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Double Domination of Complementary Prisms

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

Lamont Vaughan

August 2008

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Robert Gardner, Ph.D.

Anant Godbole, Ph.D.

Keywords: graph theory, domination, double domination and complementary prism

ABSTRACT

Double Domination of Complementary Prisms

by

Lamont Vaughan

The complementary prism of a graph G is obtained from a copy of G and its complement $\overline{G}$ by adding a perfect matching between the corresponding vertices of G and $\overline{G}$. For any graph G, a set $D \subseteq V(G)$ is a *double dominating set* (DDS) if that set dominates every vertex of G twice. The *double domination number*, denoted $\gamma_{\times 2}(G)$, is the cardinality of a minimum double dominating set of G. We have proven results on graphs of small order, specific families and lower bounds on $\gamma_{\times 2}(G\overline{G})$.

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DEDICATION

This paper is dedicated to my family that have left since I've been here. Daddy, Grandma and Uncle Bunky, you guys would have been proud. To Mommy, Tami, Chen and Lionell, put a fork in it. I'm Done!

ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. Teresa Haynes for her assistance and guidance in bringing this thesis together. I would also like to thank Dr. Robert Gardner and Dr. Anant Godbole for providing me with an enjoyable graduate experience in and out of the classroom. Finally, I would like to thank all my math colleagues, especially, John "Squared", Pius, Romy, Mel, and Lizzie. We sure did make it fun.

CONTENTS

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1 INTRODUCTION

There are several real-life applications that can be modeled with graph theory. For example, it can be used if a company wants to minimize the number of computers for a backup network or find the most cost effective way to route airplanes. Using graph parameters, such as domination, we can model these real-life problems.

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of two sets. The set $V(G)$ is called the *vertex set*, and the set $E(G)$ is called the *edge set*. Two vertices u and v are said to be *adjacent* if there is an edge between the two. This edge is denoted as uv, and u and v are called *neighbors*. The *order* of a graph G, denoted as $|V(G)|$, is the cardinality of $V(G)$. For the purposes of this paper, we will ignore graphs with *loops*, an edge between one vertex, and *multiple edges*; having more than one edge between two vertices. Also, we will omit graphs where the edge uv differs from vu, known as *digraphs*. A *trivial graph* is a graph with no edges. An *isolated vertex*, or an *isolate*, is a vertex of graph that has no neighbors. The *degree* of a vertex in a graph, denoted as $deg_v(G)$, is the number of vertices in G that are adjacent to v. The *open neighborhood* of a vertex $v \in G$ is $N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of a vertex $v \in G$ is $N[v] = \{v\} \cup N(v)$. A *leaf* of a graph is vertex of degree one, and the vertex adjacent to a leaf is called a *support vertex*. The *complement* of a graph G, denoted $\overline{G}$, is a graph with all the vertices of G and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

There are some specific families of graphs that will be discussed. For example, a *complete graph* or *clique*, K_n, is a graph on n vertices where every vertex of K_n is adjacent to every other vertex of K_n. In other words, the graph K_n has all possible edges. A *path*, denoted by P_n, is a graph G whose vertices can be ordered in a way that two vertices are adjacent if and only if they are consecutive in the ordered list. A *cycle*, denoted by C_n, is a graph with the $|V(G)| = |E(G)|$ and for each vertex the $deg_v(G) = 2$. Similar to the path, for a cycle, the set $V(G)$ is ordered and two vertices u and v in the cycle are adjacent if and only if $\{u, v\} \subseteq V(G)$. A *wheel*, denoted W_n, is a graph composed of $C_{n-1} \cup K_1$ where every vertex of C_{n-1} is adjacent to K_1. A *star* is a connected graph with no cycles consisting of a K_1 graph and every vertex of $G - K_1$ has $deg_v(G) = 1$.

A subset of vertices D in $V(G)$ is a called a *dominating set* if every vertex of $V \setminus D$ is adjacent to a vertex of D. The *domination number* of a graph G, denoted as $\gamma(G)$, is the cardinality of a minimum dominating set of G. A dominating set with cardinality $\gamma(G)$ is a $\gamma(G)$-*set*. A *total dominating set*, or TDS, is a set of vertices T where every vertex of G is adjacent to a vertex of T. Similar to domination, the *total domination number* of a graph G, denoted by $\gamma_t(G)$, is the cardinality of a minimum total dominating set of G. A total dominating set with cardinality $\gamma_t(G)$ is a $\gamma_t(G)$-*set*. Note that only graphs with no isolates will have a total domination number because a vertex is not adjacent to itself. For a graph G, a set $D \subseteq V(G)$ is a *double dominating set* (DDS) if that set dominates every vertex of G twice. Note that every vertex dominates itself and its neighbors. The *double domination number*, denoted $\gamma_{\times 2}(G)$, is the cardinality of a minimum double dominating set of G. A DDS with minimum cardinality is called a $\gamma_{\times 2}(G)$-*set*. Again the graph G can have no isolates for $\gamma_{\times 2}(G)$ to be defined.

The *Cartesian Product* of two graphs G and H, $G \square H$, has a vertex set of $V(G) \times$

$V(H)$. The edges of $G\Box H$ are formed by replacing each vertex in G with a copy of H and replacing each vertex of H with a copy of G. Now, given two graphs G and H and subsets $A \subseteq V(G)$ and $B \subseteq V(H)$, the *complementary product*, denoted $G(A)\Box H(B)$ has the vertex set $V(G) \times V(H)$. The edge set is defined as follows: There is an edge between the vertices (g_i, h_j) and (g_k, h_l) if one of the following holds [2]:

(1) $i = k$, g_i is in A and there is an edge between h_j and h_l

(2) or if $i = k$, g_i is not in A and there is no edge between h_j and h_l

(3) if $j = l$, h_j is in B and there is an edge between g_i and g_k

(4) or if $j = l$, h_j is not in B and there is no edge between g_i and g_k.

More simply, the complementary product is a graph on $V(G) \times V(H)$ vertices and for a vertex in G, we replace that vertex with a copy of H if it is in A and a copy of $\overline{H}$ if it is not in A. For a vertex in H, we replace that vertex with a copy of G if it is in B and a copy of $\overline{G}$ if it is not in B. See Figure 1 for an example. We have finally come to the type of graph that is the focus of this paper, the complementary prism, a special case of the complementary product $G\Box K_2(S)$ where $|S|$=1. The *complementary prism* of a graph G, denoted as $G\overline{G}$, is obtained from the graph $G \cup \overline{G}$ by adding a perfect matching between the corresponding vertices of G and $\overline{G}$. For examples, see Figures 2 and 3. We note that the Petersen Graph is the complementary prism $C_5\overline{C}_5$.

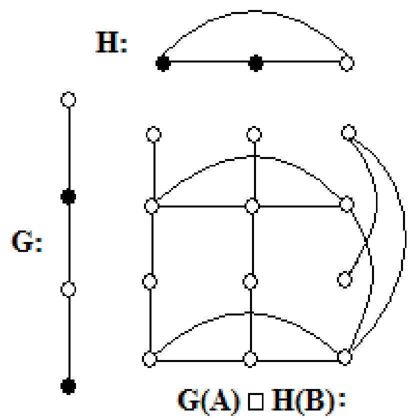

Figure 1: Complementary Product

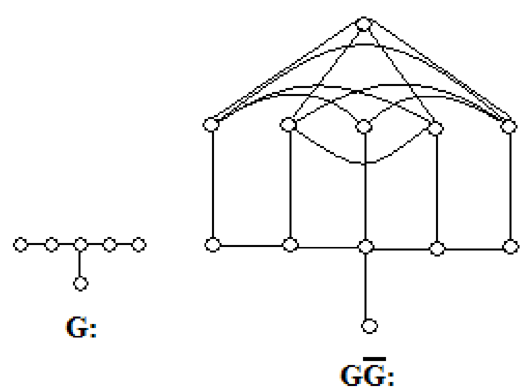

Figure 2: Complementary Prism

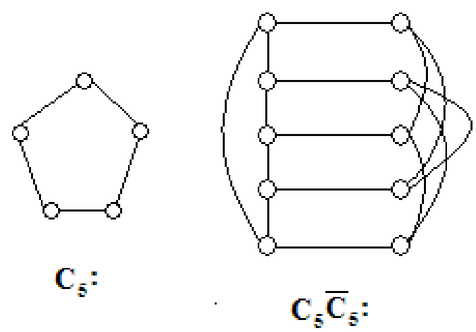

Figure 3: The Petersen Graph $C_5\overline{C}_5$

2 MOTIVATION

In this paper, we will explore double domination in complementary prisms. Several earlier results have been proven by Haynes, Henning, Slater and Van Der Merwe [3] that introduce the complementary prism and discuss parameters such as vertex degree, chromatic number, domination and total domination. We list them here. The first results on domination of the complementary prism are proven for specific families of graphs, followed by graphs or small order.

Proposition 2.1 *[3]*

(a) *If* $G = K_n$*, then* $\gamma(G\overline{G}) = n$*.*

(b) *If* $G = tK_2$*, then* $\gamma(G\overline{G}) = t + 1$*.*

(c) *If* $G = K_t \circ K_1$ *and* $t \geq 3$*, then* $\gamma(G\overline{G}) = \gamma(G) = t$*.*

(d) *If* $G = C_n$ *and* $n \geq 3$*, then* $\gamma(G\overline{G}) = \lceil (n + 4)/3 \rceil$*.*

(e) *If* $G = P_n$ *and* $n \geq 2$*, then* $\gamma(G\overline{G}) = \lceil (n + 3)/3 \rceil$*.*

Proposition 2.2 *[3]*

Let G *be a graph of order* n*. Then,*

(a) $\gamma(G\overline{G}) = 1$ *if and only if* $G = K_1$*.*

(b) $\gamma(G\overline{G}) = 2$ *if and only if* $n \geq 2$ *and* G *has a support vertex that dominates* V *or* $\overline{G}$ *has a support vertex that dominates* $\overline{V}$*.*

Next, there are general upper and lower bounds for dominating $G\overline{G}$.

Proposition 2.3 *[3]*

For any graph G*,* $\max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma(G\overline{G}) \leq \gamma(G) + \gamma(\overline{G})$*.*

Lastly, for domination we have a characterization of graphs where the domination number of the complementary prism equals the domination number of the graph. A subset P of the vertices of G is called an *open packing* if the open neighborhoods of vertices of P are pairwise disjoint [3].

Theorem 2.4 *[3]*

A graph G satisfies $\gamma(G\overline{G}) = \gamma(G) \geq \gamma(\overline{G})$ if and only if G has an isolated vertex or there exists a packing P of G such that $|P| \geq 2$ and $\gamma(G - P) = \gamma(G) - |P|$.

The paper then transitions to total domination and again presents results about specific families of graphs.

Proposition 2.5 *[3]*

(a) *If $G = K_n$, then $\gamma_t(G\overline{G}) = n$.*

(b) *If $G = tK_2$, then $\gamma_t(G\overline{G}) = n = 2t$.*

(c) *If $G = K_t \circ K_1$ and $t \geq 3$, then $\gamma_t(G\overline{G}) = \gamma_t(G) = t$.*

(d) *If $G \in \{P_n, C_n\}$ with order $n \geq 5$, then*

$$\gamma_t(G\overline{G}) = \begin{cases} \gamma_t(G) & \text{if } n \equiv 2 \,(mod\ 4) \\ \gamma_t(G) + 2 & \text{if } G = C_n \text{ and } n \equiv 0 \,(mod\ 4) \\ \gamma_t(G) + 1 & \text{otherwise.} \end{cases}$$

Results for graphs with small order and general bounds for $\gamma_t(G\overline{G})$ are proven next in the paper, as they were with domination.

Proposition 2.6 *[3]*

Let G be a graph of order $n \geq 2$ with $|E(G)| \geq |E(\overline{G})|$. Then,

(a) *$\gamma_t(G\overline{G}) = 2$ if and only if $G = K_2$.*

(b) $\gamma_t(G\overline{G}) = 3$ *if and only if* $n \geq 3$ *and* $G = K_3$ *or* G *has a support vertex that dominates* V *or* $\overline{G}$ *has a support vertex that dominates* $\overline{V}$.

Proposition 2.7 *[3]*

If G *and* $\overline{G}$ *have no isolated vertices, then* $\max\{\gamma_t(G), \gamma_t(\overline{G})\} \leq \gamma_t(G\overline{G}) \leq \gamma_t(G) + \gamma_t(\overline{G})$.

The final result in this section comes from [3]. It is a characterization of the graphs G where $\gamma_t(G) \geq \gamma_t(\overline{G})$ and G and its complementary prism $G\overline{G}$ have equal total domination numbers.

Theorem 2.8 *[3]*

Let G *be a graph such that neither* G *nor* $\overline{G}$ *has an isolated vertex. Then,* $\gamma_t(G\overline{G}) = \gamma_t(G) \geq \gamma_t(\overline{G})$ *if and only if* $G = \frac{n}{2}K_2$ *or there exists an open packing* $P = P_1 \cup P_2$ *in* G *satisfying the following conditions:*

(i) $|P| \geq 2$;

(ii) $P_1 \cap P_2 = \emptyset$;

(iii) *if* $P_1 \neq \emptyset$, *then* P_1 *is a packing in* G;

(iv) *if* $P_1 = \emptyset$, *then* $|P| \geq 3$ *or* $G[P] = \overline{K}_2$;

(v) $\gamma_t(G - N[P_1] - P_2) = \gamma_t(G) - 2|P_1| - |P_2|$.

Suppose that a network of computers needs to be designed so that each computer is backed up by two special computers and the most cost-effective way to model the computer connection has the structure of the complementary prism. Herein lies the motivation for this thesis. We will prove results for the double domination of the complementary prism similar to the ones presented above in this chapter on domination and total domination. In particular, we study double domination of complementary prisms where $\gamma_{\times 2}(G\overline{G})$ is small, for specific families of graphs such as cycles and paths, establish upper and lower bounds for double dominating all complementary prisms and give examples of graphs where $\gamma_{\times 2}(G) = \gamma_{\times 2}(G\overline{G})$.

3 RESULTS

In this section, we present results on the double domination number of complementary prisms. We begin with a key observation on double domination regarding leaves and an upper bound on the double domination number of complementary prisms of cycles.

Observation 3.1 *If D is a double dominating set of a graph G, then D contains all the leaves and support vertices of G.*

Theorem 3.2 *Let $n \geq 5$ and if $G = C_n$, then $\gamma_{\times 2}(G\overline{G}) \leq 2 + \gamma_{\times 2}(G)$.*

Proof. The result is easily verified for $n = 5$. Therefore, assume that $n \geq 6$ and let D be a $\gamma_{\times 2}(C_n)$-set such that there exists two vertices v_i,$v_j \notin D$. Since D is a $\gamma_{\times 2}(C_n)$-set we will two have disjoint paths containing v_i and v_j, then $dist_G\{v_i, v_j\} \geq 3$. Consider the vertices $\{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\} \in D$, and observe that in $\overline{G}$ each vertex $\{\overline{v}_{i-1}, \overline{v}_{i+1}, \overline{v}_{j-1}, \overline{v}_{j+1}\}$ is dominated exactly once by D. Now notice in $\overline{G}$, that $\overline{v}_i$ will dominate $\overline{G} - \{\overline{v}_{i-1}, \overline{v}_{i+1}\}$ and $\overline{v}_j$ will dominate $\overline{G} - \{\overline{v}_{j-1}, \overline{v}_{j+1}\}$. Therefore, $D \cup \{\overline{v}_i, \overline{v}_j\}$ forms a double dominating set of $G\overline{G}$ and hence $\gamma_{\times 2}(G\overline{G}) \leq 2 + \gamma_{\times 2}(G)$. ■

Next we give the domination number of $G\overline{G}$ when G is a star.

Theorem 3.3 *If G a star with order $n \geq 3$, then $\gamma_{\times 2}(G\overline{G}) = n + 1$.*

Proof. Let G be a star with order $n \geq 3$. First, we show $\gamma_{\times 2}(G\overline{G}) \leq n + 1$. Let D be a $\gamma_{\times 2}(G\overline{G})$-set. Since G is a star, then the support vertex v in G is an isolated vertex $\overline{v}$ in $\overline{G}$ and a leaf in $G\overline{G}$. Therefore, by Observation 3.1, $\{v, \overline{v}\} \in D$. Denote the $n-1$ leaves of G by $\{u_1, u_2, ..., u_{n-1}\}$ and notice that they are dominated once by v. Since G is a star, the corresponding leaf vertices in G will form a complete graph on $n-1$ vertices in $\overline{G}$. Consider the set $\{\overline{u}_1, \overline{u}_2, ..., \overline{u}_{n-1}\} \cup \{v, \overline{v}\}$. This set forms a double dominating set of $G\overline{G}$ and hence, $\gamma_{\times 2}(G\overline{G}) \leq n+1$. Now we must show that $n+1 \geq \gamma_{\times 2}(G\overline{G})$. Previously, it has been shown that $\{v, \overline{v}\} \subset D$. We must at least dominate the $n-1$ leaf vertices of G and double dominate their corresponding vertices in $\overline{G}$. However, to do this we must have at least $n-1$ vertices because the set L of leaves of G is independent in $G\overline{G}$ and no pair of them have a common neighbor in $G\overline{G}$. In other words, L is a packing in $G\overline{G}$ and needs at least $n-1$ vertices to dominate it. Therefore $n-1+2 = n+1 \leq \gamma_{\times 2}(G\overline{G})$ and $\gamma_{\times 2}(G\overline{G}) = n+1$. ■

Theorem 3.4 *If G=K_n, then $\gamma_{\times 2}(G\overline{G}) = 2n$.*

Proof. Let G=K_n and recall that to double dominate a leaf we must use that vertex and its neighbor. Since G is a complete graph, then $\overline{G}$ will be a set isolated vertices. This makes all n vertices of $\overline{G}$ leaves in $G\overline{G}$. Therefore, we need at least 2n vertices to double dominate $G\overline{G}$ and 2n vertices will double dominate $G\overline{G}$, thus $\gamma_{\times 2}(G\overline{G}) = 2n$. ■

Next we will show an upper bound for double domination of the complementary prism of a path and a wheel. Recall that a *path* is a graph G whose vertices can be ordered in such a way that two vertices are adjacent if and only if they are consecutive in the ordered list.

Theorem 3.5 *Let G be a path and $n \geq 4$, then $\gamma_{\times 2}(P_n\overline{P}_n) \leq \gamma_{\times 2}(P_{n-2}) + 2$.*

Proof. Let G be a path and $n \geq 4$. Let D be a $\gamma_{\times 2}(P_{n-2})$-set for $G \setminus \{u, v\}$ where u and v are the two ends of the path. Next, in $\overline{P}_n$ take $\overline{u}$ and $\overline{v}$. This double dominates u and v in G along with all the remaining vertices in $\overline{G}$ and $D \cup \{u, v\}$ double dominates $P_n\overline{P}_n$. Therefore $\gamma_{\times 2}(P_n\overline{P}_n) \leq \gamma_{\times 2}(P_{n-2}) + 2$. ■

Theorem 3.6 *If G is a wheel of order n, then $\gamma_{\times 2}(G\overline{G}) \leq n + 1$.*

Proof. Let G be a wheel of order n and consider $G\overline{G}$. Since the center vertex v of G is adjacent to every other vertex of G, it is an isolate in $\overline{G}$ and a leaf in $G\overline{G}$. Therefore, by Observation 1, we at least need two vertices, v and $\overline{v}$, to double dominate $G\overline{G}$. So far we have dominated all the remaining $n - 1$ vertices of G once with v and not dominated any of the remaining vertices of $\overline{G}$. Now, by using the remaining $n-1$ vertices of $\overline{G}$, we can double dominate $G\overline{G}$. Therefore $\gamma_{\times 2}(G\overline{G}) \leq n-1+2 = n+1$. ■

Next we consider complementary prisms with small double domination numbers.

Theorem 3.7 *For a graph G, $\gamma_{\times 2}(G\overline{G}) = 2$ if and only if G is the trivial graph K_1.*

Proof. Clearly if $G = K_1$ then $G\overline{G} = P_2$ and $\gamma_{\times 2}(G\overline{G}) = 2$. Next assume that $\gamma_{\times 2}(G\overline{G}) = 2$. Let D be a $\gamma_{\times 2}(G\overline{G})$-set. If $D \subset V(G)$ (respectively, $D \subset V(\overline{G})$),then $V(\overline{G})$ (respectively, $V(G)$) is not double dominated by D. Hence, let $D_1 = D \cap V(G)$, $D_2 = D \cap V(\overline{G})$, and $|D_1| = |D_2| = 1$. Moreover the vertex v in D_1 must be adjacent to the vertex in D_2 implying that $D_2 = \{\overline{v}\}$. It follows that $V - \{v\} = \emptyset$ because no vertex in $G\overline{G}$ can be double dominated by $D = \{v, \overline{v}\}$. Hence $G = K_1$ and $G\overline{G} = P_2$. ■

Theorem 3.8 *For any nontrivial graph G, $\gamma_{\times 2}(G\overline{G}) \geq 4$.*

Proof. Let G be a nontrivial graph. By Theorem 4, $\gamma_{\times 2}(G\overline{G}) \geq 3$. Assume that $\gamma_{\times 2}(G\overline{G}) = 3$, and let $D = \gamma_{\times 2}(G\overline{G})$-set. If $D \subset V(G)$ (respectively, $D \subset V(\overline{G})$), then $V(\overline{G})$ (respectively, $V(G)$) is not double dominated by D. Let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(\overline{G})$. Then $\{|D_1|, |D_2|\} = \{1, 2\}$. Without loss of generality, let $|D_1| = 1$ and $D_2 = \{\overline{u}, \overline{v}\}$. Since D is a $\gamma_{\times 2}(G\overline{G})$ set, then it follows that $\overline{u}$ is adjacent to $\overline{v}$, then $uv \notin E(G)$. Since $D_1 \subseteq V(G)$ and $|D_1| = 1$, then without loss of generality $D_1 = \{u\}$ otherwise u is not double dominated. But recall that $uv \notin E(G)$, then v is not double dominated by D, a contradiction. Therefore, $\gamma_{\times 2}(G\overline{G}) \geq 4$. ■

We define a family of graphs $\mathcal{F} = \{\text{G}|\text{G}= \{P_2, P_3\}\} \cup \{$G is a graph with an induced P_4 such that every vertex in $G - P_4$ is adjacent to the support vertices of P_4 and not adjacent to the leaves of $P_4\}$.

Theorem 3.9 *Let G be a graph. Then $\gamma_{\times 2}(G\overline{G}) = 4$ if and only if $G \in \mathcal{F}$.*

Proof. Clearly if $G \in \mathcal{F}$, Theorem 3.8 implies that $\gamma_{\times 2}(G\overline{G}) \geq 4$. It is a simple exercise to to see that $\gamma_{\times 2}(G\overline{G}) = 4$. For the converse, let G be a graph and $\gamma_{\times 2}(G\overline{G}) = 4$, and let D be a $\gamma_{\times 2}(G\overline{G})$ set. Let $D_1 = D \cap G$ and $D_2 = D \cap \overline{G}$. Note that if either D_1 or D_2 is empty, then $G\overline{G}$ is not double dominated. Therefore, either we have (1) without loss of generality $|D_1| = 1$ and $|D_2| = 3$ or (2) $|D_1| = 2$ and $|D_2| = 2$. First, suppose $|D_1| = 1$ and $|D_2| = 3$, and let $\{u\} = D_1$, then $\overline{u}$ must be an element of D_2 or u will not be double dominated. Now, let $\overline{v}, \overline{w}$ be the other two elements in D_2. Notice that v and w in G must be adjacent to u to be double dominated.

Therefore, to double dominate $\overline{v}$ and $\overline{w}$, they must be adjacent. Now, suppose there exists $x \neq u, v, w$ in G, then x must be adjacent to $u \in D_1$ and is only adjacent to $\overline{x} \in \overline{G}$ so it will only be dominated once by D. Therefore, $V(G) = \{u, v, w\}$ and $G = P_3$.

Next, suppose $|D_1| = 2 = |D_2|$, and we have three possibilities: (1) $D_1 = \{u, v\}$ and $D_2 = \{\overline{u}, \overline{v}\}$, (2) $D_1 = \{u, v\}$ and $D_2 = \{\overline{u}, \overline{w}\}$, and (3) $D_1 = \{u, v\}$ and $D_2 = \{\overline{x}, \overline{y}\}$.

Case 1: Let $D_1 = \{u, v\}$ and $D_2 = \{\overline{u}, \overline{v}\}$. Then either v and u are adjacent or $\overline{u}$ and $\overline{v}$ are adjacent. Now, suppose that $x \notin D_1$. Then x must be adjacent to u and v to double dominate it but then $\overline{x}$ will not be adjacent to $\overline{u}$ or $\overline{v}$ and thus not be double dominated by D, a contradiction. Therefore, $V(G) = \{u, v\}$ and so $G = P_2$.

Case 2: Let $D_1 = \{u, v\}$ and $D_2 = \{\overline{u}, \overline{w}\}$ and notice in G that v must be adjacent to u to be double dominated by D. Similarly, in $\overline{G}$, $\overline{w}$ must be adjacent to $\overline{u}$. Thus, w is not adjacent to u and must be adjacent to v. Therefore, $\overline{v}$ is not adjacent to $\overline{u}$ or to $\overline{w}$ and is not double dominated by D, a contradiction.

Case 3: Let $D_1 = \{u, v\}$ and $D_2 = \{\overline{x}, \overline{y}\}$. First, note that u and v must be adjacent or else they will not be double dominated, likewise $\overline{x}$ and $\overline{y}$ are adjacent. Vertices x and y must be adjacent to at least one of u or v to be double dominated by D. Suppose x is adjacent to u and v. Then $\overline{x}$ is not adjacent to $\overline{u}$ and $\overline{v}$. Implying that $\overline{u}$ and $\overline{v}$ must be adjacent to $\overline{y}$. Then y is not adjacent to u or v and hence y is not double dominated in $G\overline{G}$. Therefore, x must be adjacent to exactly one of u or v. Without loss of generality let x be adjacent to u, then $\overline{u}$ must be adjacent to $\overline{y}$ and y must be adjacent to v and $\overline{v}$ must be adjacent to $\overline{x}$. Now suppose that there exists another vertex $w \in G$. Then w must be adjacent to u and v and that w cannot be adjacent to x or y or else $\overline{w}$ will not be double dominated by D. Therefore, we can add an unlimited number of vertices to G as long as each vertex $w_1, w_2, ...w_n$ is adjacent to both u and v, and not adjacent to x and y. So, G is a graph such that there exists an induced P_4 where every vertex of $G - P_4$ is adjacent to the support vertices of P_4 and not adjacent to the leaves of the P_4 . ■

Now, we will establish upper and lower bounds on $\gamma_{\times 2}(G\overline{G})$ in terms of the double domination number of G and $\overline{G}$.

Theorem 3.10 *For any graph G with no isolated vertices,* $\max\{\gamma_{\times 2}(G), \gamma_{\times 2}(\overline{G})\} \leq \gamma_{\times 2}(G\overline{G}) \leq \gamma_{\times 2}(G) + \gamma_{\times 2}(\overline{G})$.

Proof. For $\gamma_{\times 2}(G\overline{G}) \leq \gamma_{\times 2}(G) + \gamma_{\times 2}(\overline{G})$, let D_1 be a $\gamma_{\times 2}(G)$-*set* and let D_2 be a $\gamma_{\times 2}(\overline{G})$-*set*. The set $D_1 \cup D_2$ is a double dominating set of $G\overline{G}$. Next, for $\max\{\gamma_{\times 2}(G), \gamma_{\times 2}(\overline{G})\} \leq \gamma_{\times 2}(G\overline{G})$, assume, without loss of generality, that $\gamma_{\times 2}(G) = \max\{\gamma_{\times 2}(G), \gamma_{\times 2}(\overline{G})\}$. Let D be a $\gamma_{\times 2}(G)$-*set*, and let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(\overline{G})$. If D_1 double dominates G, then we are finished, since $\gamma_{\times 2}(G) \leq |D_1| \leq |D_1| + |D_2| = |D| = \gamma_{\times 2}(G\overline{G})$. If D_1 does not double dominate G, then there is some set $S \subseteq V(G)$ that is not double dominated by D_1. Note that if $v \in S$, then either: (1) $v \notin D_1$ and is adjacent to at most one other vertex in D_1 or (2) $v \in D_1$ and v has no neighbor in D_1. Let $A = S \cap D_1$. Since the vertices in A have no neighbor in D_1, then each $v \in A$ is dominated by $\overline{v}$ in $G\overline{G}$. Hence, $\overline{A} \subseteq D_2$. Since G has no isolates, v has a neighbor in $V - D_1$, for each vertex in A, select a neighbor in $V - D_1$ and call this set A'. Then, $|A'| \leq |A| = |\overline{A}|$. Let $B = S - D_1$. Since the vertices in B are not double dominated in G by D_1, each has at most one neighbor in D_1. However, since D double dominates $G\overline{G}$ and by the definition of the complementary prism, the vertices of B are dominated exactly once by $\overline{B} \subseteq D_2$ and each vertex in B has exactly one neighbor in D_1. Hence, $D_1 \cup A' \cup B$ is a double dominating set of G. Therefore, $\gamma_{\times 2}(G) \leq |(D_1 \cup A') \cup B| \leq |D_1| + |A'| + |B| \leq |D_1| + |A| + |B| = |D_1| + |\overline{A}| + |\overline{B}| \leq |D_1| + |D_2| = |D| = \gamma_{\times 2}(G, \overline{G})$. ■

We note that the lower bound of Theorem 3.10 is sharp. For example, if $G \in \{P_7, P_{10}\}$, then $\gamma_{\times 2}(G) = \gamma_{\times 2}(G\overline{G})$. Also, for any graph G where every vertex is either a leaf or a support vertex and no component of G is a star, $\gamma_{\times 2}(G) = \gamma_{\times 2}(G\overline{G})$. On the other hand, we have not been able to show that the upper bound is the best possible.

4 CONCLUSION

We have shown various results for the double domination of the complementary prism of specific families of graphs. Also, we have proven upper and lower bounds for the double domination number of complementary prisms. We noted that the lower bound of Theorem 3.10 is sharp. Further research is needed because it is possible that the upper bound in Theorem 3.10 may be be improved to $\gamma_{\times 2}(G\overline{G}) \leq \gamma_{\times 2}(G) + \gamma_{\times 2}(\overline{G}) - 1$. Also, further study will attempt to characterize the graphs G for which the lower bound holds.

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