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Global Domination Stable Graphs

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ABSTRACT

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A set of vertices S in a graph G is a global dominating set (GDS) of G if S is a dominating set for both G and its complement $\overline{G}$. The minimum cardinality of a global dominating set of G is the global domination number of G. We explore the effects of graph modifications on the global domination number. In particular, we explore edge removal, edge addition, and vertex removal.

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1 BACKGROUND

The concept of global domination was introduced in 1989 by Sampathkumar [8] and since then, many of its properties have been explored. As with mathematics in general, and graph theory in particular, the potential for expanding our knowledge is unlimited. We investigate global domination in graphs.

1.1 Basic Graph Theory Terminology

Before we discuss advanced topics in domination we give some basic graph theory definitions. As defined in [4], a *graph* G is a finite nonempty set of objects called *vertices* (the singular is *vertex*) together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$, while the *edge set* is denoted by $E(G)$. The edge $e = \{u, v\}$ is said to *join* the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent vertices*, while u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*. It is sometimes convenient to denote an edge by uv or vu rather than by $\{u, v\}$. The cardinality of, or number of elements contained in, the vertex set of a graph G is called the *order* of G and is commonly denoted by $n(G)$, or more simply by n when the graph under consideration is clear. The cardinality of its edge set is the *size* of G and is often denoted by $m(G)$ or m. It is customary to define or describe a graph G by means of a diagram in which each vertex of G is represented by a point (which we draw as a small circle) and each edge $e = uv$ of G is represented by a line segment or curve joining the points corresponding to u and v. The *degree of a vertex* v in a graph

G is the number of edges of G incident with v, which is denoted by $deg(v)$. A vertex of degree 0 in G is called an *isolated vertex*, while a vertex of degree of 1 is referred to as a *leaf* or *pendant*. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. The *maximum degree* is defined similarly and is denoted by $\Delta(G)$. If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the *induced* subgraph $G[U]$ is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. A graph of n vertices is *complete* if every two of its vertices are adjacent. This is denoted K_n. A graph G is k-*partite*, $k \geq 1$, if it is possible to partition $V(G)$ into k subsets $V_1, V_2, \cdots, V_k$ (called *partite* sets) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. For $k = 2$, such graphs are called *bipartite graphs*. A *complete k-partite graph* G is a k-partite graph with partite sets $V_1, V_2, \cdots, V_k$ having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. A *complete bipartite graph* with partite sets V_1 and $V2$, where $|V_1| = r$ and $|V_2| = s$, is then denoted by $K_(r, s)$. The graph $K_{1,s}$ is called a *star*, often denoted S_k where the graph has one internal node and k leaves. A graph is a *complete multipartite graph* if it is a complete k-partite graph for some $k \geq 2$.

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. This means that both G and its complement $\overline{G}$ have the same vertices, but G has precisely the edges that $\overline{G}$ lacks. A graph G_1 is *isomorphic* to a graph G_2 if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency; that is $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. A graph G

is *self-complementary* if $G \cong \overline{G}$. A *u-v walk* W of G is a finite, alternating sequence $W : u = u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1}u_i$ for $i = 1, 2, \ldots, k$. The number k (the number of occurrences of edges) is called the *length* of W.

A *u-v* walk is *closed* or *open* depending on whether $u = v$ or $u \neq v$. A *u-v trail* is a *u-v* walk in which no edge is repeated, while a *u-v path* is a *u-v* walk in which no vertex is repeated. A nontrivial closed trail of a graph G is referred to as a *circuit* of G, and a circuit $v_1, v_2, \ldots, v_n$, v_1 $(n \geq 3)$ whose n vertices v_i are distinct is called a *cycle*. Paths on n vertices are denoted P_n and cycles on n vertices are denoted C_n. An *acyclic graph* has no cycles. A *tree* is an acyclic connected graph. A graph is called *triangle-free* if it contains no triangles. A *wheel graph*, W_n, on n vertices is a graph consisting of a cycle C_n and a single vertex which is adjacent to all vertices in the cycle. A vertex u is said to be *connected* to a vertex v in a graph G if there exists a *u-v* path in G. A graph G is *connected* if every two vertices are connected. A graph that is not connected is *disconnected*. The relation 'is connected to' is an equivalence relation on the vertex set of every graph G. Each subgraph induced by the vertices in a resulting equivalence class is called a *component* of G. For a connected graph G, we define the *distance* $d(u, v)$ between two vertices u and v as the minimum of the lengths of the *u-v* paths of G. The *eccentricity* $e(v)$ of a vertex v of a connected graph G is the distance between v and a vertex farthest from v. The *diameter* of G, denoted $diam(G)$ is the maximum eccentricity. The *open neighborhood* $N(v)$ of a vertex v consists of the set of vertices adjacent to v, that is, $N(v) = \{u \in V | uv \in E\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup v$. For a set $S \subseteq V$, the *open* *neighborhood* $N(S)$ is defined to be $\cup_{v \in S} N(v)$, and the *closed neighborhood* of S is $N[S] = N(S) \cup S$. For a set S of vertices and $u \in S$, we say that a vertex v is a *private neighbor* of u (with respect to S) if $N[v] \cap S = \{u\}$. Furthermore, we define the *private neighbor set of* u, called the *private neighborhood of* u, with respect to S, to be $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. The *external private neighborhood of* u with respect to S is $epn[u, S] = \{v \in V \setminus S \,:\, N(v) \cap S = \{u\}\}$.

1.2 Global Domination

Our main results are on global domination in graphs. According to [5] a subset $S \subseteq V$ is a *dominating set* of G if every vertex of $V \setminus S$ is adjacent to at least one vertex of S. The cardinality of a smallest dominating set of G, denoted $\gamma(G)$, is the *domination number* of G. A dominating set of G having cardinality $\gamma(G)$ is called a $\gamma(G)$-*set*.

Sampathkumar [8] introduced the idea of a *global dominating set* (GDS, for short) in which a subset $S \subseteq V$ is a dominating set of both G and its complement, $\overline{G}$. The *global domination number*, $\gamma_g(G)$, of G (and of $\overline{G}$) is the minimum cardinality of a global dominating set of G, and a global dominating set of this size is a $\gamma_g(G)$-*set*. The darkened vertices of Figure 1 illustrate a $\gamma_g(C_5)$-set.

1.3 Prisms

We determine global domination number of prisms and complementary prisms. First, the *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. For disjoint graphs G and H, the *join* $K = G + H$ has $V(K) = V(G) \cup V(H)$ and

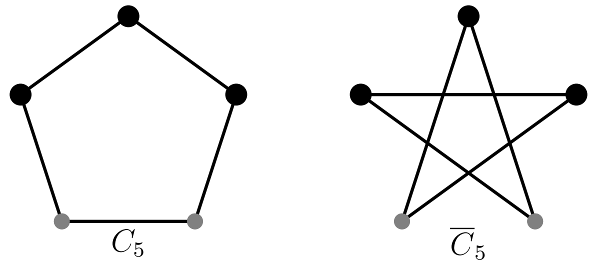

Figure 1: $\gamma_g(C_5) = 3$

$E(K) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } u \in V(H)\}$. A set of pairwise independent edges of G is called a *matching*. If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a *perfect matching* in G. A *prism* GG of G is formed from the disjoint union of G and G by adding the edges of a perfect matching between the corresponding vertices of the two copies of G. Figure 2 illustrates an example of a prism where $G = C_5$.

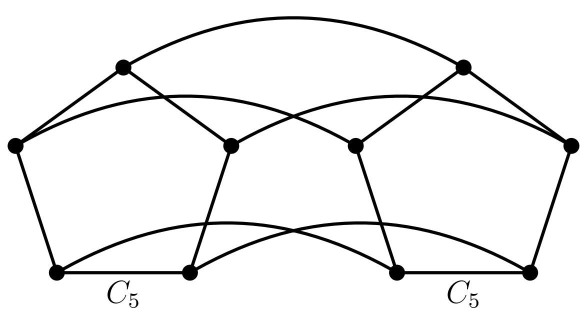

Figure 2: An example of the prism C_5C_5

The *complementary prism* $G\overline{G}$ of G is formed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and its complement $\overline{G}$. Figure 3 illustrates an example of a complementary prism where $G = C_5$.

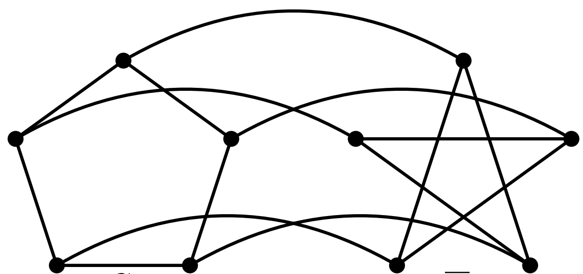

Figure 3: An example of the complementary prism $C_5\overline{C}_5$

1.4 Criticality

In [5], Brigham and Dutton explored the effects of graph modifications on the global domination number. They observed that when an edge e is removed from G, it is added to $\overline{G}$. When an edge is removed from G, the global domination number can decrease, stay the same, or increase. They define the edge e to be *minus global domination edge critical*, or simply *minus critical*, if $\gamma_g(G - e) < \gamma_g(G)$. Similarly, e is *plus global domination edge critical*, or simply *plus critical*, if $\gamma_g(G - e) > \gamma_g(G)$. If the same inequality holds for all edges of G, then G is called either *minus critical* or *plus critical* as appropriate. They also define certain classes of graphs as follows:

1. $E_{\gamma g} = \{H : \gamma_g(H - e) = \gamma_g(H) - 1 \text{ for all } e \in E(H)\}$ and
2. $V_{\gamma g} = \{H : \gamma_g(H - v) = \gamma_g(H) - 1 \text{ for all } v \in V(H)\}$.

We call a graph where the removal of any edge e does not effect the global domination number a *global domination edge minus stable* graph, or simply *EMS*. We call a graph where the addition of any edge e does not effect the global domination number a *global domination edge plus stable* graph, or simply *EPS*. We call a graph G in the family $V_{\gamma g}$ a *global domination vertex critical* graph, or simply *VC*. A graph for which the removal of any vertex v does not decrease the global domination number, we call *global domination vertex stable*, or simply *VS*. If G is a global domination edge critical graph and $\gamma_g(G) = k$, then we say G is a k_g*-edge critical graph.* Similarly, we call global domination edge minus stable graphs, global domination edge plus stable graphs, global domination vertex stable graphs, and global domination vertex critical graphs each with $\gamma_g(G) = k$, k_g*-edge minus stable graphs*, k_g*-edge plus stable graphs*, k_g*-vertex stable graphs*, and k_g*-vertex critical graphs*, respectively.

2 LITERATURE SURVEY

In this section, we survey pertinent known results about global domination, diameter, prisms, complementary prisms, and graph modification. We use the notation and terminology of [6] unless stated otherwise.

2.1 Bounds on Global Domination

Sampathkumar [8] introduced the concept of global domination in graphs in 1989. Since then, there have been many bounds defined for the global domination number of families of graphs. Brigham and Carrington [2] state the following values of γ_g for specific families:

Theorem 2.1 *[2]*

i) For the complete graph K_n, $\gamma_g(K_n) = n$.

ii) For the path P_n,

$$\gamma_g(P_n) = \begin{cases} 2 & \textit{if } n = 2, 3 \\ \lceil n/3 \rceil & \textit{if } n \geq 4. \end{cases}$$

iii) For the cycle C_n,

$$\gamma_g(C_n) = \begin{cases} 3 & \textit{if } n = 3, 5 \\ \lceil n/3 \rceil & \textit{otherwise.} \end{cases}$$

iv) For the wheel W_n,

$$\gamma_g(W_n) = \begin{cases} 4 & \text{if } n = 4 \\ \\ 3 & \text{otherwise.} \end{cases}$$

v) For the complete multipartite graph, $\gamma_g(K_{n_1,n_2,\ldots,n_r}) = r$.

Next we present bounds on the global domination number.

Theorem 2.2 *[5] For graph G,*

1. $max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma_g(G) = \gamma_g(\overline{G}) \leq \gamma(G) + \gamma(\overline{G})$*, and*

2. if G or $\overline{G}$ *is disconnected, then* $\gamma_g(G) = \gamma_g(\overline{G}) = max\{\gamma(G), \gamma(\overline{G})\}$

Theorem 2.3 *[6]For any graph G with a pendant vertex,* $\gamma_g(G) \leq \gamma(G) + 1$.

Theorem 2.4 *[5]If G is a triangle-free graph, then* $\gamma(G) \leq \gamma_g(G) \leq \gamma(G) + 1$.

Theorem 2.5 *[1]Let G be a connected bipartite graph with bipartition X,Y and* $|X| \leq |Y|$*. Then,* $\gamma_g(G) = \gamma(G) + 1$ *if and only if either G is isomorphic to* K_2 *or every vertex in X is adjacent to at least two pendant vertices and there exists a vertex in Y which is adjacent to all vertices in X.*

Theorem 2.6 *[7]Let T be a tree. Then,* $\gamma_g(T) = \gamma(T) + 1$ *if and only if T is a star or T is a tree of diameter 4 which is constructed from two or more stars, each having at least two pendant vertices, by connecting the centers of these stars to a common vertex.*

A vertex and an edge are said to *cover* each other in G if they are incident in G. A *vertex cover* of G is a set of vertices that covers all the edges. The minimum cardinality of a vertex cover is $\alpha_0(G)$.

Theorem 2.7 *[1]Let G be a connected bipartite graph with bipartition (X, Y) and $|X| \leq |Y|$. Then, $\gamma_g(G) = \alpha_0(G) + 1$ if and only if either $G = K_2$ or every vertex in X is adjacent to at least two pendant vertices and there exists a vertex in Y which is adjacent to all vertices in X.*

Theorem 2.8 *[7] Let G be a graph having diameter at least five, and let A be a subset of vertices in G. A is a minimal dominating set of G if and only if A is a minimal global dominating set of G.*

2.2 Known Results on Criticality

In [5], Brigham and Dutton study the effects of graph modifications on global domination. It is important to note that the removal of any edge of a graph G can increase or decrease the global domination number by only 1. We list results from [5] below and will refer to them as we need them in later chapters.

Theorem 2.9 *[5] For any graph $G = (V, E)$ and any edge $e \in E$, $\gamma_g(G) - 1 \leq \gamma_g(G - e) \leq \gamma_g(G) + 1$.*

Theorem 2.10 *[5] Let $G = (V, E)$ be a graph such that $\gamma_g(G - e) = \gamma_g(G) - 1$ for some edge $e \in E$. Then, $\gamma_g(G - e) \leq \gamma_g(G)$ for every edge $e \in E$.*

Theorem 2.11 *[5] If $\gamma_g(G - e) = \gamma_g(G) + 1$ for some edge $e \in E$, then $\gamma_g(G - e) \geq \gamma_g(G)$ for every edge $e \in E$.*

Theorem 2.12 *[5] If graph $G = (V, E)$ is not connected, then $\gamma_g(G - e) \geq \gamma_g(G)$ for any edge $e \in E$.*

Theorem 2.13 *[5] Let $G = (V, E)$ be a graph for which $\overline{G}$ is disconnected. Then, $G \in E_{\gamma g}$ if and only if*

1. $\overline{G} \in E_{\gamma g}$, and

2. when $\overline{G}$ has two componenets, one of which is an isolated vertex x, then $\gamma_g(\overline{G} + xy) < \gamma_g(\overline{G})$ for every edge $xy \in E$.

Brigham and Dutton [5] characterized when edge removal increases the global domination number.

Theorem 2.14 *[5] Let $G = (V, E)$ be a graph. Then, $\gamma_g(G - e) = \gamma_g(G) + 1$ for every edge $e \in E$ if and only if G is a collection of $m \geq 2$ stars.*

To help distinguish the copies of G within the prism GG, we define them as G_1 and G_2. Also, let vertices of G_1 be denoted $\{u_i | i = 1, 2, ..., n(GG)/2\}$ and vertices of G_2 be denoted $\{v_j | j = 1, 2, ..., n(GG)/2\}$, where $i = j$ represents corresponding vertices within G_1 and G_2. Figure 2 is an example of a prism GG and Figure 4 is its complement. The graph in Figure 5 is isomorphic to the graph in Figure 4, but it is drawn with dashed lines to indicate the missing edge between the copies of G.

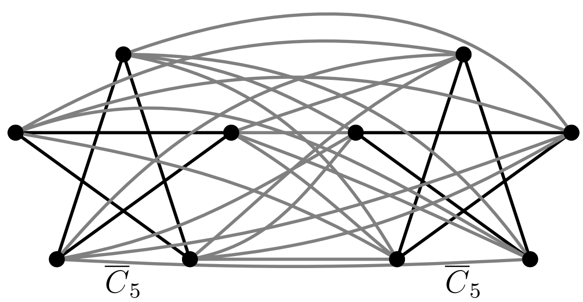

Figure 4: The complement of the prism C_5C_5

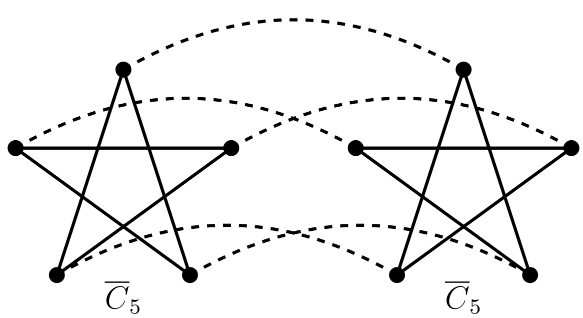

Figure 5: The complement of the prism C_5C_5

Note that the complement of a prism GG is the join $\overline{G} + \overline{G}$ minus a perfect matching between the corresponding vertices of $\overline{G}$ and $\overline{G}$. Since for a nontrivial graph G, $\gamma(GG) \geq 2$ and $\gamma(\overline{GG}) = 2$, it follows $max\{\gamma(GG), \gamma(\overline{GG})\} = \gamma(GG)$.

Theorem 3.1 *Let GG be a prism. Then, $\gamma(GG) \leq \gamma_g(GG) \leq \gamma(GG) + 1$.*

Proof. Since $\gamma(GG) \geq \gamma(\overline{GG})$, the lower bound follows directly from Theorem 2.2. To establish the upper bound, let S be a $\gamma(GG)$-set. If $|S \cap V(G_1)| \geq 2$ and $|S \cap V(G_2)\ | \geq 2$ in GG, then S is a GDS of GG. Moreover, if $|S| \geq 2$ and $S \subseteq V(G_1)$ or $S \subseteq V(G_2)$ in GG, then $|S| = |V(G_1)|$ and S is a GDS of GG. Hence, in both cases $\gamma_g(GG) \leq |S| = \gamma(GG)$ and so $\gamma_g(GG) = \gamma(GG)$. Thus, we may assume that at least one of G_1 and G_2 have exactly one vertex in S and the other has at least one. Relabeling G_1 and G_2 if necessary, assume that $S \cap V(G_1) = \{u_i\}$. If $|S \cap V(G_2)| \geq 2$, then $S \cup \{v_i\}$ is a GDS of GG implying that $\gamma_g(GG) \leq |S \cup \{v_i\}| \leq |S| + 1 = \gamma(GG) + 1$, as desired. Hence, we may assume that $|S \cap V(G_2)| = 1$. Let $S \cap V(G_2) = \{v_j\}$. If $i = j$, then u_i dominates G_1 and v_j dominates G_2 in GG, and $\{u_i, v_i\}$ dominates $\overline{GG}$. Therefore, $\gamma_g(GG) \leq |S| = \gamma(GG)$. Assume $i \neq j$. In the complement, $\overline{GG}$, u_i and v_j dominate all vertices with the possible exception of v_i and u_j. Then, $S \cup \{v_i\}$ or $S \cup \{u_j\}$ is a GDS of GG. In either case, $\gamma_g(GG) \leq |S| + 1 = \gamma(GG) + 1$. ■

To distinguish the copies of G and $\overline{G}$ within the complementary prism $G\overline{G}$, we call them G and $\overline{G}$ respectively. Also, let vertices of G be denoted $\{u_i | i = 1, 2, ..., n(G\overline{G})/2\}$ and vertices of $\overline{G}$ be denoted $\{v_j | j = 1, 2, ..., n(G\overline{G})/2\}$, where $i = j$ represents corresponding vertices within G and $\overline{G}$. Figure 3 is an example of a complementary prism $G\overline{G}$ and Figure 6 is its complement. Note that the complement of a complementary prism $G\overline{G}$ is the join $G + \overline{G}$ minus a perfect matching between the corresponding vertices of G and $\overline{G}$. It follows if G is nontrivial, then

$max\{\gamma(G\overline{G}), \gamma(\overline{G\overline{G}})\} = \gamma(G\overline{G})$.

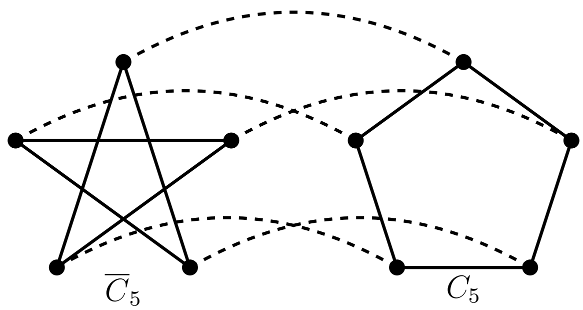

Figure 6: The complement of the complementary prism $C_5\overline{C}_5$

Theorem 3.2 *Let G be a nontrivial graph, and let $G\overline{G}$ be a complementary prism. Then, $\gamma(G\overline{G}) \leq \gamma_g(G\overline{G}) \leq \gamma(G\overline{G}) + 1$.*

Proof. Since $\gamma(G\overline{G}) \geq \gamma(\overline{G\overline{G}})$, the lower bound follows directly from Theorem 2.2. To establish the upper bound, let S be a $\gamma(G\overline{G})$-set. If $|S \cap V(G)| \geq 2$ and $|S \cap V(\overline{G})| \geq 2$ in $G\overline{G}$, then S is a DS of $\overline{G\overline{G}}$. Hence S is a GDS of $G\overline{G}$. Moreover, if $|S| \geq 2$ and $S \subseteq V(G)$ or $S \subseteq V(\overline{G})$ in $G\overline{G}$, then $|S| = |V(G)|$ and S is a DS of $\overline{G\overline{G}}$. Hence, S is a GDS of $G\overline{G}$. Thus, in both cases $\gamma_g(G\overline{G}) \leq |S| = \gamma(G\overline{G})$ and so $\gamma_g(G\overline{G}) = \gamma(G\overline{G})$. Thus, we may assume that at least one of G and $\overline{G}$ have exactly one vertex in S, and that the other has at least one. Relabeling G and $\overline{G}$ if necessary, assume that $S \cap V(G) = \{u_i\}$. If $|S \cap V(\overline{G})| \geq 2$, then $S \cup \{v_i\}$ is a GDS of $G\overline{G}$. This implies that $\gamma_g(G\overline{G}) \leq |S \cup \{v_i\}| \leq |S| + 1 = \gamma(G\overline{G}) + 1$, as desired. Hence, we may assume that $|S \cap V(\overline{G})| = 1$. Let $S \cap V(\overline{G}) = \{v_j\}$. If $i = j$, then u_i dominates G and v_i dominates $\overline{G}$, a contradiction. Thus, $i \neq j$. Then, S dominates the complement of

$G\overline{G}$ with the possible exception of u_j and v_i in $\overline{G\overline{G}}$. Then, $S \cup \{u_j\}$ or $S \cup \{v_i\}$ is a GDS of $G\overline{G}$. Thus $\gamma_g(G\overline{G}) \leq |S| + 1 = \gamma(G\overline{G}) + 1$. ■

4 BASIC RESULTS

Using some basic results about diameter, we investigate its effect on the global domination number. Because we look at both a graph and its complement, we need the following results.

Theorem 4.1 *[9] If* $diam(G) \geq 4$*, then* $diam(\overline{G}) \leq 2$*.*

Theorem 4.2 *[6] If* $\gamma(\overline{G}) \geq 3$*, then* $diam(G) \leq 2$*.*

Theorem 4.3 *[6] If a graph* G *has no isolated vertices and* $diam(G) \geq 3$*, then* $\gamma(\overline{G}) = 2$*.*

Using these three theorems, an obvious proposition follows.

Proposition 4.4 *Let* G *and* $\overline{G}$ *be connected, nontrivial graphs. If* $diam(G) \geq 3$*, then* $\gamma_g(G) \leq \gamma(G) + 2$*.*

Proof. Let G and $\overline{G}$ be connected and $diam(G) \geq 3$. It follows directly from Theorem 2.2 and Theorem 4.3 that $\gamma_g(G) \leq \gamma(G) + \gamma(\overline{G}) = \gamma(G) + 2$. ■

Theorem 4.5 *Let* G *be a graph such that* $\gamma(G) \geq \gamma(\overline{G})$*. For any graph* G*,* $\gamma_g(G) = \gamma(G)$ *if and only if there exists some* $\gamma(G)$*-set not contained in the open neighborhood of any vertex of* G*.*

Proof. Assume $\gamma_g(G) = \gamma(G)$. Let S be any $\gamma_g(G)$-set. Clearly S is a $\gamma(G)$-set. If there exists some vertex $v \in V(G)$ such that $S \subseteq N_G(v)$, then in $\overline{G}$, v is not adjacent to any vertex of S, a contradiction. Thus, S is not contained in the open neighborhood of any vertex of $V(G)$. This proves the necessary condition. For the sufficiency, assume that there exists some $\gamma(G)$-set S not contained in the open neighborhood of any vertex of G. It is easy to see that S is a dominating set for $\overline{G}$. Hence $\gamma_g(G) \leq \gamma(G)$. By Theorem 2.2 the result follows. ■

Corollary 4.6 *For any graph G, if $\gamma(G) \geq \Delta(G)$, then $\gamma_g(G) = \gamma(G)$.*

Corollary 4.7 *If G is a graph of order n and $\gamma(G) \leq \Delta(G)$, then $n \leq \Delta(G)(\Delta(G) + 1)$.*

Proof. Let S be a $\gamma(G)$-set. Then, each vertex dominates at most itself and $\Delta(G)$ vertices in $N(v)$. Since $|S| = \gamma(G) \leq \Delta(G)$, the result follows. ■

Corollary 4.8 *Let G be a graph of order n. If $\gamma_g(G) > \gamma(G)$, then $n \leq \Delta(G)(\Delta(G) + 1)$.*

Proof. If $\gamma_g(G) > \gamma(G)$, then Corollary 4.7 implies $\gamma(G) \leq \Delta(G)$. Thus, Corollary 4.8 implies our result. ■

Theorem 4.9 *If $diam(G) \geq 3$ and $diam(\overline{G}) \geq 3$, then $2 \leq \gamma_g(G) \leq 4$.*

Proof. Since $diam(G) \geq 3$ and $diam(\overline{G}) \geq 3$, no vertex dominates both G and $\overline{G}$, so $\gamma_g(G) \geq 2$ and we have the lower bound. Theorem 4.3 implies that $\gamma(G) = 2$ and $\gamma(\overline{G}) = 2$. Then, by Theorem 2.2, $\gamma_g(G) \leq \gamma(G) + \gamma(\overline{G}) = 4$. ■

5 GLOBAL DOMINATION STABLE GRAPHS

In this section we explore the effects of edge deletion, vertex deletion, and edge addition on the global domination number of a graph. We focus on graphs having global domination number 2 or 3. For simplicity purposes, we let $S = \{a, b\}$ or $S = \{a, b, c\}$ be a $\gamma_g(G)$-set for the graph G under consideration. We let A be the external private neighborhood of a with respect to S, that is $A = epn_G[a, S]$. We define B and C similarly, and so $B = epn_G(b, S)$ and $C = epn_G(c, S)$, respectively. Also, let $AB = N_G(a) \cap N_G(b) \cap (V \setminus S)$, $AC = N_G(a) \cap N_G(c) \cap (V \setminus S)$, $BC = N_G(b) \cap N_G(c) \cap (V \setminus S)$, and $ABC = N_G(a) \cap N_G(b) \cap N_G(c) \cap (V \setminus S)$. Furthermore, let $\overline{A} = epn_{\overline{G}}(\overline{a}, S)$. When $\gamma_g(G) = 3$, note that $\overline{A} = BC$. Define $\overline{B} = AC$ and $\overline{C} = AB$ similarly. Define $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, as expected.

Lemma 5.1 *For any graph G and $\gamma_g(G)$-set S, there is no vertex adjacent to all the vertices of S.*

Proof. This follows from Theorem 4.5. ■

By Lemma 5.1 we see that $ABC = \emptyset$.

Lemma 5.2 *Let T be a tree with global dominating set $S = \{a, b\}$. Then, $|AB| \leq 1$.*

Proof. Assume to the contrary, that $|AB| \geq 2$. A cycle C_4 is formed with the vertices a, b, and any 2 vertices in AB, which is a contradiction to T being a tree. ■

Lemma 5.3 *Let T be a tree with global dominating set $S = \{a, b, c\}$. Then, $|AB| \leq 1$, $|AC| \leq 1$, and $|BC| \leq 1$.*

Proof. Assume to the contrary, that any of $|AB|$, $|AC|$, or $|BC|$ is greater than 1. Without loss of generality, say $|AB| \geq 2$. A cycle C_4 is formed with the vertices a, b, and any 2 vertices in AB, which is a contradiction to T being a tree. ■

For the remainder of this thesis, we characterize the stable trees T having $\gamma_g(T) = 2$ or $\gamma_g(T) = 3$. First, we consider edge removal.

5.1 Edge Removal

In this section, we consider graphs whose global domination number does not change when any arbitrary edge is removed. We call such graphs k_g*-edge minus stable*. We focus on when $k = 2$ and when $k = 3$ and call those graphs 2_g*-edge minus stable* or 3_g*-edge minus stable*, 2_g-EMS or 3_g-EMS respectively, for short. We first consider 2_g-EMS trees. In this thesis, all of the theorems for $\gamma_g(T) = 2$ trees are proved using cases. Case 1 explores the possibilities for when S is an independent set. We look at what happens to the global domination number of the tree when a has no private neighbors, one private neighbor, two private neighbors, or many private neighbors. Similarly for b, we consider the effects of it's private neighbors on the global domination number. We also consider any combination of the private neighbors of a and the private neighbors of b. Case 2 explores the possibilities for when S is not an independent set. We, again, look at the effects of the private neighborhoods of a and b on the global domination number.

Theorem 5.4 *Let T be a tree. The tree T is a 2_g-EMS graph if and only if T is one of the paths P_2, P_3, P_4 or the star S_n.*

Proof. See Figures 7, 8, 9, and 10 to see that the paths P_2, P_3, P_4, and the star S_n are 2_g-EMS graphs. The darkened vertices are the global dominating set while the dashed lines are the removed edges. To prove the necessary condition, assume that T is a 2_g-EMS tree. Let $S = \{a, b\}$ be a $\gamma_g(T)$-set. Define the sets A, B, and AB as before.

Figure 7: $P_2 - e$

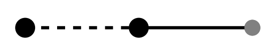

Figure 8: $P_3 - e$

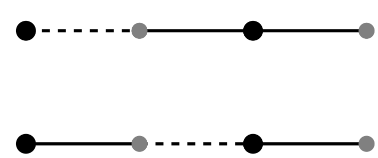

Figure 9: $P_4 - e$

By Lemma 5.1, $|AB| \leq 1$. Note also that since T is a tree, each of A and B is an independent set.

We consider the cases.

Case 1 S is an independent set.

See Figure 11. If $|AB| \neq \emptyset$, then the vertex in AB dominates S, contradicting Lemma 5.1. Hence $AB = \emptyset$. Since T is connected and $AB = \emptyset$, there must be an edge between a vertex in A and a vertex in B. Without loss of generality, say that $a' \in A$ is adjacent to $b' \in B$. Also note that since T is a tree, the only edge with both

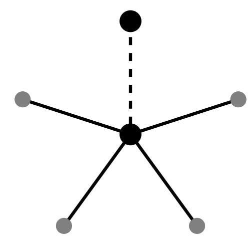

Figure 10: $S_n - e$

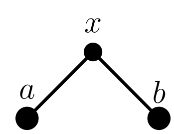

Figure 11: γ_g=2

its endpoints in $A \cup B$ is the edge $a'b'$. Thus, every vertex in $(A \cup B) \setminus \{a', b'\}$ is a leaf in T. See Figure 12.

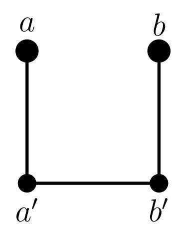

Figure 12: γ_g=2

We show that $|A| = |B| = 1$. By symmetry it suffices to show that $|A| = 1$. Note that $A \neq \emptyset$ because $a' \in A$. Assume to the contrary that $|A| \geq 2$, and delete a pendant edge e incident to a, say aa''. Since T is 2_g-EMS and a'' is in every $\gamma_g(T-e)$-set, only one vertex dominates a, a', b, b', a contradiction. Therefore, $|A| = |B| = 1$ and $T = P_4$ as desired.

Case 2 S is not independent.

We note that since T is a tree, $T[S]$ has exactly one edge and $AB = \emptyset$. Figure 13 illustrates Case 2.

Figure 13: γ_g=2 Case 2

If $|A| \geq 2$ and $B = \emptyset$ (respectively, $A = \emptyset$ and $|B| \geq 2$), then $T = S_n$ as desired. Assume,without loss of generality, that $|A| \geq 2$ and $|B| \neq \emptyset$. Then, delete a pendant edge e incident to a, say aa'. Since T is 2_g-EMS and a' is in every $\gamma_g(T-e)$-set, only one vertex dominates $a, A \setminus \{a'\}, b$, and B, a contradiction. Hence, we may assume that $0 \leq |A| \leq 1$ and $0 \leq |B| \leq 1$. If $A = B = \emptyset$, then $T = P_2$ as desired. If $|A| = 1$ and $B = \emptyset$ (respectively, $A = \emptyset$ and $|B| = 1$), then $T = P_3$ as desired. If $|A| = |B| = 1$, then $T = P_4$ as desired. ■

For the remainder of this section we focus on 3_g-EMS trees. Similarly to how we proved theorems for 2_g-EMS trees, all of the theorems for $\gamma_g(T) = 3$ trees are proved using cases. Case 1 explores the possibilities for when S is an independent set. There are three subcases, in which we consider the possibilities of the sizes of AB, AC, and BC and their effects on the global domination number. In each subcase we look at what happens to the global domination number of the tree when a has no private neighbors, one private neighbor, two private neighbors, or many private neighbors. Similarly for b and c. We also consider all combinations of the private neighborhoods of a, b, and c. Case 2 explores the possibilities for when S is not an independent set. There are two subcases, in which we consider the possibilities; $T[S] = 1$ or $T[S] = 2$.

For our results for 3_g-EMS trees we need to define a caterpillar. A *caterpillar* is a tree C for which the removal of all leafs leaves a path, which is called the *spine*, $(v_1, v_2, \cdots, v_k)$, of C. We represent its code by $(x_1, x_2, \cdots, x_k)$, where x_i is the number of leaves adjacent to v_i, $1 \leq i \leq k$. Figure 14 is an example of a caterpillar with code $(1, 0, 3, 2)$.

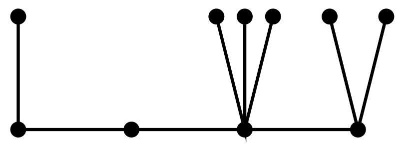

Figure 14: Caterpillar $(1, 0, 3, 2)$

Theorem 5.5 *Let T be a tree. The tree T is a 3_g-EMS graph if and only if T is the path P_7 or the caterpillar* $(1, 1, 1)$.

Proof. See Figure 15 and Figure 16 to see that P_7 and the caterpillar $(1, 1, 1)$ are 3_g-EMS graphs. To prove the necessary condition, assume that T is a 3_g-EMS tree. Let $S = \{a, b, c\}$ be a $\gamma_g(T)$-set. Define the sets A, B, C, AB, AC, BC, and ABC as before.

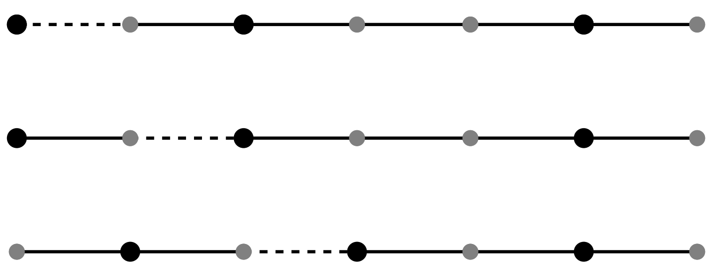

Figure 15: $P_7 - e$

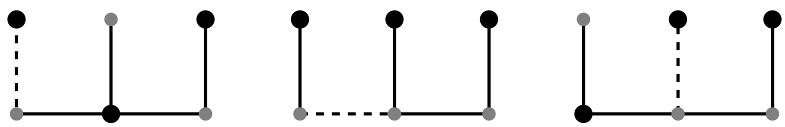

Figure 16: caterpillar $(1,1,1)-e$

Since S is a $\gamma_g(T)$-set, Theorem 4.5 implies that $ABC=\emptyset$. Since T is a tree, at least one of AB, AC, and BC is empty. Without loss of generality, assume that $AC=\emptyset$. By Lemma 5.3, $|AB|\leq 1$ and $|BC|\leq 1$. Note also that since T is a tree, each of A, B, and C is an independent set.

We consider the cases.

Case 1 S is an independent set.

Case 1(a) $|AB|=|BC|=1$.

Since T is a tree, $A\cup B\cup C$ is an independent set, that is, each vertex in $A\cup B\cup C$ is a leaf in T. Figure 17 shows the set up for Case 1(a).

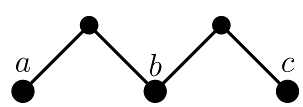

Figure 17: γ_g=3 Case 1(a)

Claim 1 $|A|\leq 1$ *and* $|C|\leq 1$*.*

Proof of Claim 1. By symmetry, it suffices to show that $|A|\leq 1$. Assume to the contrary that $|A|\geq 2$, and let $a'\in A$. Then, a' is an isolated vertex in $T'=T-aa'$, and so $a'\in S$ for every $\gamma_g(T')$-set S. Moreover, at least one additional vertex from $\{a\}\cup A$ is in S. But then one vertex must dominate both b and c implying that

$x \in BC$ is in S, and so $B = C = \emptyset$. But then $\{x, a\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. $(\Box)$

Thus, we may assume that $|A| \leq 1$ and $|C| \leq 1$.

Claim 2 $|B| \leq 1$.

Proof of Claim 2. Assume to the contrary, that $|B| \geq 2$, and let $b' \in B$. Then, b' is an isolated vertex in $T - bb'$ and so $b' \in S$ for every $\gamma_g(T - bb')$-set S. Also another vertex from $\{b\} \cup B$ is in S to dominate the leaf neighbors of b in $T - bb'$. But then no single vertex will dominate both a and c, a contradiction. $(\Box)$

Henceforth, we have $|A| \leq 1$, $|B| \leq 1$, and $|C| \leq 1$. If $|A| = |B| = |C| = 1$, then T is the caterpillar $(1, 0, 1, 0, 1)$, and T is not $3_g(T)$-EMS. If $A = B = C = \emptyset$, then $T = P_5$ and $\gamma_g(T) = 2$, a contradiction. Hence, at least one of A, B, and C is empty, and at least one has cardinality one. We consider the two possibilities for set B.

Assume first that $|B| = 1$. If $|A| = 1$ and $C = \emptyset$, then T is the caterpillar (1, 0, 1, 1), which is not 3_g-EMS, a contradiction. Similarly, we have a contradiction if $|C| = 1$ and $A = \emptyset$. Thus we may assume that $A = C = \emptyset$. Then, T is the caterpillar with code $(1, 1, 1)$ as desired.

Next, assume that $B = \emptyset$. Then, at least one of A and C has cardinality one. If A or C is empty, then $T = P_6$ and $\gamma_g(T) = 2$, a contradiction. Hence, $|A| = |C| = 1$, and so $T = P_7$, as desired. $(\Box)$

<u>Case1(b)</u> $|AB| = 1$ and $BC = \emptyset$.

Since T is connected and $BC = \emptyset$, there must be an edge between a vertex in C and a vertex in $A \cup B$. Without loss of generality, say that $b' \in B$ is adjacent to $c' \in C$. Also note that since T is a tree, the only edge with both its endpoints in

$A \cup B \cup C$ is the edge $b'c'$. Thus, every vertex in $(A \cup B \cup C) \setminus \{b', c'\}$ is a leaf in T. Figure 18 shows the set up for Case 1(b).

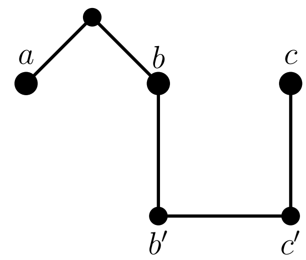

Figure 18: γ_g=3 Case 1(b)

Claim 3 $|B| = 1$.

Proof of Claim 3. Assume $|B| \geq 2$, and delete a pendant edge e incident to b, say bb''. Then, since T is 3_g-EMS and b'' is in every $\gamma_g(T - e)$-set, it follows that $A = \emptyset$ and $C = \{c'\}$. Thus, T is the caterpillar with code $(1, k, 0, 1)$ where $k \geq 1$, which is not 3_g-EMS, a contradiction. Hence, $|B| = 1$. ($\square$)

Hence, $B = \{b'\}$. If $|C| \geq 3$ or $|A| \geq 2$, or if $|C| = 2$ and $|A| = 1$, then removing a pendant edge incident to either a or c causes the global domination number to increase, contradicting the fact that T is 3_g-EMS. Hence, $1 \leq |C| \leq 2$ and $|A| \leq 1$. Moreover, we may assume that we do not have $|A| = 1$ and $|C| = 2$. If $A = \emptyset$ and $|C| = 1$, then $T = P_6$ and $\gamma_g(P_6) = 2$, a contradiction. If $A = \emptyset$ and $|C| = 2$, or if $|A| = 1$ and $|C| = 1$, then $T = P_7$, as desired.

Case1(c) $AB = BC = \emptyset$.

Because T is a tree, it must be connected via edges between vertices in $A \cup B \cup C$. Without loss of generality, assume that $a'b' \in E(T)$, where $a' \in A$ and $b' \in B$.

Furthermore, we may assume that $b''c' \in E(T)$, where $b'' \in B$ (note b'' can be b') and $c' \in C$. Figures 19 and 20 show the two possibilities for this case.

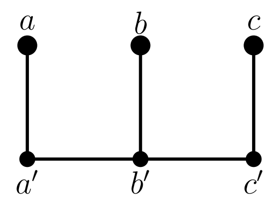

Figure 19: γ_g=3 Case 1(c), $b' = b''$

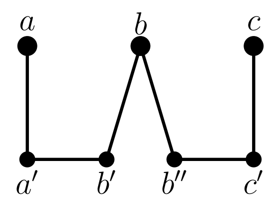

Figure 20: γ_g=3 Case 1(c), $b' \neq b''$

Now all vertices of $(A \cup B \cup C) \setminus \{a', b', b'', c'\}$ are leaves of T, otherwise a cycle is formed. But the removal of any pendant edge incident to one of a, b, and c, increases the global domination number, contradicting that T is 3_g-EMS. Hence, $A = \{a'\}$, $B = \{b', b''\}$, and $C = \{c'\}$. If $b' = b''$, then T is the caterpillar $(1, 1, 1)$ and if $b' \neq b''$, then $T = P_7$, as desired.

Case 2 S is not independent.

We note that since T is a tree, $T[S]$ has at most two edges.

Case 2(a) $T[S]$ has two edges, without loss of generality, let ab and bc be the edges of $T[S]$. Figure 21 shows the set up for Case 2(a).

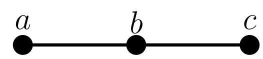

Figure 21: γ_g=3 Case 2(a)

Then, $A \cup B \cup C$ is an independent set, and so $A \cup B \cup C$ is a set of leaves in T. If either of A or C is empty, then $\gamma_g(T) = 2$. Thus we may assume $|A| \geq 1$ and $|C| \geq 1$. If $|A| = |B| = |C| = 1$, then we have the caterpillar $(1, 1, 1)$ as desired. If $B = \emptyset$ and $(|A| = 1$ or $|C| = 1)$, then $\gamma_g(T) = 2$, a contradiction. If $B = \emptyset$, $|A| \geq 2$, and $|C| \geq 2$, then T is the caterpillar $(k, 0, t)$, where $k \geq 2$ and $t \geq 2$. To see that this graph is not 3_g-EMS, note that removing an edge incident to b decreases the global domination number. Thus, assume that $B \geq 1$ and that at least one of A and C, say A, has at least two elements. Removing a pendant edge incident to a creates an isolated vertex that must be in every $\gamma_g(T-e)$-set S. Moreover, at least three additional vertices are in S, one each from $\{a\} \cup A$, $\{b\} \cup B$, and $\{c\} \cup C$, contradicting that T is 3_g-EMS.

Case 2(b) $T[S]$ has exactly one edge, without loss of generality, assume that $ab \in E(T)$.

Since T is a tree $AB = \emptyset$, and either $BC \neq \emptyset$ or there is an edge between a vertex in C and a vertex in $A \cup B$. Figures 22 and 23 show the two possibilities for Case 2(b).

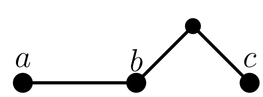

Figure 22: γ_g=3 Case 2(b), $BC \neq \emptyset$

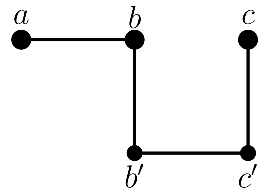

Figure 23: γ_g=3 Case 2(b), $BC = \emptyset$

Assume first that $BC \neq \emptyset$. Now the vertices of $A \cup B \cup C$ are leaves in T. If $B = \emptyset$, then $\{a, c\}$ is a GDS of T, again a contradiction. Hence, $|B| \geq 1$.

If $A = \emptyset$, then T is the caterpillar with code $(j, 0, k)$ where $j \geq 2$ and $k \geq 0$. If $k = 1$, then $\{b, c'\}$ is a GDS of T and $\gamma_g(T) = 2$, a contradiction. If $k = 0$, then $\{b, x\}$ where $BC = \{x\}$, is a GDS, again a contradiction. Hence, $k \geq 2$. But then T is not 3_g-EMS, a contradiction.

Thus, assume that $|A| \geq 1$. If $|A| \geq 2$ (respectively, $|B| \geq 2$), then removing a pendant edge incident to a (respectively, b) increases the global domination number, contradicting that T is 3_g-EMS. Thus $|A| = |B| = 1$. If $c = \emptyset$, then T is the caterpillar $(1, 1, 1)$, and the result holds. Assume $|C| \geq 1$. But now removing a pendant edge incident to c increases the global domination number, a contradiction.

Finally, assume that $BC = \emptyset$. Since T is connected $c' \in C$ has a neighbor in $A \cup B$. Without loss of generality, let $b' \in B$ be a neighbor of c'. Since T is a tree, it follows that $b'c'$ is the only edge in $T[V \setminus S]$. If $A = \emptyset$, then $\gamma_g(T) = 2$, a contradiction. Hence, $|A| \geq 1$. If $|A| \geq 2$ and ($|B| \geq 2$ or $|C| \geq 2$), then removing a pendant edge incident to a increases the global domination number. If $|A| \geq 2$ and $|B| = |C| = 1$, then $\{a, c'\}$ is a GDS of T, a contradiction. Thus, $|A| = 1$. If $C \setminus \{c'\} = \emptyset$, then $S' = \{a, b, c'\}$ is a $\gamma_g(T)$-set with properties of a previous case, namely, one edge in

$T[S']$ and $BC' \neq \emptyset$. Hence, $C \setminus \{c'\} \neq \emptyset$. Now if b has a leaf neighbor, then removing a pendant edge incident to b increases the global domination number. Thus, we may assume that $B = \{b'\}$. If $|C \setminus \{c'\}| = 1$, then $T = P_7$, as desired. Assume that $|C \setminus \{c'\}| \geq 2$. But then removing a pendant edge incident to c increases the global domination number of T, contradicting that T is a 3_g-EMS. ■

5.2 Vertex Removal

In this subsection, we consider graphs whose global domination number stays the same upon the removal of any arbitrary vertex. We call such graphs *k_g-vertex stable graphs*. We focus on when $k = 2$ and when $k = 3$ and call those graphs *2_g-vertex stable graphs* or *3_g-vertex stable graphs*, 2_g-VS or 3_g-VS respectively, for short. We first consider 2_g-VS trees. Proofs are constructed as previously described.

Theorem 5.6 *Let T be a tree. The tree T is a 2_g-vertex stable tree if and only if T is one of the paths P_3, P_4, or P_5.*

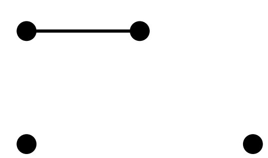

Figure 24: $P_3 - v$

Proof. See Figures 24, 25, and 26 to see that the paths P_3, P_4, and P_5 are 2_g-VS trees. To prove the necessary condition, assume that T is a 2_g-VS tree. Let $S = \{a, b\}$ be a $\gamma_g(T)$-set. Define the sets A, B, and AB as before.

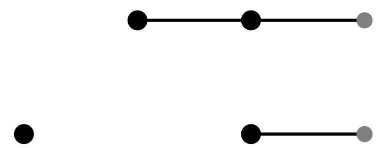

Figure 25: $P_4 - v$

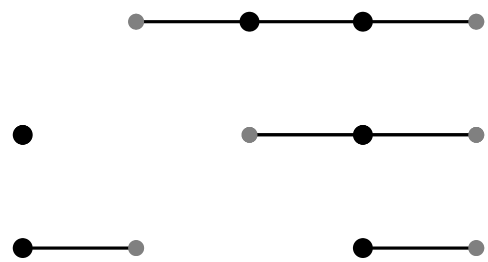

Figure 26: $P_5 - v$

By Lemma 5.1, $|AB| \leq 1$. Note also that since T is a tree, each of A and B is an independent set.

We consider the cases.

Case 1 S is an independent set.

See Figure 11. If $|AB| \neq \emptyset$, then the vertex in AB dominates S, contradicting Lemma 5.1. Hence, $AB = \emptyset$. Since T is connected and $AB = \emptyset$, there must be an edge between a vertex in A and a vertex in B. Without loss of generality, say that $a' \in A$ is adjacent to $b' \in B$. Also note that since T is a tree, the only edge with both its endpoints in $A \cup B$ is the edge $a'b'$. Thus, every vertex in $(A \cup B) \setminus \{a', b'\}$ is a leaf in T. See Figure 12.

Note that $A \neq \emptyset$ because $a' \in A$ and that $B \neq \emptyset$ because $b' \in B$. Assume that $|A| \geq 3$. Then, all the leaves adjacent to a are isolated vertices in $T - a$, and so $A \subseteq S$ for every $\gamma_g(T-a)$-set S'. Moreover, exactly one vertex is in S' to dominate a', b', b

and B, which is impossible. Hence, $1 \leq |A| \leq 2$ and $1 \leq |B| \leq 2$. If $|A| = |B| = 1$, then $T = P_4$ as desired. If $|A| = 2$ and $|B| = 1$ (or $|A| = 1$ and $|B| = 2$), then $T = P_5$ as desired.

Case 2 S is not independent.

We note that since T is a tree, $T[S]$ has exactly one edge and $AB = \emptyset$. Figure 13 illustrates Case 2.

Without loss of generality, assume that $|A| \geq 2$. Then, all the leaves adjacent to a are isolated vertices in $T - a$, and so $A \subseteq S$ for every $\gamma_g(T - a)$-set S'. Moreover, exactly one additional vertex is in S' to dominate b and B, a contradiction. Hence, $0 \leq |A| \leq 1$ and $0 \leq |B| \leq 1$. Assume that $A = B = \emptyset$. The removal of either vertex, say a, results in an isolated vertex that can be globally dominated with only one vertex, a contradiction. If $|A| = 1$ and $B = \emptyset$ (or $A = \emptyset$ and $|B| = 1$), then $T = P_3$ as desired. If $|A| = |B| = 1$, then $T = P_4$ as desired. ■

For the remainder of this section we focus on 3_g-VS trees. Proofs are constructed as previously described.

Observation 5.7 *If G is a 3_g-vertex stable graph and S is a $\gamma_g(G)$-set, then for each $x \in S$, $pn_G(x, S) \neq \emptyset$ or $pn_{\overline{G}}(x, S) \neq \emptyset$.*

Lemma 5.8 *Let G be a 3_g-vertex stable graph and S be a $\gamma_g(G)$-set. If $x \in S$, $pn_G(x, S) = \emptyset$ and $|N_G(x) \cap S| = 1$, then $|pn_{\overline{G}}(x, S)| \geq 2$.*

Proof. Let G be a 3_g-VS graph and S be a $\gamma_g(G)$-set. Assume that $x \in S$ and $pn_G(x, S) = \emptyset$ and $|N_G(x) \cap S| = 1$. Then, $|N_{\overline{G}}(x) \cap S| = 1$ and Observation 5.7 implies that $pn_{\overline{G}}(x, S) \neq \emptyset$. If $pn_{\overline{G}}(x, S) = \{y\}$, then $S - \{x\}$ is a GDS of $G - y$, contradicting that G is 3_g-VS. Hence, $|pn_{\overline{G}}(X, S)| \geq 2$. ■

For the following theorem we need a new definition. A *subdivision* of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with endpoints $\{u, v\}$ yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, $\{u, w\}$ and $\{w, v\}$. Figure 27 is the subdivided star S_3.

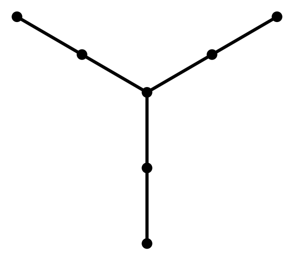

Figure 27: Subdivided star S_3

Theorem 5.9 *Let T be a tree. The tree T is a 3_g-vertex stable tree if and only if T is one of the path P_8, the caterpillar $(1,0,1,0,1)$, or the subdivided star S_3.*

Proof. See Figures 28, 29, and 30 to see that P_8, the caterpillar $(1,0,1,0,1)$, and the subdivided star S_3 are 3_g-VS trees. To prove the necessary condition, assume that T is a 3_g-VS tree. Let $S = \{a, b, c\}$ be a $\gamma_g(T)$-set. Define the sets A, B, C, AB, AC, BC, and ABC and their complements as before.

Since S is a $\gamma_g(T)$-set, Theorem 4.5 implies that $ABC = \emptyset$. Moreover since T is a tree, at least one of AB, AC, and BC is empty. Without loss of generality, assume that $AC = \emptyset$. Moreover, by Lemma 5.3, $|AB| \leq 1$ and $|BC| \leq 1$. Note also that since T is a tree, each of A, B, and C is an independent set.

We consider the cases.

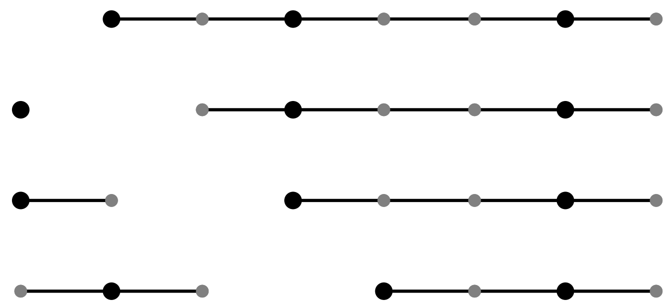

Figure 28: $P_8 - v$

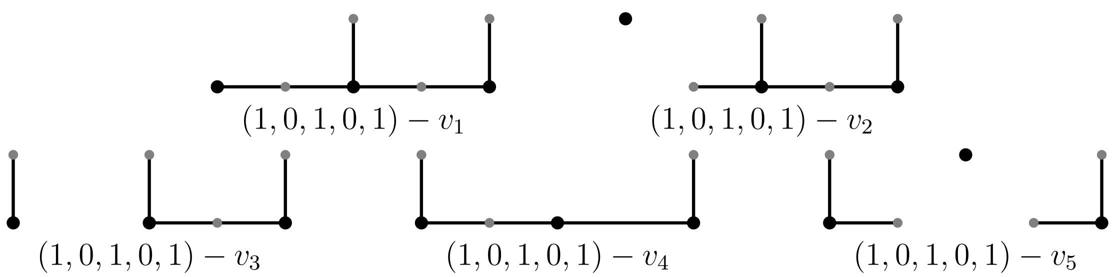

Figure 29: $(1,0,1,0,1) - v$

<u>Case 1</u> S is an independent set.

<u>Case 1(a)</u> $|AB| = |BC| = 1$.

Since T is a tree, $A \cup B \cup C$ is an independent set, that is, each vertex in $A \cup B \cup C$ is a leaf in T. Figure 17 shows the set up for Case 1(a).

Claim 4 $|A| = |C| = 1$.

Proof of Claim 4. By symmetry, it suffices to show that $|A| = 1$. Assume to the contrary, that $|A| \neq 1$. The removal of the vertex a results in all the leaves of A becoming isolated vertices. Thus $A \subseteq S'$ for every $\gamma_g(T - a)$-set S'. Moreover, at least one additional vertex is in S' to dominate b and c. Hence, $|A| \leq 2$ and so we may assume that $|A| = 2$ or $A = \emptyset$. If $|A| = 2$, then $B = \emptyset$ and $C = \emptyset$ implying that

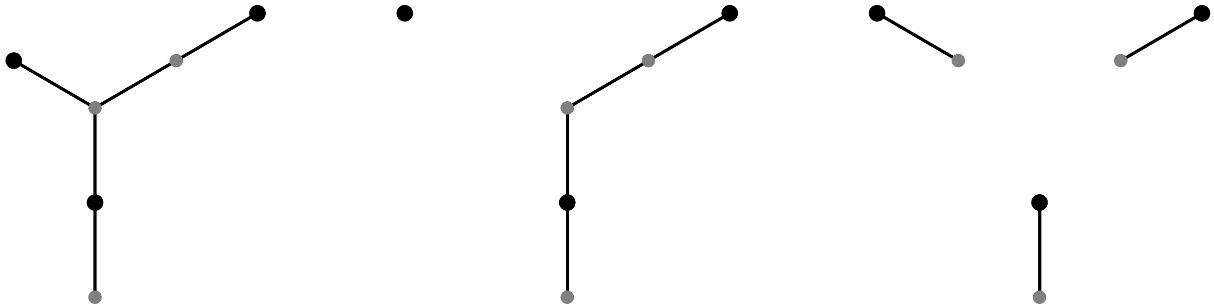

Figure 30: subdivided $S_3 - v$

T is the caterpillar $(2,0,0,1)$, which has $\gamma_g(T) = 2$, a contradiction. Next, assume that $A = \emptyset$. If $B = \emptyset$, then $\gamma_g(T-a) = 2$, a contradiction. If $|B| \geq 1$ and $|C| = 1$, then again $\gamma_g(T-a) = 2$. Hence, $|B| \geq 1$ and $|C| \geq 2$. But $\gamma_g(T-c) > \gamma_g(T)$, a contradiction.

Thus we may assume that $|A| = |C| = 1$. ($\square$)

Claim 5 $|B| = 1$.

Proof of Claim 5. Assume to the contrary, that $|B| \neq 1$. Then, all the leaves adjacent to b are isolated vertices in $T - b$, and so $B \subseteq S$ for every $\gamma_g(T-b)$-set S'. If $|B| = 2$, then another vertex must dominate both a and c implying that $x \in AC$ is in S', but we assumed $AC = \emptyset$, a contradiction. If $|B| \geq 3$, then $|S'| \geq 4 > \gamma_g(T)$, a contradiction. If $B = \emptyset$, then T is the path P_7, which is not a 3_g-VS tree, a contradiction. Hence, $|B| = 1$. ($\square$)

Therefore, we have $|A| = |B| = |C| = 1$ and T is the caterpillar $(1,0,1,0,1)$, as desired.

<u>Case 1(b)</u> $|AB| = 1$ and $BC = \emptyset$.

Since T is connected and $BC = \emptyset$, there must be an edge between a vertex in C and a vertex in $A \cup B$. Without loss of generality, say that $b' \in B$ is the neighbor to $c' \in C$. Also note that since T is a tree, the only edge with both its endpoints in $A \cup B \cup C$ is the edge $b'c'$. Thus, every vertex in $(A \cup B \cup C) \setminus \{b', c'\}$ is a leaf in T. Figure 18 shows the set up for Case 1(b).

Claim 6 $1 \leq |B| \leq 2$.

Proof of Claim 6. Assume $|B| \geq 3$. The removal of the vertex b results in at least two isolated vertices. If $|B| = 3$, then the two isolated vertices, say b'' and b''', must be in every $\gamma_g(T - b)$-set. Then, one other vertex must dominate both a and c, implying that $x \in AC$, but we assumed $AC = \emptyset$, again a contradiction. If $|B| > 3$, then there are at least 3 isolated vertices and T is not a 3_g- VS tree. Hence, $1 \leq |B| \leq 2$. ($\square$)

Claim 7 $|A| = 1$.

Proof of Claim 7. Assume to the contrary, that $|A| \neq 1$. The removal of the vertex a results in all the leaves of A becoming isolated vertices, so A is a subset of every $\gamma_g(T - a)$-set S'. Moreover, at least two additional vertices are in S' to dominate $T - (A \cup \{a\})$. If follows that $|A| \leq 1$. If $A = \emptyset$, then $\{b, c\}$ is a GDS of $T - a$, contradicting that T is a 3_g-VS tree. ($\square$)

Claim 8 $1 \leq |C| \leq 2$.

Proof of Claim 8. First, $C \neq \emptyset$ because we already said $c' \in C$. Next, assume to the contrary, that $|C| \geq 3$. The removal of the vertex c results in all the leaves of C becoming isolated vertices that are in every $\gamma_g(T-c)$-set. If $|C| \geq 3$, then $\gamma_g(T-c) > \gamma_g(T)$, a contradiction. $(\Box)$

Thus, $|A| = 1$, $1 \leq |B| \leq 2$, and $1 \leq |C| \leq 2$.

Assume $|B| = 2$, where b' is adjacent to c' and b'' is the pendant edge incident to b. If $|C| = 1$, then T is the caterpillar $(1,0,1,0,1)$, as desired. If $|C| = 2$, then $\gamma_g(T-c) > \gamma_g(T)$, a contradiction.

Next, assume that $|B| = 1$, where $b' \in B$. If $|C| = 1$, then T is the path P_7, and T is not a 3_g-VS tree. If $|C| = 2$, then T is the path P_8, as desired.

<u>Case 1(c)</u> $AB = BC = \emptyset$.

Because T is a tree, it must be connected via edges between vertices in $A \cup B \cup C$. Without loss of generality, assume that $a'b' \in E(T)$, where $a' \in A$ and $b' \in B$. Furthermore, we may assume that $b''c' \in E(T)$, where $b'' \in B$ (note b'' can be b') and $c' \in C$. Figures 19 and 20 show the two possibilities for this case. Now all vertices of $(A \cup B \cup C) \setminus \{a', b', b'', c'\}$ are leaves of T, otherwise a cycle is formed.

First consider when $b' = b''$.

Claim 9 $1 \leq |A| \leq 2$ *and* $1 \leq |C| \leq 2$.

Proof of Claim 9. By symmetry it suffices to show that $1 \leq |A| \leq 2$. Note that $A \neq \emptyset$ because $a' \in A$. Assume to the contrary that $|A| \geq 3$. Then, $\gamma_g(T-a) > \gamma_g(T)$, a contradiction. $(\Box)$

Assume $|B| = 1$. If $|A| = |C| = 1$, then $\{b', c\}$ is a GDS of $T - a$, contradicting that T is 3_g-VS. If, without loss of generality, $|A| = 1$ and $|C| = 2$, then $\gamma_g(T-b) = 2$, a contradiction. If $|A| = |C| = 2$, then T is the caterpillar $(1,0,1,0,1)$, as desired.

If $|B| \geq 3$, then $\gamma_g(T-b) > \gamma_g(T)$, a contradiction. Assume $|B| = 2$, where b' is adjacent to both a' and c' and b'' is a pendant edge incident to b. If $|A| = |C| = 1$, then T is the subdivided star S_3, as desired. If $|A| = 2$ (respectively, $|C| = 2$), then $\gamma_g(T-a) > \gamma_g(T)$ (respectively, $\gamma_g(T-c) > \gamma_g(T)$), a contradiction.

Next, consider $b' \neq b''$.

Claim 10 $1 \leq |A| \leq 2$ *and* $1 \leq |C| \leq 2$.

Proof of Claim 10. By symmetry it suffices to show that $1 \leq |A| \leq 2$. Note that $A \neq \emptyset$ because $a' \in A$. Assume to the contrary that $|A| \geq 3$. Then, $\gamma_g(T-a) > \gamma_g(T)$, a contradiction. ($\square$)

Claim 11 $2 \leq |B| \leq 3$.

Proof of Claim 11. Note that $|B| \geq 2$ because $\{b', b''\} \in B$. If $|B| \geq 4$, then $\gamma_g(T-b) > \gamma_g(T)$, a contradiction. ($\square$)

Assume $|B| = 2$. If $|A| = |C| = 1$, then T is the path P_7, and T is not 3_g-VS. If $|A| = 1$ and $|C| = 2$, (respectively, $|A| = 2$ and $|C| = 1$), then T is the path P_8, as desired. If $|A| = |C| = 2$, then T is the path P_9 and T is not 3_g-VS.

Now assume $|B| = 3$ where $b''' \in B$ is a pendant edge incident to b. If $|A| = |C| = 1$, then T is the caterpillar $(1, 0, 1, 0, 1)$, as desired. If $|A| = 2$ or $|C| = 2$, then $\gamma_g(T-b) > \gamma_g(T)$, a contradiction. This concludes the case where S is independent.

<u>Case 2</u> S is not independent.

We note that since T is a tree, $T[S]$ has at most two edges.

<u>Case 2(a)</u> $T[S]$ has two edges, without loss of generality, let ab and bc be the edges of $T[S]$. Figure 21 shows the set up for Case 2(a).

Then, $A \cup B \cup C$ is an independent set, and so $A \cup B \cup C$ is a set of leaves in T. If either of A or C is empty, then $\{b, c\}$ (respectively, $\{a, b\}$) is a GDS and $\gamma_g(T) = 2$, a contradiction. Thus, we may assume $|A| \geq 1$ and $|C| \geq 1$. If $|A| = 1$, say $A = \{a'\}$, then $\{b, c\}$ is a GDS of $T - a'$, so $\gamma_g(T - a') = 2$, a contradiction. Similarly, if $|C| = 1$, $\gamma_g(T - c') = 2$, where $C = \{c'\}$, a contradiction. Thus, we may assume $|A| \geq 2$ and $|C| \geq 2$. If $|B| \geq 2$, then $\gamma_g(T - b) > \gamma_g(T)$, a contradiction. If $|B| = 1$, then $\gamma_g(T - a) > \gamma_g(T)$, a contradiction. If $B = \emptyset$, then $\gamma_g(T - b) = 2$, a contradiction. Hence, if $T[S]$ has two edges, then no 3_g-VS tree exists.

Case 2(b) $T[S]$ has exactly one edge, without loss of generality, assume that $ab \in E(T)$.

Since T is a tree $AB = \emptyset$, and either $BC \neq \emptyset$ or there is an edge between a vertex in C and a vertex in $A \cup B$. Figures 22 and 23 show the two possibilities for Case 2(b).

Assume first that $BC \neq \emptyset$. Now the vertices of $A \cup B \cup C$ are leaves in T. If $B = \emptyset$, then $\{a, c\}$ is a GDS of T, a contradiction. Hence, $|B| \geq 1$. If $A = \emptyset$, then $\{b, c\}$ is a GDS of $T - x$, where $x \in AB$, a contradiction. Hence, $|A| \geq 1$ and $|B| \geq 1$. If $|C| \geq 2$, then $\gamma_g(T - c) > \gamma_g(T)$, a contradiction. Hence, $0 \leq |C| \leq 1$. If $|A| \geq 2$, (respectively, $|B| \geq 2$), then $\gamma_g(T - a) > \gamma_g(T)$ (respectively $\gamma_g(T - b) > \gamma_g(T)$), a contradiction. Hence, $|A| = |B| = 1$ and $0 \leq |C| \leq 1$. If $C = \emptyset$, then T is the caterpillar $(1, 1, 1)$ which is not 3_g-VS, a contradiction. If $|C| = 1$, then T is the caterpillar $(1, 1, 0, 1)$. But removing the leaf adjacent to b decreases the global domination number, a contradiction.

Finally, assume that $BC = \emptyset$. Since T is connected, $c' \in C$ has a neighbor in $A \cup B$. Without loss of generality, let $b' \in B$ be a neighbor of c'. Since T is a tree, it follows that $b'c'$ is the only edge in $T[V \setminus S]$. If $A = \emptyset$, then $\{b, c\}$ is a GDS of T and $\gamma_g(T) = 2$, a contradiction. If $|A| = \{a'\}$, then $\{b, c\}$ is a GDS of $T - a'$ and $\gamma_g(T - a') = 2$, a contradiction. If $|A| \geq 2$, then $\gamma_g(T - a) > \gamma_g(T)$, a contradiction. Hence, there are no $3_g(T)$-VS trees when $T[S]$ has exactly one edge. ■

5.3 Edge Addition

In this subsection we consider graphs whose global domination number remains the same upon the addition of any arbitrary edge. We call such graphs T with k_g-*edge plus graphs*. We focus on when $k = 2$ and when $k = 3$ and call those graphs 2_g- *edge plus graphs* or 3_g- *edge plus graphs*, 2_g-EPS or 3_g-EPS respectively, for short. We first consider 2_g-EPS trees. Proofs are constructed as previously described.

Theorem 5.10 *Let T be a tree. The tree T is a 2_g-EPS tree if and only if T is the path P_2 or P_4 or the star S_n where $n \geq 3$.*

Proof. Vacuously $T = P_2$ is a 2_g-EPS tree. Henceforth, we assume that $n \geq 3$. See Figures 31 and 32, where the gray edges are the added edges, to see that the path P_4 and the star S_n where $n \geq 3$ is are 2_g-EPS trees. To prove the necessary condition, assume that T is a 2_g-EPS tree. Let $S = \{a, b\}$ be a $\gamma_g(T)$-set. Define the sets A, B, and AB as before.

By Lemma 5.1, $|AB| \leq 1$. Note also that since T is a tree, each of A and B is an independent set.

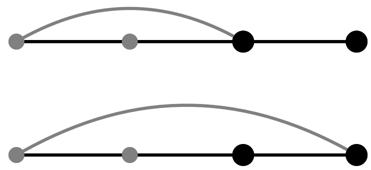

Figure 31: $P_4 + e$

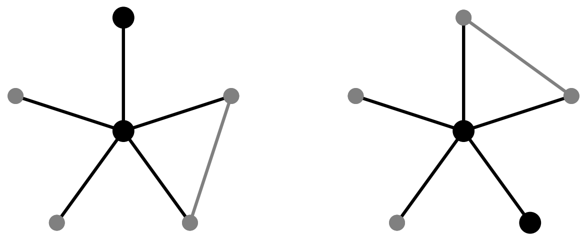

Figure 32: $S_5 + e$

We consider the cases.

<u>Case 1</u> S is an independent set.

See Figure 11. If $|AB| \neq \emptyset$, then the vertex in AB dominates S, contradicting Lemma 5.1. Hence, $AB = \emptyset$. Since T is connected and $AB = \emptyset$, there must be an edge between a vertex in A and a vertex in B. Without loss of generality, say that $a' \in A$ is adjacent to $b' \in B$. Also note that since T is a tree, the only edge with both its endpoints in $A \cup B$ is the edge $a'b'$. Thus, every vertex in $(A \cup B) \setminus \{a', b'\}$ is a leaf in T. See Figure 12.

Note that $A \neq \emptyset$ because $a' \in A$ and that $B \neq \emptyset$ because $b' \in B$. Assume that $|A| \geq 2$. Let $e = ba''$, where $a'' \in A$. Then, $\gamma_g(T + e) \geq 3$, a contradiction. Hence, $|A| \leq 1$ and analogously, $|B| \leq 1$, which implies $|A| = |B| = 1$. Thus, $T = P_4$ as desired.

Case 2 S is not independent.

We note that since T is a tree, $T[S]$ has exactly one edge and $AB = \emptyset$. Figure 13 illustrates Case 2.

Since $AB = \emptyset$ and $n \geq 3$, at least one of $A = \emptyset$ and $B = \emptyset$ is true. Without loss of generality, if $|A| = \{a'\} = 1$ and $B = \emptyset$, then $T + a'b$ results in a complete graph K_3 and $\gamma_g(K_3) = 3$, a contradiction. If $|A| = |B| = 1$, then $T = P_4$ as desired. Without loss of generality, if $|A| \geq 2$ and $B = \emptyset$ (respectively, $A = \emptyset$ and $|B| \geq 2$), then $T = S_n$ as desired. If $|A| \geq 2$ and $|B| = \{b'\} = 1$, then the addition of an edge incident to b and any leaf in a forces a GDS of at least 3, a contradiction. Similarly, if $|A| \geq 2$ and $|B| \geq 2$, then the addition of an edge incident to b and any leaf in a, say a', will result in a GDS of $\{a, a', b\}$, a contradiction. ■

For the remainder of this section we focus on 3_g-EPS trees. Proofs are constructed as previously described.

We define families $\mathcal{T}_i$, $1 \leq i \leq 7$, of caterpillars with codes as follows:

$\mathcal{T}_1$: $(i, 0, j)$, for $i \geq 3$ and $j \geq 3$,

$\mathcal{T}_2$: (i, j, k), for $i \geq 2$, $j \geq 1$, and $k \geq 2$,

$\mathcal{T}_3$: $(i, j, 0, k)$, for $j \geq 2$ and i and k are positive integers, where $i \geq 2$ or $k \geq 2$,

$\mathcal{T}_4$: $(i, 0, j, 0, k)$, for $i \geq 1$, $j \geq 1$, and $k \geq 1$,

$\mathcal{T}_5$: $(i, j, 0, 0, k)$, for $i \geq 2$, $j \geq 1$, and $k \geq 1$,

$\mathcal{T}_6$: $(i, 0, j, 0, 0, k)$, for $i \geq 1$, $j \geq 0$, and $k \geq 1$,

$\mathcal{T}_7$: $(i, 0, 0, j, 0, 0, k)$, for $i \geq 1$, $j \geq 0$, and $k \geq 1$.

Define $\mathcal{H}$ as the family of graphs obtained from the caterpillar $(1, 1, 1)$ with spine x, y, z, adjacent to leaves x', y', z', respectively, by adding $i \geq 0$ new vertices adjacent to x', $j \geq 1$ new vertices adjacent to y', and $k \geq 0$ new vertices adjacent to z'. Figure 33 is an example of a tree $H \in \mathcal{H}$ with $(i, j, k) = (0, 3, 2)$.

Let $\mathcal{F} = \bigcup_{i=1}^{7} \mathcal{T}_i \cup \mathcal{H}$.

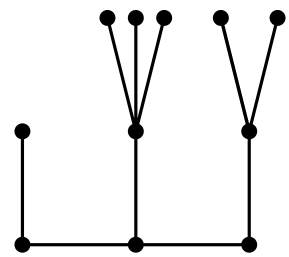

Figure 33: H, $(0, 3, 2)$

Theorem 5.11 *Let T be a tree. The tree T is a 3_g-EPS tree if and only if $T \in \mathcal{F}$.*

Proof. See Figure 34 to see that $\mathcal{T}_1$ is 3_g-EPS. It is straightforward to see that the family $\mathcal{F}$ and the graph H are 3_g-EPS trees. To prove the necessary condition, assume that T is a 3_g-EPS tree. Let $S = \{a, b, c\}$ be a $\gamma_g(T)$-set. Define the sets A, B, C, AB, AC, BC, and ABC and their complements as before.

Since S is a $\gamma_g(T)$-set, Theorem 4.5 implies that $ABC = \emptyset$. Moreover, since T is a tree, at least one of AB, AC, and BC is empty. Without loss of generality, assume that $AC = \emptyset$. Moreover, by Lemma 5.3, we have $|AB| \leq 1$ and $|BC| \leq 1$. Note also that since T is a tree, each of A, B, and C is an independent set.

We note that since T is a tree, $T[S]$ has at most two edges. We consider the cases based on the number of edges in $T[S]$.

Case 1. $T[S]$ has no edges, that is, S is an independent set.

Case 1(a). $|AB| = |BC| = 1$.

Since T is a tree, $A \cup B \cup C$ is an independent set, that is, each vertex in $A \cup B \cup C$ is a leaf in T. If $B = \emptyset$, then $\{a, c\}$ is a GDS of $T + ab$, contradicting that T is a 3_g-EPS tree. Hence, $|B| \geq 1$. If $|A| \geq 1$ and $|C| \geq 1$, then T is the caterpillar $(i, 0, j, 0, k)$, where $i \geq 1$, $j \geq 1$, and $k \geq 1$, and so $T \in \mathcal{T}_4 \subseteq \mathcal{F}$.

Therefore, we may assume that at least one of A and C is empty. Without loss of generality, let $A = \emptyset$. If $B = \{b'\}$, then $\{x, c\}$ is a GDS of $G + xb'$ where $AB = \{x\}$, contradicting that T is a 3_g-EPS tree. Thus, $|B| \geq 2$. If $C = \emptyset$, then $\{x, b\}$ is a GDS of $T + bc$, a contradiction. If $C = \{c'\}$, then $\{b, c'\}$ is a GDS of $T + ba$, again a contradiction. Thus, $|C| \geq 2$, and so T is the caterpillar $(1, j, 0, k)$, where $j = |B| \geq 2$ and $k = |C| \geq 2$. Hence, $T \in \mathcal{T}_3 \subseteq \mathcal{F}$.

Case 1(b). $|AB| = 1$ and $BC = \emptyset$.

Since T is connected and $BC = \emptyset$, there is an edge between a vertex in C and a vertex in $A \cup B$. Without loss of generality, say that $b' \in B$ is adjacent to $c' \in C$. Also note that since T is a tree, the only edge with both its endpoints in $A \cup B \cup C$ is the edge $b'c'$. Thus every vertex in $(A \cup B \cup C) \setminus \{b', c'\}$ is a leaf in T. If $A = \emptyset$, then $\{b, c\}$ is a GDS of $T + ab$, contradicting that T is a 3_g-EPS tree. Hence, $|A| \geq 1$. If $|C| \geq 2$, then T is the caterpillar $(i, 0, j, 0, 0, k)$, where $i = |A| \geq 1$, $j = |B| - 1 \geq 0$, and $k = |C| - 1 \geq 1$. Thus, $T \in \mathcal{T}_6 \subseteq \mathcal{F}$.

Thus, we may assume that $|C| = 1$, that is, $C = \{c'\}$. If $B = \{b'\}$, then $\{a, b'\}$ is a GDS of $G + b'c$, contradicting that T is a 3_g-EPS tree. Thus, $|B| \geq 2$. But then, T is the caterpillar $(i, 0, j, 0, k)$, where $i = |A| \geq 1$, $j = |B| - 1 \geq 1$, and $k = 1$, and so

$T \in \mathcal{T}_4 \subseteq \mathcal{F}$.

Case 1(c). $AB = BC = \emptyset$.

Since T is a tree, it is connected via edges between vertices in $A \cup B \cup C$. Without loss of generality, assume that $a'b' \in E(T)$, where $a' \in A$ and $b' \in B$. Furthermore, we may assume that $b''c' \in E(T)$, where $b'' \in B$ (note b'' can be b') and $c' \in C$. Now all vertices of $(A \cup B \cup C) \setminus \{a', b', b'', c'\}$ are leaves of T, otherwise a cycle is formed. Note that none of A, B, and C is empty because $a' \in A$, $b' \in B$, and $c' \in C$.

First, consider when $b' = b''$. If $|B| \geq 2$, then $T \in \mathcal{H} \subseteq \mathcal{F}$, and the result holds. Thus, we may assume that $|B| = 1$. If $|A| = 1$ (respectively, $|C| = 1$), then $\{a', c\}$ (respectively $\{c', a\}$) is a GDS of $T + a'b$ (respectively, $T + c'b$), a contradiction. Thus, $|A| \geq 2$ and $|C| \geq 2$, and so, T is the caterpillar $(i, 0, 1, 0, k)$, where $i = |A| - 1 \geq 1$ and $k = |C| - 1 \geq 1$. Hence, $T \in \mathcal{T}_4 \subseteq \mathcal{F}$.

Next, assume $b' \neq b''$. Then, $|B| \geq 2$. Assume that $|B| = 2$, that is, $B = \{b', b''\}$. If $|A| = |C| = 1$, then $T = P_7$, which is not a 3_g-EPS tree. Thus, $|A| \geq 2$ or $|C| \geq 2$. Assume, without loss of generality, that $|A| \geq 2$. Then, depending on $|C|$, T is either the caterpillar $(i, 0, 0, 0, 0, 1)$ and $T \in \mathcal{T}_6 \subseteq \mathcal{F}$, or the caterpillar $(i, 0, 0, 0, 0, 0, k)$ and $T \in \mathcal{T}_7 \subseteq \mathcal{F}$.

Hence, we may assume that $|B| \geq 3$. Depending on $|A|$ and $|C|$, we have that T is one of the following caterpillars: $(1, 0, j, 0, 1)$, $(1, 0, j, 0, 0, k)$, $(i, 0, 0, j, 0, 1)$, or $(i, 0, 0, j, 0, 0, k)$, where $i = |A| - 1 \geq 1$, $j = |B| - 2 \geq 1$, and $k = |C| - 1 \geq 1$. Hence, $T \in \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7 \subseteq \mathcal{F}$.

Case 2(a). $T[S]$ has exactly two edges.

Without loss of generality, let ab and bc be the edges of $T[S]$. Since T is a tree, $A \cup B \cup C$ is an independent set, and so $A \cup B \cup C$ is a set of leaves in T. If either of A or C is empty, then $\{b, c\}$ (respectively $\{a, b\}$) is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Thus, we may assume $|A| \geq 1$ and $|C| \geq 1$. If $|A| = 1$, say $A = \{a'\}$, then $\{b, c\}$ is a GDS of $T + ba'$, contradicting that T is a 3_g-EPS tree. Thus, we may assume that $|A| \geq 2$, and analogously, $|C| \geq 2$. If $|B| \geq 1$, then T is the caterpillar (i, j, k), where $i \geq 2$, $j \geq 1$, and $k \geq 2$, and so $T \in \mathcal{T}_2 \subseteq \mathcal{F}$.

We may assume that $B = \emptyset$. If $A = \{a', a''\}$, then $\{a', c\}$ is a GDS of $T + a'a''$, a contradiction. Therefore, $|A| \geq 3$, and analogously, $|C| \geq 3$. Then, T is the caterpillar $(i, 0, j)$ where $i \geq 3$ and $j \geq 3$, and so $T \in \mathcal{T}_1 \subseteq \mathcal{F}$.

<u>Case 2(b)</u>. $T[S]$ has exactly one edge.

Without loss of generality, assume that $ab \in E(T)$. Since T is a tree $AB = \emptyset$, and either $BC \neq \emptyset$ or there is an edge between a vertex in C and a vertex in $A \cup B$. Assume first that $BC \neq \emptyset$. Now the vertices of $A \cup B \cup C$ are leaves in T. If $B = \emptyset$, then $\{a, c\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Hence, $|B| \geq 1$. If $B = \{b'\}$, then $\{a, c\}$ is a GDS of $T + ab'$, contradicting that T is a 3_g-EPS tree. Hence, $|B| \geq 2$.

If $C = \emptyset$, then $\{a, b\}$ is a GDS of $G + bc$, contradicting that T is a 3_g-EPS tree. Hence, $C \neq \emptyset$. Assume that $A = \emptyset$. If $|C| \geq 3$, then T is the caterpillar $(i, 0, j)$, where $i \geq 3$ and $j \geq 3$. Thus, $T \in \mathcal{T}_1 \subseteq \mathcal{F}$. Then, we may assume that $1 \leq |C| \leq 2$. If $C = \{c'\}$, then $\{b, x\}$ is a GDS of $T + bc'$, a contradiction. Hence, we may assume that $C = \{c', c''\}$. But then $\{b, c'\}$ is a GDS of $T + c'c''$, again contradicting that T is a 3_g-EPS tree.

Hence, $A \neq \emptyset$ and $C \neq \emptyset$. If $|A| = |C| = 1$, then $\{b, c'\}$ is a GDS of $T + ba'$, a contradiction. We conclude that $|A| \geq 2$ or $|C| \geq 2$. Therefore, T is the caterpillar $(i, j, 0, k)$, where i, j, and k are positive integers satisfying $j \geq 2$ and at least one of i and j is at least two. Thus, $T \in \mathcal{T}_3 \subseteq \mathcal{F}$.

Finally, assume that $BC = \emptyset$. Since T is connected, $c' \in C$ has a neighbor in $A \cup B$. Without loss of generality, let $b' \in B$ be a neighbor of c'. Since T is a tree, it follows that $b'c'$ is the only edge in $T[V \setminus S]$. If $A = \emptyset$, then $\{b, c\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Hence, $|A| \geq 1$. If $A = \{a'\}$, then $\{b, c\}$ is a GDS of $T + a'b$, where $a' \in A$, contradicting that T is a 3_g-EPS tree. Hence, $|A| \geq 2$. If $|B| = 1$, then $\{a, c\}$ is a GDS of $T + b'c$, again a contradiction. Thus, $|B| \geq 2$. If $|B| = 2$ and $|C| = 1$, then $\{a, c'\}$ is a GDS of $T + b''c'$, where $b'' \in B \setminus \{b'\}$, a contradiction. Thus, $|B| \geq 3$ or $|C| \geq 2$. If $|C| \geq 2$, then T is the caterpillar $(i, j, 0, 0, k)$, where $i = |A| \geq 2$, $j = |B| - 1 \geq 1$, and $k = |C| - 1 \geq 1$, and so $T \in \mathcal{T}_5 \subseteq \mathcal{F}$. If $|B| \geq 3$, then depending on the $|C|$, T is either the caterpillar $(i, j, 0, 1)$ or the caterpillar $(i, j, 0, 0, k)$, where $i = |A| \geq 2$, $j = |B| - 1 \geq 2$, and $k = |C| - 1 \geq 1$. Hence, $T \in \mathcal{T}_3 \cup \mathcal{T}_5 \subseteq \mathcal{F}$. ■

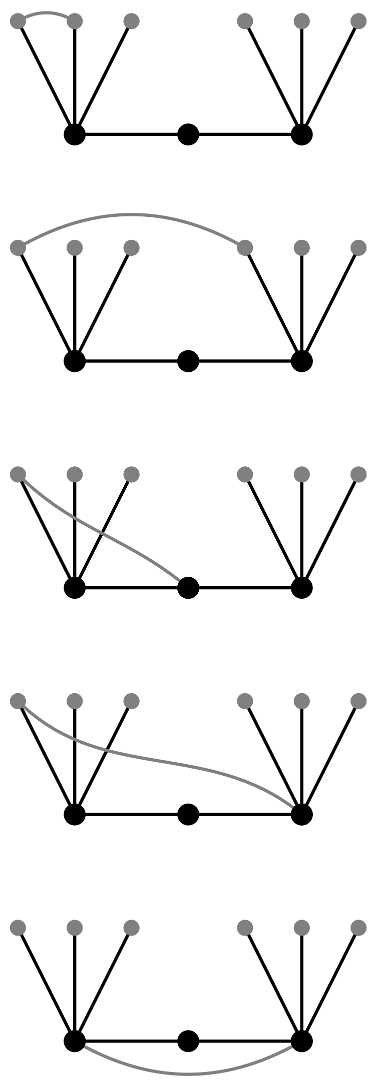

Figure 34: $\mathcal{T}_1$+e

6 CONCLUDING REMARKS

We have characterized the trees T with $\gamma_g(T) =$ and $\gamma_g(T) = 3$ for which edge addition, edge removal, and vertex removal had no effect on the global domination number. This raises the question about trees for larger values of $\gamma_g(T)$ trees. We conclude this thesis with the following open problems:

1. Characterize k_g-VS trees, $k \geq 4$.

2. Characterize k_g-EMS trees, $k \geq 4$.

3. Characterize k_g-EPS trees, $k \geq 4$.

4. Characterize k_g-VS graphs, $k \geq 3$.

5. Characterize k_g-EMS graphs, $k \geq 3$.

6. Characterize k_g-EPS graphs, $k \geq 3$.

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