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On the Attainability of Upper Bounds for the Circular Chromatic Number of K_4-Minor-Free Graphs.

Tracy Lance Holt
East Tennessee State University

On the Attainability of Upper Bounds for the Circular Chromatic Number of K_4-Minor-Free Graphs

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by

Tracy Holt

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Yared Nigussie, Ph.D., Chair

Robert Beeler, Ph.D.

Robert Gardner, Ph.D.

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ABSTRACT

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Let G be a graph. For $k \geq d \geq 1$, a $\frac{k}{d}$-coloring of G is a coloring c of vertices of G with colors $0, 1, 2, \ldots, k-1$, such that $d \leq |c(x) - c(y)| \leq k - d$, whenever xy is an edge of G. We say that the circular chromatic number of G, denoted $\chi_c(G)$, is equal to the smallest $\frac{k}{d}$ where a $\frac{k}{d}$-coloring exists. In [6], Pan and Zhu have given a function $\mu(g)$ that gives an upper bound for the circular-chromatic number for every K_4-minor-free graph G_g of odd girth at least g, $g \geq 3$. In [7], they have shown that their upper bound in [6] can not be improved by constructing a sequence of graphs approaching $\mu(g)$ asymptotically. We prove that for every odd integer $g = 2k+1$, there exists a graph $G_g \in \mathcal{G}/K_4$ of odd girth g such that $\chi_c(G_g) = \mu(g)$ if and only if k is not divisible by 3. In other words, for any odd g, the question of attainability of $\mu(g)$ is answered for all g by our results. Furthermore, the proofs [6] and [7] are long and tedious. We give simpler proofs for both of their results.

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1 INTRODUCTION

A *graph* is a pair $G = (V, E)$ of sets such that the elements of E are 2-element subsets of V [1]. The set V is the set of vertices (points) and the set E is the set of edges (lines). We assume graphs are finite and simple (no multiple edges or loops). Let G and G' be graphs. A *homomorphism* from G to G' is a mapping $f : V(G) \to V(G')$ which preserves adjacency, i.e., $uv \in E(G)$ implies $f(u)f(v) \in E(G')$. To illustrate this, Figure 1 shows how a 5-cycle can be mapped by homomorphism f to a graph G. The notation $G \leq G'$ means there is a homomorphism from G to G'. Note that " $\leq$ " is a reflexive and transitive relation. Also, the notation $G \sim G'$ means that $G \leq G' \leq G$. Other terminology we use is from [1].

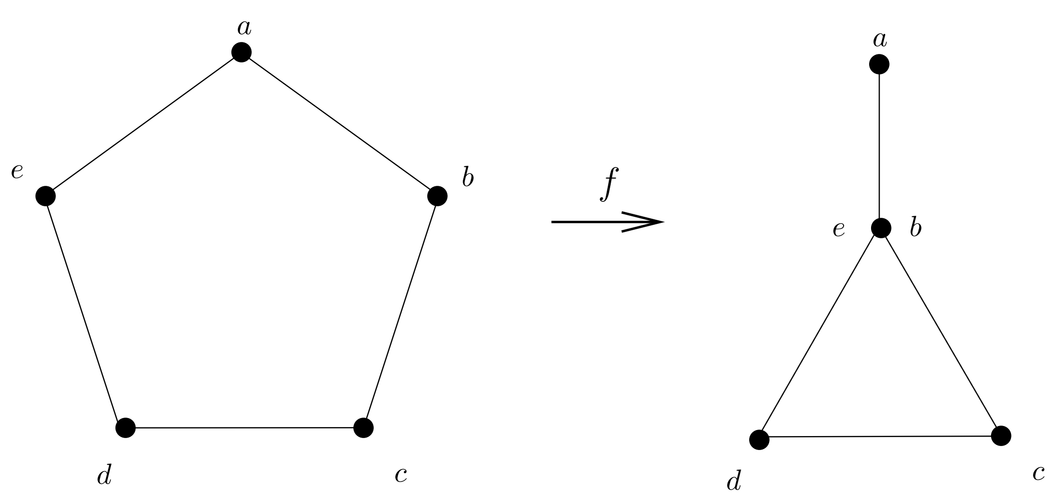

Figure 1: A Homomorphism Mapping C_5 to Graph G.

$\frac{0}{1}$										$\frac{1}{1}$
$\frac{0}{1}$					$\frac{1}{2}$					$\frac{1}{1}$
$\frac{0}{1}$			$\frac{1}{3}$		$\frac{1}{2}$		$\frac{1}{3}$			$\frac{1}{1}$
$\frac{0}{1}$		$\frac{1}{4}$	$\frac{1}{3}$		$\frac{1}{2}$		$\frac{1}{3}$	$\frac{3}{4}$		$\frac{1}{1}$
$\frac{0}{1}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{1}{1}$

Table 1: The First Five Farey Sequences.

1.1 Farey Sequences

Farey sequences are useful in helping to understand relationships between rational numbers. In particular, for our purposes, we find them very useful in proving the attainability of upper bounds. From [5] we get the following definition of Farey sequences. Every rational number between two integers can be generated using Farey sequences. It suffices to show the values in the sequence between 0 and 1. Construct a table in the following way. For the first row, write $\frac{0}{1}$ and $\frac{1}{1}$. For $n \in \{2, 3, ...\}$ use the following rule: Form the nth row by copying the $(n-1)st$ row in order, but insert the fractions $\frac{a+a'}{b+b'}$ between consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ if $b+b' \leq n$. Since $1+1 \leq 2$, $\frac{0+1}{1+1} = \frac{1}{2}$ is inserted between $\frac{0}{1}$ and $\frac{1}{1}$ giving the second row $\frac{0}{1}, \frac{1}{2}, \frac{1}{1}$. Likewise, the third row is $\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{1}$. The first five rows are depicted in Table 1.

A Farey sequence of order n is the nth row of the table described above. Some useful properties of Farey sequences are the following.

Theorem 1.1 *[5] If $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive fractions in the nth row, say with $\frac{a}{b}$ to the left of a $\frac{a'}{b'}$, then $a'b - ab' = 1$.*

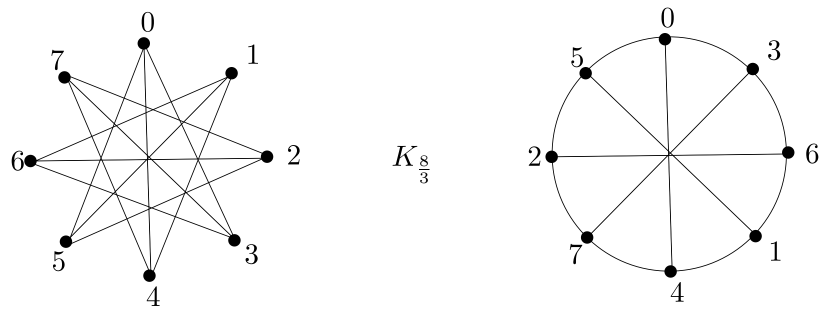

Figure 2: Two Equivalent Representations for $K_{\frac{8}{3}}$.

Corollary 1.2 *[5] Every $\frac{a}{b}$ in the table is in reduced form, that is $gcd(a, b) = 1$.*

Corollary 1.3 *[5] The fractions in each row are listed in order of their size.*

It is also important to note that a Farey sequence of order n is the sequence of all fractions in reduced form, with denominators not exceeding n [5].

1.2 Circular Graphs

A circular graph is defined in [4] as a graph $K_{\frac{k}{d}}$ with the vertex set $V = \{0, 1, 2, ..., k-1\}$ and the edge set $E = \{ij : d \leq |i-j| \leq k-d\}$. To illustrate, $K_{\frac{8}{3}}$ has $V = \{0, 1, 2, \ldots, 7\}$ and $E = \{ij : 3 \leq |i-j| \leq 5\}$. Two equivalent representations for $K_{\frac{8}{3}}$ are depicted in Figure 2.

For the graph $K_{\frac{2k-1}{k}}$, the vertex 0 is adjacent to exactly two vertices, k and $k+1$, and likewise all other vertices are adjacent to exactly two vertices, i.e., $K_{\frac{2k-1}{k}}$ is 2-connected. So, $K_{\frac{2k-1}{k}}$ is a cycle equivalent to C_{2k-1}. See Figure 3 for the example of $K_{\frac{7}{3}}$.

We are interested in the circular graphs with the following properties:

(i) $\frac{k}{d}$ is in reduced form

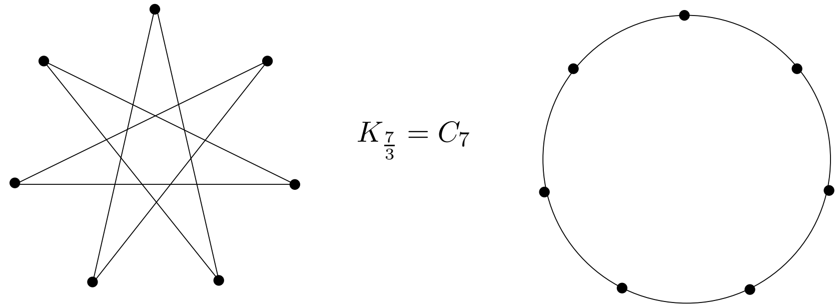

Figure 3: Two Equivalent Representations for $K_{\frac{7}{3}}$ (C_7).

$K_{\frac{2}{1}}$										$K_{\frac{3}{1}}$
$K_{\frac{2}{1}}$					$K_{\frac{5}{2}}$					$K_{\frac{3}{1}}$
$K_{\frac{2}{1}}$			$K_{\frac{7}{3}}$		$K_{\frac{5}{2}}$		$K_{\frac{8}{3}}$			$K_{\frac{3}{1}}$
$K_{\frac{2}{1}}$		$K_{\frac{9}{4}}$	$K_{\frac{7}{3}}$		$K_{\frac{5}{2}}$		$K_{\frac{8}{3}}$	$K_{\frac{11}{4}}$		$K_{\frac{3}{1}}$
$K_{\frac{2}{1}}$	$K_{\frac{11}{5}}$	$K_{\frac{9}{4}}$	$K_{\frac{7}{3}}$	$K_{\frac{12}{5}}$	$K_{\frac{5}{2}}$	$K_{\frac{13}{5}}$	$K_{\frac{8}{3}}$	$K_{\frac{11}{4}}$	$K_{\frac{14}{5}}$	$K_{\frac{3}{1}}$

Table 2: Sequences of Circular Graphs Ordered Using Farey Sequences.

(ii) $gcd(k, d) = 1$

(iii) $K_{\frac{k}{d}} \leq K_{\frac{k'}{d'}}$ if and only if $\frac{k}{d} \leq \frac{k'}{d'}$

Table 2 shows sequences of circular graphs. As will be seen later, circular graphs with $\frac{2}{1} \leq \frac{k}{d} \leq \frac{3}{1}$ are of particular interest to us.

The third row of Table 2 is depicted in Figure 4. Notice that when $d = 1$, $K_{\frac{k}{d}}$ is equal to the complete graph K_k.

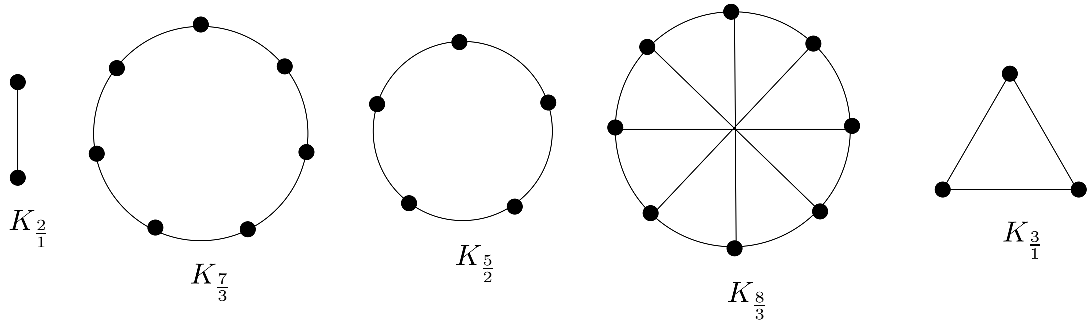

Figure 4: Graphs for the Third Row of Table 2.

1.3 Chromatic Numbers and Circular Chromatic Numbers

One definition for the chromatic number of a graph G denoted $\chi(G)$ is the following. Graph G is said to be k-chromatic if k is the smallest integer such that $G \leq K_k$. In [4], a similar definition is given for the circular chromatic number of a graph G denoted $\chi_c(G)$. $\chi_c(G) = \frac{k}{d}$ where $\frac{k}{d}$ is the smallest rational number such that $G \leq K_{\frac{k}{d}}$.

We are interested in circular chromatic numbers because they give us more information about graphs than chromatic numbers. For example, for any cycle C_{2k-1}, $\chi(C_{2k-1}) = 3$, but $\chi_c(C_{2k-1}) = \chi_c(K_{\frac{2k-1}{k}}) = \frac{2k-1}{k}$. Notice that $\frac{2k-1}{k} = 2 - \frac{1}{k}$. From this we can see that as k gets large, $2 - \frac{1}{k}$ approaches 2. For example, $\chi(C_{1001}) = 3$, but $\chi_c(C_{1001}) = \chi_c(K_{\frac{1001}{500}}) = 2.001$. An important theorem for circular chromatic numbers it the following.

Theorem 1.4 *[9] For any finite graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$.*

Zhu gives the following example [9]. Consider the problem of traffic flow at an intersection. Each lane of traffic needs to be assigned an interval of time during which it has a green light. A complete traffic period is a period of time in which each traffic

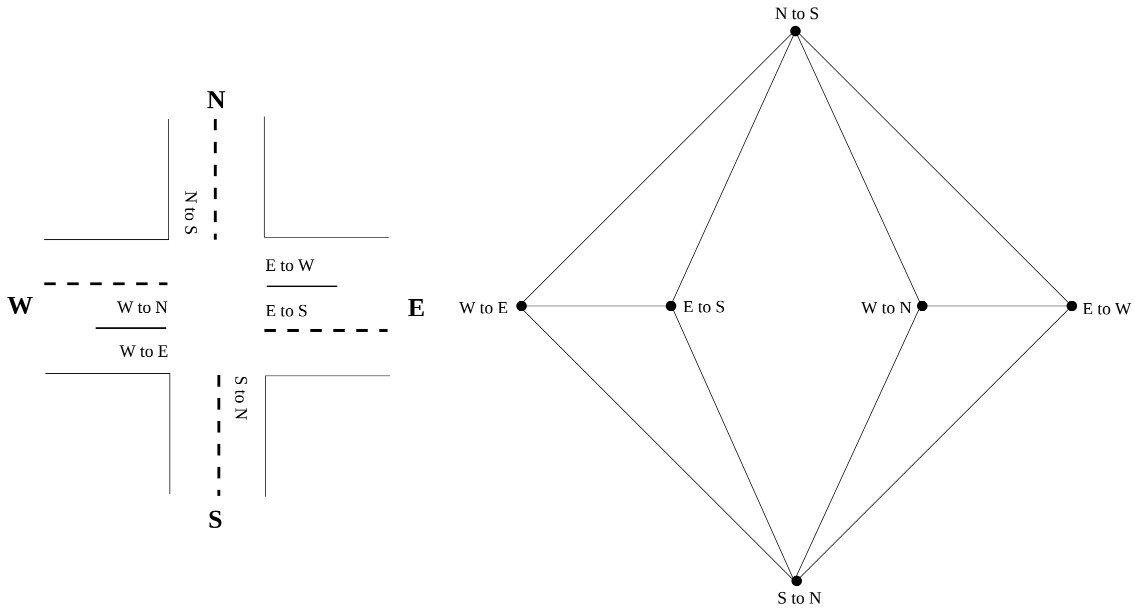

Figure 5: Traffic lanes at an Intersection.

lane gets a turn at a green light. A pattern of red and green lights needs to be designed for a complete traffic period, where each green light is of unit length.

For this problem, a graph makes an ideal model. Let each traffic lane be represented by a vertex, and there is an edge between two vertices if the two corresponding traffic lanes would inhibit one another, e.g. north-south traffic, and east-west traffic would inhibit each other. A simple example of a graph like this is depicted in Figure 5.

This problem can be solved by finding the chromatic number of the corresponding graph G. The graph would be partitioned into the minimum number of sets of non-adjacent vertices with a period of green light of unit length assigned to each set. This would give a complete traffic period of kt where k is the number of sets, and t is the length of unit time. However, kt would not be an optimal solution. The optimal solution is obtained using the circular chromatic number of the graph χ_c. The number of sets, k, is equal to the chromatic number of the graph, χ. By Theorem 1.4, $\chi_c(G) \leq \chi_c(G)$, so $\chi_c t$ would be the optimal solution.

1.4 Main Results

Finding a tight upper bound for $\chi_c(G)$, in a class of K_n-minor-free graphs, $\mathcal{G}/K_n$, is a difficult problem even for small values of n. The case $n \geq 5$ remains unsolved. Even for planar graphs the problem is open. See [2] and [8]. To date the best known circular chromatic number upper bound for planar graphs is given by Zhu [5]. Pan and Zhu [6, 7] have given a function $\mu(G)$, which settles this problem for $\mathcal{G}/K_4$, and proved that their bound is indeed the best possible by asymptotically constructing $\chi_c(G)$. The following are the theorems of Pan and Zhu, for which we give new proofs.

Theorem 1.5 (Pan, Zhu [6]) *Suppose $r \in \{-1, 1, 3\}$ and $G \in \mathcal{G}/K_4$ has odd-girth g. If $g \geq 6k+r$, then $\chi_c(G) \leq \mu(g)$, where $\mu(g) = 2\sigma/(\sigma-1)$ and $\sigma = 4k+(r+|r|)/2$.*

The proof of Theorem 1.5, in Chapter 2, is from an unpublished paper by Yared Nigussie.

Theorem 1.6 (Pan, Zhu [7]) *For every $\epsilon > 0$, and every odd integer g, there exists a graph $G_g \in \mathcal{G}/K_4$ of odd girth g such that $\chi_c(G_g) > \mu(g) - \epsilon$.*

The results by Pan and Zhu are based on the so-called *labeling method*. Although the labeling method has been quite useful in several proofs, it leads to long case analysis and calculations.

Our proof technique is based on structural methods: We show a minimal counterexample G to Theorem 1.5 must have a certain configuration, which we prove to be reducible. For Theorem 1.6, we give a different construction which in fact obtains a stronger result. The following are the main results of this thesis:

Theorem 1.7 *For every odd integer g, there exists a graph $H_g^r \in \mathcal{G}/K_4$ of odd girth g such that $\chi_c(H_g^r) = \mu(g)$, if and only if $g \ncong 1 \mod 6$.*

Theorems 1.8 and 1.9 consider the remaining case where $g \cong 1 \mod 6$. Define $K_m^1 = K_{\frac{4k+3+m(4k+1)}{2k+1+m2k}}$.

Theorem 1.8 *For every graph $G \in \mathcal{G}/K_4$ of odd girth at least $6k+1$, there exists $m \in \mathbb{N}$ such that $G \leq K_m^1$.*

Theorem 1.9 *For every $m \in \mathbb{N}$, there exists a graph $G \in \mathcal{G}/K_4$ of odd girth $6k+1$ such that $G \nleq K_m^1$.*

Theorem 1.8 implies that $\chi_c(G) < \mu(g)$. Theorem 1.9 implies that $\mu(g)$ is the least upper bound, since $K_m^1 = K_{\frac{4k+3+m(4k+1)}{2k+1+m2k}}$ converges to $\mu(g) = \frac{4k+1}{2k}$.

The proof of Theorem 1.5 is given in Chapter 2. The result for Theorem 1.6 is implied in the results of Theorems 1.7, 1.8 and 1.9. The proofs of Theorems 1.7, 1.8 and 1.9 will be given in Chapter 3.

2 THE EXISTENCE OF UPPER BOUNDS

A *thread* in G is a path $P \subseteq G$ such that the two endpoints of P have degree at least 3 and all internal vertices of P are degree 2 in G. We shall often use the fact that if P and P' are two edge-disjoint paths and if the lengths of P and P' have the same parity such that P is a thread and has length at least the length of P', then there is a homomorphism that maps P to P' sending the two ends of P to the two ends of P'. Such a homomorphism is said to *fold* P to P'. Let G be a graph and let G^s denote the multi-graph we obtain from G by "smoothing" all degree 2 vertices of G. For each edge e of G^s, let P_e denote the thread of G represented by e in G^s, and let l_e denote the length of P_e. The graph G^* is obtained by identifying the parallel edges of G^s. We need the following Folding Lemma of [4]. The Folding Lemma is a key lemma which we use in the next section.

Lemma 2.1 (Edge folding lemma [4]) *Let $G \in \mathcal{G}/K_4$ be of odd girth $2k+1$ and let e and e' be parallel edges in G^s with common end vertices x, y. If G is not homomorphic to a strictly smaller graph of the same odd girth, then $l_e + l_{e'} = 2k+1$. Moreover, $P_e \cup P_{e'}$ is the unique cycle of length $2k+1$ containing both x and y.*

Lemma 2.2 *[4] Let $G \in \mathcal{G}/K_4$ have odd girth $g = 2k+1$ such that $G \nsim C_{2+1}$ and G is not homomorphic to a strictly smaller graph of the same odd girth in $\mathcal{G}/K_4$. Then, for any $y \in V(G^*)$, if $d_{G^*}(y) = 2$, then $d_{G^*}(y) = d_G(y) = 4$. Moreover, if such a y exists then G has a configuration of Figure 6, where $P_{e_1} \cup P_{e_2}$, $P_{e_3} \cup P_{e_4}$ and $P_{e_5} \cup P$ are pairwise edge-disjoint cycles of length $2k+1$, such that $l_{e_i} \geq 2$, for each i, $1 \leq i \leq 5$.*

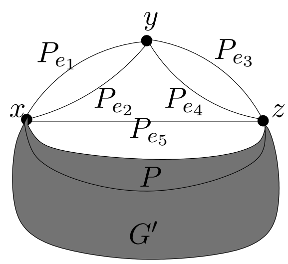

Figure 6: Unavoidable Configuration of G.

For this proof, we will be referring to Figure 6, and using the notation $L_1 = l_{e_1}$, $l_1 = l_{e_2}$, $L_2 = l_{e_3}$, $l_2 = l_{e_4}$, $l_3 = l_{e_5}$ and l'_3 is the length of P. Also, we will be using the notation $V_{8k} = K_{8k/(4k-1)}$.

Theorem 1.5 *Suppose* $r \in \{-1, 1, 3\}$ *and* $G \in \mathcal{G}/K_4$ *has odd-girth* g*. If* $g \geq 6k + r$*, then* $\chi_c(G) \leq \mu(g)$*, where* $\mu(g) = 2\sigma/(\sigma - 1)$ *and* $\sigma = 4k + (r + |r|)/2$*.*

Proof. Let $G \in \mathcal{G}/K_4$ of odd girth at least g be a counterexample with $|V(G)|$ as small as possible. It suffices to show that $G \leq K_{\mu(g)}$. We prove that if $r \neq -1$ then $G \leq C_{4k+r}$ and that if $r = -1$, $G \leq K_{8k/(4k-1)}$, which contradicts the choice of G.

It is easy to see that G must be 2-connected, because $K_{\mu(g)}$ is vertex-transitive and so inductively a homomorphism $f_i : H_i \leq K_{\mu(g)}$ for each 2-connected component $H_i, i = 1, 2, \ldots, m \geq 2$, can be extended to $f : G \leq K_{\mu(g)}$, a contradiction. By Lemma 2.1, G has odd-girth $g = 6k + r$, $r \in \{-1, 1, 3\}$. Note that $L_i + l_i = 6k + r$, $i = 1, 2$.

Let $G' = (G \ \bigcup_{i=1}^{4} P_{e_i}) \cup \{x, y\}$ be obtained by deleting. Then by induction, $f : V(G') \leq K_{\mu(g)}$ exists. Note that if $r = -1$, then $f(v_1)$ and $f(v_2)$ can be found on some C_{4k+1}. We may assume $f(v_1) = 0$ and $f(v_2) = l_3$, $l_3 + l'_3 = 4k + |r|$, $l_3 < l'_3$ and that $L_1 \geq L_2 > l_2$. Then $l_2 \geq l_1$. We may also assume that $L_1 < 4k + (r + |r|)/2$, for otherwise $G \backslash P_{L_1} \leq C_{4k+|r|}$ can be extended to $G \leq C_{4k+|r|}$, and we are done. Then, we have $l_1 > 2k$, if $r \neq -1$ and $l_1 \geq 2k$, if $r = -1$. In addition, we can assume $l_3 \geq 2$, for if $0 \leq l_3 \leq 1$, we clearly have $G \leq C_{4k+|r|}$. It follows that, $l_1 + l_2 > l_3'$.

Let $\{\alpha, \beta\} = \{L_2, l_2\}$ such that $L_1 \cong \beta + l_3 \mod 2$. Then $\beta > L_1 - l_3$, for otherwise we have $L_1 \geq \beta + l_3$ and $\alpha \geq l_1 + l_3$. Hence, we may identify P_{L_1} with $P_\beta \cup P_{l_3}$ and P_α with $P_{l_1} \cup P_{l_3}$. Since $l_1 + \beta \geq l_1 + l_2 > l_3'$, we get $G \leq C_{4k+|r|}$. We now extend f by f^* as follows: If $r = -1$, $L_1 = L_2$, and $l_3 = 2k$ or $2k+1$, it is easy to see we map G to V_{8k} by letting $f^*(v) = 8k - L_1$. Otherwise, we map G to $C_{4k+|r|}$, by showing each path: $P_{L_1}, P_{l_1}, P_\beta$ and P_α, can be identified with their corresponding subpaths of the same parity in $C_{4k+|r|}$. For P_{L_1}, we have $L_1 < 4k + |r|$, because $L_1 < 4k + (r + |r|)/2 \leq 4k + |r|$. Note also that $L_1 > l_3$ since $l_3 < l_3'$ and $l_3 + l_3' \leq L_1 + l_1$. For P_{l_1}, we have $l_1 \geq 4k + |r| - L_1$, because $L_1 + l_1 = 6k + r$. For P_β, we have $\beta > L_1 - l_3$, as shown above. For P_α we show, $\alpha \geq l_3 + (4k + |r| - L_1)$. Substituting $\alpha + \beta = 6k + r$ and rearranging we shall verify:

$$L_1 - \beta \geq (l_3 - 2k) + (|r| - r) \qquad (*)$$

Note that $L_1 - \beta \geq 0$. If $r \neq -1$, then $|r| - r = 0$, and so if $2k \geq l_3$, we are done. Otherwise, $l_3 = 2k + 1$, then $L_1 \ncong \beta \mod 2$, i.e., $L_1 > \beta$, which implies $(*)$ holds. Next, let $r = -1$, i.e., $|r| - r = 2$ and $l_3 \leq 2k$. By assumption if $l_3 = 2k$, then $L_1 \neq \beta$ and so $L_1 - \beta = 2t, t \geq 1$ and if $l_3 < 2k$, we see once more $(*)$ holds.

Note that the case $l_3 > 2k + 1$ is symmetric, since $l_3' = 4k + 1 - l_3 \leq 2k$. This concludes the proof that no counterexample exists to Theorem 1.5. □

3 THE ATTAINABILITY OF THE UPPER BOUNDS

In this Chapter the upper bounds for the three cases, graphs of girth greater than or equal to $6k + r$ for $r \in \{-1, 1, 3\}$, will be classified. That is to say, it will be shown that when girth is greater than or equal to $6k - 1$ or $6k + 3$, the upper bound is attainable, and when girth is greater than or equal to $6k + 1$, the upper bound is unattainable.

3.1 Proof of Theorem 1.7

In this section we assume graph G has odd girth $g = 6k+r$, where $r \in \{-1, 1, 3\}$. We define the following: $L_a = \frac{g+1}{2}$, $L_b = \frac{g-1}{2}$, $l_a = \frac{g+1}{2} - k + 1$, and $l_b = \frac{g-1}{2} - k + 1$. For short, $K_m^{-1} = K_{\frac{4k+1+8km}{2k+(4k-1)m}}$ and $K_m^3 = K_{\frac{4k+5+(4k+3)(2m)}{2k+2+(2k+1)(2m)}}$. We also need to define $\beta_m^{-1} = 2k + (4k-1)m$ and $\beta_m^3 = 2k + 2 + (2k+1)(2m)$, where β_m^r generates the Hamiltonian cycle for K_m^r depicted in Figure 7.

Let G_g^r be the graph depicted in Figure 8. Notice that the odd girth of G_g^r is less than $6k+r$ for $k > 1$. However, we find G_g^r to be useful in the following sense: Suppose H is a graph for which we know $\chi_c(H) < \mu(g)$. Assuming f is a homomorphism that maps graph G to H, if we show G_g^r to be a subgraph of $f(G)$, then we deduce that $\chi_c(G) \geq \chi_c(G_g^r)$, contrary to the assumption that $\chi_c(H) < \mu(g)$. This is the key method of proof for Theorem 1.7.

Lemma 3.1 *For* $r = \{-1, 3\}$, $\chi_c(G_g^r) = \mu(g)$.

Proof. For simplicity, we prove the case $r = 3$, (the case $r = -1$ is similar). Note that $G_g^3 \nleq C_{4k+5}$, for otherwise, identifying c to any vertex of l_a or l_b creates an shorter

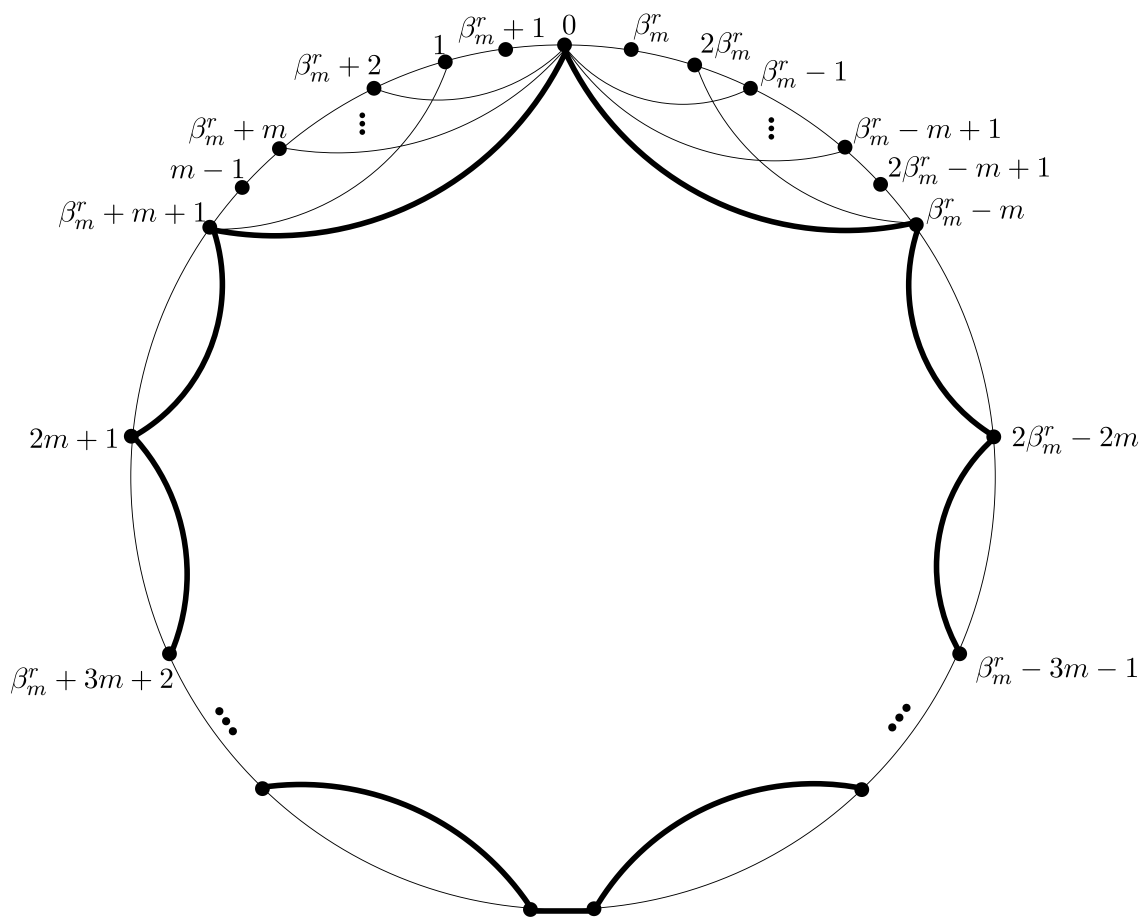

Figure 7: K_m^r with the Hamiltonian Cycle Generated by β_m^r, Where the Vertices $\{\beta_m^r - m, \beta_m^r - m + 1, \ldots, \beta_m^r + m + 1\}$ are Incident to 0. Note that the Thick Edges Depict a Cycle of Length C_{4k+r+2}.

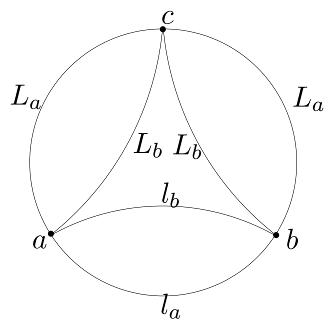

Figure 8: Graph G_g^r.

odd cycle. By Theorem 1.5, $G_g^3 \leq C_{4k+3}$. Therefore, $\frac{4k+5}{2k+2} < \chi_c(G_g^3) \leq \frac{4k+3}{2k+1}$. To prove the second inequality is actually an equality, we use the following.

From basic number theory [5], using what is known as the Farey sequence, we can see that any rational $\frac{p}{q}$ strictly between $\frac{4k+5}{2k+2}$ and $\frac{4k+3}{2k+1}$ has numerator $p \geq 8k+8$. It is well known [9] that for a graph G with a circular chromatic number a/b, the numerator a is at most the circumference of G [9], if $gcd(a,b) = 1$. But then the circumference of G_g^3 is $8k+7 < p$. Thus, $\chi_c(G_g^3) = \frac{4k+3}{2k+1}$. □

Remark. For the case $r = 1$ note that $G_g^1 \not\leq C_{4k+3} = K_0^1$, for otherwise, identifying c to any vertex of l_a or l_b creates an shorter odd cycle. However, G_g^1 does not attain $\mu(g)$, because $G_g^1 \leq K_1^1$.

The following lemma is used to help show that the desired subgraph G_g^r appears, whenever we attempt to map H_g^r to K_m^r for some $m \in \mathbb{N}$.

Lemma 3.2 *Any three vertices of K_m^r are contained in an odd cycle of length at most $4k+r+6$ when $r = \{-1, 3\}$.*

Proof. Note that the odd girth of K_m^r is $4k+r+2$, (depicted by thick curves on Figure 7). First, any two vertices v_1 and v_2 are on a $4k+r+4$-cycle. We may assume

$v_1 = 0$ and if v_2 is not on the thick cycle then, it can be reached by replacing a thick edge with 3 thin edges. Then v_3 can be found similarly. □

Theorem 1.7 follows from the following Lemma.

Lemma 3.3 *For any* $g \not\equiv 1 \mod 6$ *let* H_g^r *be the graph in Figure 9, then* $\chi_c(H_g^r) = \mu(g)$.

Proof. Assume $\chi_c(H_g^r) < \mu(g)$. Then there exists some m, such that $H_g^r \leq K_m^r < \mu(g)$. When mapping H_g^r to some K_m^r by a homomorphism f, the distance between $f(a)$ and $f(b_i)$ for some $0 \leq i \leq 2k-2$ is l_a, for if $dist(f(a), f(b_i)) < l_a$ for all i, then $f(H_g^r)$ would have an odd cycle shorter than odd-girth of K_m^r, a contradiction. By Lemma 3.2, a, b_i and c_i are on a cycle of length at most $4k+r+6$. This forces either one of the two shortest paths from a and c_i or one of the two shortest paths from b_i and c_i to be folded to a path of length either $k+1$ or $k+2$. Hence, vertices c_i and d_{ij}, $1 \leq j \leq 4$, (see Figure 10) will be on a cycle of length $4k+r+2$ at distance l_a (See Figure 8). But then this induces a G_g^r subgraph in $f(H_g^r)$, contrary to Lemma 3.1. □

3.2 Proof of Theorems 1.8 and 1.9

In this section, we study the remaining case, $r = 1$. Recall that $K_m^1 = K_{\frac{4k+3+m(4k+1)}{2k+1+m2k}}$. In contrast to the cases $r = -1$ and $r = 3$, for the case $r = 1$, we prove that $\mu(g)$ is not attainable. However, we also prove that $\mu(g)$ is the best bound that exists. Analogous to Lemma 3.2, the following lemma is useful.

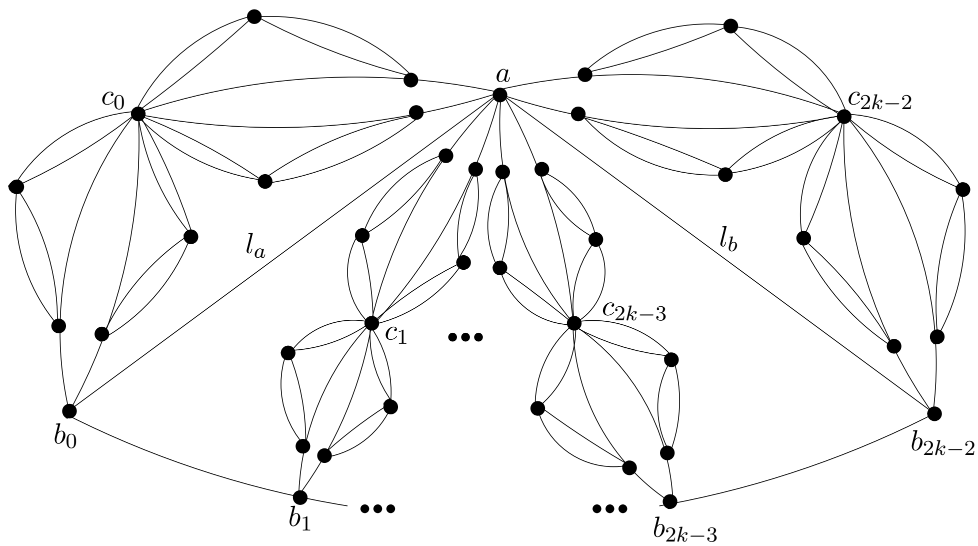

Figure 9: This Graph Attains the Upperbound for Theorem1.7 When $g \not\cong 1 \mod 6$.

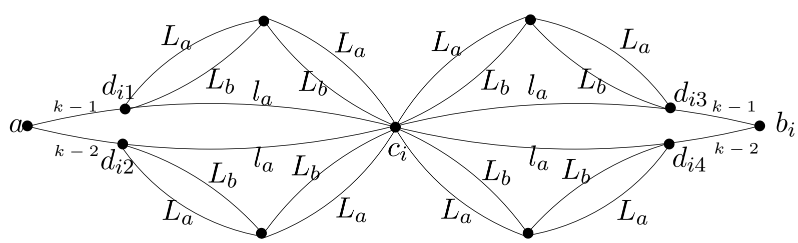

Figure 10: A Close-up View of Part of Figure 9 Between Vertices a and b_i for any i.

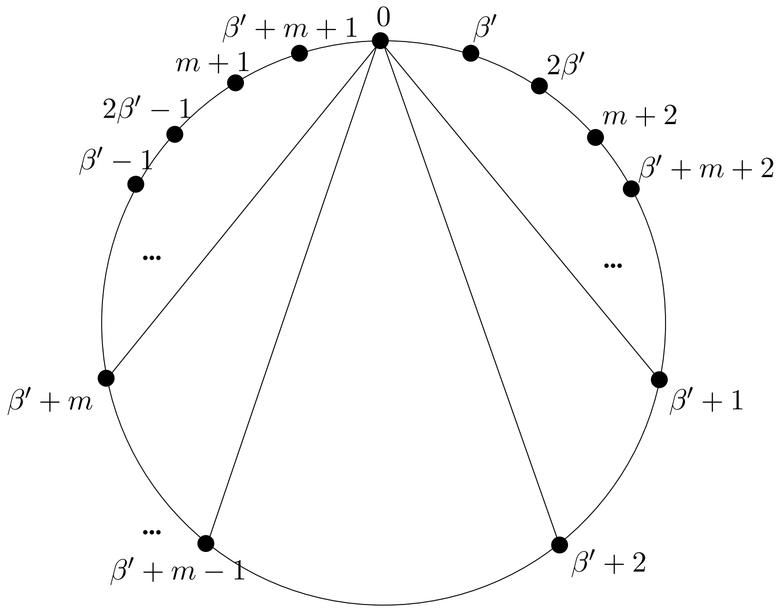

Figure 11: An Alternate Representation of K_m^1 Where $\beta' = 2k + 2 + m(2k + 1)$ Generates the Hamiltonian Cycle.

For the following lemma, note that the cycle $D \bigcup d$ is a $4k + 3$-cycle.

Lemma 3.4 *Let H be the graph depicted in Figure 13. Then, $H \leq K_1^1$.*

Proof. Consider the graph H in Figure 13. Notice first, that l and l' must be at least $2k$. If not, then $|P_L| \geq 4k+2$ or $|P_L'| \geq 4k+2$ respectively. Delete the respective path, then the remaining graph maps to C_{4k+3}. From here the deleted path can be added back and mapped to C_{4k+3} as well. Now, let $\delta = dist(f(a), f(b))$. Without loss of generality, assume l is the smallest of l, L, l', L' and let $\{\alpha, \beta\} = \{L', l'\}$ such that l and $\beta + \delta$ are the same parity. Then, $\beta < l + \delta$, otherwise we have $\beta > l + \delta$

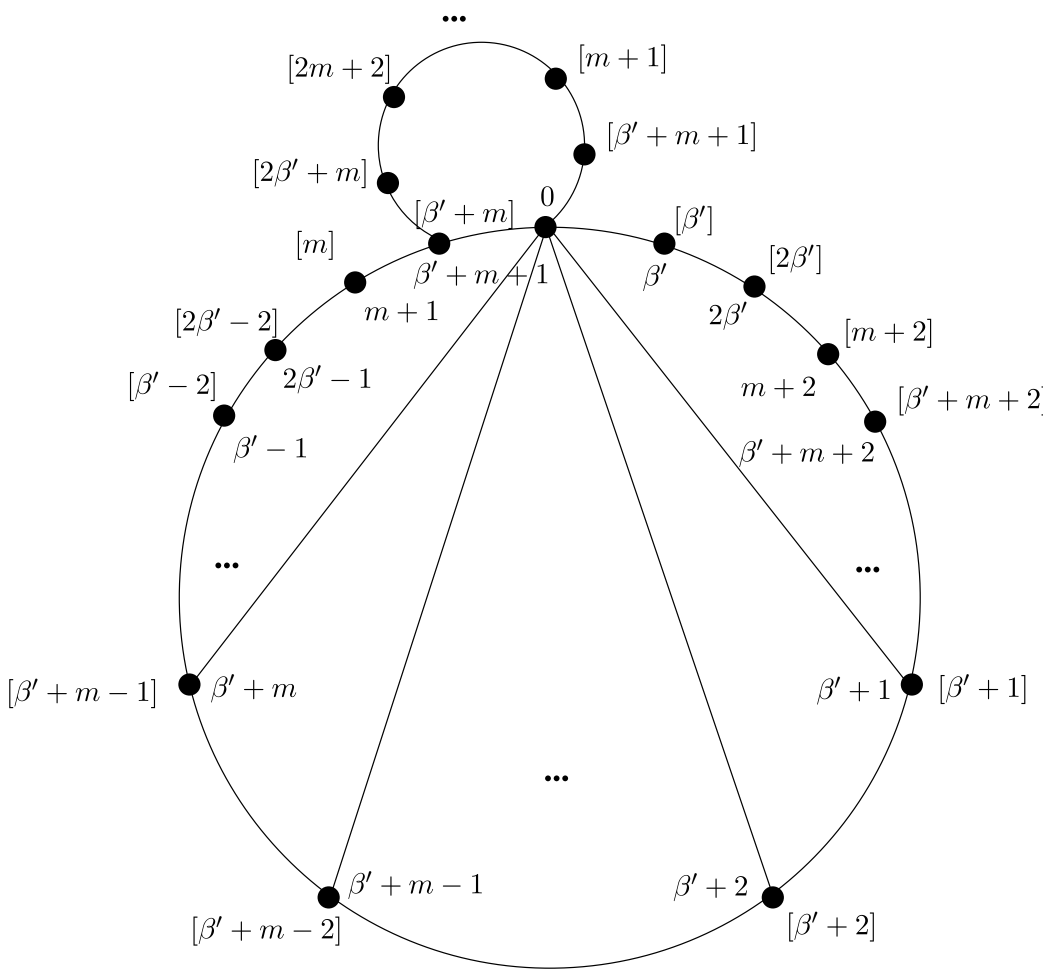

Figure 12: Extending K^1_m to K^1_{m+1}. Vertices of K^1_{m+1} are Distinguished from Vertices of K^1_m by the "[ ]" Symbol.

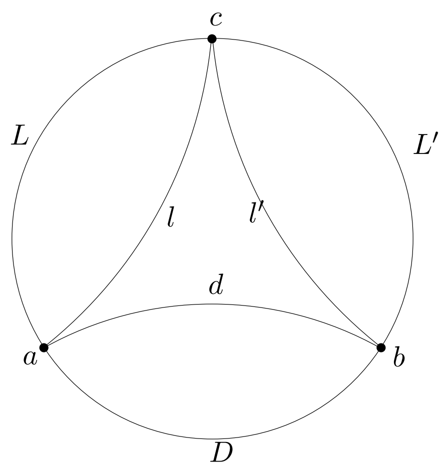

Figure 13: Graph H Used in Lemma 3.4.

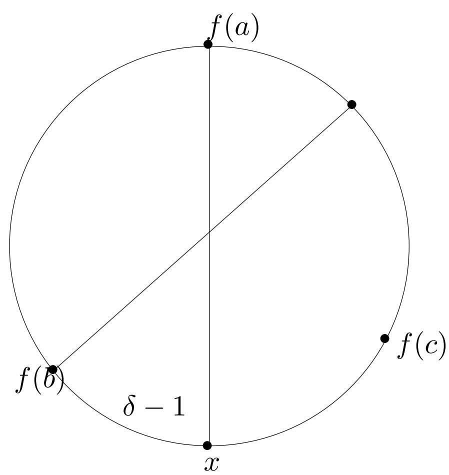

Figure 14: How H is Mapped to K_1^1.

and $L \geq \alpha + \delta$. Hence we may identify P_β with $P_l \bigcup P_\delta$ and P_L with $P_\alpha \bigcup P_\delta$, a contradiction.

Let $f(c)$ be at distance $\beta - \delta + 1$ from x. Since l and $\beta + \delta$ are the same parity, then so are l and $\beta - \delta + 2$. Also, since $\beta < l + \delta$, then $\beta + 2 \leq l + \delta$, and so $\beta - \delta + 2 \leq l$. Now, we can identify P_l with the path from a to c which includes x. Thus, $H \leq K_1^1$.

□

Theorem 1.8 *For every graph $G \in \mathcal{G}/K_4$ of odd girth at least $6k+1$, there exists $m \in \mathbb{N}$ such that $G \leq K_m^1$.*

Proof. Assume a graph $G \in \mathcal{G}/K_4$ of girth at least $6k+1$ is a minimal counterexample. Then $G \nleq K_m^1$ for all $m \geq 0$. By Lemma 2.2, G has the configuration depicted in Figure 6. Let $G' = (G \backslash \bigcup_{i=1}^4 P_{e_i}) \cup \{x, y\}$. Now, by minimality of G, we know that a homomorphism f maps G' to K_m^1, for some $m > 0$. Note that $f(x)$ and $f(z)$ are on a C_{4k+3} subgraph. We now extend f to a homomorphism f^* mapping G to K_{m+1}^1. If the shortest distance between $f(x)$ and $f(z)$ is not on the Hamiltonian cycle of K_m^1, then $G \leq K_m^1$ by Lemma 3.4. Assume the shortest distance between $f(x)$ and $f(z)$ is on the Hamiltonian cycle of K_m^1, then $G \nleq K_m^1$. In this case, we extend K_m^1 using an edge on the Hamiltonian cycle between $f(x)$ and $f(z)$, as depicted in Figure 12, to obtain K_{m+1}^1. Then in K_{m+1}^1 the shortest distance between $f(x)$ and $f(z)$ is not on the Hamiltonian cycle. By what we just proved for K_m^1, we deduce that $G \leq K_{m+1}^1$. Hence, $G \leq K_m^1$ for some $m \in \mathbb{N}$. □

We prove the remaining Theorem using the following graph. Define recursively the following. G^0 is the graph depicted in Figure 15 (left) and let H_k be the "hook-graph" depicted on the right. Then, G^j is constructed by taking a copy of G^{j-1} and

$2k-1$ copies of H_k, and identifying vertex e_{ji} to vertex a and identifying vertex d_{ji} to a vertex at distance $2k+1$ from a on the thread of length $3k+1$ between a and $c_{(j-1)i}$, for all $j \in 0, 1, \ldots, 2k-2$. (see Figure 15 for G_1^1).

Theorem 1.9 *For every $m \in \mathbb{N}$, there exists a graph $G \in \mathcal{G}/K_4$ of odd girth $6k+1$ such that $G \not\leq K_m^1$.*

Proof. By the Remark after Lemma 3.1, $G^0 \not\leq C_{4k+3}$. Notice that $C_{4k+3} = K_0^1$. Applying Lemma 3.4 $2k-1$ times implies that $G^0 \leq K_1^1$. Let f be a homorphism form G^0 to K_1^1. For each $i \in \{0, 1, \ldots, 2k-2\}$, a, b_i and c_{0i} are contained in a subgraph of G^0 of the form of Figure 13 with the specific values $L = L' = 3k+1$, $l = l' = 3k$ and $d + D = 6k+1$. For some $i \in \{0, 1, \ldots, 2k-2\}$, $f(b_i)$ is at distance $2k+1$ from $f(a)$, if not we get a cycle shorter than $4k+1$. Note that f maps some thread of length $3k+1$ between vertices a and c_{0i} injectively to a path on the Hamiltonian cycle of K_1^1.

Inductively, let m be minimal such that there is a G^m, such that $G^m \not\leq K_m^1$ and $G^m \leq K_{m+1}^1$. Further we may inductively assume, similar to the mapping of G^0 to K_1^1, when G^m is mapped to K_{m+1}^1 for some $i \in \{0, 1, \ldots, 2k-2\}$, a thread t of length $3k+1$ between a and c_{mi}, is mapped injectively to K_{m+1}^1. We extend G^m to G^{m+1} (recall the recursive construction), so that $G^{m+1} \not\leq K_{m+1}^1$. We extend K_{m+1}^1 to K_{m+2}^1 at an edge of $f(t)$ so that $f(t)$ is not on the Hamiltonian cycle of K_{m+2}^1. Now, $G^{m+1} \leq K_{m+2}^1$, and similar to the mapping of G^0 to K_1^1, when G^{m+1} is mapped to K_{m+2}^1 for some $i \in \{0, 1, \ldots, 2k-2\}$, a thread of length $3k+1$ between a and $c_{(m+1)i}$, is mapped injectively to K_{m+2}^1. By induction, the result follows. □

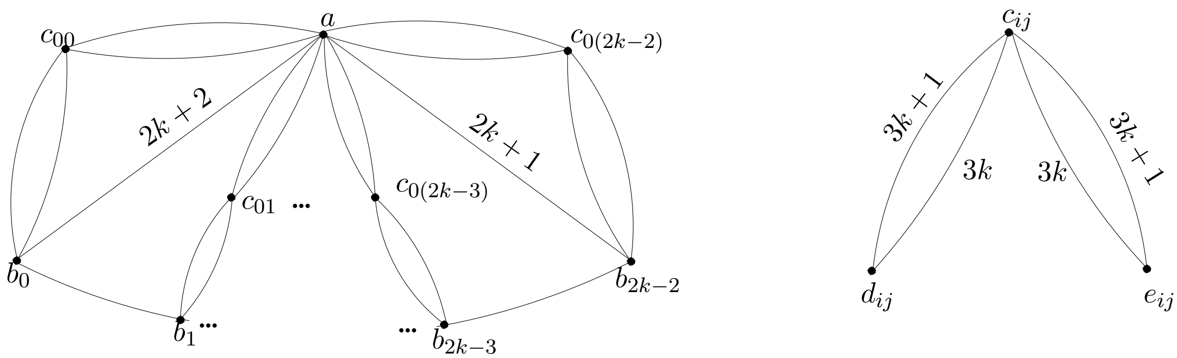

Figure 15: Graph G^0 and H_k.

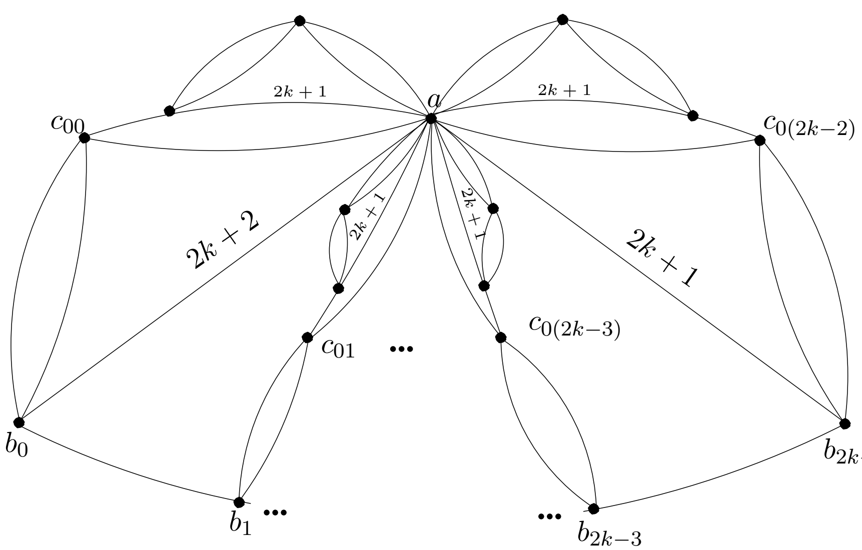

Figure 16: Graph G^1.

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VITA

TRACY HOLT

Education: M.S. Mathematics, East Tennessee State University,
Johnson City, Tennessee 2008

B.S. Mathematics, East Tennessee State University,
Johnson City, Tennessee 2006

Papers: T. Holt and N. Nigussie, "Short Proofs for Two Theorems of Pan and Zhu,"
In preparation for publication.

R. Gardner and T. Holt, "Decompositions of the Complete Symmetric
Digraph into Orientaions of the 4-Cycle with a Pendant Edge,"
In preparation for publication.