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A Variation of the Carleman Embedding Method for Second Order Systems.

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by

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ABSTRACT

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The Carleman Embedding is a method that allows us to embed a finite dimensional system of nonlinear differential equations into a system of infinite dimensional linear differential equations. This technique works well when dealing with first-order nonlinear differential equations. However, for higher order nonlinear ordinary differential equations, it is difficult to use the Carleman Embedding method. This project will examine the Carleman Embedding and a variation of the method which is very convenient in applying to second order systems of nonlinear equations.

DEDICATION

I would like to dedicate this piece of work to my sweet mother Victoria Dzacka for her continuous prayers and support to me. To my siblings especially my brothers Daniel, Eric, Patrick, Samuel (of blessed memory) and my sister Lilian for their encouragements and prayers. I love you family. Finally, to my son Charles Dzacka Jr. I hope you make me proud someday.

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1 INTRODUCTION

The theory of differential equations is a fundamental tool in physics, engineering and other mathematically based sciences. Many natural laws and models of natural phenomena are described by nonlinear systems of differential equations. Many nonlinear problems are difficult or impossible to solve in closed form and therefore the construction of such solutions is nontrivial. Carleman developed a technique to linearize nonlinear systems of equations that could lead to analytical solutions of nonlinear problems [1]. The Carleman Embedding is a method that allows us to embed a nonlinear finite dimensional ordinary differential equation into a system of linear infinite dimensional differential equation.

Despite the great deal of work done with the Carleman method in linearizing nonlinear differential equations, there are associated shortfalls with the use of this method as well. The shortfalls are that, (i) the matrix of the linear system is unbounded, thus truncation to a finite system may not be possible, and (ii) it is difficult to extend this technique to higher-order ordinary differential equations.

The major objective of this project is to examine the Carleman Embedding method and a variation of the method on second-order nonlinear differential equations. This project is divided into five chapters. This introductory chapter is followed by the second chapter in which we consider the historical development and relevant literature of the Carleman Embedding method. The third chapter constitutes the Carleman Embedding method in which the method is applied to the Van der Pol's equation. This is a result of previous work done by others prior to this research. In the fourth chapter we use a variation of the Carleman method to solve second-order nonlinear equations. We will basically consider the

Duffing's equation with no forcing and no dumping. The Duffings equation is a nonlinear second-order differential equation and an example of a dynamical system that exhibits chaotic behavior [2]. Simulations of the solutions in this chapter will be done with Maple. Finally, in our last chapter we present our conclusion.

2 HISTORY OF THE CARLEMAN LINEARIZATION METHOD

A theoretical technique was developed in the 1930's by the mathematician Torsten Carleman to globally linearize systems of nonlinear differential equations. His article, which introduced the linearization method was entitled "Application of the Theory of Linear Integral Equations to Systems of Nonlinear Differential Equations" [3]. Carleman's ideas were motivated by remarks made by Henri Poincare [4]. Poincare is known for his studies in celestial mechanics and studying oscillatory motion in celestial bodies. Poincare remarked at a 1908 conference in Rome, that one should be able to apply the theory of linear integral equations to the study of ordinary non-linear differential equations. From that remark, Carleman worked on an approach to embed a system of nonlinear differential equations into an infinite set of linear equations [1].

The Carleman technique essentially remained unused for a little over thirty years before Bellman and Richardson applied the method to approximate solutions of a nonlinear ODE [5]. Thirteen years later Montroll and Hellman [6] studied the embedding technique in relation to small denominators and secular terms. In 1980, Steeb and Wilhelm [7] used Carleman Embedding to approximate the solution of the Lotka-Volterra problem. The Lotka-Volterra model is represented by systems of nonlinear equations that have periodic solutions. The Carleman technique was successfully applied to solve the Lotka-Volterra problem [4].

In 1981, Kerner [8] studied the technique for embedding nonlinear systems into polynomial systems. Also, in 1981 Andrade and Rauh [9], and Brenig and Fairen [10] studied the Lorenz model and power series expansions for nonlinear systems, respectively, using the

Carleman Embedding technique. In 1982, Wong [11] demonstrated that a linear operator acting on a Banach space could be related to analytical vector fields. This became known as the Carleman linearization or transformation of a vector field.

Moreover, a number of other results were discovered about linearization. In 1987, Kowalski [12] related finite dimensional nonlinear systems to problems in Hilbert space. Tsiligianis and Lyberatos [13] studied steady state bifurcation and exact multiplicity conditions using the Carleman method. Finally, by 1989, Steeb showed that there is a one-to-one correspondence between solutions of the infinite linear system and solutions of the associated nonlinear finite system for the analytic solutions [4]. Fortunately, Kowalski and Steeb summarized a large portion of this work into one book [14]. This book is the main reference from which most of the history of the Carleman method is outlined. A variation of Carleman Embedding technique is the theoretical method used to approximate solutions of second-order nonlinear systems studied in this research.

The Carleman Embedding technique applied to the Van der Pol's equation in this research is inspired by a work by Azamed Gezahagne- "Qualitative Models of Neural Activity and the Carleman Embedding Technique" [1].

3 CARLEMAN EMBEDDING TECHNIQUE

3.1 Introduction

The purpose of this section is to introduce the analysis of the Carleman Embedding technique and its application to finite dimensional systems of nonlinear differential equations [1]. The objective for this section is to review previous results from the Carleman Embedding so that we could see the difference between the method and the variation of the method which is the 'center piece' of this research. In particular, we consider Van der Pol's equation.

Before describing the application of the technique, we first recall the general scheme of Carleman linearization. Consider the system with analytic nonlinearities

$$\frac{du}{dt} = V(u,t) \tag{1}$$

where

$$V : R^k \times R \to R^k$$

and V is analytic in u.

We should mention that the original Carleman approach dealt with autonomous polynomial systems (1). Following Carleman we define the function

$$u_n\left(t\right) = \prod_{i=1}^{k} \left(u_i\left(t\right)\right)^{n_i} \tag{2}$$

where $u(t)$ satisfies (1) and $n \in z_+^k$. Here Z_+^k denotes k-tuples of nonnegative integers. The system (1) implies the following linear differential-difference equation

$$\frac{du}{dt} = \sum_{n' \in Z_1^k} M_{nn'}(t) u_{n'}. \tag{3}$$

Note that the differential-difference equation (3) is finite order only in the case of V polynomial in u. It should also be noted that in the case of autonomous systems (1) the coefficient matrix $[M_{nn'}]$ is constant. In view of the fact that the set Z_+^k is countable, one finds easily that (3) is equivalent to an infinite dimensional system of linear differential equations. Obviously, the solution of the system (1) is linked to the solution of (3) by

$$u_i = ue_i, i = 1, ..., k$$

where $e_i = (0, ..., 0, 1_i, 0, ..., 0)$ is unit column vector. So the finite dimensional nonlinear system (1) is embedded into the infinite dimensional linear system (3). Such an embedding is called a Carleman embedding [1].

3.2 Carleman Embedding of Van der Pol's Equation

The Van der Pol equation, proposed by Balthasar Van der Pol in 1920 as a model of relaxation oscillations with nonlinear damping, is governed by the second order differential equation

$$\frac{d^2x}{dt^2} - \epsilon\left(1 - x^2\right)\frac{dx}{dt} + x = 0$$

where x is the dynamical variable and ϵ is a small parameter. When ϵ is small, the quadratic term x^2 is very small and the system becomes a linear differential equation with a negative damping. Thus, the fixed point ($x = 0, \frac{dx}{dt} = 0$) is unstable (an unstable focus when $0 < \epsilon < 2$ and an unstable node, otherwise). On the other hand, when x is large, the term x^2 becomes dominant and the damping becomes positive. Therefore, the dynamics of the system are expected to be restricted in some area around the fixed point. Actually, the Van der Pol system satisfies Liẽnard's theorem ensuring that there is a stable limit cycle in the phase space. The Van der Pol system is therefore a Liẽnard system [15, 16].

Using Liẽnard's transformation $y = x - \frac{x^3}{3} - \frac{1}{\epsilon}\frac{dx}{dt}$, the above equation can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= y + \epsilon\left(x - \frac{1}{3}x^3\right) \\ \frac{dy}{dt} &= -x \end{aligned} \tag{4}$$

which can be regarded as a special case of the FitzHugh-Nagumo model (also known as Bonhoeffer-Van der Pol model) [3].

By using Maple, one can easily generate a numerical solution of the initial valued problem (4). For $x(0) = \alpha = 1$, $y(0) = \beta = 0$, and $\epsilon = 0.001$ we get the plot in Figure 1.

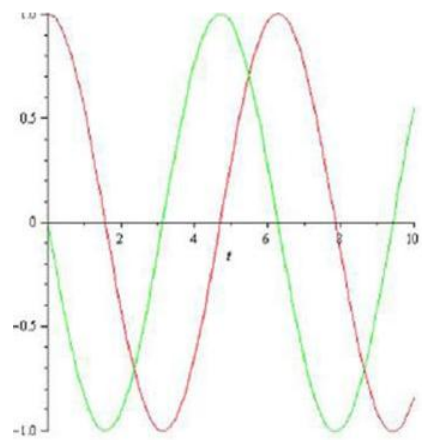

Figure 1: Numerical solution of Van der Pol's equation for $\epsilon = 0.001$

To get an approximate solution to the above systems of finite nonlinear equations, we apply the Carleman Embedding technique and solve in terms of an infinite set of linear equations as follows. Following the scheme proposed by Carleman and truncating at $n = 3$, we set

$$v_1 = x$$

$$v_2 = y$$

$$v_3 = x^2$$

$$v_4 = xy$$

$$v_5 = yx$$

$$v_6 = y^2$$

$$v_7 = x^3$$

$$v_8 = x^2 y$$

$$v_9 = yx^2$$

$$v_{10} = xy^2$$

$$v_{11} = yx^2$$

$$v_{12} = yx^2$$

$$v_{13} = y^2 x$$

$$v_{14} = y^3.$$

Applying the derivatives of each $v_i's$ followed by substituting and rearranging the equations, we get

$\frac{dv_1}{dt}=\frac{dx}{dt}=y+\epsilon x-\frac{\epsilon}{3}x^3=v_2+\epsilon v_1-\frac{\epsilon}{3}v_7$

$\frac{dv_2}{dt}=\frac{dy}{dt}=-x=-v_1$

$\frac{dv_3}{dt}=\frac{dx^2}{dt}=2x(y+\epsilon x-\frac{\epsilon}{3}x^3)=2v_4+2\epsilon v_3$

$\frac{dv_4}{dt}=\frac{d(dy)}{dt}=x(-x)+y(y+\epsilon x-\frac{\epsilon}{3}x^3)=-v_3+\epsilon v_5+v_6$

$\frac{dv_5}{dt}=\frac{d(yx)}{dt}=x(-x)+y(y+\epsilon x-\frac{\epsilon}{3}x^3)=-v_3+\epsilon v_5+v_6$

$\frac{dv_6}{dt}=\frac{dy^2}{dt}=-2yx=-2v_5$

$\frac{dv_7}{dt}=\frac{dx^3}{dt}=3x^2(y+\epsilon x-\frac{\epsilon}{3}x^3)=-v_8+3\epsilon v_7$

$\frac{dv_8}{dt}=\frac{d(x^2y)}{dt}=(-x)^3+2yx(y+\epsilon x-\frac{\epsilon}{3}x^3)=-v_7+2\epsilon v_{11}+2v_{13}$

$\frac{dv_9}{dt}=\frac{d(yx^2)}{dt}=(-x)^3+2yx(y+\epsilon x-\frac{\epsilon}{3}x^3)=v_7+v_{10}+2\epsilon v_{11}+v_{12}$

$\frac{dv_{10}}{dt}=\frac{d(xy^2)}{dt}=x(-2yx)+y^2(y+\epsilon x-\frac{\epsilon}{3}x^3)=-2v_9+\epsilon v_{13}+v_{14}$

$\frac{dv_{11}}{dt}=\frac{d(yx^2)}{dt}=(-x)^3+y(2yx+2(y+\epsilon x-\frac{\epsilon}{3}x^3))=-v_7+2\epsilon v_{11}+v_{12}$

$\frac{dv_{12}}{dt}=\frac{d(xy^2)}{dt}=xy(-x)+y(-x^2+y^2+\epsilon yx)=-v_8-v_{11}+\epsilon v_{13}+v_{14}$

$\frac{dv_{13}}{dt}=\frac{d(xy^2)}{dt}=x(-2yx)+y^2(y+\epsilon x-\frac{\epsilon}{3}x^3)=-2v_9+\epsilon_{13}+v_{14}$

$\frac{dv_{14}}{dt}=\frac{dy^3}{dt}=-3y^2x=-3v_{13}.$

One can write the above linear system of equation in the matrix form as

$$\frac{dV}{dt}=AV \tag{5}$$

where V $=[v_1,v_2,...,v_{14}]^T$ and

$$A = \begin{bmatrix} \epsilon & 1 & 0 & 0 & 0 & 0 & \frac{-\epsilon}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\epsilon & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3\epsilon & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2\epsilon & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 2\epsilon & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & \epsilon & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2\epsilon & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & \epsilon & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & \epsilon & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \end{bmatrix}.$$

The solution to (7) is given by

$$V(t) = V_0 e^{At} \tag{6}$$

where V_0 is the initial value

$$V_0 = \begin{bmatrix} v_1(0) \\ v_2(0) \\ v_3(0) \\ v_4(0) \\ v_5(0) \\ v_6(0) \\ v_7(0) \\ v_8(0) \\ v_9(0) \\ v_{10}(0) \\ v_{11}(0) \\ v_{12}(0) \\ v_{13}(0) \\ v_{14}(0) \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \\ x^2(0) \\ x(0)y(0) \\ y(0)x(0) \\ y^2(0) \\ x^3(0) \\ x^2(0)y(0) \\ x^2(0)y(0) \\ x(0)y^2(0) \\ x^2(0)y(0) \\ x(0)y^2(0) \\ x(0)y^2(0) \\ y3(0) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \alpha^2 \\ \alpha\beta \\ \beta\alpha \\ \beta^2 \\ \alpha^3 \\ \alpha^2\beta \\ \alpha^2\beta \\ \alpha\beta^2 \\ \alpha^2\beta \\ \alpha\beta^2 \\ \alpha\beta^2 \\ \beta^3 \end{bmatrix}.$$

We now use Maple to plot the solution of Van der Pol's equation obtained by Carleman Embedding technique. Figure 2 shows the result of Carleman Embedding for $x(0) = \alpha = 1, y(0) = \beta = 0$ and $\epsilon = 0.001$.

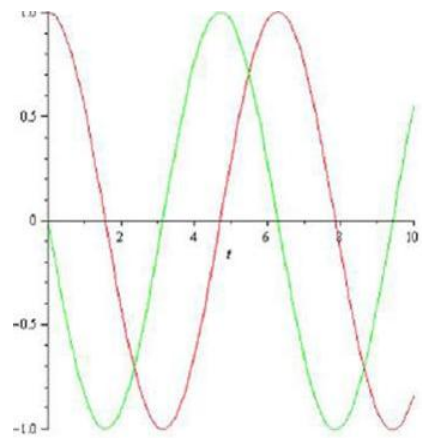

Figure 2: Solution by Carleman Embedding technique for $\epsilon = 0.001$

Now we can compare the two results above and it is shown that the Carleman Embedding technique gives the best approximation to the solution of the Van der Pol's equation [1].

4 THE SECOND ORDER APPROACH

4.1 Introduction

As noted in the previous section, nonlinear systems of higher order are difficult to solve with the Carleman Embedding method. Therefore we introduce a variation of the Carleman embedding method to solving higher nonlinear systems.

Basically, we would consider a Duffing equation

$$\frac{d^2x}{dt^2} = -x + 2\epsilon x^3, \quad x(0) = 1, x^{'}(0) = 0. \tag{7}$$

Before we use the variation of the Carleman Embedding on the above system, we first consider a Classical Perturbation of the system.

4.2 Nonlinear Systems

A known mathematical method like the Perturbation technique could also be used to find approximate solutions to problems which cannot be solved exactly. This is done by starting from the exact solution of a related problem [17].

Now, let's try to solve the system (7)

$$\frac{d^2x}{dt^2} = -x + 2\epsilon x^3, \quad x(0) = 1, x^{'}(0) = 0.$$

The Perturbation assumption is that

$$x(t, \epsilon) = \sum_{n=0}^{\infty} x_n(t)\epsilon^n$$

converges uniformly for ϵ in the neighborhood of 0. We thus have

$$x^{''}(t,\epsilon) = \sum_{n=0}^{\infty} x_n^{''}(t)\epsilon^n$$

where $x^{''}$ is the second derivative of x with respect to t.

Upon substitution into (7), we obtain

$$\sum_{n=0}^{\infty}[x_n^{''}(t) + x_n(t)]\epsilon^n = 2\epsilon[\sum_{n=0}^{\infty} x(t)\epsilon^n]^3. \tag{8}$$

For a first order Perturbation, we consider our Perturbation assumption is of the form

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2)$$

where $O(\epsilon^2)$ denotes all terms of the infinite series with ϵ to a power of 2 or higher.

For a first order Perturbation, (8) reduces to

$$\begin{aligned} \sum_{n=0}^{1}(x_n^{''} + x_n)\epsilon^n + O(\epsilon^2) &= 2\epsilon(x_0 + x_1\epsilon)^3 + O(\epsilon^2) \\ x_0^{''} + x_0 + (x_1^{''} + x_1)\epsilon + O(\epsilon^2) &= 2\epsilon(x_0^3 + 3x_0^2 x_1\epsilon + O(\epsilon^2)). \end{aligned}$$

The result is that we have

$$x_0^{''} + x_0 + (x_1^{''} + x_1)\epsilon + O(\epsilon^2) = 2\epsilon x_0^3 + O(\epsilon^2).$$

Thus we have

$$x_0^{''} + x_0 = 0, \quad x_0(0) = 1, x_0^{'} = 0 \tag{9}$$

and

$$x_1^{''} + x_1 = 2x_0^3, \quad x_1(0) = 0, x_1^{'} = 0. \tag{10}$$

But the solution to (9) is

$$x_0(t) = \cos(t).$$

Substituting the solution of (9) into (10), we have

$$x_1^{''} + x_1 = 2\cos^3(t), \quad x_0(0) = 0, x_1(0) = 0. \tag{11}$$

Variation of parameters implies that the solution to (11) is

$$x_1(t) = \frac{1}{4}\cos(t) - \frac{1}{4}\cos^3(t) + \frac{3}{4}t\sin(t).$$

Hence, the system (7) to the first order in ϵ is

$$x(t,\epsilon) = \cos(t) + \epsilon[\frac{1}{4}\cos(t) - \frac{1}{4}\cos^3(t) + \frac{3}{4}t\sin(t)].$$

The approximate solution to (7) by the Perturbation approach is shown in figure 3.

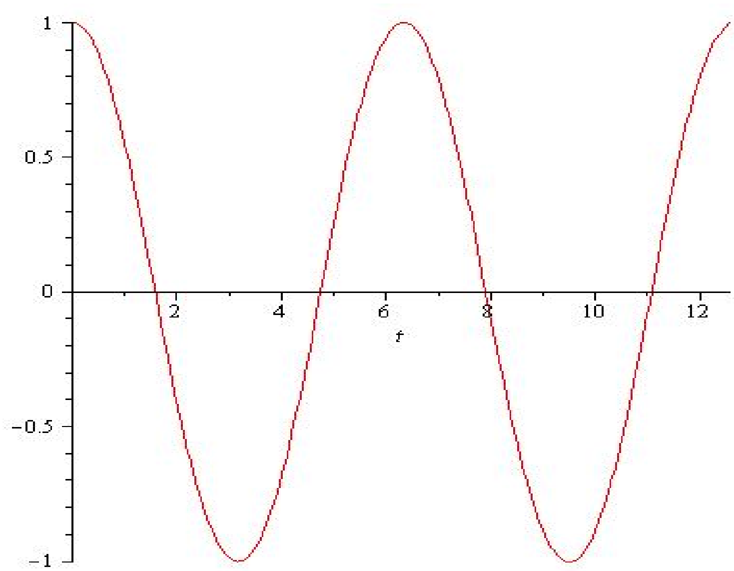

Figure 3: Solution by Perturbation technique for $\epsilon = 0.01$

We will later compare the solution by the Perturbation approach to the solution by the variation of Carleman Embedding approach as well as the solution of the original system for different values of ϵ. We will also combine the Perturbation theory with Carleman linearization.

4.3 Carleman Embedding for Second Order Systems

The objective for this section is to show how a variation of the Carleman Embedding method could be used to approximate solutions to second order systems.

Again, let's consider the system (7)

$$x^{''} + x = 2\epsilon x^3, \quad x(0) = 1, x^{'}(0) = 0.$$

If we integrate the above system we have

$$(x^{'})^2 = -x^2 + \epsilon x^4 + c.$$

But for the initial condition we have

$$(x^{'})^2 = -x^2 + \epsilon x^4 + 1 - \epsilon.$$

Implying that,

$$(x^{'})^2 = -x^2 + \epsilon x^4 + \epsilon^{'},$$

where $\epsilon^{'} = 1 - \epsilon$. Now, we want to do this transformation. Let

$$u_n(t) = [x(t)]^n.$$

Then,

$$u_n^{'}(t) = nx^{n-1}x^{'}$$

and

$$\begin{aligned}
u_n^{''} &= n(n-1)x^{n-2}(x^{'})^2 + nx^{n-1}(x^{''}) \\
&= n(n-1)x^{n-2}(-x^2 + \epsilon x^4 + \epsilon^{'}) + nx^{n-1}(-x + 2\epsilon x^3) \\
&= -n(n-1)x^n + \epsilon n(n-1)x^{n+2} + \epsilon^{'} n(n-1)x^{n-2} - nx^n + 2n\epsilon x^{n+2}.
\end{aligned}$$

Therefore, the system (7) could be transformed into

$$u_n^{''}(t) = -n^2 u_n(t) + \epsilon n(n-1)u_{n+2}(t) + n(1+\epsilon)(n-1)u_{n-2}(t) \tag{12}$$

$$u_n(0) = 1, u_n^{'} = 0.$$

If $\epsilon = 0$, we have that

$$u_n^{''} = -n^2 u_n(t) + n(n-1)u_{n-2}(t). \tag{13}$$

But again we know that the solution to system (13) is

$$u_n(t) = [\cos(t)]^n.$$

Now, truncating at n=10, we can generate 10×10 matrix for (12)

$$
\begin{bmatrix} u_1'' \\ u_2'' \\ u_3'' \\ u_4'' \\ u_5'' \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_{10}'' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2(1)\epsilon & 0 & 0 & 0 & ... \\ 0 & -4 & 0 & 3(2)\epsilon & 0 & 0 & ... \\ 6(1-\epsilon) & 0 & -9 & 0 & 4(3)\epsilon & 0 & ... \\ 0 & 12(1-\epsilon) & 0 & -16 & 0 & 5(4)\epsilon & ... \\ 0 & 0 & 20(1-\epsilon) & 0 & -25 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & ... \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_{10} \end{bmatrix}
$$

and with initial conditions $u_j(0) = 1, u_j'(0) = 0, j = 1, ..., 10.$

For $\epsilon = 0.01$, we find the matrix

$$
M_1 = \begin{bmatrix} -1 & 0 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5.94 & 0 & -9 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11.88 & 0 & -16 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 19.8 & 0 & -25 & 0 & 0.3 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 89.1 & 0 & -1 \end{bmatrix}
$$

We now find the matrix P, whose rows and columns are the eigenvectors of matrix M_1

$$
P = \begin{bmatrix} -0.63 & -0.10 & -0.21 & 6.93 * 10^{-9} & 3.59 * 10^{-11} & 0 & ... & 0 \\ 0 & ... & 0 & -1.76 * 10^{-10} & 2.56 * 10^{-8} & -0.54 & -0.16 & 0.48 \\ -0.47 & 0.40 & 0.002 & -0.16 & -1.45 * 10^{-7} & 0 & ... & 0 \\ 0 & ... & 0 & 2.87 * 10^{-7} & -0.25 & 0.003 & 0.31 & 0.48 \\ -0.39 & 0.51 & -.33 & 0.005 & 0.0001 & 0 & ... & 0 \\ 0 & ... & 0 & -0.12 & 0.006 & -0.27 & 0.48 & 0.45 \\ -0.35 & 0.54 & -0.58 & -0.42 & -0.17 & 0 & ... & 0 \\ 0 & ... & 0 & 0.02 & -0.38 & -0.56 & 0.56 & 0.42 \\ -0.31 & 0.53 & -0.74 & -0.91 & 0.99 & 0 & ... & 0 \\ 0 & ... & 0 & -0.99 & -0.92 & -0.78 & 0.59 & 0.39 \end{bmatrix}
$$

Let D be the diagonal matrix with the eigenvalues of M_1 on the diagonal

$$D = \begin{bmatrix} -0.9849 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8.8649 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -24.63 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -48.32 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -82.21 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -101.72 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -63.12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -35.46 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We find the inverse matrix of P denoted P^{-1}.

The Jordan form for M_1 is

$$M_1 = P \cdot D \cdot P^{-1}.$$

Also, let Φ be a vector-valued function

$$\Phi = \begin{bmatrix} \Phi_1(t) \\ \Phi_2(t) \\ \Phi_3(t) \\ \vdots \\ \Phi_{10}(t) \end{bmatrix}.$$

We determine the initial conditions by

$$P \cdot < 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 > .$$

Thus, our diagonal system of differential equations are

$$\begin{aligned} \frac{d^2}{dt^2}\Phi_1(t) &= (-0.98499048239789924)\Phi_1(t) \\ \frac{d^2}{dt^2}\Phi_2(t) &= (-8.86491485276673430)\Phi_2(t) \\ &\vdots \\ \frac{d^2}{dt^2}\Phi_{10}(t) &= (-3.93996193010770712)\Phi_{10}(t). \end{aligned}$$

Now, we solve the diagonal system of differential equations with its initial conditions and we have

$$\begin{aligned}
\Phi_1(t) &= -0.6344988093\cos(0.9924668672t) \\
\Phi_2(t) &= 0.4800726627\cos(2.97740067t) \\
&\vdots \\
\Phi_{10}(t) &= -1.716430144\cos(1.984933734t).
\end{aligned}$$

But Φ is the inverse of P applied to the vector $\mathbf{u}$ of the Embedding. Therefore

$$\mathbf{u} = P^{-1} \cdot \Phi.$$

Again, we are interested in $u_1(t)$ so we have

$$u_1(t) = 0.99\cos(0.99t) + 0.000759\cos(4.97t) + 0.00000357\cos(9.06t) +$$
$$0.0000000963\cos(7.95t) + 0.000000000288\cos(3.97t).$$

Also, for $\epsilon = 0.1$ we have

$$u_1(t) = 0.99\cos(0.99t) + 0.000255\cos(4.96t) + 0.00000427\cos(9.07t) +$$
$$0.0000000105\cos(7.94t) + 0.000000000379\cos(3.97t).$$

The Maple implementation of this process is in the appendix.

Now we plot the solution of the system obtained by Carleman linearization. Figure 4 shows the result of Carleman linearization for $\epsilon = 0.01$.

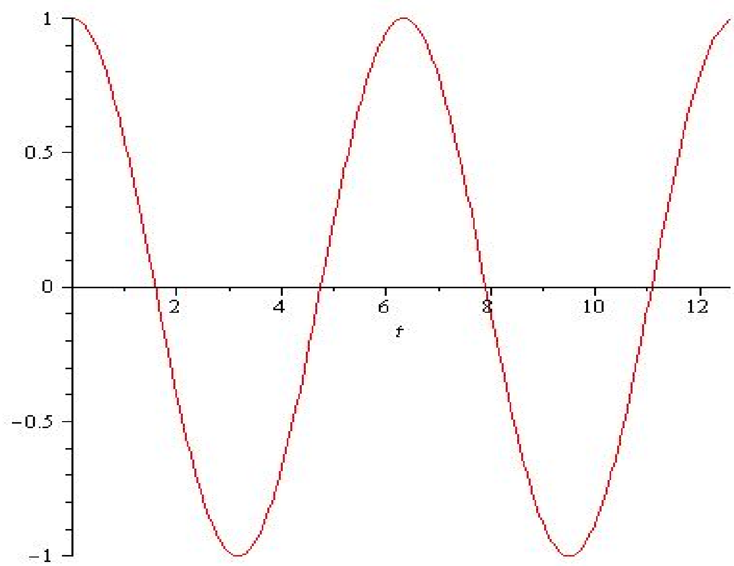

Figure 4: Solution by Carleman Embedding for $\epsilon = 0.01$

4.4 Perturbation theory combined with Carleman linearization

Now we will combine Perturbation theory with Carleman linearization. Let's consider (12) above

$$u_n'' = -n^2 u_n + \epsilon n(n+1)u_{n+2} + (1-\epsilon)n(n-1)u_{n-2}.$$

Let's suppose that

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \\ \vdots \end{bmatrix}.$$

Then we can say that

$$\mathbf{u} = \sum_{k=0}^{\infty} \mathbf{u}_k(t)\epsilon^k.$$

Now, we will see how $\mathbf{u}_0(t) + \epsilon \mathbf{u}_1(t)$, where

$$\mathbf{u}_0 = \begin{bmatrix} \cos(t) \\ \cos^2(t) \\ \vdots \\ \cos^n(t) \\ \vdots \end{bmatrix}$$

and

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ \vdots \end{bmatrix}$$

would turn.

If we write

$$u_n'' = -n^2 u_n + n(n+1)\epsilon u_{n+2} + (1-\epsilon)n(n-1)u_{n-2}$$

in matrix form then we get

$$\begin{bmatrix} u_1^{''} \\ u_2^{''} \\ u_3^{''} \\ u_4^{''} \\ u_5^{''} \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2(1)\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 3(2)\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 6(1-\epsilon) & 0 & -9 & 0 & 4(3)\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 12(1-\epsilon) & 0 & -16 & 0 & 5(4)\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 20(1-\epsilon) & 0 & -25 & 0 & 6(5)\epsilon & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \vdots \\ \vdots \end{bmatrix}$$

The above system can be rewritten as

$$\mathbf{u}_1^{''} = A\mathbf{u}_1 + \epsilon B\mathbf{u}_0.$$

From the system we have

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & .. \\ 0 & -4 & 0 & 0 & 0 & .. \\ 3(2) & 0 & -9 & 0 & 0 & .. \\ 0 & 4(3) & 0 & -16 & 0 & .. \\ 0 & 0 & 5(4) & 0 & -25 & .. \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & .. \\ 0 & 0 & 0 & 3(2) & 0 & .. \\ -3(2) & 0 & 0 & 0 & 4(3) & .. \\ 0 & -4(3) & 0 & 0 & 0 & \ddots \\ 0 & 0 & -5(4) & 0 & 0 & .. \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We now have our equations as

$$\begin{aligned}
u_1'' &= -u_1 + 2\epsilon \cos^3(t) \\
u_2'' &= -4u_2 + 6\epsilon \cos^4(t) \\
u_3'' &= 6u_1 - 9u_3 - 6\epsilon \cos(t) + 12\epsilon \cos^5(t)
\end{aligned}$$

$$\vdots$$

But our interest is the first equation. Therefore, taking $\epsilon = 0.01$ with initial conditions $u_1(0) = 1$ and $u_1'(0) = 0$, we use Maple to solve for u_1 and have

$$u_1 = \frac{401}{400}\cos(t) - \frac{1}{400}\cos^3(t) + \frac{3}{400}t\sin(t).$$

Now we plot the solution of Pertubation theory combined with Carleman linearization. Figure 5 shows the solution by this approach.

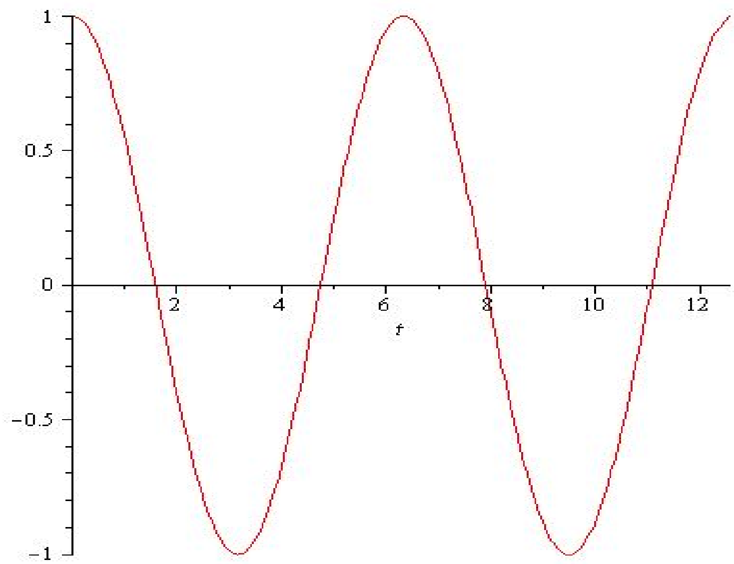

Figure 5: Solution by Perturbation theory combined with Carleman linearization for $\epsilon = 0.01$

5 CONCLUSION

Now, let's compare the solution by Perturbation theory applied to Carleman, the solution by variation of Carleman Embedding and the actual solution of the system. We observed that all the approaches produced nice approximations of the solution to the original system. The figures below show the comparison of the solutions by the different approaches for $\epsilon = 0.01$ and 0.1 respectively.

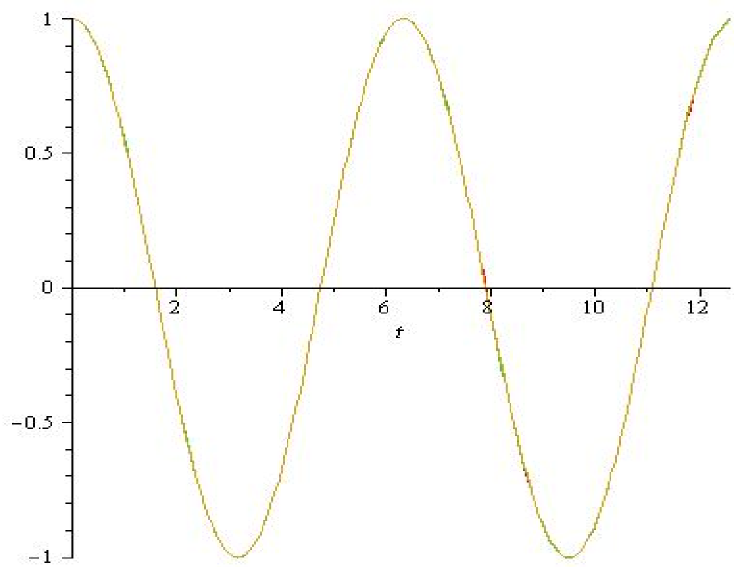

Figure 6: Comparison of solutions by Carleman Embedding, Numerical approach and Classical Perturbation for $\epsilon = 0.01$

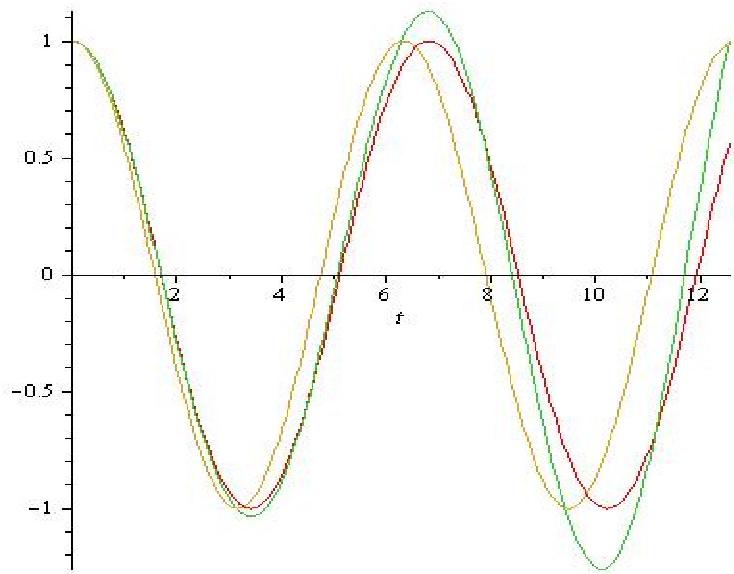

Figure 7: Comparison of solutions by Carleman Embedding (red), Numerical approach (yellow) and Classical Perturbation (green) for $\epsilon = 0.1$

In the discussion above, we see that the Classical Perturbation and the Perturbation theory applied to Carleman linearization produced similar result. We also realize that there is a secular term $t\sin(t)$ in the solutions by Classical Perturbation and Perturbation theory combined with Carleman linearization, where a secular term is a term whose 'lim sup' approaches ∞. That is,

$$\lim_{t\to\infty} \sup(t\sin(t)) = \infty$$

which makes the solutions unbounded. However, Liẽnard's theorem shows that all solutions to Duffing's equation are bounded [18].

In this thesis, we have used a modified Carleman Embedding to produce bounded approximation to the solution to a Duffing equation. In contrast, Classical Perturbation applied to both the original equation and the Embedding produced unbounded approximations. Thus, in this case, the modified Carleman Embedding proved superior to the standard approaches.

An immediate next step would be to determine if the modified Carleman Embedding produces bounded approximations to any second-order ODE that by Liẽnard's theorem has bounded solutions. However, an equally important future direction is that of applying the modified Carleman Embedding to second-order equations in Mathematical Finance. For example, Probit model is important in Statistics, Mathematical Finance and Economics, and the modified Carleman Embedding extends to Probit models in a natural way. Indeed, bounded approximations are very important in Finance and Statistics, thus making this a desirable approach.

In conclusion, we have extended the ordinary Carleman Embedding to second-order systems in a way that produces bounded solutions. This makes Carleman Embedding an interesting approach and one that merits further study.

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APPENDIX: Maple Code

```
with(LinearAlgebra):

epsilon := 0.01:

M1:=Matrix([[-1.0,0,0.02,0,0,0,0,0,0,0],
   -4.0,0,0.06,0,0,0,0,0,0

, [3*2*(1-epsilon),0,-9.0,0,0.12,0,0,0,0,0], [0,4*3*(1-epsilon),0,-16.0,0,0.2,0,0,0,0], [0,0,5*4*(1-
epsilon),0,-25.0,0,0.3,0,0,0], [0,0,0,6*5*(1-epsilon),0,-36.0,0,0.42,0,0], [0,0,0,0,7*6*(1-epsilon),0,-
49.0,0,0.56,0], [0,0,0,0,0,8*7*(1-epsilon),0,-64.0,0,0.72], [0,0,0,0,0,0,9*8*(1-epsilon),0,-81.0,0],
[0,0,0,0,0,0,0,10*9*(1-epsilon),0,-100.0]]):

LinearAlgebra:-Eigenvectors( M1 ):

P:=M1[2]:

DD:=DiagonalMatrix( M1[1] ):

Re(DD):

P. DD. MatrixInverse(P) :

simplify(P. DD. MatrixInverse(P)):

Re(simplify(P. DD. MatrixInverse(P))):

P.[1,1,1,1,1,1,1,1,1,1]:

inits:=Re(P.[1,1,1,1,1,1,1,1,1,1]:):

ODE:= seq( diff(phi[i](t),t,t) = DD(i,i)*phi[i](t), i=1..10):

inits := seq( [phi[i](0) = inits[i], D(phi[i])(0)=0][],i=1..10) :

dsolve(ODE union inits, seq(phi[i](t),i=1..10)):

Sols:=evalf(dsolve(ODE union inits, seq(phi[i](t),i=1..10))):
```

MatrixInverse(P).[seq(rhs(Sols[i]), i=1..10)]:

EmbeddingApprox:=Re(simplify(MatrixInverse(P).[seq(rhs(Sols[i]), i=1..10)[1])) assuming t is greater than 0:

dsolve(diff(x(t),t,t) = -x(t) + 2 * 0.01 * x(t)$^3, x(0) = 1, D(x)(0) = 0)$:

$ActualSol := rhs(dsolve(diff(x(t), t, t) = -x(t) + 2 * 0.01 * x(t)^3, x(0) = 1, D(x)(0) = 0)$:

$ClassicalPerturb := 401/400 * cos(t) - cos(t)^3/400 + 3 * t * sin(t)/400$:

$plot(ClassicalPerturb, EmbeddingApprox, ActualSol, t = 0..4 * Pi)$:

$Perturbation := (cos(t) + 0.01 * 1/4 * cos(t) - 0.01 * 1/4 * (cos(t))^3 + 0.1 * 3/4 * t * sin(t))$:

$plot(1.025000000 * cos(t) - 0.2500000000e - 1 * cos(t)^3 + 0.7500000000e - 2 * t * sin(t), t = 0..4 * Pi)$:

$plot(ActualSol, t = 0..4 * Pi)$:

$plot(ClassicalPerturb, t = 0..4 * Pi)$:

$plot(EmbeddingApprox, t = 0..4 * Pi)$:

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