Packings and Coverings of Various Complete Digraphs with the Orientations of a 4-Cycle.

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Packings and Coverings of Various Complete Digraphs with the Orientations of a 4−Cycle

A thesis

presented to

the faculty of the Department of Mathematics

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In partial fulfillment

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Master of Science in Mathematical Sciences

by

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ABSTRACT

Packings and Coverings of Various Complete Digraphs with the Orientations of a 4-Cycle

by

Melody Cooper

There are four orientations of cycles on four vertices. Necessary and sufficient conditions are given for covering complete directed digraphs $D_v$, packing and covering complete bipartite digraphs, $D_{m,n}$, and packing and covering the complete digraph on $v$ vertices with hole of size $w$, $D(v,w)$, with three of the orientations of a 4-cycle, including $C_4$, $X$, and $Y$. 
DEDICATION

This paper is dedicated to my parents, Ronnie and Audrey Cooper, for being the best cheerleaders in the world! From kindergarten to graduate school, they have always pushed me to do my best. Without their constant encouragement and support, I truly believe I would not have made it this far. To my mom, your Melzy Welzy finally made it. To my dad, thanks for helping find the light at the end of the tunnel. I love you!
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1 INTRODUCTION

Within the field of combinatorial mathematics exists a branch of mathematics known as design theory. Design theory can be used in professions related to computer science, telecommunications, traffic management, and environmental conservation [2]. More specifically, within design theory we can study decompositions, packings, and coverings of graphs.

A \textit{graph} is a mathematical representation of a relationship. There are finite and infinite graphs. In this paper we only consider finite graphs. That is, a graph $G$ consists of two sets—a nonempty finite set $V$ of \textit{vertices} and a finite set $E$ of \textit{edges} consisting of unordered pairs of distinct vertices from $V$. There also exist directed graphs $D$ called \textit{digraphs}. In these digraphs, we no longer have edges but instead we have \textit{arcs} that are assigned a direction. In a non-directed graph, two vertices are \textit{adjacent} if they have an edge in common. A graph on $v$ vertices in which every vertex is adjacent to every other vertex is a \textit{complete graph on $v$ vertices} and is denoted $K_v$. For directed complete graphs, $D_v$, we replace each edge with two arcs of opposite orientations. The degree of a vertex, $v$, is defined as the number of edges adjacent to $v$. In directed graphs, each vertex has an \textit{out degree}, $od(v)$, which is the number of vertices that $v$ is adjacent to and \textit{in degree}, $id(v)$, which is the number of vertices that $v$ is adjacent from. Equivalently, $od(v)$ is the number of arcs that point away from $v$ and $id(v)$ is the number of arcs that point toward $v$. The \textit{total degree} of a vertex in a directed graph is $od(v)+id(v)$. In this paper, if we have two graphs, $G$ and $H$, $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

A \textit{decomposition} of a simple graph $H$ with isomorphic copies of a graph $G$ is a set
\{G_1, G_2, \ldots, G_n\} where \(G_i \cong G\) and \(V(G_i) \subset V(H)\) for all \(i\), \(E(G_i) \cap E(G_j) = \emptyset\) if \(i \neq j\), \(\bigcup_{i=1}^{n} G_i \subset H\). The vertex set of a graph \(G\) is denoted \(V(G)\) and the edge set of graph \(G\) is denoted \(E(G)\). Figure 1 is an example of a basic graph decomposition. In Figure 1, we have decomposed \(K_5\) into two 5-cycles.

![Graph Diagram]

Figure 1: Decomposition of a \(K_5\) into 5–cycles

The first decompositions studied were the Steiner triple systems of order \(v\) [9], which are decompositions of \(K_v\) into isomorphic copies of \(C_3'\)s. The \(C_3'\)s are non-oriented three cycles. We denote these as \(STS(v)\). These systems exist if and only if \(v \equiv 1 \text{ or } 3 \pmod{6}\). Decompositions also exist for directed digraphs. However, instead of having an edge set, \(E(G)\), we have an arc set \(A(G)\). So orientations were given to the edges of the \(C_3'\)s. There are two orientations of \(C_3\), the 3–circuit and the transitive triple as depicted in Figure 2.

This led to the study of Mendelsohn triple systems of order \(v\), \(MTS(v)\) [10], which are decompositions of \(D_v\) into 3–circuits, and directed triple system or order \(v\), \(DTS(v)\), which are decompositions of \(D_v\) into transitive triples [7]. A \(MTS(v)\) exists if and only if \(v \equiv 0 \text{ or } 1 \pmod{3}, v \neq 6\) [10]. A \(DTS(v)\) exists if and only if
Then several other directed and non-directed graphs were decomposed, among those the 4-cycle. In Figure 3, notice that there are 4 orientations of a 4-cycle.

These digraphs are denoted in the following way: \([a,b,c,d]_C\), \([a,b,c,d]_X\), \([a,b,c,d]_Y\), and \([a,b,c,d]_Z\) respectively. In this paper, we will only be considering \(C_4\), \(X\), and \(Y\).

The following are the results of the decompositions of each of the orientations of the four cycle:

- An \(X\)-decomposition of \(D_v\) exists if and only if \(v \equiv 0 \) or \(1 \) (mod 3) \([7]\).
- A \(Y\)-decomposition of \(D_v\) exists if and only if \(v \equiv 0 \) or \(1 \) (mod 3)\(, v \neq 5\) \([6]\); and
A $C_4$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 4$ [12].

However, not all graphs can be decomposed. Now consider how close we can get to a graph decomposition. One way is to pack a graph. That is, we remove isomorphic copies of $G$ without removing any repeated arcs until there are no more copies of $G$ left. The arcs that remain are known as the leave of the graph. Formally, a maximal packing of a simple graph $H$ with isomorphic copies of a graph $G$ is a set $\{G_1, G_2, \ldots, G_n\}$ where $G_i \cong G$ and $V(G_i) \subset V(H)$ for all $i$, $E(G_i) \cap E(G_j) = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^{n} G_i \subset H$, and

$$|E(L)| = \left| E(H) \setminus \bigcup_{i=1}^{n} E(G_i) \right|$$

is minimal. The set $L$ is the leave of the packing. Figure 4 is an example of a graph packing. In this figure, we have packed $K_5$ with $C_3'$s.

![Figure 4: Packing of a $K_5$ with 3-cycles ($C_3'$s)](image)

Packings of the complete graph on $v$ vertices of $K_v$, with a graph $G$, have been studied when $G$ is a 3-cycle [11], a 4-cycle [12], a $K_4$1, and a 6-cycle [8]. Packings of the orientations of a 3-cycle, that is $DTS(v)$ and $MTS(v)$ have also been studied
We also know the packings of the orientations of a 4-cycle. They are as follows:

**Theorem 1.1** An optimal packing of $D_v$ with copies of $C_4$ and leave $L$ satisfies:

1. $L = \emptyset$ if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 4$,
2. $|A(L)|=4$ if $v=4$,
3. $L=D_2$ if $v \equiv 2$ or $3 \pmod{4}$[\ref{5}].

**Theorem 1.2** An optimal packing of $D_v$ with copies of $X$ and leave $L$ satisfies:

1. $L = \emptyset$ if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 5$,
2. $|A(L)|=4$ if $v=5$,
3. $L=D_2$ if $v \equiv 2$ or $3 \pmod{4}$[\ref{5}].

**Theorem 1.3** An optimal packing of $D_v$ with copies of $Y$ and leave $L$ satisfies:

1. $L = \emptyset$ if $v \equiv 0$ or $1 \pmod{4}$, $v \not\in \{4,5\}$,
2. $|A(L)|=4$ if $v \in \{4,5\}$,
3. $L=D_2$ if $v \equiv 2$ or $3 \pmod{4}$[\ref{5}].

**Theorem 1.4** An optimal packing of $D_v$ with copies of $Z$ and leave $L$ satisfies:

1. $L = \emptyset$ if $v \equiv 1 \pmod{4}$,
2. $|A(L)|=v$ and the arcs of $L$ are arranged in a collection of disjoint circuits if $v \equiv 0$ or $2 \pmod{4}$,
3. $|A(L)|=6$ and the arcs of $L$ are arranged in such a way that each vertex of the leave has in-degree $=$ out-degree $\equiv 0 \pmod{2}$ and $v \equiv 3 \pmod{4}$[\ref{5}].

Another option, if a graph cannot be decomposed, is to cover the graph. That is, we cover the graph with isomorphic copies of $G$ until all arcs are covered. The arcs that
are repeated are known as the padding of the graph. Formally, a minimal covering of a simple graph $H$ with isomorphic copies of a graph $G$ is a set $\{G_1, G_2, \ldots, G_n\}$, where $G_i \cong G$ and $V(G_i) \subset V(H)$ and $E(G_i) \subset E(H)$ for all $i$, $H \subset \bigcup_{i=1}^{n} G_i$, and

$$|E(P)| = \left| \bigcup_{i=1}^{n} E(G_i) \setminus E(H) \right|$$

is minimal. However, the graph $\bigcup_{i=1}^{n} G_i$ may not be simple and $\bigcup_{i=1}^{n} E(G_i)$ may be a multiset. The graph $P$ is the padding of the covering. We can see what a graph covering is in Figure 5. In this figure, we cover $K_5$ with copies of $C_5$.

Coverings of the complete graph on $v$ vertices of, $K_v$ with a graph $G$ have been studied when $G$ is a 3-cycle [3], a 4-cycle [13], and a 6-cycle. Coverings of the orientations of a 3-cycle [8], that is $DTS(v)$ and $MTS(v)$, have been studied as well [4].

In this paper, we will cover complete directed digraphs $D_v$, pack and cover complete bipartite digraphs, $D_{m,n}$, and pack and cover the complete digraph on $v$ vertices with hole of size $w$, $D(v, w)$, with three of the orientations of a 4-cycle, including $C_4$, $X$, and $Y$. 
Figure 5: Covering a $K_5$ with 3-cycles ($C_3'$s)

2 PACKING AND COVERING $D_{m,n}$ WITH THE $C_4$, $X$, AND $Y$ ORIENTATIONS OF THE 4-CYCLE

A complete bipartite digraph is a graph that can be partitioned into two nonempty subsets $V_m$ and $V_n$, called the partite sets, such that $(a, b)$ is an arc of $D_{m,n}$, if and only if $a$ and $b$ belong to different partitions \cite{2}. We use the notation $V_m=\{0_1, 1_1, \ldots, (m - 1)_1\}$ and $V_n=\{0_2, 1_2, \ldots, (n - 1)_2\}$. Decompositions of these graphs have been studied in the past, with the following results, beginning with $C_4$. 
Theorem 2.1 [5] A $C_4$ decomposition of $D_{m,n}$ exists if and only if $m, n \geq 2$ and $mn \equiv 0 \pmod{2}$.

We now want to know the results of packing these graphs.

Theorem 2.2 A maximal $C_4$ packing of $D_{m,n}$ satisfies:

1. $|A(L)| = 0$ when $mn \equiv 0 \pmod{2}$, $m, n \geq 2$,
2. $|A(L)| = 2$, otherwise.

For clarity, throughout this thesis we consider values of parameter $m$ and $n$ modulo 4. We do these modulo 4 because we are considering orientations of a 4-cycle.

Proof. Therefore, we present sixteen cases. Since $|A(C_4)| = 4$, it is necessary that the leave, $L$, satisfy $|A(L)| \equiv |A(D_{m,n})|(\pmod{4})$. Therefore, for $m \equiv n \equiv 0 \pmod{4}$, $|A(L)| \geq 0$ and for the other cases $|A(L)| \geq 2$.

Case 1: If $m \equiv n \equiv 0 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.1, a decomposition exists.

Case 2: If $m \equiv 0 \pmod{4}$ and $n \equiv 1 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Thus, by Theorem 2.1, a decomposition exists.

Case 3: If $m \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.1, a decomposition exists.

Case 4: If $m \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Thus, by Theorem 2.1, a decomposition exists.

Case 5: If $m \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.1, a decomposition exists.

Case 6: If $m \equiv n \equiv 1 \pmod{4}$, where $m, n \geq 2$, $D_{m,n}=D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5}$. Now pack each smaller graph with $C_4$. For $D_{5,5}$, the partite sets are
{0_1, 1_1, 2_1, 3_1, 4_1} and {0_2, 1_2, 2_2, 3_2, 4_2}. One possible way of packing D_{5,5} so that the leave is minimal is: \{[0_1, 0_2, 1_1, 1_2]_C, [0_1, 1_2, 2_1, 2_2]_C, [0_1, 2_2, 4_1, 4_2]_C, [0_1, 3_2, 4_1, 0_2]_C, [0_1, 4_2, 1_1, 3_2]_C, [1_1, 0_2, 4_1, 1_2]_C, [1_1, 4_2, 2_1, 3_2]_C, [2_1, 0_2, 3_1, 2_2]_C, [2_1, 1_2, 3_1, 0_2]_C, [2_1, 4_2, 3_1, 3_2]_C, [3_1, 1_2, 4_1, 3_2]_C, [3_1, 4_2, 4_1, 2_2]_C\} and |A(L)|=\{(1_1, 2_2),(2_2, 1_1)\}. The partite sets for D_{m-5,5} are \{5_1, 6_1, \ldots, (m - 1)_1\} and \{0_2, 1_2, 2_2, 3_2, 4_2\}.

For D_{m,n-5}, the partite sets are \{0_1, 1_1, 2_1, 3_1, 4_1\} and \{5_2, 6_2, \ldots, (n-1)_2\}. Finally, for D_{m-5,n-5}, the partite sets are \{5_1, 6_1, \ldots, (m-1)_1\} and \{5_2, 6_2, \ldots, (n-1)_2\}.

Since \(m \equiv 1 \pmod{4}\) and \(n \equiv 1 \pmod{4}\) and \(m, n \geq 2\), then \(m - 5 \equiv 0 \pmod{2}\) and \(n - 5 \equiv 0 \pmod{2}\). Therefore, \(D_{m-5,5}, D_{m,n-5}, \text{ and } D_{m-5,n-5}\) can be decomposed by Theorem 2.1. Thus, the leave \(L\) satisfies \(|A(L)|=2\).

Case 7: If \(m \equiv 1 \pmod{4}\) and \(n \equiv 2 \pmod{4}\), then \(mn \equiv 0 \pmod{2}\). By Theorem 2.1, it follows that a decomposition exists.

Case 8: If \(m \equiv 1 \pmod{4}\) and \(n \equiv 3 \pmod{4}\), where \(m, n \geq 2\), then \(D_{m,n}=D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3}\). Now pack each smaller graph with \(C_4\). For \(D_{5,3}\), the partite sets are \{0_1, 1_1, 2_1, 3_1, 4_1\} and \{0_2, 1_2, 2_2\}. One possible way of packing \(D_{5,3}\) so that the leave is minimal is: \{[0_2, 0_1, 1_2, 4_1]_C, [0_1, 0_2, 1_1, 2_2]_C, [1_1, 0_2, 4_1, 1_2]_C, [0_2, 2_1, 2_2, 3_1]_C, [0_2, 3_1, 1_2, 2_1]_C, [2_2, 2_1, 1_2, 0_1]_C, [2_2, 1_1, 1_2, 3_1]_C\} and \(|A(L)|=\{(4_1, 2_2),(2_2, 4_1)\}\). The partite sets for \(D_{m-5,3}\) are \{5_1, 6_1, \ldots, (m-1)_1\} and \{0_2, 1_2, 2_2\}.

For \(D_{m,n-3}\), the partite sets are \{0_1, 1_1, 2_1, 3_1, 4_1\} and \{3_2, 4_2, \ldots, (n-1)_2\}. Finally, for \(D_{m-5,n-3}\), the partite sets are \{5_1, 6_1, \ldots, (m-1)_1\} and \{3_2, 4_2, \ldots, (n-1)_2\}.

Since \(m \equiv 1 \pmod{4}\) and \(n \equiv 3 \pmod{4}\) and \(m, n \geq 2\), then \(m - 5 \equiv 0 \pmod{2}\) and \(n - 3 \equiv 0 \pmod{2}\). Therefore, \(D_{m-5,3}, D_{m,n-3}, \text{ and } D_{m-5,n-3}\) can be decomposed by Theorem 2.1. Thus, the leave \(L\) satisfies \(|A(L)|=2\).
Case 9: If \( m \equiv 2 \pmod{4} \) and \( n \equiv 0 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1, a decomposition exists.

Case 10: If \( m \equiv 2 \pmod{4} \) and \( n \equiv 1 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). By Theorem 2.1, it follows that a decomposition exists.

Case 11: If \( m \equiv n \equiv 2 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1, a decomposition exists.

Case 12: If \( m \equiv 2 \pmod{4} \) and \( n \equiv 3 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Thus, by Theorem 2.1, a decomposition exists.

Case 13: If \( m \equiv 3 \pmod{4} \) and \( n \equiv 0 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). By Theorem 2.1, it follows that a decomposition exists.

Case 14: If \( m \equiv 3 \pmod{4} \) and \( n \equiv 1 \pmod{4} \), where \( m, n \geq 2 \), then \( D_{m,n} = D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5} \). Now pack each smaller graph with \( C_4 \). For \( D_{3,5} \), the partite sets are \( \{0, 1, 2, 3, 4\} \) and \( \{0, 1, 2, 3, 4\} \). One possible way of packing \( D_{3,5} \) so that the leave is minimal is: \( \{(0, 1, 2, 3, 4)\} \) and \( \{(0, 1, 2, 3, 4)\} \). The partite sets for \( D_{m-3,5} \) are \( \{3, 4, \ldots, (m-1)\} \) and \( \{0, 1, 2, 3, 4\} \). For \( D_{m,n-5} \), the partite sets are \( \{0, 1, 2\} \) and \( \{0, 1, 2\} \). Finally, for \( D_{m-3,n-5} \), the partite sets are \( \{3, 4, \ldots, (m-1)\} \) and \( \{0, 1, 2, 3, 4\} \). Since \( m \equiv 3 \pmod{4} \) and \( n \equiv 1 \pmod{4} \) and \( m, n \geq 2 \), then \( m - 3 \equiv 0 \pmod{2} \) and \( n - 3 \equiv 0 \pmod{2} \). Therefore, \( D_{m-3,5}, D_{m,n-5}, \) and \( D_{m-3,n-5} \) can be decomposed by Theorem 2.1. Thus, the leave \( L \) satisfies \( |A(L)| = 2 \).

Case 15: If \( m \equiv 3 \pmod{4} \) and \( n \equiv 2 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1 a decomposition exists.
Case 16: If \( m \equiv n \equiv 3 \pmod{4} \), where \( m, n \geq 2 \), then decompose \( D_{m,n} \). That is \( D_{m,n}=D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3} \). Now pack each smaller graph with \( C_4 \). For \( D_{3,3} \), the partite sets are \{0_1, 1_1, 2_1\} and \{0_2, 1_2, 2_2\}. One possible way of packing \( D_{3,3} \) so that the leave is minimal is: \{[0_1, 0_2, 1_1, 1_2][C, 0_1, 1_2, 2_1, 2_2][C, 0_1, 2_2, 2_1, 0_2][C, 1_1, 0_2, 2_1, 1_2][C] \} and \(|A(L)|=\{(1_1, 2_2), (2_2, 1_1)\}\). The partite sets for \( D_{m-3,3} \) are \{3_1, 4_1, \ldots, (m-1)_1\} and \{0_2, 1_2, 2_2\}. For \( D_{m,n-3} \), the partite sets are \{0_1, 1_1, 2_1\} and \{3_2, 4_2, \ldots, (n-1)_2\}. Finally, for \( D_{m-3,n-3} \), the partite sets are \{3_1, 4_1, \ldots, (m-1)_1\} and \{3_2, 4_2, \ldots, (n-1)_2\}. Since \( m \equiv 1 \pmod{4} \) and \( n \equiv 1 \pmod{4} \) and \( m, n \geq 2 \), then \( m-3 \equiv 0 \pmod{2} \) and \( n-3 \equiv 0 \pmod{2} \). Therefore, \( D_{m-3,3}, D_{m,n-3}, \) and \( D_{m-3,n-3} \) can be decomposed by Theorem 2.1 Thus, the leave \( L \) satisfies \(|A(L)|=2\). Q.E.D.

Thus, each case for \( C_4 \) satisfies the conditions of the theorem. Decomposing \( D_{m,n} \) with copies of \( X \) has the following results.

**Theorem 2.3** [5] An \( X \) decomposition of \( D_{m,n} \) exists if and only if either \( m \equiv n \equiv 0 \pmod{2} \) or \( m \equiv 1 \pmod{2}, m \geq 3 \), and \( n \equiv 0 \pmod{4} \).

Now we need to pack \( D_{m,n} \) with copies of \( X \).

**Theorem 2.4** A maximal \( X \) packing of \( D_{m,n} \) satisfies:

1. \(|A(L)|=0 \) when \( m \equiv n \equiv 0 \pmod{2} \) or \( m \equiv 1 \pmod{2}, m \geq 3, \)
2. \(|A(L)|=2, \) when \( m \equiv 1 \pmod{2} \) and \( n \equiv 1 \pmod{2}, m \geq 3. \)
3. \(|A(L)|=4, \) when \( m \equiv 1 \pmod{2} \) and \( n \equiv 2 \pmod{4}, m \geq 3. \)

Proof. As in Theorem 2.2, it is necessary that \(|A(L)| \equiv |A(D_{m,n})|(\pmod{4}) \). So for \( m \equiv n \equiv 0 \pmod{2} \), we have that \(|A(L)| \geq 0 \). For \( m \equiv n \equiv 1 \pmod{2} \), we have
\(|A(L)| \geq 2\). For the remaining cases, \(|A(D_{m,n})| \equiv 0 \pmod{4}\). However, by Theorem 2.3, a decomposition does not exist. Thus, \(|A(L)| \geq 4\) in these cases.

Case 1: If \(m \equiv n \equiv 0 \pmod{4}\), then \(m \equiv n \equiv 0 \pmod{2}\). Therefore, by Theorem 2.3, a decomposition exists.

Case 2: If \(m \equiv 0 \pmod{4}\) and \(n \equiv 1 \pmod{4}\), then \(m \equiv 0 \pmod{4}\) and \(n \equiv 1 \pmod{2}\). By Theorem 2.3, it follows that a decomposition exists.

Case 3: If \(m \equiv 0 \pmod{4}\) and \(n \equiv 2 \pmod{4}\), then \(m \equiv n \equiv 0 \pmod{2}\). Therefore, by Theorem 2.3, a decomposition exists.
Case 4: If \( m \equiv 0 \) (mod 4) and \( n \equiv 3 \) (mod 4), then \( m \equiv 0 \) (mod 4) and \( n \equiv 1 \) (mod 2). Thus, by Theorem 2.3, a decomposition exists.

Case 5: If \( m \equiv 1 \) (mod 4) and \( n \equiv 0 \) (mod 4), then \( m \equiv 1 \) (mod 2) and \( n \equiv 0 \) (mod 4). Therefore, by Theorem 2.3, a decomposition exists.

Case 6: If \( m \equiv n \equiv 1 \) (mod 4), where \( m \geq 3 \), then \( D_{m,n}=D_{5,5} \cup D_{m−5,5} \cup D_{m,n−5} \cup D_{m−5,n−5} \). Now pack each smaller graph with \( X \). For \( D_{5,5} \), the partite sets are \( \{0, 1, 3, 1, 4 \} \) and \( \{0, 1, 3, 1, 4 \} \). One possible way of packing \( D_{5,5} \) so that the leave is minimal is: \( \{[0, 1, 3, 1, 4]X, [2, 1, 3, 1, 4]X, [1, 2, 0, 3, 1]X, [2, 3, 1, 4, 1]X \} \). For \( D_{m,n−5} \), the partite sets are \( \{0, 1, 3, 1, 4 \} \) and \( \{0, 1, 3, 1, 4 \} \). One possible way of packing \( D_{m,n−5} \) so that the leave is minimal is: \( \{[0, 1, 3, 1, 4]X, [2, 1, 3, 1, 4]X, [1, 2, 3, 1, 4]X \} \). Finally, for \( D_{m−5,n−5} \), the partite sets are \( \{0, 1, 3, 1, 4 \} \) and \( \{0, 1, 3, 1, 4 \} \). Since \( m \equiv 1 \) (mod 4), \( n \equiv 1 \) (mod 4), then \( m−5 \equiv 0 \) (mod 2) and \( n−5 \equiv 0 \) (mod 2). Therefore, \( D_{m−5,5} \), \( D_{m−5,n−5} \), and \( D_{m−5,n−5} \) can be decomposed by Theorem 2.3. Thus, the leave, \( L \), satisfies \( |A(L)|=2 \).

Case 7: If \( m \equiv 1 \) (mod 4) and \( n \equiv 2 \) (mod 4), where \( m \geq 3 \), then \( D_{m,n}=D_{5,2} \cup D_{m−5,2} \cup D_{m−2,2} \cup D_{m−5,n−2} \). Now pack each smaller graph with \( X \). For \( D_{5,2} \), the partite sets are \( \{0, 1, 3, 1, 4 \} \) and \( \{0, 1, 3, 1, 4 \} \). One possible way of packing \( D_{5,2} \) so that the leave is minimal is: \( \{[0, 1, 3, 1, 4]X, [2, 1, 3, 1, 4]X, [3, 1, 3, 1, 4]X \} \). Finally, for \( D_{m−5,n−2} \), the partite sets are \( \{0, 1, 3, 1, 4 \} \) and \( \{0, 1, 3, 1, 4 \} \).
sets are \{5_1, 6_1, \ldots, (m - 1)_1\} and \{2_2, 3_2, \ldots, (n - 1)_2\}. Since \(m \equiv 1 \pmod{4}\) and \(n \equiv 3 \pmod{4}\), then \(m - 5 \equiv 0 \pmod{2}\) and \(n - 2 \equiv 0 \pmod{2}\). Therefore, 
\(D_{m-5,2}, D_{m,n-2}\), and \(D_{m-5,n-2}\) can be decomposed by Theorem 2.3. Thus, the leave, 
\(L\), satisfies \(|A(L)|=4\).

Case 8: If \(m \equiv 1 \pmod{4}\) and \(n \equiv 3 \pmod{4}\), where \(m \geq 3\), then \(D_{m,n}=D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3}\). Now pack each smaller graph with \(X\). For \(D_{5,3}\), the partite sets are \{0_1, 1_1, 2_1, 3_1, 4_1\} and \{0_2, 1_2, 2_2\}. One possible way of packing \(D_{5,3}\) so that the leave is minimal is: \{[0_2, 0_1, 1_2, 4_1]_X, [2_2, 3_1, 4_2, 4_1]_X, [1_2, 0_1, 4_2, 2_1]_X, [2_1, 3_2, 3_1, 2_2]_X, [0_2, 0_1, 3_2, 4_1]_X, [3_2, 0_1, 2_2, 2_1]_X, [4_1, 0_2, 2_1, 4_2]_X\} and \(|A(L)|=\{(4_1, 2_2),(2_2, 4_1)\}\) The partite sets for \(D_{m-5,3}\) are \{5_1, 6_1, \ldots, (m - 1)_1\} and \{0_2, 1_2, 2_2\}. For \(D_{m,n-3}\), the partite sets are \{0_1, 1_1, 2_1, 3_1, 4_1\} and \{3_2, 4_2, \ldots, (n - 1)_2\}. Finally, for \(D_{m-5,n-3}\), the partite sets are \{5_1, 6_1, \ldots, (m - 1)_1\} and \{3_2, 4_2, \ldots, (n - 1)_2\}. Now, note that since \(m \equiv 1 \pmod{4}\) and \(n \equiv 3 \pmod{4}\), then \(m - 5 \equiv 0 \pmod{2}\) and \(n - 3 \equiv 0 \pmod{2}\). Therefore \(D_{m-5,3}, D_{m,n-3}\), and \(D_{m-5,n-3}\) can be decomposed by Theorem 2.3. Therefore, the leave, \(L\), satisfies \(|A(L)|=2\).

Case 9: If \(m \equiv 2 \pmod{4}\) and \(n \equiv 0 \pmod{4}\), then \(m \equiv n \equiv 0 \pmod{2}\). Therefore, by Theorem 2.3, a decomposition exists.

Case 10: If \(m \equiv 2 \pmod{4}\) and \(n \equiv 1 \pmod{4}\), where \(m \geq 3\), then \(D_{m,n}=D_{2,5} \cup D_{m-2,5} \cup D_{m,n-5} \cup D_{m-2,n-5}\). Now pack each smaller graph with \(X\). For \(D_{2,5}\), the partite sets are \{0_1, 1_1\} and \{0_2, 1_2, 2_2, 3_2, 4_2\}. One possible way of packing \(D_{2,5}\) so that the leave is minimal is: \{[0_2, 0_1, 1_2, 1_1]_X, [2_2, 1_1, 4_2, 0_1]_X, [3_2, 1_1, 1_2, 0_1]_X, [0_1, 4_2, 1_1, 3_2]_X\} and \(|A(L)|=\{(0_1, 0_2),(0_1, 2_2), (1_1, 0_2), (1_1, 2_2)\}\). The partite sets for \(D_{m-2,2}\) are \{2_1, 3_1, \ldots, (m - 1)_1\} and \{0_2, 1_2, 2_2, 3_2, 4_2\}. For \(D_{m,n-5}\), the partite sets
are \(\{0_1, 1_1\}\) and \(\{5_2, 6_2, \ldots, (n-1)_2\}\). Finally, for \(D_{m-2,n-5}\), the partite sets are \(\{2_1, 3_1, \ldots, (m-1)_1\}\) and \(\{5_2, 6_2, \ldots, (n-1)_2\}\). Since \(m \equiv 2 \pmod{4}\) and \(n \equiv 1 \pmod{4}\), then \(m - 2 \equiv 0 \pmod{2}\) and \(n - 5 \equiv 0 \pmod{2}\). Therefore, \(D_{m-2,5}\), \(D_{m,n-5}\), and \(D_{m-2,n-5}\) can be decomposed by Theorem 2.3. Thus, the leave, \(L\), satisfies \(|A(L)|=4\).

Case 11: If \(m \equiv n \equiv 2 \pmod{4}\), then \(m \equiv n \equiv 0 \pmod{2}\). Therefore, by Theorem 2.3, a decomposition exists.

Case 12: If \(m \equiv 2 \pmod{4}\) and \(n \equiv 3 \pmod{4}\), where \(m \geq 3\), then \(D_{m,n}=D_{2,3} \cup D_{m-2,3} \cup D_{m,n-3} \cup D_{m-2,n-3}\). Now pack each smaller graph with \(X\). For \(D_{2,3}\), the partite sets are \(\{0_1, 1_1\}\) and \(\{0_2, 1_2, 2_2\}\). One possible way of packing \(D_{2,3}\) so that the leave is minimal is: \(\{[0_2, 0_1, 1_2, 1_1]_X, [1_1, 1_2, 0_1, 2_2]_X\}\) and \(|A(L)|=\{(2_2, 0_1), (2_2, 1_1), (0_1, 0_2), (1_1, 0_2)\}\). The partite sets for \(D_{m-2,3}\) are \(\{2_1, 3_1, \ldots, (m-1)_1\}\) and \(\{0_2, 1_2, 2_2\}\). For \(D_{m,n-3}\), the partite sets are \(\{0_1, 1_1\}\) and \(\{3_2, 4_2, \ldots, (n-1)_2\}\). Finally, for \(D_{m-2,n-3}\), the partite sets are \(\{2_1, 3_1, \ldots, (m-1)_1\}\) and \(\{3_2, 4_2, \ldots, (n-1)_2\}\). Since \(m \equiv 2 \pmod{4}\) and \(n \equiv 3 \pmod{4}\), then \(m - 2 \equiv 0 \pmod{2}\) and \(n - 3 \equiv 0 \pmod{2}\). Therefore, \(D_{m-2,3}\), \(D_{m,n-3}\), and \(D_{m-2,n-3}\) can be decomposed by Theorem 2.3. Thus, the leave, \(L\), satisfies \(|A(L)|=4\).

Case 13: If \(m \equiv 3 \pmod{4}\) and \(n \equiv 0 \pmod{4}\), then \(m \equiv 1 \pmod{2}\) and \(n \equiv 0 \pmod{4}\). Therefore, by Theorem 2.3, a decomposition exists.

Case 14: If \(m \equiv 3 \pmod{4}\) and \(n \equiv 1 \pmod{4}\), where \(m \geq 3\), then \(D_{m,n}=D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5}\). Now pack each smaller graph with \(X\). For \(D_{3,5}\), the partite sets are \(\{0_1, 1_1, 2_1\}\) and \(\{0_2, 1_2, 2_2, 3_2, 4_2\}\). One possible way of packing \(D_{3,5}\) so that the leave is minimal is: \(\{[0_1, 0_2, 1_1, 4_2]_X, [0_2, 0_1, 1_2, 2_1]_X, [1_2, 0_1, 4_2, 1_1]_X, [0_1, 2_2, 2_1, 3_2]_X, [0_1, 3_2, 1_1, 2_2]_X, [2_1, 2_2, 1_1, 0_2]_X, [2_1, 1_2, 1_1, 3_2]_X\}\) and \(|A(L)|=\{(2_1,
4_2),(4_2, 2_1)}. The partite sets for D_{m-3,5} are \{3_1, 4_1, \ldots, (m-1)_1\} and \{0_2, 1_2, 2_2, 3_2, 4_2\}. For D_{m,n-5}, the partite sets are \{0_1, 1_1, 2_1\} and \{5_2, 6_2, \ldots, (n-1)_2\}. Finally, for D_{m-3,n-5}, the partite sets are \{3_1, 4_1, \ldots, (m-1)_1\} and \{5_2, 6_2, \ldots, (n-1)_2\}. Since m \equiv 3 \pmod 4 and n \equiv 1 \pmod 4, then m - 3 \equiv 0 \pmod 2 and n - 2 \equiv 0 \pmod 2. Therefore, D_{m-3,5}, D_{m,n-5}, and D_{m-3,n-5} can be decomposed by Theorem 2.3. Thus, the leave, \(L\), satisfies \(|A(L)|=2\).

Case 15: If m \equiv 3 \pmod 4 and n \equiv 2 \pmod 4, where m \geq 3, then \(D_{m,n}=D_{3,2} \cup D_{m-3,2} \cup D_{m,n-2} \cup D_{m-3,n-2}\). Now pack each smaller graph with \(X\). For \(D_{3,2}\), the partite sets are \{0_1, 1_1, 2_1\} and \{0_2, 1_2\}. One possible way of packing \(D_{3,2}\) so that the leave is minimal is: \[[0_1, 0_2, 1_1, 1_2]_X, [1_2, 1_1, 0_2, 2_1]_X\} and \(|A(L)|=\{(2_1, 0_2),(2_1, 1_2), (0_2, 0_1), (1_2, 0_1)\}\. The partite sets for \(D_{m-3,2}\) are \{3_1, 4_1, \ldots, (m-1)_1\} and \{0_2, 1_2\}. For \(D_{m,n-2}\), the partite sets are \{0_1, 1_1, 2_1\} and \{2_2, 3_2, \ldots, (n-1)_2\}. Finally, for \(D_{m-3,n-2}\), the partite sets are \{3_1, 4_1, \ldots, (m-1)_1\} and \{2_2, 3_2, \ldots, (n-1)_2\}. Since m \equiv 3 \pmod 4 and n \equiv 2 \pmod 4, then m - 3 \equiv 0 \pmod 2 and n - 2 \equiv 0 \pmod 2. Therefore, \(D_{m-3,2}, D_{m,n-2},\) and \(D_{m-3,n-2}\) can be decomposed by Theorem 2.3. Thus, the leave, \(L\), satisfies \(|A(L)|=4\).

Case 16: If m \equiv n \equiv 3 \pmod 4, where m \geq 3, then \(D_{m,n}=D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3}\). Now pack each smaller graph with \(X\). For \(D_{3,3}\), the partite sets are \{0_1, 1_1, 2_1\} and \{0_2, 1_2, 2_2\}. One possible way of packing \(D_{3,3}\) so that the leave is minimal is: \[[0_1, 0_2, 2_1, 1_2]_X, [1_1, 0_2, 0_1, 2_2]_X, [2_2, 2_1, 1_2, 1_1]_X, [1_2, 2_1, 2_2, 0_1]_X\} and \(|A(L)|=\{(2_1, 0_2),(0_2, 2_1)\}\. The partite sets for \(D_{m-3,3}\) are \{3_1, 4_1, \ldots, (m-1)_1\} and \{0_2, 1_2, 2_2\}. For \(D_{m,n-3}\), the partite sets are \{0_1, 1_1, 2_1\} and \{3_2, 4_2, \ldots, (n-1)_2\}. Finally, for \(D_{m-3,n-3}\), the partite sets are \{3_1, 4_1, \ldots, (m-1)_1\} and \{3_2, 4_2, \ldots,
\((n - 1)^2\). Since \(m \equiv 3 \pmod{4}, n \equiv 3 \pmod{4}\), then \(m - 3 \equiv 0 \pmod{2}\) and \(n - 3 \equiv 0 \pmod{2}\). Therefore, \(D_{m-3,3}, D_{m,n-3}\), and \(D_{m-3,n-3}\) can be decomposed by Theorem 2.3. Thus, the leave, \(L\), satisfies \(|A(L)|=2\). Q.E.D

Thus, each case for \(X\) satisfies the conditions of the theorem. Decomposing \(D_{m,n}\) with copies of \(Y\) has the following results.

**Theorem 2.5** [5] A \(Y\) decomposition of \(D_{m,n}\) exists if and only if \(m, n \geq 2\) and \(mn \equiv 0 \pmod{2}\).

**Theorem 2.6** A maximal \(Y\) packing of \(D_{m,n}\) satisfies:

1. \(|A(L)|=0\) when \(mn \equiv 0 \pmod{2}\), \(m, n \geq 2\),
2. \(|A(L)|=2\), otherwise.

Proof. The necessary conditions follow as in Theorem 2.2.

Case 1: If \(m \equiv n \equiv 0 \pmod{4}\), then \(mn \equiv 0 \pmod{2}\). Therefore, by Theorem 2.5, a decomposition exists.

Case 2: If \(m \equiv 0 \pmod{4}\) and \(n \equiv 1 \pmod{4}\), then \(mn \equiv 0 \pmod{2}\). By Theorem 2.5, it follows that a decomposition exists.

Case 3: If \(m \equiv 0 \pmod{4}\) and \(n \equiv 2 \pmod{4}\), then \(mn \equiv 0 \pmod{2}\). Thus, by Theorem 2.5, a decomposition exists.

Case 4: If \(m \equiv 0 \pmod{4}\) and \(n \equiv 3 \pmod{4}\), then \(mn \equiv 0 \pmod{2}\). Therefore, by Theorem 2.5, a decomposition exists.

Case 5: If \(m \equiv 1 \pmod{4}\) and \(n \equiv 0 \pmod{4}\), then \(mn \equiv 0 \pmod{2}\). By Theorem 2.5, it follows that a decomposition exists.
Case 6: If \( m \equiv n \equiv 1 \pmod{4} \), where \( m, n \geq 2 \), then \( D_{m,n} = D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5} \). Now pack each smaller graph with \( Y \). For \( D_{5,5} \), the partite sets are \( \{0_1, 1_1, 2_1, 3_1, 4_1\} \) and \( \{0_2, 1_2, 2_2, 3_2, 4_2\} \). One possible way of packing \( D_{5,5} \) so that the leave is minimal is: \( \{[3_1, 0_2, 2_1, 3_2]_Y, [2_1, 3_1, 4_2]_Y, [3_2, 0_1, 2_2, 4_1]_Y, [1_1, 1_2, 0_1, 2_2]_Y, [4_2, 1_1, 0_2, 1_2]_Y, [0_2, 1_1, 2_1]_Y, [4_2, 0_1, 2_2]_Y, [0_1, 1_2, 1_1, 3_2]_Y, [2_2, 1_2, 1_2, 3_1]_Y, [0_2, 4_1, 2_2, 3_1]_Y, [1_2, 2_1, 2_2, 3_1]_Y\} \) and \( |A(L)| = \{(4_1, 1_2), (1_2, 4_1)\} \). The partite sets for \( D_{m,5-5} \) are \( \{5_1, 6_1, \ldots, (m-1)_1\} \) and \( \{0_2, 1_2, 2_2, 3_2, 4_2\} \). For \( D_{m,n-5} \), the partite sets are \( \{0_1, 1_1, 2_1, 3_1, 4_1\} \) and \( \{5_2, 6_2, \ldots, (n-1)_2\} \). Finally, for \( D_{m-5,n-5} \), the partite sets are \( \{5_1, 6_1, \ldots, (m-1)_1\} \) and \( \{5_2, 6_2, \ldots, (n-1)_2\} \). Since \( m \equiv 1 \pmod{4} \), \( n \equiv 1 \pmod{4} \) and \( m, n \geq 2 \), then \( m-5 \equiv 0 \pmod{2} \) and \( n-5 \equiv 0 \pmod{2} \). Therefore, \( D_{m-5,5}, D_{m,n-5}, \) and \( D_{m-5,n-5} \) can be decomposed by Theorem 2.5. Thus, the leave, \( L \), satisfies \( |A(L)| = 2 \).

Case 7: If \( m \equiv 1 \pmod{4} \) and \( n \equiv 2 \pmod{4} \), then \( mn \equiv n \equiv 0 \pmod{2} \). Therefore, by Theorem 2.5, a decomposition exists.

Case 8: If \( m \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{4} \), where \( m, n \geq 2 \), then \( D_{m,n} = D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3} \). Now pack each smaller graph with \( Y \). For \( D_{5,3} \), the partite sets are \( \{0_1, 1_1, 2_1, 3_1, 4_1\} \) and \( \{0_2, 1_2, 2_2\} \). One possible way of packing \( D_{5,3} \) so that the leave is minimal is: \( \{[4_1, 0_2, 0_1, 1_2]_Y, [3_1, 0_2, 4_1, 2_2]_Y, [0_2, 3_1, 1_2, 2_1]_Y, [0_2, 3_1, 1_2, 2_1]_Y, [0_1, 2_1, 1_2]_Y, [1_1, 1_2, 3_1, 2_2]_Y, [1_2, 2_1, 2_2, 4_1]_Y\} \) and \( |A(L)| = \{(0_1, 2_2), (2_2, 0_1)\} \). The partite sets for \( D_{m,5-3} \) are \( \{5_1, 6_1, \ldots, (m-1)_1\} \) and \( \{0_2, 1_2, 2_2\} \). For \( D_{m,n-3} \), the partite sets are \( \{0_1, 1_1, 2_1, 3_1, 4_1\} \) and \( \{3_2, 4_2, \ldots, (n-1)_2\} \). Finally, for \( D_{m-5,n-3} \), the partite sets are \( \{5_1, 6_1, \ldots, (m-1)_1\} \) and \( \{3_2, 4_2, \ldots, (n-1)_2\} \). Since \( m \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{4} \) and \( m, n \geq 2 \), then \( m-5 \equiv 0 \pmod{2} \) and
$n - 3 \equiv 0 \pmod{2}$. Therefore, $D_{m-5,3}$, $D_{m,n-3}$, and $D_{m-5,n-3}$ can be decomposed by Theorem 2.5. Thus, the leave, $L$, satisfies $|A(L)| = 2$.

Case 9: If $m \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. By Theorem 2.5, it follows that a decomposition exists.

Case 10: If $m \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.5, a decomposition exists.

Case 11: If $m \equiv n \equiv 2 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Thus, by Theorem 2.5, a decomposition exists.

Case 12: If $m \equiv 2 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. By Theorem 2.5, it follows that a decomposition exists.

Case 13: If $m \equiv 3 \pmod{4}$ and $n \equiv 0 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.5 a decomposition exists.

Case 14: If $m \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{4}$, where $m, n \geq 2$, then $D_{m,n-5} = D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5}$. Now pack each smaller graph with $Y$. For $D_{3,5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of packing $D_{3,5}$ so that the leave is minimal is: $\{(4_2, 0_1, 0_2, 1_1, 2_1)_Y, (3_2, 0_1, 4_2, 2_1)_Y, (0_1, 3_2, 1_1, 2_2)_Y, (0_2, 0_1, 1_2, 1_1)_Y, (1_2, 1_1, 3_2, 2_1)_Y, (1_1, 2_2, 2_1, 4_2)_Y, (2_1, 1_2, 0_1, 2_2)_Y\}$ and $|A(L)| = \{(2_1, 0_2), (0_2, 2_1)\}$. The partite sets for $D_{m-3,5}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-5}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{4}$ and $m, n \geq 2$, then $m - 3 \equiv 0 \pmod{2}$ and $n - 5 \equiv 0 \pmod{2}$. Therefore, $D_{m-3,5}$, $D_{m,n-5}$, and $D_{m-3,n-5}$ can be decomposed by Theorem 2.5. Thus, the leave, $L$, satisfies $|A(L)| = 2$. 

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Case 15: If \( m \equiv 3 \pmod{4} \) and \( n \equiv 2 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). By Theorem 2.5, it follows that a decomposition exists.

Case 16: If \( m \equiv n \equiv 3 \pmod{4} \), where \( m, n \geq 2 \), then \( D_{m,n} = D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3} \). Now pack each smaller graph with \( Y \). For \( D_{3,3} \), the partite sets are \{0,1,2\} and \{0,1,2\}. One possible way of packing \( D_{3,3} \) so that the leave is minimal is: \{[2, 0, 1], [1, 2, 1], [2, 1, 0]\} and \(|A(L)|=\{(1,0),(0,1)\}\). The partite sets for \( D_{m-3,3} \) are \{3, 4, \ldots, (m-1)\} and \{0, 1, 2, 2\}. For \( D_{m,n-3} \), the partite sets are \{0, 1, 2\} and \{3, 2, 4, \ldots, (n-1)\}. Finally, for \( D_{m-3,n-3} \), the partite sets are \{3, 4, \ldots, (m-1)\} and \{3, 2, 4, \ldots, (n-1)\}. Now, note that since \( m \equiv 3 \pmod{4} \), \( n \equiv 3 \pmod{4} \) and \( m, n \geq 2 \), then \( m - 3 \equiv 0 \pmod{2} \) and \( n - 3 \equiv 0 \pmod{2} \). Therefore, \( D_{m-3,3} \), \( D_{m,n-3} \), and \( D_{m-3,n-3} \) can be decomposed by Theorem 2.5. Thus, the leave, \( L \), satisfies \(|A(L)|=2\). Q.E.D

Thus, each case for \( Y \) satisfies the conditions of the theorem.

Now we will consider coverings of \( D_{m,n} \).

**Theorem 2.7** A minimal \( C_4 \) covering of \( D_{m,n} \) satisfies:

1. \(|A(P)|=0 \) when \( mn \equiv 0 \pmod{2} \), \( m, n \geq 2 \),

2. \(|A(P)|=2 \), otherwise.

Proof. Since \(|A(C_4)|=4\), it is necessary that the padding \( P \) satisfy \(|A(D_{m,n})| + |A(P)| \equiv 0 \pmod{4} \). Therefore, for \( m \equiv n \equiv 0 \pmod{4} \), \(|A(P)| \geq 0 \) and for the other cases \(|A(P)| \geq 2 \).

Case 1: If \( m \equiv n \equiv 0 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1, a decomposition exists.
Case 2: If \( m \equiv 0 \pmod{4} \) and \( n \equiv 1 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Thus, by Theorem 2.1, a decomposition exists.

Case 3: If \( m \equiv 0 \pmod{4} \) and \( n \equiv 2 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1, a decomposition exists.

Case 4: If \( m \equiv 0 \pmod{4} \) and \( n \equiv 3 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). By Theorem 2.1, it follows that a decomposition exists.

Case 5: If \( m \equiv 1 \pmod{4} \) and \( n \equiv 0 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1 a decomposition exists.

Case 6: If \( m \equiv n \equiv 1 \pmod{4} \), where \( m, n \geq 2 \), then

Case 7: If \( m \equiv 1 \pmod{4} \) and \( n \equiv 2 \pmod{4} \), then \( m \equiv n \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1 a decomposition exists.
Case 8: If \( m \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{4} \), where \( m, n \geq 2 \), then \( D_{m,n} = D_{5,3} \cup D_{m-5,3} \cup D_{m-5,n-3} \). Now cover each smaller graph with \( C_4 \). For \( D_{5,3} \), the partite sets are \( \{0_1, 1_1, 2_1, 3_1, 4_1\} \) and \( \{0_2, 1_2, 2_2\} \). One possible way of covering \( D_{5,3} \) so that the padding is minimal is: \[ \{[0_2, 0_1, 1_2, 4_1]_C, [0_2, 1_1, 2_2, 3_1]_C, [1_2, 1_1, 0_2, 2_1]_C, [1_2, 3_1, 2_2, 4_1]_C, [2_2, 0_1, 0_2, 4_1]_C, [1_2, 2_1, 2_2, 3_1]_C, [2_2, 2_1, 0_2, 3_1]_C, [1_2, 0_1, 2_2, 3_1]_C \} \] and \( |A(P)| = \{(3_1, 2_2), (2_2, 3_1)\} \). The partite sets for \( D_{m-5,3} \) are \( \{5_1, 6_1, \ldots, (m-1)_1\} \) and \( \{0_2, 1_2, 2_2\} \). For \( D_{m,n-3} \), the partite sets are \( \{0_1, 1_1, 2_1, 3_1, 4_1\} \) and \( \{3_2, 4_2, \ldots, (n-1)_2\} \). Finally, for \( D_{m-5,n-3} \), the partite sets are \( \{5_1, 6_1, \ldots, (m-1)_1\} \) and \( \{3_2, 4_2, \ldots, (n-1)_2\} \). Since \( m \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{4} \) and \( m, n \geq 2 \), then \( m - 5 \equiv 0 \pmod{2} \) and \( n - 3 \equiv 0 \pmod{2} \). Therefore, \( D_{m-5,3}, D_{m,n-3}, \) and \( D_{m-5,n-3} \) can be decomposed by Theorem 2.1. Thus, the padding, \( P \), satisfies \( |A(P)| = 2 \).

Case 9: If \( m \equiv 2 \pmod{4} \) and \( n \equiv 0 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1, a decomposition exists.

Case 10: If \( m \equiv 2 \pmod{4} \) and \( n \equiv 1 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). By Theorem 2.1, it follows that a decomposition exists.

Case 11: If \( m \equiv n \equiv 2 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Thus, by Theorem 2.1, a decomposition exists.

Case 12: If \( m \equiv 2 \pmod{4} \) and \( n \equiv 3 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.1, a decomposition exists.

Case 13: If \( m \equiv 3 \pmod{4} \) and \( n \equiv 0 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). By Theorem 2.1, it follows that a decomposition exists.

Case 14: If \( m \equiv 3 \pmod{4} \) and \( n \equiv 1 \pmod{4} \), where \( m, n \geq 2 \), then decompose \( D_{m,n} \). That is \( D_{m,n} = D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5} \). Now cover each smaller
graph with $C_4$. For $D_{3,5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{3,5}$ so that the padding is minimal is: $\{(0_1, 0_2, 1_1, 4_2) \in C, [0_1, 1_2, 2_1, 3_2] \in C, [1_1, 2_2, 0_1, 0_2] \in C, [1_1, 2_1, 3_2] \in C, [1_1, 2_2, 0_1, 3_2] \in C, [1_1, 2_2, 3_2] \in C, [1_1, 2_2, 0_1, 3_2] \in C\}$ and $|A(P)| = \{(2_1, 3_2), (3_2, 2_1)\}$. The partite sets for $D_{m-3,5}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-5}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{4}$ and $m, n \geq 2$, then $m - 3 \equiv 0 \pmod{2}$ and $n - 5 \equiv 0 \pmod{2}$. Therefore, $D_{m-3,5}$, $D_{m,n-5}$, and $D_{m-3,n-5}$ can be decomposed by Theorem 2.1. Thus, the padding, $P$, satisfies $|A(P)| = 2$.

Case 15: If $m \equiv 3 \pmod{4}$ and $n \equiv 2 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.1, a decomposition exists.

Case 16: If $m \equiv n \equiv 3 \pmod{4}$, where $m, n \geq 2$, then $D_{m,n} = D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3}$. Now cover each smaller graph with $C_4$. For $D_{3,3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{3,3}$ so that the padding is minimal is: $\{(0_1, 0_2, 1_1, 1_2) \in C, [2_2, 2_1, 1_2, 1_1] \in C, [0_2, 0_1, 2_2, 1_1] \in C, [1_2, 1_1, 0_2, 2_1] \in C, [2_1, 0_2, 1_1, 2_2] \in C\}$ and $|A(P)| = \{(0_1, 2_2), (2_2, 0_1)\}$. The partite sets for $D_{m,3}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod{4}$, $n \equiv 3 \pmod{4}$ and $m, n \geq 2$, then $m - 3 \equiv 0 \pmod{2}$ and $n - 3 \equiv 0 \pmod{2}$. Therefore $D_{m-3,3}$, $D_{m,n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.1. Thus, the padding, $P$, satisfies $|A(P)| = 2$. Q.E.D.
Thus, each case for $C_4$ satisfies the conditions of the theorem.

**Theorem 2.8** A minimal $X$ covering of $D_{m,n}$ satisfies:

1. $|A(P)| = 0$, when $m \equiv n \equiv 0 \pmod{2}$ or $m \equiv 1 \pmod{2}$, and $m \geq 3$;

2. $|A(P)| = 2$, when $m \equiv 1 \pmod{2}$ and $n \equiv 1 \pmod{2}$, and $m \geq 3$;

3. $|A(P)| = 4$, when $m \equiv 1 \pmod{2}$ and $n \equiv 2 \pmod{4}$, and $m \geq 3$;

Proof. As in Theorem 2.9, for $m \equiv n \equiv 0 \pmod{4}$ we have $|A(P)| \geq 0$. For $m \equiv n \equiv 1 \pmod{2}$, we have $|A(P)| \geq 2$. For $m \equiv 1 \pmod{2}$ and $n \equiv 2 \pmod{4}$, a decomposition does not exist for $|A(D_{m,n})| \equiv 0 \pmod{4}$. So $|A(P)| \geq 4$ is necessary.

Case 1: If $m \equiv n \equiv 0 \pmod{4}$, then $m \equiv n \equiv 0 \pmod{2}$. Therefore, by Theorem 2.3, a decomposition exists.

Case 2: If $m \equiv 0 \pmod{4}$ and $n \equiv 1 \pmod{4}$, then $m \equiv 0 \pmod{4}$ and $n \equiv 1 \pmod{2}$. Thus, by Theorem 2.3, a decomposition exists.

Case 3: If $m \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$, then $m \equiv n \equiv 0 \pmod{2}$. Therefore, by Theorem 2.3, a decomposition exists.

Case 4: If $m \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then $m \equiv 0 \pmod{4}$ and $n \equiv 0 \pmod{2}$. By Theorem 2.3, it follows that a decomposition exists.

Case 5: If $m \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{4}$, then $m \equiv 1 \pmod{2}$ and $n \equiv 0 \pmod{4}$. Therefore, by Theorem 2.3, a decomposition exists.

Case 6: If $m \equiv n \equiv 1 \pmod{4}$, where $m \geq 3$, then $D_{m,n} = D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5}$. Now cover each smaller graph with $X$. For $D_{5,5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{5,5}$ so that the padding is minimal is: $\{[0_1, 1_2, 0_1]_X, [0_1, 2_2, 1_2]_X, [2_2, 0_1, 4_2, 1_1]_X, [0_2, 2_1, 1_2, 0_1]_X, [3_2, 4_1, 4_2, 0_1]_X, [3_1, 1_2, 4_1, 0_2]_X, [3_1, 2_2, 4_1, 3_2]_X, [4_2, 2_1, 0_2, 3_1]_X, [4_2,
4, 2, 1]_X, [3, 3, 4, 2, 1]_X, [1, 2, 1, 2, 3]_X, [1, 3, 2, 1, 4]_X, [1, 2, 4]_X, [1, 0, 4, 1, 2]_X \} and \ |A(P)| = \{(1, 1, 2), (1, 1, 2)\}. The partite sets for \( D_{m-5,5} \) are \{5, 6, \ldots , (m-1)_1\} and \{0, 2, 1, 2, 3, 4\}. For \( D_{m,n-5} \), the partite sets are \{0, 1, 1, 2, 3, 4\} and \{5, 6, \ldots , (n-1)_2\}. Finally, for \( D_{m-5,n-5} \), the partite sets are \{5, 6, \ldots , (m-1)_1\} and \{5, 6, \ldots , (n-1)_2\}. Since \( m \equiv 1 \pmod{4} \), \( n \equiv 1 \pmod{4} \) and \( m,n \geq 5 \), then \( m-5 \equiv 0 \pmod{2} \) and \( n-5 \equiv 0 \pmod{2} \). Therefore, \( D_{m-5,5}, D_{m,n-5}, \) and \( D_{m-5,n-5} \) can be decomposed by Theorem 2.3. Thus, the padding, \( P \), satisfies \( |A(P)| = 2 \).

Case 7: If \( m \equiv 1 \pmod{4} \) and \( n \equiv 2 \pmod{4} \), where \( m \geq 3 \), then \( D_{m,n} = D_{5,2} \cup D_{m-5,2} \cup D_{m,n-2} \cup D_{m-5,n-2} \). Now cover each smaller graph with \( X \). For \( D_{5,2} \), the partite sets are \{0, 1, 1, 2, 3, 4\} and \{0, 2, 1\}. One possible way of covering \( D_{5,2} \) so that the padding is minimal is: \{0, 1, 1, 2]_X, [2, 1, 2, 4, 1, 0]_X, [3, 1, 1, 2, 0]_X, [0, 2, 1, 0, 2, 1]_X, [0, 1, 0, 2, 1]_X, [1, 0, 1, 2, 1]_X \} and \( |A(P)| = \{(0, 1), (0, 2), (1, 0), (1, 2), (0, 2, 1), (1, 2)\} \). The partite sets for \( D_{m-5,2} \) are \{5, 6, \ldots , (m-1)_1\} and \{0, 2, 1\}. For \( D_{m,n-2} \), the partite sets are \{0, 1, 1, 2, 3, 4\} and \{2, 3, \ldots , (n-1)_2\}. Finally, for \( D_{m-5,n-2} \), the partite sets are \{5, 6, \ldots , (m-1)_1\} and \{2, 3, \ldots , (n-1)_2\}. Since \( m \equiv 1 \pmod{4} \) and \( n \equiv 2 \pmod{4} \) and \( m,n \geq 5 \), then \( m-5 \equiv 0 \pmod{2} \) and \( n-2 \equiv 0 \pmod{2} \). Therefore, \( D_{m-5,2}, D_{m,n-2}, \) and \( D_{m-5,n-2} \) can be decomposed by Theorem 2.3. Thus, the padding, \( P \), satisfies \( |A(P)| = 4 \).

Case 8: If \( m \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{4} \), where \( m \geq 3 \), then \( D_{m,n} = D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3} \). Now cover each smaller graph with \( X \). For \( D_{5,3} \), the partite sets are \{0, 1, 1, 2, 3, 4\} and \{0, 2, 1, 2\}. One possible way of covering \( D_{5,3} \) so that the padding is minimal is: \{0, 1, 1, 2]_X, [2, 1, 2, 4, 1]_X, [4, 1, 0, 2, 1]_X, [3, 1, 1, 2]_X, [1, 2, 0]_X, [1, 2, 1, 2, 3]_X, [2, 1, 0, 2, 1]_X, [3, 1, 0, 2, 1]_X \}
and $|A(P)|=\{(3_1, 0_2), (0_2, 3_1)\}$. The partite sets for $D_{m-5,3}$ are $\{5_1, 6_1, \ldots, (m - 1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{3_2, 4_2, \ldots, (n - 1)_2\}$. Finally, for $D_{m-5,n-3}$, the partite sets are $\{5_1, 6_1, \ldots, (m - 1)_1\}$ and $\{3_2, 4_2, \ldots, (n - 1)_2\}$. Since $m \equiv 1 \pmod{4}$ and $n \equiv 3 \pmod{4}$ and $m,n \geq 5$, then $m - 3 \equiv 0 \pmod{2}$ and $n - 3 \equiv 0 \pmod{2}$. Therefore, $D_{m-5,3}$, $D_{m,n-3}$, and $D_{m-5,n-3}$ can be decomposed by Theorem 2.2. Thus, the padding, $P$, satisfies $|A(P)|=2$.

Case 9: If $m \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$, then $m \equiv n \equiv 0 \pmod{2}$. Therefore, by Theorem 2.3, a decomposition exists.

Case 10: If $m \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{4}$, where $m \geq 3$, then $D_{m,n}=D_{2,5} \cup D_{m-2,5} \cup D_{m,n-5} \cup D_{m-2,n-5}$. Now cover each smaller graph with $X$. For $D_{2,5}$, the partite sets are $\{0_1, 1_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{2,5}$ so that the padding is minimal is: $\{[0_2, 0_1, 2_1, 3_1]_X, [2_2, 1_1, 4_2, 0_1]_X, [3_2, 1_1, 1_2, 0_1]_X, [0_1, 4_2, 1_2, 3_2]_X, [1_1, 0_2, 1_2, 2_2]_X, [0_1, 0_2, 1_1, 2_2]_X\}$ and $|A(P)|=\{(0_2, 0_1), (0_2, 1_1), (0_1, 2_2), (1_1, 2_2)\}$. The partite sets for $D_{m-2,5}$ are $\{2_1, 3_1, \ldots, (m - 1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1\}$ and $\{5_2, 6_2, \ldots, (n - 1)_2\}$. Finally, for $D_{m-2,n-5}$, the partite sets are $\{2_1, 3_1, \ldots, (m - 1)_1\}$ and $\{5_2, 6_2, \ldots, (n - 1)_2\}$. Since $m \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{4}$ and $m,n \geq 5$, then $m - 2 \equiv 0 \pmod{2}$ and $n - 5 \equiv 0 \pmod{2}$. Therefore, $D_{m-2,5}$, $D_{m,n-5}$, and $D_{m-2,n-5}$ can be decomposed by Theorem 2.3. Therefore, the padding, $P$, satisfies $|A(P)|=4$.

Case 11: If $m \equiv n \equiv 2 \pmod{4}$, then $m \equiv n \equiv 0 \pmod{2}$. Therefore, by Theorem 2.3, a decomposition exists.

Case 12: If $m \equiv 2 \pmod{4}$ and $n \equiv 3 \pmod{4}$, where $m \geq 3$, then $D_{m,n}=D_{2,3} \cup D_{m-2,3} \cup D_{m,n-3} \cup D_{m-2,n-3}$. Now cover each smaller graph with $X$. For $D_{2,3}$,
the partite sets are \{0_1, 1_1\} and \{0_2, 1_2, 2_2\}. One possible way of covering \(D_{2,3}\) so that the padding is minimal is: \{[0_2, 0_1, 1_2, 1_1]_X, [1_1, 1_2, 0_1, 2_2]_X, [2_2, 0_1, 0_2, 1_1]_X, [2_2, 1_1, 0_2, 0_1]_X\} and \(|A(P)|=\{(2_2, 0_1), (2_2, 1_1), (0_1, 0_2), (1_1, 0_2)\}\) The partite sets for \(D_{m-2,3}\) are \{2_1, 3_1, \ldots, (m - 1)_1\} and \{0_2, 1_2, 2_2\}. For \(D_{m,n-3}\), the partite sets are \{0_1, 1_1\} and \{3_2, 4_2, \ldots, (n - 1)_2\}. Finally, for \(D_{m-2,n-3}\), the partite sets are \{2_1, 3_1, \ldots, (m - 1)_1\} and \{3_2, 4_2, \ldots, (n - 1)_2\}. Since \(m \equiv 2 \pmod{4}\) and \(n \equiv 3 \pmod{4}\) and \(m, n \geq 5\), then \(m - 2 \equiv 0 \pmod{2}\) and \(n - 3 \equiv 0 \pmod{2}\). Therefore, \(D_{m-2,3}\), \(D_{m,n-3}\), and \(D_{m-2,n-3}\) can be decomposed by Theorem 2.3. Thus, the padding, \(P\), satisfies \(|A(P)|=4\).

Case 13: If \(m \equiv 3 \pmod{0}\) and \(n \equiv 0 \pmod{4}\), then \(m \equiv n \equiv 0 \pmod{2}\). Therefore, by Theorem 2.3, a decomposition exists.

Case 14: If \(m \equiv 3 \pmod{4}\) and \(n \equiv 1 \pmod{4}\), where \(m \geq 3\), then \(D_{m,n}=D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5}\). Now cover each smaller graph with \(X\). For \(D_{3,5}\), the partite sets are \{0_1, 1_1, 2_1\} and \{0_2, 1_2, 2_2, 3_2, 4_2\}. One possible way of covering \(D_{3,5}\) so that the padding is minimal is: \{[0_1, 0_2, 1_1, 4_2]_X, [0_1, 1_2, 2_1, 3_2]_X, [1_1, 1_2, 0_1, 2_2]_X, [1_1, 3_2, 2_1, 4_2]_X, [2_1, 0_2, 0_1, 4_2]_X, [1_1, 0_2, 2_1, 3_2]_X, [1_1, 2_2, 1_2, 3_2]_X, [2_1, 2_2, 0_1, 3_2]_X\} and \(|A(P)|=\{(2_1, 3_2), (3_2, 2_1)\}\). The partite sets for \(D_{m-3,5}\) are \{3_1, 4_1, \ldots, (m - 1)_1\} and \{0_2, 1_2, 2_2, 3_2, 4_2\}. For \(D_{m,n-5}\), the partite sets are \{0_1, 1_1, 2_1\} and \{5_2, 6_2, \ldots, (n - 1)_2\}. Finally, for \(D_{m-3,n-5}\), the partite sets are \{3_1, 4_1, \ldots, (m - 1)_1\} and \{5_2, 6_2, \ldots, (n - 1)_2\}. Since \(m \equiv 3 \pmod{4}\) and \(n \equiv 1 \pmod{4}\) and \(m, n \geq 5\), then \(m - 3 \equiv 0 \pmod{2}\) and \(n - 5 \equiv 0 \pmod{2}\). Therefore, \(D_{m-3,5}, D_{m,n-5},\) and \(D_{m-3,n-5}\) can be decomposed by Theorem 2.3. Thus, the padding, \(P\), satisfies \(|A(P)|=2\).
Case 15: If $m \equiv 3 \pmod{4}$ and $n \equiv 2 \pmod{4}$, where $m \geq 3$, then $D_{m,n}=D_{3,2} \cup D_{m-3,2} \cup D_{m,n-2} \cup D_{m-3,n-2}$. Now cover each smaller graph with $X$. For $D_{3,2}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2\}$. One possible way of covering $D_{3,2}$ so that the padding is minimal is: $\{(0_1, 0_2, 1_1, 1_2)_X, [1_2, 1_1, 0_2, 2_1]_X, [2_1, 0_2, 0_1, 1_2]_X, [2_1, 1_2, 0_1, 2_2]_X\}$ and $|A(P)|=\{(2_1, 0_2), (2_1, 1_2), (0_2, 0_1), (1_2, 0_1)\}$. The partite sets for $D_{m-3,2}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2\}$. For $D_{m,n-2}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-2}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod{4}$ and $n \equiv 2 \pmod{4}$ and $m, n \geq 5$, then $m-3 \equiv 0 \pmod{2}$ and $n-2 \equiv 0 \pmod{2}$. Therefore, $D_{m-3,2}$, $D_{m,n-2}$, and $D_{m-3,n-2}$ can be decomposed by Theorem 2.3. Thus, the padding, $P$, satisfies $|A(P)|=4$.

Case 16: If $m \equiv n \equiv 3 \pmod{4}$, where $m \geq 3$, then $D_{m,n}=D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3}$. Now cover each smaller graph with $X$. For $D_{3,3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{3,3}$ so that the padding is minimal is: $\{(0_1, 0_2, 2_1, 1_2)_X, [2_2, 1_1, 1_2, 2_1]_X, [1_1, 0_2, 0_1, 2_2]_X, [1_2, 0_1, 0_2, 1_1]_X, [2_1, 2_2, 0_1, 2_2]_X\}$ and $|A(P)|=\{(0_1, 0_2, 0_2, 0_1)\}$. The partite sets for $D_{m-3,3}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod{4}$, $n \equiv 3 \pmod{4}$ and $m, n \geq 5$, then $m-3 \equiv 0 \pmod{2}$ and $n-3 \equiv 0 \pmod{2}$. Therefore, $D_{m-3,3}$, $D_{m,n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.3. Thus, the padding, $P$, satisfies $|A(P)|=2$. Q.E.D.

Thus, each case for $X$ satisfies the conditions of the theorem.
Theorem 2.9 A minimal $Y$ covering of $D_{m,n}$ satisfies:

1. $|A(P)| = 0$, when $mn \equiv 0 \pmod{2}$, $m, n \geq 2$,

2. $|A(P)| = 2$, otherwise.

Proof. The necessary conditions follow as in Theorem 2.9.

Case 1: If $m \equiv n \equiv 0 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.5 a decomposition exists.

Case 2: If $m \equiv 0 \pmod{4}$ and $n \equiv 1 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. By Theorem 2.5, it follows that a decomposition exists.

Case 3: If $m \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.5, a decomposition exists.

Case 4: If $m \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Thus, by Theorem 2.5, a decomposition exists.

Case 5: If $m \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{4}$, then $mn \equiv 0 \pmod{2}$. Therefore, by Theorem 2.5, a decomposition exists.

Case 6: If $m \equiv n \equiv 1 \pmod{4}$, where $m, n \geq 2$, then $D_{m,n} = D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5}$. Now cover each smaller graph with $Y$. For $D_{5,5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{5,5}$ so that the padding is minimal is: $\{[2_1, 0_2, 3_1, 1_2], Y, [3_2, 0_1, 2_2, 4_1]Y, [1_1, 2_1, 3_2]Y, [2_1, 2_2, 1_1, 3_2]Y, [0_1, 0_2, 1_1, 1_2]Y, [2_2, 3_1, 1_2, 4_1]Y, [3_1, 0_2, 2_1, 3_2]Y, [2_1, 3_2, 3_1, 4_2]Y, [0_2, 0_1, 3_2, 4_1]Y, [4_2, 4_1, 0_2, 1_1]Y, [1_1, 4_2, 0_1, 2_2]Y, [1_2, 0_1, 4_2, 4_1]Y, [3_1, 2_2, 2_1, 4_2]Y\}$ and $|A(P)| = \{(2_1, 3_2), (3_2, 2_1)\}$. The partite sets for $D_{m-5,5}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-5}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$.
and \( \{5, 6, \ldots, (n-1)\} \). Since \( m \equiv 1 \pmod{4}, n \equiv 1 \pmod{4} \) and \( m, n \geq 2 \), then \( m - 5 \equiv 0 \pmod{2} \) and \( n - 5 \equiv 0 \pmod{2} \). Therefore, \( D_{m-5, 5}, D_{m,n-5}, \) and \( D_{m-5,n-5} \) can be decomposed by Theorem 2.5. Thus, the padding, \( P \), satisfies \( |A(P)| = 2 \).

**Case 7:** If \( m \equiv 1 \pmod{4} \) and \( n \equiv 2 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). By Theorem 2.5, it follows that a decomposition exists.

**Case 8:** If \( m \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{4} \), where \( m, n \geq 2 \), then \( D_{m,n} = D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \). Now cover each smaller graph with \( Y \). For \( D_{5,3} \), the partite sets are \( \{0, 1, 1, 2, 3, 1\} \) and \( \{0, 2, 1, 2\} \). One possible way of covering \( D_{5,3} \) so that the padding is minimal is: \( \{[0, 0, 1, 1, 2]_Y, [1, 2, 3, 1, 2]_Y, [2, 1, 0, 2, 2]_Y, [2, 1, 2, 0, 1, 2]_Y, [4, 0, 2, 1, 1, 2]_Y, [1, 0, 2, 3, 1, 2]_Y, [3, 0, 1, 1, 1, 2]_Y, [0, 1, 2, 4, 1, 2]_Y\} \) and \( |A(P)| = \{(1, 0, 2), (0, 1, 1)\} \). The partite sets for \( D_{m-5,3} \) are \( \{5, 6, \ldots, (m-1)\} \) and \( \{0, 2, 1, 2, 2\} \). For \( D_{m,n-3} \), the partite sets are \( \{0, 1, 1, 2, 3, 1\} \) and \( \{3, 2, 4, \ldots, (n-1)\} \). Finally, for \( D_{m-5,n-3} \), the partite sets are \( \{5, 6, \ldots, (m-1)\} \) and \( \{3, 2, 4, \ldots, (n-1)\} \). Since \( m \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{4} \) and \( m, n \geq 2 \), then \( m - 5 \equiv 0 \pmod{2} \) and \( n - 3 \equiv 0 \pmod{2} \). Therefore, \( D_{m-5,3}, D_{m,n-3}, \) and \( D_{m-5,n-3} \) can be decomposed by Theorem 2.5. Thus, the padding, \( P \), satisfies \( |A(P)| = 2 \).

**Case 9:** If \( m \equiv 2 \pmod{4} \) and \( n \equiv 0 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.5, a decomposition exists.

**Case 10:** If \( m \equiv 2 \pmod{4} \) and \( n \equiv 1 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Thus, by Theorem 2.5, a decomposition exists.

**Case 11:** If \( m \equiv n \equiv 2 \pmod{4} \), then \( mn \equiv 0 \pmod{2} \). Therefore, by Theorem 2.5, a decomposition exists.
Case 12: If \( m \equiv 2 \) (mod 4) and \( n \equiv 3 \) (mod 4), then \( mn \equiv 0 \) (mod 2). By Theorem 2.5, it follows that a decomposition exists.

Case 13: If \( m \equiv 3 \) (mod 4) and \( n \equiv 0 \) (mod 4), then \( mn \equiv 0 \) (mod 2). Therefore, by Theorem 2.5, a decomposition exists.

Case 14: If \( m \equiv 3 \) (mod 4) and \( n \equiv 1 \) (mod 4), where \( m, n \geq 2 \), then \( D_{m,n} = D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5} \). Now cover each smaller graph with \( Y \). For \( D_{3,5} \), the partite sets are \( \{0_1, 1_1, 2_1\} \) and \( \{0_2, 1_2, 2_2, 3_2, 4_2\} \). One possible way of covering \( D_{3,5} \) so that the padding is minimal is: \( \{[0_1, 0_2, 1_1, 1_2]_Y, [1_1, 3_2, 2_1, 4_2]_Y, [2_1, 1_2, 0_1, 2_2]_Y, [2_2, 2_1, 0_2, 1_1]_Y, [4_2, 0_1, 2_2, 1_1]_Y, [0_2, 2_1, 4_2, 0_1]_Y, [1_2, 0_1, 3_2, 2_1]_Y, [3_2, 0_1, 1_2, 1_1]_Y \} \) and \( |A(P)| = \{(0_1, 1_2), (1_2, 0_1)\} \). The partite sets for \( D_{m-3,5} \) are \( \{3_1, 4_1, \ldots, (m-1)_1\} \) and \( \{0_2, 1_2, 2_2, 3_2, 4_2\} \). For \( D_{m,n-5} \), the partite sets are \( \{0_1, 1_1, 2_1\} \) and \( \{5_2, 6_2, \ldots, (n-1)_2\} \). Finally, for \( D_{m-3,n-5} \), the partite sets are \( \{3_1, 4_1, \ldots, (m-1)_1\} \) and \( \{5_2, 6_2, \ldots, (n-1)_2\} \). Since \( m \equiv 3 \) (mod 4) \( n \equiv 1 \) (mod 4) and \( m, n \geq 2 \), then \( m-3 \equiv 0 \) (mod 2) and \( n-5 \equiv 0 \) (mod 2). Therefore, \( D_{m-3,5} \), \( D_{m,n-5} \), and \( D_{m-3,n-5} \) can be decomposed by Theorem 2.5. Thus, the padding, \( P \), satisfies \( |A(P)| = 2 \).

Case 15: If \( m \equiv 3 \) (mod 4) and \( n \equiv 2 \) (mod 4), then \( m \equiv n \equiv 0 \) (mod 2). Therefore, by Theorem 2.5, a decomposition exists.

Case 16: If \( m \equiv n \equiv 3 \) (mod 4), where \( m, n \geq 2 \), then \( D_{m,n} = D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3} \). Now cover each smaller graph with \( Y \). For \( D_{3,3} \), the partite sets are \( \{0_1, 1_1, 2_1\} \) and \( \{0_2, 1_2, 2_2\} \). One possible way of covering \( D_{3,3} \) so that the padding is minimal is: \( \{[2_1, 0_2, 0_1, 1_2]_Y, [0_1, 2_2, 1_1, 0_2]_Y, [1_1, 0_2, 2_1, 2_2]_Y, [1_2, 2_1, 2_2, 1_1]_Y, [2_2, 1_1, 1_2, 0_1]_Y \} \) and \( |A(P)| = \{(1_1, 2_2), (2_2, 1_1)\} \). The partite sets for \( D_{m-3,3} \) are \( \{3_1, 4_1, \ldots, (m-1)_1\} \) and \( \{0_2, 1_2, 2_2\} \). For \( D_{m,n-3} \), the partite sets are \( \{0_1, 1_1, 2_1\} \) and \( \{0_2, 1_2, 2_2\} \).
Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod{4}$, $n \equiv 3 \pmod{4}$ and $m, n \geq 2$, then $m - 3 \equiv 0 \pmod{2}$ and $n - 3 \equiv 0 \pmod{2}$. Therefore, $D_{m-3,3}$, $D_{m,n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.5. Thus, the padding, $P$, satisfies $|A(P)| = 2$. Q.E.D.

Thus, each case for $Y$ satisfies the conditions of the theorem. Therefore, theorems 2.1-2.8 provide the necessary and sufficient conditions for maximal packings and minimal coverings of the complete bipartite graph with $C_4$, $X$, and $Y$. Thus, packings and coverings of $D_{m,n}$ have been completed. Now we will pack and cover the complete directed graphs.
3 COVERING $D_v$ WITH THE $C_4$, $X$, AND $Y$ ORIENTATIONS OF A 4–CYCLE

If $\{d_1, d_2, \ldots, d_n\}$ is a covering of $D_v$ with copies of $d$, then we define the digraph $P_v$ with arc set $A(P) = \bigcup_{i=1}^{n} A(g_i) \setminus A(G)$, as the padding of the covering. Therefore a minimal covering of $D_v$ minimizes $|A(P)|$. We will now give necessary and sufficient conditions for minimizing $|A(P)|$ with a minimal covering of $D_v$ for the various orientations of the 4–cycle. We need to only consider graphs where $v \geq 4$. In this chapter we denote $V(D_v)$ as $\{0, 1, \ldots, v-1\}$.

**Theorem 3.1** A minimal $C_4$ covering of $D_v$ satisfies:

1. $|A(P)|=0$ if $v \equiv 0$ or 1 (mod 4), $v \neq 4$,
2. $|A(P)|=4$ if $v=4$,
3. $P=D_2$ if $v \equiv 2$ or 3 (mod 4).

**Proof.** Since $|A(C_4)|=4$, it is necessary that the padding $P$ satisfy $|A(D_v)| + |A(P)| \equiv 0$ (mod 4). Therefore, for $v \equiv 0$ or 1 (mod 4) we have $|A(P)| \geq 0$. For $v \equiv 2$ or 3 (mod 4), we need $|A(P)| \geq 2$. Since a decomposition does not exist for $D_4$, it is necessary that $|A(P)| \geq 4$ when $v=4$.

A $C_4$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 4$. [12].

If $v = 4$, then we have a covering of $D_4$ with copies of $C_4$: $\{[0,3,2,1]_C, [0,2,1,3]_C, [0,3,1,2]_C, [0,1,2,3]_C\}$. Then the padding is $A(P)=\{(0, 3), (1, 2), (2, 1), (3, 0)\}$ and $|A(P)|=4$.

For $v \equiv 2$ (mod 4), we know $D_v= D_{v-6} \cup D_{v-6,6} \cup D_6$. The vertex set for $D_{v-6}$ is $\{0, 1, \ldots, (v-5)\}$. The partite sets for $D_{v-6,6}$ are $\{0, 1, \ldots, (v-5)\}$ and $\{0, 1, \ldots, 5\}$.
For $D_6$, the vertex set is $\{0, 1, \ldots, 5\}$. Since $v-6 \equiv 0 \pmod{4}$, $D_{v-6}$ decomposes with the exception of $v=10$. By Theorem 2.1, $D_{v-6,6}$ can be decomposed. If $v = 6$, then we have a covering of $D_6$ with copies of $C_4$: $\{[0,1,2,3]_C, [0,2,4,1]_C, [0,3,1,4]_C, [0,4,2,5]_C, [0,5,3,2]_C, [1,3,5,2]_C, [1,3,4,5]_C, [1,5,4,3]_C\}$. The padding is then $A(P) = \{(1, 3), (3, 1)\}$.

Thus $|A(P)| = 2$.

If $v = 10$, then we have a covering of $D_{10}$ with copies of $C_4$: $\{[0,1,3,4]_C, [0,2,1,5]_C, [0,3,1,6]_C, [0,4,1,7]_C, [0,5,1,8]_C, [0,6,1,9]_C, [0,7,8,1]_C, [0,9,1,2]_C, [1,4,2,7]_C, [2,3,5,9]_C, [2,4,7,5]_C, [2,5,3,8]_C, [2,6,3,7]_C, [2,8,9,3]_C, [2,9,4,6]_C, [3,6,4,8]_C, [3,9,5,7]_C, [4,5,8,7]_C, [4,9,6,5]_C, [5,6,9,8]_C, [6,7,9,8]_C, [6,8,9,7]_C\}$. The padding is then $A(P) = \{(8, 9), (9, 8)\}$ and $|A(P)|= 2$.

For $v \equiv 3 \pmod{4}$, we know $D_v = D_{v-7} \cup D_{v-7,7} \cup D_7$. The vertex set for $D_{v-7}$ is $\{0, 1, \ldots, (v-6)\}$. The partite sets for $D_{v-7,7}$ are $\{0, 1, \ldots, (v-6)\}$ and $\{0, 1, \ldots, 6\}$. For $D_7$, the vertex set is $\{0, 1, \ldots, 6\}$. Since $v - 7 \equiv 0 \pmod{4}$, $D_{v-7}$ can be decomposed. By Theorem 2.1, $D_{v-7,7}$ can be decomposed. If $v = 7$, then we have a covering of $D_7$ with copies of $C_4$: $\{[0,1,2,3]_C, [0,2,4,6]_C, [0,3,4,5]_C, [0,4,3,2]_C, [0,5,2,1]_C, [0,6,1,4]_C, [1,3,5,4]_C, [1,5,6,3]_C, [1,6,2,5]_C, [2,6,5,3]_C, [3,6,4,2]_C\}$. The padding is then $A(P) = \{(2, 3), (3, 2)\}$. Then $|A(P)| = 2$. Q.E.D.

Thus, each case for $C_4$ satisfies the conditions of the theorem.

**Theorem 3.2** 1. $|A(P)| = 0$ if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 5$,

2. $|A(P)| = 4$ if $v = 5$,

3. $P = D_2$ if $v \equiv 2$ or $3 \pmod{4}$.

Proof. There necessary conditions follow similar to those in Theorem 3.1.

An $X$- decomposition of $D_v$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 5$.
If \( v = 5 \), then we have a covering of \( D_5 \) with copies of \( X \): \([3,0,1,4]_X, [4,0,2,3]_X, [2,0,3,1]_X, [1,0,4,2]_X, [4,1,3,2]_X, [3,2,4,1]_X\}. The padding is then \( A(P) = \{(3, 1), (3, 2), (4, 1), (4, 2)\} \) and \( |A(P)| = 4 \).

For \( v \equiv 2 \pmod{4} \), we know \( D_v = D_{v-6} \cup D_{v-6,6} \cup D_6 \). The vertex set for \( D_{v-6} \) is \( \{0, 1, \ldots, (v - 5)\} \). The partite sets for \( D_{v-6,6} \) are \( \{0, 1, \ldots, (v - 5)\} \) and \( \{0, 1, \ldots, 5\} \). For \( D_6 \), the vertex set is \( \{0, 1, \ldots, 5\} \). Since \( v - 6 \equiv 0 \pmod{4} \), \( D_{v-6} \) decomposes except when \( v = 10 \). By Theorem 2.3, \( D_{v-6,6} \) can be decomposed.

If \( v = 6 \), then we have a covering of \( D_6 \) with copies of \( X \): \([5,0,1,2]_X, [4,0,2,3]_X, [2,0,3,4]_X, [1,0,4,5]_X, [3,0,5,1]_X, [4,1,3,2]_X, [5,3,1,4]_X, [2,1,3,5]_X\}. Then the padding is \( A(P) = \{(1, 3), (3, 1)\} \). Then \( |A(P)| = 2 \).

If \( v = 10 \), then we have a covering of \( D_{10} \) with copies of \( X \): \([3,0,1,2]_X, [6,0,4,5]_X, [9,0,7,8]_X, [8,0,2,4]_X, [7,0,3,5]_X, [2,0,5,7]_X, [5,0,6,9]_X, [1,0,8,3]_X, [4,0,9,1]_X, [2,1,4,9]_X, [6,1,5,2]_X, [5,1,6,8]_X, [3,1,7,6]_X, [7,1,8,9]_X, [8,1,9,5]_X, [9,2,5,3]_X, [8,2,6,7]_X, [4,2,8,6]_X, [6,3,9,4]_X, [7,2,3,4]_X, [9,4,3,7]_X, [5,4,9,6]_X, [4,7,3,8]_X\}. Then the padding is \( A(P) = \{(4,9), (9,4)\} \) and \( |A(P)| = 2 \).

For \( v \equiv 3 \pmod{4} \), we know \( D_v = D_{v-7} \cup D_{v-7,7} \cup D_7 \). The vertex set for \( D_{v-7} \) is \( \{0, 1, \ldots, (v - 6)\} \). The partite sets for \( D_{v-7,7} \) are \( \{0, 1, \ldots, (v - 6)\} \) and \( \{0, 1, \ldots, 6\} \). For \( D_7 \), the vertex set is \( \{0, 1, \ldots, 6\} \). Since \( v - 7 \equiv 0 \pmod{4} \), \( D_{v-7} \) can be decomposed. By Theorem 2.3, \( D_{v-7,7} \) can be decomposed. If \( v = 7 \), then we have a covering of \( D_7 \) with copies of \( X \): \([3,0,1,2]_X, [6,0,2,3]_X, [3,1,0,4]_X, [3,5,0,6]_X, [2,5,1,6]_X, [4,5,2,0]_X, [1,5,2,4]_X, [0,5,4,3]_X, [6,4,2,5]_X, [5,6,1,3]_X, [4,6,2,1]_X\}. The padding is then \( A(P) = \{(2,5), (5,2)\} \) and \( |A(P)| = 2 \). Q.E.D

Thus, each case for \( X \) satisfies the conditions of the theorem.
Theorem 3.3 1. $|A(P)| = 0$ if $v \equiv 0$ or $1 \pmod{4}$, $v \not\in \{4, 5\}$,

2. $|A(P)| = 4$ if $v \in \{4, 5\}$,

3. $P = D_2$ if $v \equiv 2$ or $3 \pmod{4}$.

Proof. The necessary conditions follow similar to those of Theorem 3.1.

A Y-decomposition of $D_v$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$, $v \not\in \{4, 5\}$. [6]

If $v = 4$, then we have a covering of $D_4$ with copies of $Y$: $\{[0,1,2,3]_Y, [2,3,0,1]_Y, [2,1,3,0]_Y, [3,1,2,0]_Y\}$. The padding is then $A(P) = \{(0, 3), (0, 2), (2, 0), (3, 0)\}$ and $|A(P)| = 4$. 
If \( v = 5 \), then we have a covering of \( D_5 \) with copies of \( Y \): \{[3,0,1,2]_Y, [4,0,2,1]_Y, [2,1,4,0]_Y, [4,2,3,1]_Y, [3,4,0,1]_Y, [2,4,3,0]_Y \}. Then the padding is \( A(P) = \{(2, 0), (2, 1), (4, 0), (4, 1)\} \) and \( |A(P)| = 4 \).

For \( v \equiv 2 \pmod{4} \), we know \( D_v = D_{v-6} \cup D_{v-6,6} \cup D_6 \). The vertex set for \( D_{v-6} \) is \{0, 1, \ldots, (v - 5)\}. The partite sets for \( D_{v-6,6} \) are \{0, 1, \ldots, (v - 5)\} and \{0, 1, \ldots, 5\}. For \( D_6 \), the vertex set is \{0, 1, \ldots, 5\}. Since \( v - 6 \equiv 0 \pmod{4} \), \( D_{v-6} \) decomposes except when \( v = 10 \). By Theorem 2.5, \( D_{v-6,6} \) can be decomposed. If \( v = 6 \), then we have a covering of \( D_6 \) with copies of \( Y \): \{[3,0,1,2]_Y, [4,0,2,5]_Y, [2,0,5,3]_Y, [0,3,1,4]_Y, [5,1,2,4]_Y, [1,4,3,5]_Y, [1,2,4,3]_Y, [2,1,0,5]_Y \}. The padding is then \( A(P) = \{(1, 2), (2, 1)\} \) and \( |A(P)| = 2 \).

If \( v = 10 \), then we have a covering of \( D_{10} \) with copies of \( Y \): \{[3,0,1,2]_Y, [6,3,4,5]_Y, [7,1,2,8]_Y, [8,4,5,9]_Y, [2,5,6,3]_Y, [6,1,0,9]_Y, [0,4,3,7]_Y, [8,1,3,5]_Y, [3,5,8,1]_Y, [1,7,4,9]_Y, [1,4,0,5]_Y, [7,5,1,9]_Y, [7,0,8,2]_Y, [4,1,6,2]_Y, [0,5,2,6]_Y, [5,4,7,9]_Y, [0,2,9,3]_Y, [6,0,9,4]_Y, [2,4,6,7]_Y, [9,3,8,6]_Y, [8,3,7,6]_Y, [9,2,0,8]_Y, [4,8,7,5]_Y \}. The padding is then \( A(P) = \{(4,5), (5,4)\} \) and \( |A(P)| = 2 \).

For \( v \equiv 3 \pmod{4} \), we know \( D_v = D_{v-7} \cup D_{v-7,7} \cup D_7 \). The vertex set for \( D_{v-7} \) is \{0, 1, \ldots, (v - 6)\}. The partite sets for \( D_{v-7,7} \) are \{0, 1, \ldots, (v - 6)\} and \{0, 1, \ldots, 6\}. For \( D_7 \), the vertex set is \{0, 1, \ldots, 6\}. Since \( v - 7 \equiv 0 \pmod{4} \), \( D_{v-7} \) can be decomposed. By Theorem 2.5, \( D_{v-7,7} \) can be decomposed. If \( v = 7 \), then we have a covering of \( D_7 \) with copies of \( Y \): \{[3,0,1,2]_Y, [5,0,2,4]_Y, [6,0,2,3]_Y, [1,0,4,3]_Y, [4,0,5,1]_Y, [2,0,6,5]_Y, [6,3,1,5]_Y, [5,2,6,3]_Y, [1,6,4,2]_Y, [1,3,5,4]_Y, [4,6,1,3]_Y \}. The padding is then \( A(P) = \{(1, 3), (3, 1)\} \) and \( |A(P)| = 2 \). Q.E.D.
Thus, each case for $Y$ satisfies the conditions of the theorem. Therefore, Theorems 3.1-3.3 provide the necessary and sufficient conditions for maximal packings and minimal coverings of the complete directed graph with $C_4$, $X$, and $Y$. Now we will pack and cover the complete digraph on $v$ vertices and a hole of size $w$. 
4 PACKING AND COVERING $D(v, w)$ WITH THE $C_4$, $X$, AND $Y$

ORIENTATIONS OF THE 4–CYCLE

With the completion of packings and coverings of the complete bipartite graph and the complete directed graph, we can pack and cover complete directed graphs with holes. The complete directed graph on $v$ vertices with a hole of size $w$ is a graph with $v$ vertices that are mutually adjacent and a set of $w$ vertices that are adjacent to the $v$ vertices and incident to the $w$ vertices. We will begin with the maximal $C_4$-packings.

Figure 8: Example of $D(v, w)$
Theorem 4.1 [5] A $C_4$-decomposition of $D_v$ with a hole of size $w$ exists if and only if $\{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\}$ or $\{v \pmod{4}, w \pmod{4}\} \subset \{2, 3\}$ and $v - w > 3$.

Theorem 4.2 [5] An optimal packing of $D_v$ with copies of $C_4$ and leave $L$ satisfies:

1. $L = \emptyset$ if $v \equiv 0$ or 1 (mod 4), $v \neq 4$,
2. $|A(L)| = 4$ if $v = 4$,
3. $L = D_2$ if $v \equiv 2$ or 3 (mod 4).

Theorem 4.3 A maximal $C_4$ packing of $D(v, w)$ satisfies:

1. $|A(L)| = 0$ if $\{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\}$ or $\{v \pmod{4}, w \pmod{4}\} \subset \{2, 3\}, v - w > 3$,
2. $|A(L)| = 2$, otherwise.

Proof. Since $|A(C_4)| = 4$, it is necessary that the leave, $L$, satisfy $|A(L)| \equiv |A(D(v, w))| \pmod{4}$. The necessary conditions on $|A(L)|$ follow.

Case 1: If $v \equiv w \equiv 0$ (mod 4), then $v \equiv w \equiv 0$ (mod 2). Therefore, by Theorem 4.1, a decomposition exists.

Case 2: If $v \equiv 0$ (mod 4) and $w \equiv 1$ (mod 4), then $\{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\}$. Thus, by Theorem 4.1, a decomposition exists.

Case 3: If $v \equiv 0$ (mod 4) and $w \equiv 2$ (mod 4), then we know the number of arcs is $v(v - 1) - w(w - 1) \equiv 2$ (mod 4). Then $D(v, w) = D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w - 1)_2)\}$. Since $v \equiv 0$ (mod 4) and $w \equiv 2$ (mod 4), $D_{v-w,w}$ can be decomposed by Theorem 2.1. $D_{v-w}$ can be packed with leave $L = D_2$ by Theorem 4.2. Thus $|A(L)| = 2$. 

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Case 4: If \( v \equiv 0 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w-1)_2)\} \). It follows from Theorem 2.2, that \( D_{v-w,w} \) can be packed and \( |A(L)|=2 \). Thus \( D_{v-w} \) can be decomposed by Theorem 4.1.

Case 5: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{0,1\} \). Therefore, by Theorem 4.1, a decomposition exists.

Case 6: If \( v \equiv w \equiv 1 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{0,1\} \). Thus, by Theorem 4.1, a decomposition exists.

Case 7: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w-1)_2)\} \). Since \( v \equiv 1 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then \( D_{v-w,w} \) can be decomposed by Theorem 2.1. \( D_{v-w} \) can be packed with leave \( L=D_2 \) by Theorem 4.2. Thus \( |A(L)|=2 \).

Case 8: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w-1)_2)\} \). Note that since, \( v \equiv 1 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then \( D_{v-w,w} \) can be decomposed by Theorem 2.1. \( D_{v-w} \) can be packed with leave \( L=D_2 \) by Theorem 4.2. Thus \( |A(L)|=2 \).
Case 9: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v - w - 1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v - w - 1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w - 1)_2)\} \). Note that since, \( v \equiv 2 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( D_{v-w,w} \) can be decomposed by Theorem 2.1. \( D_{v-w} \) can be packed with leave \( L = D_2 \) by Theorem 4.2. Then \( |A(L)| = 2 \).

Case 10: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 1 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v - w - 1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v - w - 1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w - 1)_2)\} \). By Theorem 2.2, \( D_{v-w,w} \) can be packed and \( |A(L)| = 2 \). \( D_{v-w} \) can be decomposed by Theorem 4.1.

Case 11: If \( v \equiv w \equiv 2 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2, 3\} \). Therefore, by Theorem 4.1, a decomposition exists.

Case 12: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2, 3\} \). Thus, by Theorem 4.1, a decomposition exists.

Case 13: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Thus \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v - w - 1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v - w - 1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w - 1)_2)\} \). Since \( v \equiv 3 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( D_{v-w,w} \) can be decomposed by Theorem 2.1. \( D_{v-w} \) can be packed with leave \( L = D_2 \) by Theorem 4.2. Thus \( |A(L)| = 2 \).

Case 14: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 1 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for
For $D_{v-w}$, the vertex set is $\{(0, 1, \ldots, (v-w-1))\}$. For $D_{v-w,w}$, the partite sets are $\{(0, 1, \ldots, (v-w-1))\}$ and $\{(0, 1, \ldots, (w-1))\}$. Since $v \equiv 3 \pmod{4}$ and $w \equiv 0 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. $D_{v-w}$ can be packed with leave $L=D_2$ by Theorem 4.2. Thus $|A(L)|=2$.

Case 15: If $v \equiv 3 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$. Therefore, by Theorem 4.1, a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$. Then, by Theorem 4.1, a decomposition exists. Q.E.D

Thus, each case for $C_4$ satisfies the conditions of the theorem.

**Theorem 4.4** [5] An $X$ decomposition of $D_v$ with a hole of size $w$ exists if and only if $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$ or $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$ and $v-w \neq 3$ in the case of $v \equiv 2 \pmod{4}$ and $w \equiv 3 \pmod{4}$.

**Theorem 4.5** [5] An optimal packing of $D_v$ with copies of $X$ and leave $L$ satisfies:

1. $L=\emptyset$ if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 5$,
2. $|A(L)|=4$ if $v = 5$,
3. $L=D_2$ if $v \equiv 2$ or $3 \pmod{4}$.

**Theorem 4.6** A maximal $X$ packing of $D(v, w)$ satisfies:

1. $L=\emptyset$ if $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$ or $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$,
2. $|A(L)|=2$, otherwise.

Proof. The necessary conditions follow similarly to that of Theorem 4.3.

Case 1: If $v \equiv w \equiv 0 \pmod{4}$, then $v \equiv w \equiv 0 \pmod{2}$. Thus, by Theorem 4.4, a decomposition exists.
Case 2: If $v \equiv 0 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subseteq \{0,1\}$. Therefore, by Theorem 4.4, a decomposition exists.

Case 3: If $v \equiv 0 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then we know the number of arcs is $v(v - 1) - w(w - 1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2)\}$. Since $v \equiv 0 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.3. Then $D_{v-w}$ can be packed with leave $L = D_2$ by Theorem 4.5. Thus $|A(L)| = 2$.

Case 4: If $v \equiv 0 \pmod{4}$ and $w \equiv 3 \pmod{4}$, then we know the number of arcs is $v(v - 1) - w(w - 1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2)\}$. By Theorem 2.4, $D_{v-w,w}$ can be packed and $|A(L)| = 2$. It follows that $D_{v-w}$ can be decomposed by Theorem 4.5. Thus $|A(L)| = 2$.

Case 5: If $v \equiv 1 \pmod{4}$ and $w \equiv 0 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subseteq \{0,1\}$. Therefore, by Theorem 4.4 a decomposition exists.

Case 6: If $v \equiv 1 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subseteq \{0,1\}$. Therefore, by Theorem 4.4 a decomposition exists.

Case 7: If $v \equiv 1 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then we know the number of arcs is $v(v - 1) - w(w - 1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{w,w}$. Note that the vertex set for $D_{v-w-2}$ is $\{0_1, 1_1, \ldots, (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-3)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+1)_2\}$. For $D_{w,w}$, the partite sets are $\{(v-2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. By Theorem 4.5,
$D_{v-w-2}$ can be decomposed. $D_{v-w-2,w+2}$ can be decomposed by Theorem 2.3. $D_{2,w}$ can be decomposed by Theorem 2.3. Therefore $|A(L)|=2$.

Case 8: If $v \equiv 1 \pmod{4}$ and $w \equiv 3 \pmod{4}$, then we know the number of arcs is $v(v - 1) - w(w - 1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{2,w}$. Note that the vertex set for $D_{v-w-2}$ is $\{0_1, 1_1, \ldots, (v - w - 3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v - w - 3)_1\}$ and $\{0_2, 1_2, \ldots, (v - w - 1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v - 2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w - 1)_2\}$. Now note by Theorem 4.5, $D_{v-w-2}$ can be decomposed. $D_{v-w-2,w+2}$ by Theorem 2.3. $D(w + 2, 2)$ can be packed by case 7 with $L=D_2$. Therefore $|A(L)|=2$.

Case 9: If $v \equiv 2 \pmod{4}$ and $w \equiv 0 \pmod{4}$, then we know the number of arcs is $v(v - 1) - w(w - 1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}$ and $\{0_2, 1_2, \ldots, (w - 1)_2\}$. Since $v \equiv 2 \pmod{4}$ and $w \equiv 0 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.3. $D_{v-w}$ can be packed with leave $L=D_2$ by Theorem 4.5. Then $|A(L)|=2$.

Case 10: If $v \equiv 2 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then we know the number of arcs is $v(v - 1) - w(w - 1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}$ and $\{0_2, 1_2, \ldots, (w - 1)_2\}$. By Theorem 2.4, $D_{v-w,w}$ can be packed and $|A(L)|=2$. Thus $D_{v-w}$ can be decomposed by Theorem 4.5. Then $|A(L)|=2$.

Case 11: If $v \equiv w \equiv 2 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$. Therefore, by Theorem 4.4, a decomposition exists.
Case 12: If $v \equiv 2 \pmod{4}$ and $w \equiv 3 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$. Thus, by Theorem 4.4, a decomposition exists.

Case 13: If $v \equiv 3 \pmod{4}$ and $w \equiv 0 \pmod{4}$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0, 1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0, 1_2, \ldots, (w-1)_2)\}$. Since $v \equiv 3 \pmod{4}$ and $w \equiv 0 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.3. Thus $D_{v-w}$ can be packed with leave $L = D_2$ by Theorem 4.5. Then $|A(L)| = 2$.

Case 14: If $v \equiv 3 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w-3} \cup D_{v-w-3,3} \cup D_{v-w-3,w} \cup D(w+3, w) = D_{v-w-3} \cup D_{v-w-3,w+3} \cup D(w+3, w)$. Note that the vertex set for $D_{v-w-3}$ is $\{0, 1, \ldots, (v-w-4)_1\}$. For $D_{v-w-3,w+3}$, the partite sets are $\{0, 1, \ldots, (v-w-4)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+2)_2\}$. For $D(w+3, w)$, the partite sets are $\{0_2, 1_2, \ldots, (w+2)_2\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. $D_{v-w-3}$ can be packed by Theorem 4.4. By Theorem 2.3, $D_{v-w-3,w+3}$ can be decomposed. $D(w+3, w)$ can be decomposed by Theorem 4.5. Therefore $|A(L)| = 2$.

Case 15: If $v \equiv 3 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$. Therefore, by Theorem 4.4 a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$. Therefore, by Theorem 4.4 a decomposition exists. Q.E.D

Thus, each case for $X$ satisfies the conditions of the theorem.

**Theorem 4.7** [5] A Y decomposition of $D_v$ with a hole of size $w$ exists if and only if $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$ or $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$ and $v - w \neq 3$. 53
Theorem 4.8 [5] An optimal packing of $D_v$ with copies of $Y$ and leave $L$ satisfies:

1. $L=\emptyset$ if $v \equiv 0$ or $1 \pmod{4}$, $v \not\in \{4,5\}$,
2. $|A(L)|=4$ if $v \in \{4,5\}$,
3. $|A(L)|=2$ if $v \equiv 2$ or $3 \pmod{4}$.

Theorem 4.9 A maximal $Y$ packing of $D(v,w)$ satisfies:

1. $|A(L)|=0$ if $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$ or $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$,
2. $|A(L)|=2$, otherwise.

Proof. The necessary conditions follow similarly to those of Theorem 4.3.

Case 1: If $v \equiv w \equiv 0 \pmod{4}$, then $v \equiv w \equiv 0 \pmod{2}$. Therefore, by Theorem 4.7 a decomposition exists.

Case 2: If $v \equiv 0 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$. Therefore, by Theorem 4.7 a decomposition exists.

Case 3: If $v \equiv 0 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod{4}$. Then $D(v,w)=D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1,1_1,\ldots,(v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,1_1,\ldots,(v-w-1)_1)\}$ and $\{(0_2,1_2,\ldots,(w-1)_2)\}$. Note that since , $v \equiv 0 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.5. $D_{v-w}$ can be packed with leave $L=D_2$ by Theorem 4.8. Then $|A(L)|=2$.

Case 4: If $v \equiv 0 \pmod{4}$ and $w \equiv 3 \pmod{4}$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod{4}$. Then $D(v,w)=D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1,1_1,\ldots,(v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are
{(0, 1, 1, \ldots, (v - w - 1)1)} and \{(0, 1, 1, \ldots, (w - 1)2\}. By Theorem 2.6, D_{v-w,w} can be packed and |A(L)|=2. D_{v-w} can be decomposed by Theorem 4.8.

Case 5: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{0,1\} \). Therefore, by Theorem 4.7 a decomposition exists.

Case 6: If \( v \equiv w \equiv 1 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{0,1\} \). Therefore, by Theorem 4.7 a decomposition exists.

Case 7: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w} = D_{v-w-2} \cup D_{v-w-2,2} \cup D_{2,w} \cup D_2 \). Note that the vertex set for \( D_{v-w-2} \) is \{0, 1, 1, \ldots, (v-w-3)1\}. For \( D_{v-w-2,2} \), the partite sets are \{0, 1, 1, \ldots, (v-w-3)1\} and \{0, 1, 1, \ldots, (w+1)2\}. For \( D_{2,w} \), the partite sets are \{(v-2)1, (v-1)1\} and \{0, 1, 1, \ldots, (w-2)1, (w-1)2\}. By Theorem 4.8, \( D_{v-w-2} \) can be decomposed. \( D_{v-w-2,2} \) can be decomposed by Theorem 2.5. \( D_{2,w} \) can be decomposed by Theorem 2.5. Therefore |A(L)|=2.

Case 8: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w} = D_{v-w-2} \cup D_{v-w-2,2} \cup D_{2,w} \cup D_2 \). Note that the vertex set for \( D_{v-w-2} \) is \{0, 1, 1, \ldots, (v-w-3)1\}. For \( D_{v-w-2,2} \), the partite sets are \{0, 1, 1, \ldots, (v-w-3)1\} and \{0, 1, 1, \ldots, (w+1)2\}. For \( D_{2,w} \), the partite sets are \{(v-2)1, (v-1)1\} and \{0, 1, 1, \ldots, (w-1)2\}. Now note by Theorem 4.8, \( D_{v-w-2} \) can be decomposed. \( D_{v-w-2,2} \) can be decomposed by Theorem 2.5. \( D(w+2,2) \) can be packed by Case 7 with \( L=D_2 \). Therefore |A(L)|=2.
Case 9: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w-1)_2)\} \). Since \( v \equiv 2 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( D_{v-w,w} \) can be decomposed by Theorem 2.5. \( D_{v-w} \) can be packed with leave \( L = D_2 \) by Theorem 4.8. Then \( |A(L)| = 2 \).

Case 10: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 1 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w-1)_2)\} \). By Theorem 2.6, \( D_{v-w,w} \) can be packed and \( |A(L)| = 2 \). \( D_{v-w} \) can be decomposed by Theorem 4.8.

Case 11: If \( v \equiv w \equiv 2 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2,3\} \). Therefore, by Theorem 4.7 a decomposition exists.

Case 12: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2,3\} \). Thus, by Theorem 4.7 a decomposition exists.

Case 13: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v-w-1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w-1)_2)\} \). Since \( v \equiv 3 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( D_{v-w,w} \) can be decomposed by Theorem 2.5. \( D_{v-w} \) can be packed with leave \( L = D_2 \) by Theorem 4.8. Then \( |A(L)| = 2 \).

Case 14: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 1 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w-2} \cup D_{v-w-2,2} \cup \ldots \).
\[ D_{v-w-2,w} = D_{v-w-2} \cup D_{v-w-2,w+2} \cup D(w+2, w). \] Note that the vertex set for \( D_{v-w-2} \) is \( \{0, 1, \ldots, (v-w-3)\} \). For \( D_{v-w-2,w+2} \), the partite sets are \( \{0, 1, \ldots, (v-w-3)\} \) and \( \{0, 1, \ldots, (v-w+1)\} \). For \( D_{2,w} \), the partite sets are \( \{(v-2)\), \( \{(v-1)\} \) and \( \{0, 1, \ldots, (w-1)\} \). \( D_{v-w-2} \) can be decomposed by Theorem 4.8. By Theorem 2.5, \( D_{v-w-2,w+2} \) can be decomposed. \( D(w+2, w) = D_{2,w} \cup D_2 \). \( D_{2,w} \) can be decomposed by Theorem 4.7. Therefore \( |A(L)| = 2 \).

Case 15: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2, 3\} \). Therefore, by Theorem 4.7 a decomposition exists.

Case 16: If \( v \equiv w \equiv 3 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2, 3\} \). Therefore, by Theorem 4.7 a decomposition exists. Q.E.D

Thus, each case for \( Y \) satisfies the conditions of the theorem. Now we want to cover \( D(v, w) \) with \( C_4 \).

**Theorem 4.10** A minimal \( C_4 \)-covering of \( D(v, w) \) satisfies:

1. \( |A(L)| = 0 \) if \( \{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\} \) or \( \{v \pmod{4}, w \pmod{4}\} \subset \{2, 3\}, \) and \( v-w > 3 \),
2. \( |A(L)| = 2 \), otherwise.

Proof. The necessary conditions follow similar to those of Theorem 2.9.

Case 1: If \( v \equiv w \equiv 0 \pmod{4} \), then \( v \equiv w \equiv 0 \pmod{2} \). Therefore, by Theorem 4.1, a decomposition exists.

Case 2: If \( v \equiv 0 \pmod{4} \) and \( w \equiv 1 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\} \). Therefore, by Theorem 4.1, a decomposition exists.

Case 3: If \( v \equiv 0 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for
$D_{v-w}$, the vertex set is \{(0_1, 1_1, \ldots, (v-w-1)_1)\}. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2)\}$. Since $v \equiv 0 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. $D_{v-w}$ can be covered with padding $P=D_2$ by Theorem 3.2. Thus $|A(P)|=2$.

**Case 4:** If $v \equiv 0 \pmod{4}$ and $w \equiv 3 \pmod{4}$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w)=D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2)\}$. Since $v \equiv 0 \pmod{4}$ and $w \equiv 3 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. $D_{v-w}$ can be covered with padding $P=D_2$ by Theorem 3.2. Thus $|A(P)|=2$.

**Case 5:** If $v \equiv 1 \pmod{4}$ and $w \equiv 0 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$. Therefore, by Theorem 4.1 a decomposition exists.

**Case 6:** If $v \equiv w \equiv 1 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$. Thus, by Theorem 4.1 a decomposition exists.

**Case 7:** If $v \equiv 1 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w)=D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2)\}$. Since $v \equiv 1 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. $D_{v-w}$ can be covered with padding $P=D_2$ by Theorem 3.2. Thus $|A(P)|=2$.

**Case 8:** If $v \equiv 1 \pmod{4}$ and $w \equiv 3 \pmod{4}$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w)=D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are
\{(0_1, 1_1, \ldots, (v-w-1)_1)\} \text{ and } \{(0_2, 1_2, \ldots, (w-1)_2)\}. \text{ Note that since, } v \equiv 1 \pmod{4} \text{ and } w \equiv 3 \pmod{4}, \text{ then } D_{v-w,w} \text{ can be decomposed by Theorem 2.1. } D_{v-w} \text{ can be covered with padding } P = D_2 \text{ by Theorem 3.2. Then } |A(P)| = 2.

Case 9: If \(v \equiv 2 \pmod{4}\) and \(w \equiv 0 \pmod{4}\), then we know the number of arcs is \(v(v-1) - w(w-1) \equiv 2 \pmod{4}\). Then \(D(v, w) = D_{v-w} \cup D_{v-w,w}\). Note that for \(D_{v-w}\), the vertex set is \\{(0_1, 1_1, \ldots, (v-w-1)_1)\}. For \(D_{v-w,w}\), the partite sets are \\{(0_1, 1_1, \ldots, (v-w-1)_1)\} \text{ and } \{(0_2, 1_2, \ldots, (w-1)_2)\}. \text{ Since } v \equiv 2 \pmod{4} \text{ and } w \equiv 0 \pmod{4}, \text{ then } D_{v-w,w} \text{ can be decomposed by Theorem 2.1. } D_{v-w} \text{ can be covered with padding } P = D_2 \text{ by Theorem 3.2. Thus } |A(P)| = 2.

Case 10: If \(v \equiv 2 \pmod{4}\) and \(w \equiv 1 \pmod{4}\), then we know the number of arcs is \(v(v-1) - w(w-1) \equiv 2 \pmod{4}\). Then \(D(v, w) = D_{v-w} \cup D_{v-w,w}\). Note that for \(D_{v-w}\), the vertex set is \\{(0_1, 1_1, \ldots, (v-w-1)_1)\}. For \(D_{v-w,w}\), the partite sets are \\{(0_1, 1_1, \ldots, (v-w-1)_1)\} \text{ and } \{(0_2, 1_2, \ldots, (w-1)_2)\}. \text{ Since } v \equiv 2 \pmod{4} \text{ and } w \equiv 1 \pmod{4}, \text{ then } D_{v-w,w} \text{ can be decomposed by Theorem 2.1. } D_{v-w} \text{ can be covered with padding } P = D_2 \text{ by Theorem 3.2. Thus } |A(P)| = 2.

Case 11: If \(v \equiv w \equiv 2 \pmod{4}\), then \\{v (mod 4), w (mod 4)\} \subset \{2, 3\}. Therefore, by Theorem 4.1, a decomposition exists.

Case 12: If \(v \equiv 2 \pmod{4}\) and \(w \equiv 3 \pmod{4}\), then \\{v (mod 4), w (mod 4)\} \subset \{2, 3\}. Thus, by Theorem 4.1, a decomposition exists.
0 (mod 4), then $D_{v-w,w}$ can be decomposed by Theorem 2.1. $D_{v-w}$ can be covered with padding $P=D_2$ by Theorem 3.2. Thus $|A(P)|=2$.

Case 14: If $v \equiv 3 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w)=D_{v-w} \cup D_{v-w,w}$. Note that for $D_{v-w}$, the vertex set is $\{(0, 1, \ldots, (v - w - 1)\}}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1, \ldots, (v - w - 1))\}$ and $\{(0_2, 1, \ldots, (w - 1))\}$. Since $v \equiv 3 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. $D_{v-w}$ can be covered with padding $P=D_2$ by Theorem 3.2. Thus $|A(P)|=2$.

Case 15: If $v \equiv 3 \pmod{4}$ and $w \equiv 2 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$. Therefore, by Theorem 4.1, a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$. Thus, by Theorem 4.1, a decomposition exists. Q.E.D

Thus, each case for $C_4$ satisfies the conditions of the theorem.

**Theorem 4.11** A minimal $X$ covering of $D(v, w)$ satisfies:

1. $|A(L)|=0$ if $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$ or $\{v \pmod{4}, w \pmod{4}\} \subset \{2,3\}$,
2. $|A(L)|=2$, otherwise.

Proof. The necessary conditions follow similar to those of Theorem 2.9.

Case 1: If $v \equiv w \equiv 0 \pmod{4}$, then $v \equiv w \equiv 0 \pmod{2}$. Therefore, by Theorem 4.4 a decomposition exists.

Case 2: If $v \equiv 0 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then $\{v \pmod{4}, w \pmod{4}\} \subset \{0,1\}$. Therefore, by Theorem 4.4 a decomposition exists.
Case 3: If \( v \equiv 0 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w, w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0, 1, \ldots, (v - w - 1)_1)\} \). For \( D_{v-w, w} \), the partite sets are \( \{(0, 1, \ldots, (v - w - 1)_1)\} \) and \( \{(0, 1, \ldots, (w - 1)_2)\} \). Since \( v \equiv 0 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then \( D_{v-w, w} \) can be decomposed by Theorem 2.3. \( D_{v-w} \) can be covered with padding \( P = D_2 \) by Theorem 3.1. Thus \( |A(P)| = 2 \).

Case 4: If \( v \equiv 0 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w, w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0, 1, \ldots, (v - w - 1)_1)\} \). For \( D_{v-w, w} \), the partite sets are \( \{(0, 1, \ldots, (v - w - 1)_1)\} \) and \( \{(0, 1, \ldots, (w - 1)_2)\} \). Since \( v \equiv 0 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), \( D_{v-w, w} \) can be packed by Theorem 2.9 with padding \( P = D_2 \). \( D_{v-w} \) can be decomposed by Theorem 4.5. Thus \( |A(P)| = 2 \).

Case 5: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\} \). Therefore, by Theorem 4.4, a decomposition exists.

Case 6: If \( v \equiv w \equiv 1 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\} \). Thus, by Theorem 4.4, a decomposition exists.

Case 7: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w-2} \cup D_{v-w-2, 2} \cup D_{v-w-2, w} \cup D_{v-w-2, w+2} \cup D_{2, w} \cup D_2 \). Note that the vertex set for \( D_{v-w-2} \) is \( \{0, 1, \ldots, (v - w - 3)_1\} \). For \( D_{v-w-2, w+2} \), the partite sets are \( \{0, 1, \ldots, (v - w - 3)_1\} \) and \( \{0, 1, \ldots, (v - w + 1)_2\} \). For \( D_{2, w} \), the partite sets are \( \{(v - 2)_1, (v - 1)_1\} \) and \( \{0, 1, \ldots, (w - 1)_2\} \). Now note by Theorem 4.5, \( D_{v-w-2} \) can be decomposed. \( D_{v-w-2, w+2} \) by Theorem 2.3. \( D_{2, w} \) can be decomposed by Theorem 2.3.
When covering $D_2$ the padding is $D_2$. Therefore $|A(P)|=2$.

Case 8: If $v \equiv 1 \pmod{4}$ and $w \equiv 3 \pmod{4}$, then we know the number of arcs is $v(v-1)-w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w, 2} \cup D_{v-w, 2} \cup D_{v-w, 2, w} = D_{v-w, 2} \cup D_{v-w, 2, w} \cup D_{2, w} \cup D_2$. Note that the vertex set for $D_{v-w, 2}$ is $\{0, 1, 1, \ldots, (v-w-3)\}$. For $D_{v-w, 2, w+2}$, the partite sets are $\{0, 1, 1, \ldots, (v-w-3)\}$ and $\{0, 1, 2, \ldots, (v-w+1)\}$. For $D_{2, w}$, the partite sets are $\{(v-2)\}$, $\{(v-1)\}$ and $\{0, 1, 2, \ldots, (w-1)\}$. By Theorem 4.5, $D_{v-w, 2}$ can be decomposed. $D_{v-w, 2, w+2}$ can be decomposed by Theorem 2.3. $D(w+2, 2)$ can be covered by case 7 with $P=D_2$. Therefore $|A(P)|=2$.

Case 9: If $v \equiv 2 \pmod{4}$ and $w \equiv 0 \pmod{4}$, then we know the number of arcs is $v(v-1)-w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w} \cup D_{v-w, w}$. Note that for $D_{v-w}$, the vertex set is $\{(0, 1, 1, \ldots, (v-w-1)\}$. For $D_{v-w, w}$, the partite sets are $\{(0, 1, 1, \ldots, (v-w-1)\}$ and $\{0, 1, 2, \ldots, (v-1)\}$. Since $v \equiv 0 \pmod{4}$ and $w \equiv 2 \pmod{4}$, $D_{v-w, w}$ can be decomposed by Theorem 2.3. $D_{v-w}$ can be covered with padding $P=D_2$ by Theorem 3.1. Thus $|A(P)|=2$.

Case 10: If $v \equiv 2 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then we know the number of arcs is $v(v-1)-w(w-1) \equiv 2 \pmod{4}$. Then $D(v, w) = D_{v-w} \cup D_{v-w, w}$. Note that for $D_{v-w}$, the vertex set is $\{(0, 1, 1, \ldots, (v-w-1)\}$. For $D_{v-w, w}$, the partite sets are $\{(0, 1, 1, \ldots, (v-w-1)\}$ and $\{0, 1, 2, \ldots, (v-1)\}$. Since $v \equiv 2 \pmod{4}$ and $w \equiv 1 \pmod{4}$, then $D_{v-w, w}$ can be covered by Theorem 2.9 with padding $P=D_2$. $D_{v-w}$ can be decomposed by Theorem 4.5. Thus $|A(P)|=2$.

Case 11: If $v \equiv w \equiv 2 \pmod{4}$, then $\{v (\pmod{4}), w (\pmod{4})\} \subset \{2, 3\}$. Therefore, by Theorem 4.4, a decomposition exists.
Case 12: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2,3\} \). Thus, by Theorem 4.4, a decomposition exists.

Case 13: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0_1, 1_1, \ldots, (v - w - 1)_1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0_1, 1_1, \ldots, (v - w - 1)_1)\} \) and \( \{(0_2, 1_2, \ldots, (w - 1)_2)\} \). Since \( v \equiv 3 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( D_{v-w,w} \) can be decomposed by Theorem 2.3. \( D_{v-w} \) can be covered with padding \( P = D_2 \) by Theorem 3.1. Thus \( |A(P)| = 2 \).

Case 14: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 1 \pmod{4} \), then we know the number of arcs is \( v(v-1) - w(w-1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w-3} \cup D_{v-w-3,3} \cup D_{v-w-3,w} = D_{v-w-3} \cup D_{v-w-3,w+3} \cup D(w + 3, w) \). Note that the vertex set for \( D_{v-w-3} \) is \( \{0_1, 1_1, \ldots, (v - w - 4)_1\} \). For \( D_{v-w-3,w+3} \), the partite sets are \( \{0_1, 1_1, \ldots, (v - w - 4)_1\} \) and \( \{0_2, 1_2, \ldots, (v - w + 2)_2\} \). For \( D(w + 3, w) \), the partite sets are \( \{0_2, 1_2, \ldots, (w + 2)_2\} \) and \( \{0_2, 1_2, \ldots, (w - 1)_2\} \). \( D_{v-w-3} \) can be covered by Theorem 3.2. By Theorem 2.3, \( D_{v-w-3,w+3} \) can be decomposed. \( D(w + 3, w) \) can be decomposed by Theorem 4.7. Therefore \( |A(P)| = 2 \).

Case 15: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2,3\} \). Therefore, by Theorem 4.4, a decomposition exists.

Case 16: If \( v \equiv w \equiv 3 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2,3\} \). Thus, by Theorem 4.4, a decomposition exists. Q.E.D

Thus, each case for \( X \) satisfies the conditions of the theorem.

**Theorem 4.12** A minimal \( Y \) covering of \( D(v, w) \) satisfies:

1. \( |A(L)| = 0 \) if \( \{v \pmod{4}, w \pmod{4}\} \subset \{0,1\} \) or \( \{v \pmod{4}, w \pmod{4}\} \subset \{2,3\} \).
\{2, 3\},

2. \(|A(L)|=2, \text{ otherwise.}\)

Proof. The necessary conditions follow as in Theorem 2.9.

Case 1: If \(v \equiv w \equiv 0 \pmod{4}\), then \(v \equiv w \equiv 0 \pmod{2}\). Therefore, by Theorem 4.7 a decomposition exists.

Case 2: If \(v \equiv 0 \pmod{4}\) and \(w \equiv 1 \pmod{4}\), then \(\{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\}\). Therefore, by Theorem 4.7 a decomposition exists.

Case 3: If \(v \equiv 0 \pmod{4}\) and \(w \equiv 2 \pmod{4}\), then we know the number of arcs is \(v(v - 1) - w(w - 1) \equiv 2 \pmod{4}\). Then \(D(v, w) = D_{v-w} \cup D_{v-w, w}\). Note that for \(D_{v-w}\), the vertex set is \(\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}\). For \(D_{v-w, w}\), the partite sets are \(\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}\) and \(\{(0_2, 1_2, \ldots, (w - 1)_2)\}\). Since \(v \equiv 0 \pmod{4}\) and \(w \equiv 2 \pmod{4}\), then \(D_{v-w, w}\) can be decomposed by Theorem 2.5. \(D_{v-w}\) can be covered with padding \(P = D_2\) by Theorem 3.3. Thus \(|A(P)|=2\).

Case 4: If \(v \equiv 0 \pmod{4}\) and \(w \equiv 3 \pmod{4}\), then we know the number of arcs is \(v(v - 1) - w(w - 1) \equiv 2 \pmod{4}\). Then \(D(v, w) = D_{v-w} \cup D_{v-w, w}\). Note that for \(D_{v-w}\), the vertex set is \(\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}\). For \(D_{v-w, w}\), the partite sets are \(\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}\) and \(\{(0_2, 1_2, \ldots, (w - 1)_2)\}\). Since \(v \equiv 0 \pmod{4}\) and \(w \equiv 3 \pmod{4}\), then \(D_{v-w, w}\) can be covered by Theorem 2.11 with padding \(P = D_2\). \(D_{v-w}\) can be decomposed by Theorem 4.8. Thus \(|A(P)|=2\).

Case 5: If \(v \equiv 1 \pmod{4}\) and \(w \equiv 0 \pmod{4}\), then \(\{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\}\). Therefore, by Theorem 4.7, a decomposition exists.

Case 6: If \(v \equiv w \equiv 1 \pmod{4}\), then \(\{v \pmod{4}, w \pmod{4}\} \subset \{0, 1\}\). Thus, by Theorem 4.7, a decomposition exists.
Case 7: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,2,w} = D_{v-w-2} \cup D_{v-w-2,2} \cup D_{2,w} \cup D_2 \). Note that the vertex set for \( D_{v-w-2} \) is \( \{0, 1, \ldots, (v-w-3)\} \). For \( D_{v-w-2,2} \), the partite sets are \( \{0, 1, \ldots, (v-w-3)\} \) and \( \{0, 2, \ldots, (v-w+1)\} \). For \( D_{2,w} \), the partite sets are \( \{(v-2), (v-1)\} \) and \( \{0, 2, \ldots, (w-1)\} \). By Theorem 3.3, \( D_{v-w-2} \) can be decomposed. \( D_{v-w-2,2,w} \) by Theorem 2.5. \( D_{2,w} \) can be decomposed by Theorem 2.5. When covering \( D_2 \) the padding is \( D_2 \). Therefore \( |A(P)| = 2 \).

Case 8: If \( v \equiv 1 \pmod{4} \) and \( w \equiv 3 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). The \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0, 1, \ldots, (v-w-1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0, 1, \ldots, (v-w-1)\} \) and \( \{0, 2, \ldots, (w-1)\} \). Since \( v \equiv 2 \pmod{4} \) and \( w \equiv 1 \pmod{4} \), then \( D_{v-w,w} \) can be covered by Theorem 2.11 with padding \( P = D_2 \). \( D_{v-w} \) can be decomposed by Theorem 4.8. Thus \( |A(P)| = 2 \).

Case 9: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0, 1, \ldots, (v-w-1)\} \). For \( D_{v-w,w} \), the partite sets are \( \{(0, 1, \ldots, (v-w-1)\} \) and \( \{0, 2, \ldots, (w-1)\} \). Since \( v \equiv 2 \pmod{4} \) and \( w \equiv 0 \pmod{4} \), then \( D_{v-w,w} \) can be decomposed by Theorem 2.5. \( D_{v-w} \) can be covered with padding \( P = D_2 \) by Theorem 3.3. Thus \( |A(P)| = 2 \).

Case 10: If \( v \equiv 2 \pmod{4} \) and \( w \equiv 1 \pmod{4} \), then we know the number of arcs is \( v(v - 1) - w(w - 1) \equiv 2 \pmod{4} \). Then \( D(v, w) = D_{v-w} \cup D_{v-w,w} \). Note that for \( D_{v-w} \), the vertex set is \( \{(0, 1, \ldots, (v-w-1)\} \). For \( D_{v-w,w} \), the partite sets are
\{(0_1, 1_1, \ldots, (v - w - 1)_1)\} and \{(0_2, 1_2, \ldots, (w - 1)_2\}. Since \(v \equiv 2 \mod 4\) and \(w \equiv 1 \mod 4\), then \(D_{v-w,w}\) can be covered by Theorem 2.11 with padding \(P=D_2\). \(D_{v-w}\) can be decomposed by Theorem 4.8. Thus \(|A(P)|=2\).

Case 11: If \(v \equiv w \equiv 2 \mod 4\), then \(\{v \mod 4, w \mod 4\} \subset \{2,3\}\). Therefore, by Theorem 4.7, a decomposition exists.

Case 12: If \(v \equiv 2 \mod 4\) and \(w \equiv 3 \mod 4\), then \(\{v \mod 4, w \mod 4\} \subset \{2,3\}\). Thus, by Theorem 4.7, a decomposition exists.

Case 13: If \(v \equiv 3 \mod 4\) and \(w \equiv 0 \mod 4\), then we know the number of arcs is \(v(v - 1) - w(w - 1) \equiv 2 \mod 4\). Then \(D(v, w)=D_{v-w} \cup D_{v-w,w}\). Note that for \(D_{v-w}\), the vertex set is \(\{(0_1, 1_1, \ldots, (v - w - 1)_1)\}\). For \(D_{v-w,w}\), the partite sets are \(\{(0_1, 1_1, \ldots, (w - 1)_1)\} \) and \(\{(0_2, 1_2, \ldots, (w - 1)_2\}\). Since \(v \equiv 3 \mod 4\) and \(w \equiv 0 \mod 4\), then \(D_{v-w,w}\) can be decomposed by Theorem 2.5. \(D_{v-w}\) can be covered with padding \(P=D_2\) by Theorem 3.3. Thus \(|A(P)|=2\).

Case 14: If \(v \equiv 3 \mod 4\) and \(w \equiv 1 \mod 4\), then we know the number of arcs is \(v(v - 1) - w(w - 1) \equiv 2 \mod 4\). Then \(D(v, w)=D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w}=D_{v-w-2} \cup D_{v-w-2,w+2} \cup D(w+2, w)\). Note that the vertex set for \(D_{v-w-2}\) is \(\{0_1, 1_1, \ldots, (v - w - 3)_1\}\). For \(D_{v-w-2,w+2}\), the partite sets are \(\{0_2, 1_2, \ldots, (v - w + 1)_2\}\). For \(D_{2,w}\), the partite sets are \(\{(v - 2)_1, (v - 1)_1\}\) and \(\{0_2, 1_2, \ldots, (w - 1)_2\}\). \(D_{v-w-2}\) can be decomposed by Theorem 4.8. By Theorem 2.5, \(D_{v-w-2,w+2}\) can be decomposed. \(D(w+2, w)=D_{2,w} \cup D_2\). \(D_{2,w}\) can be decomposed by Theorem 4.7. When covering \(D_2\) the padding is \(D_2\). Therefore \(|A(P)|=2\).
Case 15: If \( v \equiv 3 \pmod{4} \) and \( w \equiv 2 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2,3\} \). Therefore, by Theorem 4.7, a decomposition exists.

Case 16: If \( v \equiv w \equiv 3 \pmod{4} \), then \( \{v \pmod{4}, w \pmod{4}\} \subset \{2,3\} \). Thus, by Theorem 4.7, a decomposition exists. \( Q.E.D \)

Thus, each case for \( Y \) satisfies the conditions of the theorem. Therefore, Theorems 4.1-4.12 provide the necessary and sufficient conditions for maximal packings and minimal coverings of the complete directed graph on \( v \) vertices with holes of size \( w \) with \( C_4, X, \) and \( Y \).
5 CONCLUSION

Within this paper we have considered $C_4$, $X$, and $Y$ orientations of the 4-cycle. For $C_4$ and $Y$ packings and coverings of complete bipartite graphs, the leave or padding of the graph was of size 0 or 2. If $m$ and $n$ were both even or both odd, then the size of the leave or padding was 0 or 2. For $X$, we have found that if $m$ was even and $n$ was odd or $m$ was odd and $n$ was even then the leave and padding was of size 4. In packing and covering $D_v$ with $C_4$ our leave or padding was of size 0 or 2 unless $v=4$. If $v=4$ then the size of the leave or padding was 4. For $X$, our leave or padding was of size 0 or 2 unless $v=5$. If $v=5$ then the size of the leave or padding was 4. For $Y$, our leave or padding was of size 0 or 2 unless $v=4$ or 5. If $v=4$ or 5, then the size of the leave or padding was 4. When we packed and covered $D(v, w)$, if a decomposition existed then the leave or padding was zero. If a decomposition did not exist, then the leave or padding was of size 2. We have shown the necessary and sufficient conditions for finding the maximal padding and minimal covering of the complete bipartite graph, $D_{m,n}$, the complete directed graph, $D_v$, and the complete directed graph on $v$ vertices with a hole of size $w$, through several proofs.
BIBLIOGRAPHY


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