 1\) (mod 4), and \(|A(L)| = v\) if \(v \equiv 0\) or 2 (mod 4), and \(|A(L)| = 2\) if \(v \equiv 3\) (mod 4).

And here are R. Gardner’s results from [5]:

Theorem 5.2 [5] A minimal covering of \(D_v\), where \(v \neq 6\), with copies of the transitive triple, \(T\), and padding, \(P\), satisfies:

1) \(|A(P)| = 0\) if \(v \equiv 0\) or 1 (mod 3), or

2) \(|A(P)| = 4\) if \(v \equiv 2\) (mod 3) and \(P\) may be two disjoint copies of \(C_2\), any orientation of a 4-cycle or two osculating 2-circuits \(OC_2\).

Dr. Gardner’s transitive triple results were quite different from minimal \(d_1\) through \(d_4\) coverings of \(D_v\) because we found \(|A(P)| = 2\) if \(v \equiv 2\) or 3 (mod 4) instead of \(|A(P)| = 4\) if \(v \equiv 2\) (mod 3). Again, minimal \(d_5\) or \(d_6\) coverings of \(D_v\) differed because of the structure of \(d_5\) and \(d_6\). We found \(|A(P)| = v\) if \(v \equiv 0\) or 2 (mod 4). These results differ significantly and could warrant further study.
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