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Level Crossing Times in Mathematical Finance

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Level Crossing Times in Mathematical Finance

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by

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ABSTRACT

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Level crossing times and their applications in finance are of importance, given certain threshold levels that represent the "desirable" or "sell" values of a stock. In this thesis, we make use of Wald's lemmas and various deep results from renewal theory, in the context of finance, in modelling the growth of a portfolio of stocks. Several models are employed.

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1 INTRODUCTION

Level crossing times in finance modeling are first-occurrence or threshold levels that the variables in the model in question should satisfy, [2]. In this research, the assumption is that there is a positive drift in the overall behavior of the models explored. The increments in the models possess characteristics of a random walk i.e., they are independent, identically distributed random variables. Generally, there exist overshoots or residual life of the random walk which surpasses the level crossing time, i.e., the level at which it was to be stopped. For example, consider a stock model with increments which are independent and identically distributed. The stock has an initial value of $B = \$0$, and the level crossing time, t, at which the model stops is $S = \$20$. We seek to find the expectation and variance for the increments and the waiting times. We could then deduce a central limit theorem for the waiting time. If the increment is always either zero or \$0.03, then we see that there is going to be an overshoot of \$0.02 when the stock value surpasses \$20.

First, we take a look at some basic distributions, give background terminology and then introduce and consider the models. We later look into renewal processes and their applications to the models treated in this research. Probability models are heavily employed in Mathematical Finance and its applications. An overview of some of the probability models and their applications include the following:

Binomial Distributions

Binomial distributions are discrete probability models which involve n independent trials each of which is a "success" with probability p or a "failure" with probability $1-p$, [1]. The probability mass function of a binomial distribution is given by:

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, 2, 3, ...n$$

The binomial distribution describes the behavior of a ***count variable*** X if the following conditions apply:

1. The number of trials n is fixed.
2. The trials are independent.
3. Each observation representing the outcome of any trial can be one of two outcomes (*"success"* or *"failure"*).
4. The probability of success p is the same for each trial.

If these conditions are met, then X, the number of successes, has a binomial distribution with parameters n and p, abbreviated $B(n,p)$.

The Binomial distribution yields a Bernoulli distribution if the the number of independent trials n equals 1, and if we set $X = 1$ with a probability of p if the outcome is a success and $X = 0$ with probability $1-p$ if the outcome is a failure. Thus, the probability mass function is given by:

$$p(0) = P\{X = 0\} = 1 - p$$

$$p(1) = P\{X = 1\} = p$$

where $p \in (0,1)$.

The Binomial Model for Stock Pricing.

The binomial model for pricing stocks in discrete time is given by the Markov chain given by the recursion $S_{n+1} = S_n K_{n+1}$ where the K_i's are identically, independently distributed random variables with $p = P(K_i = u)$ and $1 - p = P(K_i = d)$, $n \geq 0$. Thus

$$S_{n+1} = \begin{cases} uS_n, \text{with probability } p \\ dS_n, \text{with probability } 1-p \end{cases}$$

An arbitrage is a trading strategy with a positive probability of earning money and zero probability of losing with an initial wealth of zero. The two-state binomial option pricing model is based on the assumption of zero arbitrage and its mathematical properties are simple.

Binomial Trees

The behavior of portfolios can be described by the binomial tree model, which can be used in pricing options and its derivatives. For a binomial tree with initial price S_0 with a probability of "p" for going up by a factor u and a probability "$1-p$" for going down by a factor d, we have the following schematic diagram (Figure 1):

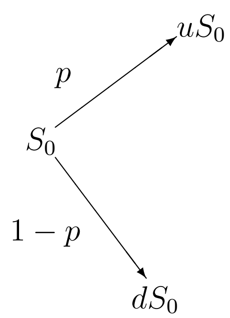

Figure 1: Generic binomial tree

- The expected value of S_1 is given by $E[S_1] = puS_0 + qdS_0$.

- The drift of the stock price is measured by the expression, $pu + qd$.

- An upward price drift is said to exist if $pu + qd > 1$, a downward price drift occurs if $pu + qd < 1$, and there is no price drift if $pu + qd = 1$

- Figure 2 is an extension of the generic binomial tree with $t = 2$

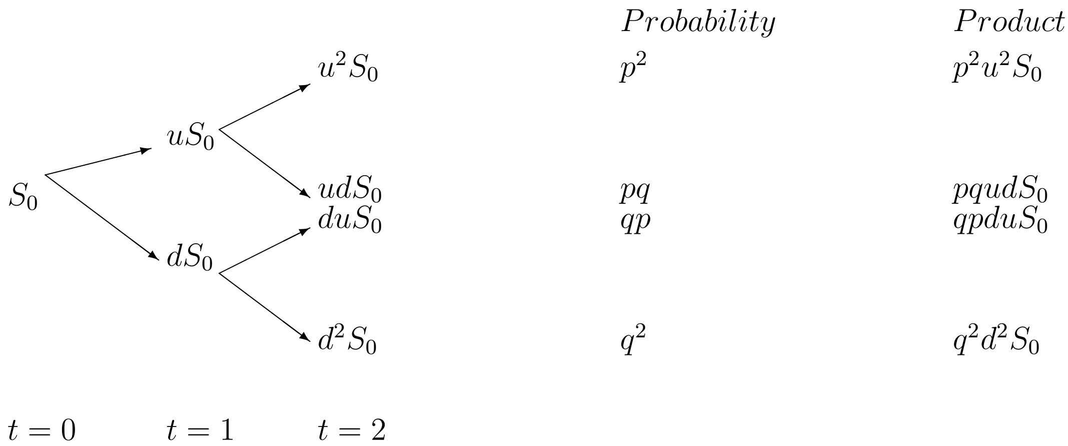

Figure 2: Binomial tree for time $t = 2$

Thus,

$$
\begin{aligned}
E[S_2] &= p^2u^2S_0 + pqudS_0 + pqudS_0 + q^2d^2S_0 \\
&= p^2u^2S_0 + 2pqudS_0 + q^2d^2S_0 \\
&= (pu + qd)^2 S_0 \\
&= (pu + qd)E[S_1]
\end{aligned}
$$

More generally, $E[S_{k+1}] = (pu+qd)E[S_k]$, and we observe that the drift term $pu+qd$ multiplies the expected value for any time period to give the expected price for the next period. It follows that $E[S_k] = (pu + qd)^k E[S_k]$.

1.1 Wald's Lemma

Definition 1.1 *For a given stochastic process* $\boldsymbol{X} = \{X_n : n \geq 0\}$*, a stopping time* τ *with respect to* $\boldsymbol{X}$ *is a random time* t *such that for each* $t \geq 0$*,* τ *is completely determined by the history of the process up to and including time* t*. More specifically, the event* $\{\tau = t\}$ *can be determined to either occur or not depending on the history of the process up till time* t*.*

Example 1.2 *(*Hitting time/Passage time/Gambler's ruin problem)

$\tau_a = \inf\{t \geq 0 | X = a\}$, i.e the first time the random process hits a. This is because we can observe the value of the process X till time t and decide whether to stop or not solely on the basis of information available to us at every time prior to and including t.

Example 1.3 (i) A gambler plays roulette and gambles until he is broke and (ii) A person flips a coin until s/he gets 5 successive tails are other examples of stopping times.

Example 1.4 (Example of a non-stopping time)

Let $\tau_b = \sup\{t \geq 0 | X = b\}$, i.e., τ_b is the last time the random process hits b. Consider a stock with a random price behavior; the last time the stock hits a certain price, b, is not a stopping time because we don't know the future evolution of the random process and looking into the future will be a violation of the very definition of a stopping time.

Wald's Lemma relates the expectation of the sum of finite random variables X_i which are independent and identically distributed to the expected number of terms in the sum and the expectation of the random variables under the condition that the number of terms in the sum and summands are independent.

Theorem 1.5 Wald's Lemma:[4] Given a sequence $\{X_i\}_{i=1}^{\infty}$ of i.i.d. random variables with distribution the same as that of X, and a stopping time τ, we have

$$\mathbb{E}(\sum_{i=1}^{\tau} X_i) = \mathbb{E}(\tau)\mathbb{E}(X).$$

1.2 Consequences of Wald's Lemma

Model 1: Additive Model

With $X_0 = B$, we consider the following model:

- $\tau = \inf\{n > 0 | X(n) \geq S\}$
- τ is a stopping time
- $X_{n+1} = X_n$ with probability $1 - p$
- $X_{n+1} = X_n + \varepsilon$ with probability p
- With $I_n = 1$ if the stock goes up at time n (zero otherwise), Wald's lemma yields

$$\begin{aligned} \lceil \frac{S-B}{\varepsilon} \rceil &= \sum_{n=1}^{\tau} I_n, \text{ so that} \\ \mathbb{E}(\tau) &= \frac{1}{p} \lceil \frac{S-B}{\varepsilon} \rceil. \end{aligned} \tag{1}$$

Proof of Equation (1), $\mathbb{E}(\tau)$ for Model 1

Suppose the random variables I_1,I_2,I_3,... are independent and identically distributed, having finite expectations, then if τ is a stopping time for I_1, I_2, I_3,... for which $E[\tau] < \infty$, then by Wald's lemma we know that with $\lceil \frac{S-B}{\epsilon} \rceil$ upwards steps, the value of the stock, starting at \$S, will be $B + \epsilon \cdot \lceil \frac{S-B}{\epsilon} \rceil \approx S$, so that, in terms of 0-1 indicators, we see that

$$\mathbb{E}(\tau) \approx \frac{S-B}{p\epsilon}.$$

The exact calculation, with an exact overshoot term, is given by (1).

Model 2: Additive Model with Increments and Decrements

Consider an additive model where the next iteration either increases the stock value by ε or decreases it by δ, both with a probability of p, and keeps it unchanged with probability $1-2p$. After buying the stock at \$$B$, we seek to find the expected value of the stopping time $T = \inf\{n \geq 1 : X_n \geq S\}$. The positive drift condition is satisfied if we assume that $\varepsilon > \delta$.

$$X_{n+1} = \begin{cases} X_n + \varepsilon & \text{with probability } p \\ X_n - \delta & \text{with probability } p \\ X_n & \text{with probability } 1-2p\text{, where } \varepsilon > \delta \end{cases}$$

Proof

If $\varepsilon \leq \delta$ we have no drift or negative drift and $E(T) = \infty$, so we assume the positive drift model $\varepsilon > \delta$. As before, ignoring ceilings and floors of quantities, we see that subject to some rounding error as in the previous case we have the equation,

$$\begin{aligned} S - B &= \sum_{n=1}^{T} I_n \\ &= E(T)E(I_j) \\ &= E(T)[p\epsilon + p(-\delta) + 0(1-2p)] \\ &= E(T)p(\epsilon - \delta), \text{ so that} \\ E(T) &= \frac{S-B}{p(\varepsilon - \delta)}. \end{aligned}$$

Let us next reconsider the model where the next iteration increases by ε or remains the same with probabilities p and $1-p$ respectively.

Model 1 reconsidered

$$X_{n+1} = \begin{cases} X_n + \varepsilon & \text{with probability } p \\ X_n & \text{with probability } 1-p \end{cases}$$

The waiting time, τ^+ defined as how long it takes X_n to become positive is of critical importance, as is the ladder variable H, which is defined as the value of X_n at the first positive amount. In general, H is the rather complex "overshoot" term, but in this example it is clear that $H = \varepsilon$. It is also clear that τ^+ is a geometric variable with parameter p, so

$$E(\tau^+) = \frac{1}{p}.$$

The variance of τ^+ can be found easily too:

$$\mathbb{V}(\tau) = \frac{1-p}{p^2} = \frac{q}{p^2},$$

and an easy calculation reveals that

$$\mathbb{V}(I_j) = \varepsilon^2 pq.$$

From Lai and Siegmund's work [1], for a nonlattice X with independent, identical distribution with positive drift and variance, μ and σ^2 respectively, where $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$, the level crossing time $\tau(b) = \{\inf n : S_n > b\}$ has variance given as

$$\mathbb{V}\tau(b) = \frac{b\sigma^2}{\mu^3} + \frac{K}{\mu^2} + o(1) \quad \text{as} \quad b \to \infty,$$

where K is a complicated expression. The constant K is deduced from the moments of the ladder height from Spitzer's identity[6] and is given as shown below; note that we can calculate the constant K exactly for this model. We will later see that even in the simplest of other models, this calculation presents a great challenge.

$$\begin{aligned} K &= \frac{\sigma^2 EH^2}{2\mu EH} + \frac{3}{4}\left(\frac{EH^2}{EH}\right)^2 - \frac{2}{3}\left(\frac{EH^3}{EH}\right) - \left(\frac{EH^2 EH\tau^+}{EHE\tau^+}\right) + \left(\frac{EH^2\tau^+}{E\tau^+}\right) \\ &= \frac{(\varepsilon^2 pq)\varepsilon^2}{2\varepsilon p\varepsilon} + \frac{3}{4}\left(\frac{\varepsilon^2}{\varepsilon}\right)^2 - \frac{2}{3}\left(\frac{\varepsilon^3}{\varepsilon}\right) - \left(\frac{\varepsilon^3\frac{1}{p}}{\varepsilon\frac{1}{p}}\right) + \varepsilon^2, \text{ and thus} \end{aligned}$$

$$\begin{aligned} \mathbb{V}(T) &= \frac{(S-B)\sigma^2}{\mu^3} + \frac{K}{\varepsilon^2 p^2} + o(1), \quad (K = \frac{\varepsilon^2 q}{2} + \frac{3\varepsilon^2}{4} - \frac{2\varepsilon^2}{3} = \frac{\varepsilon^2 q}{2} + \frac{\varepsilon^2}{12}) \\ &= \frac{(S-B)\varepsilon^2 pq}{\varepsilon^3 p^3} + \frac{\frac{\varepsilon^2 q}{2} + \frac{\varepsilon^2}{12}}{\varepsilon^2 p^2} + o(1) \\ &= \frac{(S-B)q}{\varepsilon p^2} + \frac{q}{2p^2} + \frac{1}{12p^2} + o(1) \;\; (S-B \to \infty). \end{aligned}$$

Central Limit Theorem: As $S - B \to \infty$, i.e $S \to \infty$, we expect that a central limit theorem would hold. Specifically, Theorem 3.3.2 of [5] states that:

Suppose $\mu = EY_1 = \int_0^\infty xF(dx) < \infty$ is the mean inter-arrival time in a renewal process.

(1) If $P[Y_0, \infty] = 1, then\ \ almost\ \ surely\ \ \frac{N(t)}{t} \to \mu^{-1}, as\ \ t \to \infty$

(2) If $\sigma^2 = Var(Y_1) < \infty$

$then\ \ M(t)\ \ is\ \ approximately\ N(\mu t^{-1}, t\sigma^2\mu^{-3});\ i.e$

$$\lim_{t\to\infty} P\left[\frac{(M(t) - t\mu^{-1})}{(t\sigma^2\mu^{-3})^{\frac{1}{2}}} \leq x\right] = N(0, 1, x)$$

where $N(0, 1, x)$ is the standard normal distribution function and $M(t)$ is the level crossing time for the level t. Notice the beginning of a trend; all deep results hold only if the level to be crossed grows large. We, on the other hand, have modest goals: to cross the level $S \not\to \infty$.

For a Monte Carlo simulation with an initial value of 0, a final value of 20, a probability of "taking a step" of 0.2, and step distance of 0.1, we ran 100 Monte Carlo simulations 1000 times, producing a mean with a lower bound of 999.642500 and an upper bound of 1000.294630 (Figure 3), and a standard deviation with a lower bound of 62.764970 and upper bound of 63.306784 (Figure 4). Notice that $\mathbb{E}(\tau) = \frac{S-B}{\varepsilon p} = 1000$ and $\sigma(\tau_b) = \sqrt{\frac{(S-B)q}{\varepsilon p^2} + \frac{q}{2p^2} + \frac{1}{12p^2}} = \sqrt{4012.08}$, $\sigma(\tau_b) = 63.34$ This discrepancy in the variance or deviation exists because $S - B = 20 \not\to \infty$. From[5], it suffices to say that $\frac{\tau - \mathbb{E}(\tau)}{\sigma(\tau)} \to N(0, 1)$, so that $\frac{\tau - \frac{S-B}{p\varepsilon}}{\sqrt{\frac{(S-B)q}{\varepsilon p^2}}} \to N(0, 1)$.

Histograms for the mean and variance are displayed below.

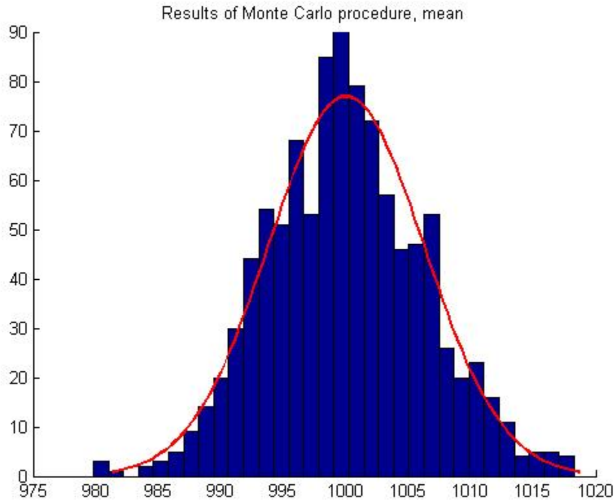

Figure 3: Reults of Monte Carlo procedure showing the distribution of mean

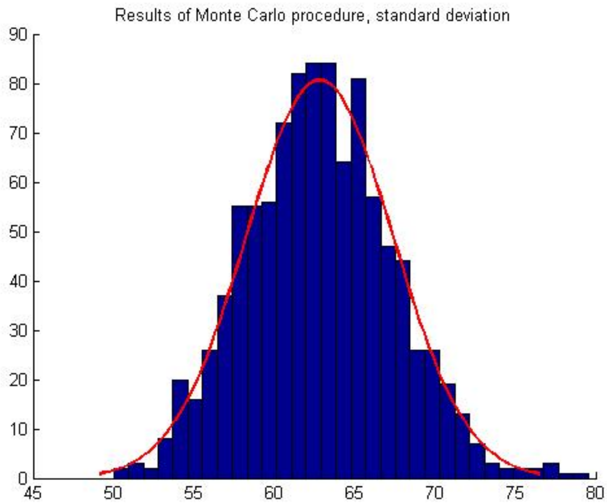

Figure 4: Monte Carlo procedure showing the distribution of standard deviation

2 ASYMMETRIC RANDOM WALKS

In this chapter we focus on the asymmetric random walk, where the increments from one period to another vary as follows:

$$X_n = \begin{cases} \varepsilon & \text{with probability } p > \frac{1}{2} \\ -\varepsilon & \text{with probability } 1-p \end{cases}$$

Notice that we are now referring to X_n as the change in portfolio value as opposed to the value at time n; this is because we will now be adopting more standard random walk terminology and renaming the portfolio value as $S_n = \sum_{j=0}^{n} X_j$, with $X_0 = B$. The assumption that $p > 1/2$ ensures a positive drift.

Model 3

$$X_{n+1} = \begin{cases} X_n + \varepsilon & \text{with probability } p > \frac{1}{2} \\ X_n - \varepsilon & \text{with probability } q = 1-p;\ p > q \end{cases}$$

In this case, it is easy to see that $H = \varepsilon$, and finding $\mathbb{V}(\tau)$ is not a trite calculation.

$$\begin{aligned}
\mathbb{E}(\tau)E(X_i) &= S - B, \text{ and since} \\
\mathbb{E}(\tau) &= \frac{S-B}{\varepsilon(p-q)} \text{ we conclude that} \\
\mathbb{E}(X_i) &= \varepsilon(p-q) = \mu. \text{ Similarly,} \\
\mathbb{E}(X_i^2) &= \varepsilon^2 p + \varepsilon^2 q = \varepsilon^2 \text{ and thus} \\
\mathbb{V}(X_i) &= \varepsilon^2 - \varepsilon^2(p-q)^2 = \varepsilon^2\{1-(p-q)^2\} = \varepsilon^2(1-p^2-q^2+2pq).
\end{aligned}$$

As for Model 1 (and unlike Model 4 that appears later), the value of K can be found exactly,

$$\begin{aligned}
K &= \frac{\sigma^2\varepsilon^2}{2\mu\varepsilon} + \frac{3}{4}\left(\frac{\varepsilon^2}{\varepsilon}\right)^2 - \frac{2}{3}\left(\frac{\varepsilon^3}{\varepsilon}\right) - \left(\frac{\varepsilon^2\frac{1}{p-q}}{\varepsilon\frac{1}{p-q}}\right) + \frac{\varepsilon^2\frac{1}{p-q}}{\frac{1}{p-q}} \\
&= \frac{\sigma^2\varepsilon^2}{2\mu} + \frac{3\varepsilon^2}{4} - \frac{2\varepsilon^2}{3} - \varepsilon + \varepsilon^2 \\
&= \frac{\sigma^2\varepsilon}{2\mu} + \frac{13\varepsilon^2}{12} - \varepsilon,
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{V}(\tau_b) &= \frac{(S-B)\varepsilon^2(1-p^2-q^2+2pq)}{\varepsilon^3(p-q)^3} + \frac{\frac{\varepsilon^2 q}{p-q} + \frac{13\varepsilon^2}{12} - \varepsilon}{\varepsilon^2(p-q)^2} \\
&= \frac{(S-B)2(1-p^2-q^2+2pq)}{\varepsilon(p-q)^3} + \frac{\frac{\varepsilon q}{p-q} + \frac{13\varepsilon}{12} - 1}{\varepsilon(p-q)^2} \\
&= \frac{(S-B)2(1-p^2-q^2+2pq) + \varepsilon q + \frac{13\varepsilon(p-q)}{12} - (p-q)}{\varepsilon(p-q)^3} \\
&= \frac{(S-B)2(1-p^2-q^2+2pq) - \frac{\varepsilon q}{12} + \frac{13\varepsilon p}{12} - (p-q)}{\varepsilon(p-q)^3}
\end{aligned}$$

and thus as $S \to \infty$, we have

$$\mathbb{E}(\tau) = \frac{S-B}{\varepsilon(p-q)}, \text{ which leads to}$$

$$\sigma(\tau) \approx \sqrt{\frac{(S-B)2(1-p^2-q^2+2pq)}{\varepsilon(p-q)^3}}.$$

Suppose that in an election year, candidate Q has q votes and candidate P has p votes where q is always greater than q. We could compute the number of ways the ballot is ordered by thinking of the ballot permutation as a lattice path starting at the origin with upward steps for Q as $(1,1)$ and downward steps for P as $(1,-1)$.

Theorem 2.1 *[3]* **Ballot Theorem**. *Let n and x be positive integers. There are exactly $\frac{x}{n}N_{n,x}$ paths $(s_1, ..., s_n = x)$ from the origin to the point (n,x) such that $s_1 > 0, ..., s_n > 0$.*

Let us arrange p +1's, for player P and $q < p$ $-1's$ for player Q so that P always leads. The desired probability is given as $\frac{p-q}{p+q}$. If p=5 and q=3, then the probability that p always leads is $\frac{5-3}{5+3} = \frac{1}{4}$. $\binom{p+q}{p} = \binom{p+q}{q} = \binom{8}{5} = \binom{8}{3} = 56$ arrangements of +1's and -1's. In 14 of them P always leads. This illustrates the ballot theorem with $N_{n,x} = \binom{8}{3}$ and $x = 2, n = 8$.

The probability of the first positive gain being at the jth trial among the n trials with success probability p is given as

$$\phi_{2k-1} = \frac{(-1)^{k-1}}{2q}\binom{\frac{1}{2}}{k}(4pq)^k; \qquad \Phi_{2k} = 0;$$

when $k = 1$, note that the above reduces to

$$\phi_1 = \frac{(-1)^0}{2q}\binom{\frac{1}{2}}{1}(4pq)^1 = p;$$

when $k = 2$ we see that the only possibilities are $\downarrow\uparrow\uparrow = qp^2$, which is verified by the equation:

$$\begin{aligned}\phi_3 &= \frac{(-1)^1}{2q}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{1}{2!}\right)16p^2q^2; \\ &= qp^2\end{aligned}$$

The generating function Φ of the first positive gain and other useful statistics are given [3] as

$$\begin{aligned}
\Phi(s) &= \sum P(w=n)s^n \\
E(s) &= \Phi'(s)|_{s=1} = \sum P(w=n)n \\
\Phi(s) - ps &= qs\Phi^2(s) \\
\Phi'(s) - p &= q\Phi^2(s) + qs2\Phi(s)\Phi'(s) \\
\Phi'(1) - p &= q\Phi^2(1) + 2q\Phi(1)\Phi'(1); \qquad \Phi(1) = 1, \quad p > q \\
\Phi'(1)(1-2q) &= p + q \\
\Phi'(1) &= \frac{p+q}{1-2q} = \frac{1}{1-q-q} = \frac{1}{p-q}.
\end{aligned}$$

Wald equations yield the same result, as seen in our initial analysis of Model 2 in Chapter 1. W have

$$\begin{aligned}
\sum_1^W X_i &= 1, \text{ but since} \\
E(W)E(X_i) &= 1, \text{ we finally have} \\
E(X_i) &= p - q, \text{ or}
\end{aligned}$$

$$E(W) = \frac{1}{p-q}.$$

A unique bounded solution of the generating function $\Phi(s)$ that satisfies the equation

$$\Phi(s) - ps = qs\Phi^2(s)$$

is given as

$$\begin{aligned}
\Phi(s) &= \frac{1-\sqrt{1-4pqs^2}}{2qs}, \text{ so} \\
\Phi(1) &= \frac{1-\sqrt{1-4pq}}{2q}. \text{ Since} \\
1 &= (p+q)^2 \text{ and} \\
\sqrt{1-4pq} &= p-q, \text{ it follows that} \\
\Phi(1) &= \frac{1-|p-q|}{2q}.
\end{aligned}$$

When $p \geq q$

$$\frac{1-(p-q)}{2q} = \frac{q+q}{2q} = 1$$

and when $p < q$

$$\frac{1-(q-p)}{2q} = \frac{p+p}{2q} = \frac{p}{q};$$

this fact proves that for $p \geq q$, the waiting time is finite with probability one, whereas for $p < q$, the earnings reach the level 1 with probability p/q, which, e.g., equals $1/2$ when $p = 1/3$, and tends to zero as $p \to 0$. Also we have

$$\begin{aligned}
\Phi(s) &= \sum \Phi_n S^n \\
&= \frac{1-\sqrt{1-4pqs^2}}{2qs}, \text{ so}
\end{aligned}$$

$$
\begin{aligned}
\Phi'(s) &= \frac{2qs\frac{1}{2\sqrt{1-4pqs^2}}8pqs - (1-\sqrt{1-4pqs^2})2q}{4q^2s^2} \\
&= \frac{8pq^2 - 2q(p-q)(1-(p-q))}{4q^2(p-q)}, \quad \sqrt{1-4pq} = |p-q|, \text{ and thus} \\
\Phi'(s)|_{s=1} &= \frac{1}{p-q}, \text{ yielding} \\
E(W) &= \Phi'(s)|_{s=1} = \frac{1}{p-q}
\end{aligned}
$$

If there are $p : +1's$ and $q : -1's$ and $S_j = \sum_j^i \varepsilon_i$ and $(s_1, s_2, s_3..., s_n)$ is a path with $s_1 = (0,0) \quad s_n = (n, p-q) = (n, x)$, then there are 2^n paths (total), with

$$
\begin{aligned}
n &= p+q \\
x &= p-q
\end{aligned}
$$

The number of legal paths equals $\binom{p+q}{p} = \binom{p+q}{q}$.

Refection Principle [3]. The number of paths from A to B which touches or cross the X-axis equals the number of all paths from A' to B, where A' is a reflection of A along the X-axis. The reflection principle is used to prove the ballot theorem. Let S_n be the position of the particle at time n with upward steps of $+1's$ and downward steps of $-1's$. The following equations follow easily:

$$
\begin{aligned}
S_n &= \text{number of } (+1's = H) - \text{number of } (-1's = T) = r \\
x - (n-x) &= r \\
2x - n &= r \\
x &= \frac{n+r}{2} \\
P\{S_n = r\} &= \binom{n}{\frac{n+r}{2}} \frac{1}{2^{2n}}
\end{aligned}
$$

If $S_\upsilon = 0$ then there is a return to the origin at time υ where υ is even. We set

$$\begin{aligned} f_{2\upsilon} &= P(S_1 \neq 0, S_2 \neq ..., S_{2\upsilon-1} \neq 0, S_{2\upsilon} = 0), \text{ and note that} \\ P(\#H = \#T) &= \binom{2\upsilon}{\upsilon}\frac{1}{2^{2\upsilon}}. \end{aligned}$$

Using Stirling's formula,

$$n! \approx \sqrt{2\pi\upsilon}\binom{\upsilon}{e}^{\upsilon}$$

the above simplifies to

$$P(H = T) \approx \frac{1}{\sqrt{\pi\upsilon}}$$

Example 2.2

$$\begin{aligned} P(S_1 \neq 0, S_2 \neq ..., S_{2\upsilon-1} \neq 0, S_{2\upsilon} = 0) &= P(S_{2\upsilon} = 0) \\ \text{For } \upsilon &= 2 \text{ we have} \\ P(S_4 = 0) &= \binom{4}{2}\left(\frac{1}{2^2}\right)^2 = \frac{6}{16}, \end{aligned}$$

verified as follows: For $S_1 \neq 0, S_2 \neq ..., S_{2\upsilon-1} \neq 0$, there are 6 possibilites, 3 above the X-axis and the remainder by reflection. The possibilities above the X-axis are given below as:

1	1	1	1
1	1	1	-1
1	1	-1	1

There are 3 more paths by reflection that are below the X-axis

-1	-1	-1	-1
-1	-1	-1	1
-1	-1	1	-1

$$
\begin{aligned}
f_{2\upsilon} &= P(\text{1st return to origin occurs at } 2\upsilon) \\
&= \frac{1}{2\upsilon - 1}\binom{2\upsilon}{\upsilon}\frac{1}{2^{2\upsilon}}, \text{ and since} \\
f_{2\upsilon} &= u_{2\upsilon-2} - u_{2\upsilon}, \text{ it follows that} \\
f_{2\upsilon} &= \binom{2\upsilon-2}{\upsilon-1}\left(\frac{1}{2}\right)^{2\upsilon-2} - \binom{2\upsilon}{\upsilon}\left(\frac{1}{2}\right)^{2\upsilon} \\
&= \frac{1}{2\upsilon-1}u_{2\upsilon}.
\end{aligned}
$$

The first passage of the random walk through r is at n

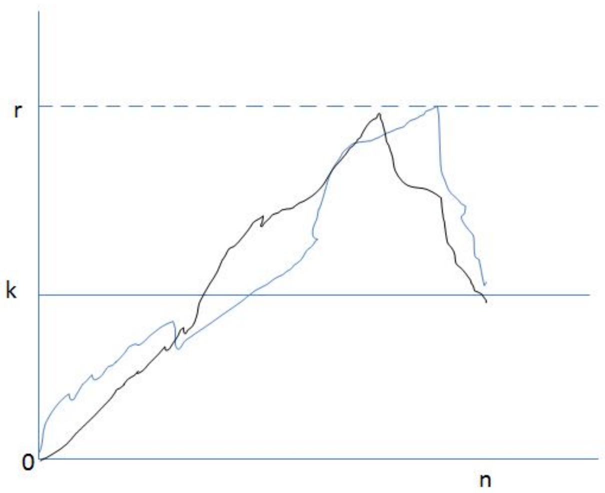

Figure 5: The first passage of the random walk through r is at n

$$
\begin{aligned}
P\left(\max \leq r|\text{ending at } \binom{n}{k}\right) &= P(S_n = 2r-k), \text{ and since} \\
P_{n,r} &= \binom{n}{\frac{n+r}{2}}\frac{1}{2^n}, \\
P(\text{max of a path of length n is r} \geq 0) &= \max(P_{n,r}, P_{n,r+1}). \\
P_{n,r} &= \mathbb{P}(S-F=r) \\
&= \mathbb{P}(S-(n-S)=r). \text{ With} \\
2S-n &= r, \text{ we have that} \\
S &= \frac{n+r}{2}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Thus } \mathbb{P}(\text{1st passage through r is at n}) &= \frac{1}{2}(P_{n-1,r-1}, P_{n-1,r+1}) \\
&= \frac{1}{2}\left[\binom{n-1}{\frac{n-1+r-1}{2}}\frac{1}{2^{n-1}} - \binom{n-1}{\frac{n-1+r+1}{2}}\frac{1}{2^{n-1}}\right] \\
&= \frac{1}{2^n}\left[\binom{n-1}{\frac{n+r}{2}-1} - \binom{n-1}{\frac{n+r}{2}}\right] \\
&= \frac{1}{2^n}\frac{r}{n}\binom{n}{\frac{n+r}{2}}.
\end{aligned}
$$

Let us next consider a model where the next iterations are given as below:

Model 4

$$X_{n+1} = \begin{cases} X_n + 2\varepsilon & \text{with probability } \frac{1}{2} \\ X_n - \varepsilon & \text{with probability } \frac{1}{2} \end{cases}$$

In this case, it is easy to see that $H = \varepsilon$ or $H = 2\varepsilon$, but writing the probabilities of these events is quite difficult, so that $\mathbb{V}(T)$ is difficult to write down. It is easy to see that

$$\begin{aligned} \mu &= \frac{2\varepsilon}{2} - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}, \text{ and} \\ \mathbb{E}(X_{i+1} - X_i)^2 &= 4\varepsilon^2\frac{1}{2} + \varepsilon^2\frac{1}{2} = \frac{5\varepsilon^2}{2}, \text{ so} \\ \mathbb{V}(X_i) &= \frac{5}{2}\varepsilon^2 - \frac{\varepsilon^2}{4} = \frac{9\varepsilon^2}{4}, \\ \text{or } \sigma &= \frac{3\varepsilon}{2}. \end{aligned}$$

We can use Keener's work [4] to find upper and lower bounds for the variance. For this model, $\varepsilon \leq \ EH \leq 2\varepsilon$ and (for example) the lower bound of the first term is bounded by ε when it appears in the numerator and by $1/2\varepsilon$ when it appears in the numerator; the inverse is true for the upper bound. For the first term of K, we thus have,

$$\begin{aligned} \frac{\sigma^2\varepsilon^2}{2\mu 2\varepsilon} &\leq \frac{\sigma^2 EH^2}{2\mu EH} \leq \frac{\sigma^2 4\varepsilon^2}{2\mu\varepsilon} = \frac{2\sigma^2\varepsilon}{\mu} \\ \frac{\sigma^2\varepsilon}{4\mu} &\leq \frac{\sigma^2 EH^2}{2uEH} \leq \frac{2\sigma^2\varepsilon}{\mu} \\ \frac{9\varepsilon^2}{16} &\leq \frac{\sigma^2 EH^2}{2uEH} \leq 9\varepsilon^2 \end{aligned}$$

$$\frac{9\varepsilon^2}{16} \leq \frac{\sigma^2 EH^2}{2uEH} \leq \ 9\varepsilon^2 \qquad (1)$$

The 2nd term of K is bounded as:

$$\frac{3}{4}\left(\frac{\varepsilon^2}{2\varepsilon}\right)^2 \quad \leq \quad \frac{3}{4}\left(\frac{EH^2}{EH}\right)^2 \leq \frac{3}{4}\left(\frac{4\varepsilon^2}{\varepsilon}\right)^2, \text{yielding}$$

$$\frac{3}{16}\varepsilon^2 \leq \frac{3}{4}\left(\frac{EH^2}{EH}\right)^2 \leq 12\varepsilon^2 \qquad (2)$$

The rest of the calculation is as follows:

3rd term of K:

$$\frac{-2}{3}\frac{8\varepsilon^3}{\varepsilon} \quad \leq \quad \frac{-2}{3}\frac{EH^3}{EH} \leq \frac{-2}{3}\frac{\varepsilon^3}{2\varepsilon}$$

$$\frac{-16\varepsilon^2}{3} \leq \frac{-2EH^3}{3EH} \leq \frac{-\varepsilon^2}{3} \qquad (3)$$

4th term of K:

$$\frac{-8\varepsilon^3}{\varepsilon} \quad \leq \quad -\frac{EH^2EH\tau+}{EHE\tau+} \leq \frac{-\varepsilon^3}{2\varepsilon}$$

$$-8\varepsilon^2 \leq -\frac{EH^2EH\tau+}{EHE\tau+} \leq \frac{-\varepsilon^2}{2} \qquad (4)$$

5th term of K:

$$\varepsilon^2 \leq -\frac{EH^2H_{\tau+}}{E_{\tau+}} \leq 4\epsilon^2 \qquad (5)$$

Putting it all together, our bounds for K are:

$$-\frac{57}{4}\varepsilon^2 \leq K \leq \frac{145}{6}\varepsilon^2;$$

this is a wide range and later we use simulations to estimate the exact value of K.

Bound for the variance are thus as follows,

$$\frac{\frac{9(S-B)\varepsilon^2}{4}}{\frac{\varepsilon^3}{8}} - \frac{\frac{57\varepsilon^2}{4}}{\frac{\varepsilon^2}{4}} \leq \mathbb{V}(\tau_b) \leq \frac{\frac{9(S-B)\varepsilon^2}{4}}{\frac{\varepsilon^3}{8}} + \frac{\frac{145\varepsilon^2}{6}}{\frac{\varepsilon^2}{4}}, \text{ which gives}$$

$$\frac{18(S-B)}{\varepsilon} - \frac{57}{4} \leq \mathbb{V}(\tau_b) \leq \frac{18(S-B)}{\varepsilon} + \frac{290}{3}, \text{i.e.,}$$

$$\boxed{\frac{18(S-B)}{\varepsilon} - \frac{91}{3} \leq \mathbb{V}(\tau_b) \leq \frac{18(S-B)}{\varepsilon} + \frac{290}{3}}$$

According to Theorem 3.3.2 in [5] we have

$$\frac{\tau - \frac{S-B}{\frac{\varepsilon}{2}}}{\sqrt{\frac{18(S-B)}{\varepsilon}}} \to N(0,1).$$

Simulations with $\varepsilon = 0.1$, $S = 50$, $B = 10$ produce a mean time of 800.00125 and a standard deviation ≈ 28. It is worth noting that the computed mean $\frac{S-B}{\frac{\varepsilon}{2}} = 800$ is quite close to the simulated value. Likewise we know,

$$\sqrt{\frac{18(S-B)}{\varepsilon} + \frac{K}{\mu^2}} \approx 28.$$

Solving for K with $S - B = 40$, $\varepsilon = 0.1$, $\mu^2 = \frac{\varepsilon^2}{4} = \frac{0.01}{4} = 0.0025$, we obtain a value for $K = -16.04$, which falls in the range $-57 \leq K \leq \frac{145}{6}$.

Model 5: Multiplicative Model

This model is as follows:

$$X_{n+1} = \begin{cases} uX_n & \text{if probability is } p \\ X_n & \text{if probability is } 1-p \end{cases}$$

We define the stopping time

$$T = \inf\left\{ n : \prod_{j=0}^{n} \frac{X_{j+1}}{X_j} \geq \frac{S}{B} \right\},$$

or,

$$T = \inf\left\{ n : \sum_{j=0}^{n} \log \frac{X_{j+1}}{X_j} \geq \log \frac{S}{B} \right\}.$$

Since $\mathbb{E}(Z_j) := \mathbb{E}(\log \frac{X_{j+1}}{X_j}) = p \log u$, we see by Wald's lemma that

$$\mathbb{E}(T) = \frac{\log \frac{S}{B}}{p \log u}.$$

Let

$$\begin{aligned} Y_j &= \log \frac{X_{j+1}}{X_j} = \log \mu; \qquad \text{with probability } p \\ Y_j &= 0; \qquad \text{with probability } 1-p \\ \text{It follows that } \mathbb{E}Y_j &= p \log u \\ \text{and } \mathbb{E}Y_j^2 &= p \log^2 u, \text{ so that} \\ \mathbb{V}(Y_j) &= p \log^2 u - p^2 \log^2 u. \text{ Since} \\ \mathbb{E}(T)\mathbb{E}(Y) &= \log \frac{S}{B}, \end{aligned}$$

the expectation of the stopping time is thus given as,

$$\mathbb{E}(T) = \frac{\log \frac{S}{B}}{p \log u}.$$

Using Keener's work to find a value of K (which can be found exactly) and solving for the variance, we get

$$\begin{aligned}
K &= \frac{\sigma^2\varepsilon^2}{2\varepsilon p\log u} + \frac{3}{4}\left(\frac{\varepsilon^2}{\varepsilon}\right)^2 - \frac{2}{3}\left(\frac{\varepsilon^3}{\varepsilon}\right) - \left(\frac{\varepsilon^2\frac{1}{p\log u}}{\varepsilon\frac{1}{p\log u}}\right) + \frac{\varepsilon^2\frac{1}{p\log u}}{\frac{1}{p\log u}} \\
&= \frac{\sigma^2\varepsilon^2}{2p\log u} + \frac{3\varepsilon^2}{4} - \frac{2\varepsilon^2}{3} - \varepsilon + \varepsilon^2 \\
&= \frac{\sigma^2\varepsilon}{2p\log u} + \frac{13\varepsilon^2}{12} - \varepsilon, \text{ and so}
\end{aligned}$$

$$\mathbb{V}(\tau_b) = \frac{\log\frac{S}{B}(p\log^2 u)(1-p)}{p^3\log^3 u} + \frac{1}{p^2\log^2 u}\left\{\frac{\sigma^2\varepsilon}{2p\log u} + \frac{13\varepsilon^2}{12} - \varepsilon\right\}.$$

Thus as $\log\frac{S}{B} \to \infty$, we can find the expectation and variance of the stopping time

$$\begin{aligned}
\mathbb{E}(T) &= \frac{\log\frac{S}{B}}{p\log u} \\
\sigma(\tau_b) &= \sqrt{\frac{\log\frac{S}{B}(p\log^2 u)(1-p)}{p^3\log^3 u}}.
\end{aligned}$$

Also, the Central Limit Theorem [5], holds and thus

$$\begin{aligned}
\frac{\tau - \mathbb{E}(\tau)}{\sigma(\tau)} &\to N(0,1), \text{ or} \\
\frac{\tau - \frac{\log\frac{S}{B}}{p\log u}}{\sqrt{\frac{\log\frac{S}{B}(1-p)}{p^2\log u}}} &\to N(0,1).
\end{aligned}$$

Model 6: Multiplicative Model with Increments and Decrements

This model is a slight modification of Model 5:

$$X_{n+1} = \begin{cases} uX_n & \text{with probability } p \\ dX_n & \text{with probability is } p \\ X_n & \text{with probability is } 1-2p \end{cases}$$

Now, when $u > 1 > d$, $ud > 1$ and

$$\prod_{n=1}^{T} \frac{X_{j+1}}{X_j} = \frac{S}{B},$$

we see as before by Wald's lemma that

$$\mathbb{E}(T)\mathbb{E}(\log \frac{X_{j+1}}{X_j}) = \log \frac{S}{B},$$

which yields

$$\mathbb{E}(T) = \frac{\log \frac{S}{B}}{p(\log ud)},$$

which reduces to the previous model on setting $d = 1$. Now,

$$\begin{aligned} \sum \log \frac{X_{j+1}}{X_j} &\geq \sum \log \frac{S}{B} = \text{“}b\text{”. With} \\ Y_j &= \log \frac{X_{j+1}}{X_j}, \end{aligned}$$

either $Y_j = \log u > 0$ or $\log d < 0$ or 0. It is diffficult to compute bounds on this model so we use simulations:

Using values of $u = 1.05$, $d = 0.97$, $B = 5$, $S = 20$, we can find a simulated standard deviation and use it as an estimate of σ_T. Thus, we can estimate $\mathbb{V}T$.

$$\mathbb{V}(\tau) = \frac{b\sigma^2}{\mu^3} + \frac{k}{\mu^2},$$

We can estimate a value for k, since all the other parameters are known, and $\mathbb{V}(\tau)$ could be derived through simulation.

From the model,

$$\begin{aligned}
\mu &= p\log u + p\log d \\
&= p\log ud; \\
\mathbb{E}X_i^2 &= p\log^2 u + p\log^2 d, \text{ so} \\
\mathbb{V}(X_i) &= p(\log^2 u + \log^2 d) - p^2(\log u + \log d)^2 \\
&= p\log^2 u + p\log^2 d - p^2\log^2 u - p^2\log^2 d - 2p^2\log u\log d \\
&= (p-p^2)\log^2 u + (p-p^2)\log^2 d - 2p^2\log u\log d \\
&= (p-p^2)\{\log^2 u + \log^2 d\} - 2p^2\log u\log d \\
&= (p-p^2)(\log^2 ud) - 2p^2\log u\log d.
\end{aligned}$$

Setting

$$S_\tau^2 = \frac{\log\frac{S}{B}\sigma^2}{\mu^3} + \frac{k}{\mu^2},$$

we can estimate a value for k.

Model 7: Damped Additive Model

This is the same as the additive model, but the jth increase is by $\frac{\varepsilon}{j}$. Specifically X_{j+1} equals X_j with probability $1-p$, and increases to $X_j+\frac{\varepsilon}{j}$ with probability p. In general, we can do the following for any divergent series $\sum f(j)$ for which $\sum f^2(j)<\infty$; the reasons for these conditions will become clear later. Let T be the same stopping time, i.e., $T=\inf\{n : X_n \geq S\}$. Set

$$\sum_{i=1}^{T}\frac{1}{j}\geq S-B, \text{ or,}$$

$$\sum_{i=1}^{T}\frac{1}{j}\approx\int_1^T\frac{dx}{x}\approx\ln T\geq S-B.$$

More exactly, with γ representing Euler's constant,

$$\begin{aligned}\sum_{i=1}^{T}\frac{1}{j} &= \ln T+\gamma+o(1) \text{ so that}\\ \varepsilon p(\ln\mathbb{E}(T)+\gamma+o(1)) &= S-B \text{ i.e.,}\\ \varepsilon p\ln\mathbb{E}(T) &= S-B-\varepsilon p\gamma-o(1) \text{ i.e.,}\\ \mathbb{E}(T) &= e^{\frac{S-B-\varepsilon p\gamma}{\varepsilon p}}.\end{aligned}$$

From the above model, if the next iteration increases by either the ratio $\frac{\varepsilon}{j}$ or remains the same then we have

$$X_{j+1}=\begin{cases} X_j+\frac{\varepsilon}{j} & \text{with probability } p\\ X_j & \text{with probability } 1-p\end{cases}$$

Suppose the increment in the $X_j's$ is given as Y_j where,

$$\begin{aligned} Y_j &= X_{j+1}-X_j\\ \text{Then }\ \mathbb{E}(Y_j) &= \frac{\varepsilon p}{j}\\ \mathbb{E}\Big(\sum_1^T Y_i\Big) &= \mathbb{E}\Big(\sum_1^T\mathbb{E}(Y_i)\Big).\end{aligned}$$

Under certain conditions, we know that Wald's lemma holds even when the variables are non-i.i.d. Thus if these conditions are satisfied, we have

$$\begin{aligned}\mathbb{E}\bigg(\sum_1^T \frac{\varepsilon}{j}\bigg) &= S - B \\ \mathbb{E}\bigg(\sum_1^T \frac{1}{j}\bigg) &= \frac{S-B}{\varepsilon}\end{aligned}$$

Conditions: From work in [7], we see that we need, with

$$Z_j = Y_j - \frac{\varepsilon p}{j},$$

$$\mathbb{E}\bigg|\sum Z_j\bigg| = \mathbb{E}\bigg|\sum_1^T Y_j - \frac{\varepsilon p}{j}\bigg| < \infty$$

Using the triangle inequality, and Hölder's inequality, we need to show that

$$\sum_1^T \mathbb{E}\bigg|Y_j - \frac{\varepsilon p}{j}\bigg| \leq \sqrt{\sum_1^T \mathbb{E}\bigg|Y_j - \frac{\varepsilon p}{j}\bigg|^2} = \sqrt{\sum \mathbb{V}(Y_j)} < \infty.$$

Now,

$$\begin{aligned}\mathbb{E}(Y_j) &= \frac{\varepsilon p}{j} \text{ and} \\ \mathbb{E}(Y_j^2) &= \frac{\varepsilon^2 p}{j^2} \text{ so} \\ \mathbb{V}(Y_j) &= \frac{\varepsilon^2 p}{j^2} - \frac{(\varepsilon p)^2}{j^2} \\ &= \frac{\varepsilon^2 p(1-p)}{j^2}. \\ \therefore \sqrt{\sum \mathbb{V}(Y_j)} &\leq \sqrt{\frac{\varepsilon^2 p(1-p)}{j^2}} < \infty.\end{aligned}$$

Notice for the square summability condition appears.

Other Models for Future Work

The bulleted models have different distributions in comparison to the models incorporated in this research, and they can yield interesting results as to the expectations and variances of the increments with respective to their models.

- Consider a divergent series $\sum f(j)$ by an increase of $\varepsilon \cdot f(j)$ at the jth step. Let F be the antiderivative of f. Then we get

$$E[T] \approx F^{-1}\left(\frac{S-B}{p\varepsilon}\right).$$

- Consider positive random variables, $X_1, X_2, X_3, ...$ that are independent and identically distributed such that

$$X_{n+1} = \begin{cases} X_n + \varepsilon \\ X_n \end{cases}$$

If

$$X_{n+1} = X_n + \varepsilon \text{ then } \mathrm{Y_{n+1}} = 1$$

and

$$X_{n+1} = X_n \text{ then } \mathrm{Y_{n+1}} = 0.$$

We can construct a Markov matrix for Y_{n+1} and $\mathrm{Y_n}$ with associated transition probabilities α and β given in the table below.

Table 1: Markov Matrix

		Y_{n+1}	
		1	0
	1	α	$1-\alpha$
Y_n			
	0	β	$1-\beta$

- More complicated Markov Chains related to Models 2, 3, 4, 5, 6, 7.

- Time Series models, e.g., $X_{n+1} = \alpha X_n + \beta X_{n-1} + N[0, \varepsilon^2]$

- Nonlattice Models such as $X_{n+1} = X_n + U[0, \theta]$ on uniform distributions – where it can be established that finding bounds on the variance is not a trite calculation

- Exponential Families, e.g., $X_{n+1} = X_i + N(\theta, 1)$.

4 THEORETICAL RESULTS FROM RENEWAL PROCESS

In this chapter, we look at theoretical results and sufficient conditions from the theory of renewal processes (sums of non negative random variables) and random walks with positive drift that are applicable to the models that have been treated in earlier Chapters. The reader will notice that, in all cases, the level we need to to cross must tend to infinity for the results to be valid, which is definitely not the case for us.

Let $\{X_n, n \geq 0\}$ be a random walk,$\{N(t), t \geq 0\}$ be a family of random indices and the family of randomly indexed random walk be $\{X_{N(t)}, t \geq 0\}$. A key assumption made is that the random walk drifts to $+\infty$ and the increments of the random variables $\{X_k, k \geq 1\}$have positive, finite mean. The first passage time is $\tau = \min\{n|X_n > t\}$ and the family of random indices in the case of renewal counting processes is defined by $N(t) = \max\{n : X_n \leq t\}$, thus $N(t)$ is not a stopping time. Clearly, $\tau = N(t)+1$ for renewal processes, since we consider nonnegative increments in renewal processes.

The renewal function is given by $U(t) = \sum_1^\infty P(X_n \leq t) = EN(t)$, which is finite for all t. A sufficient and necessary condition for this to hold for all random walks is that $E(X_1^-)^2 < \infty$ i.e the second moment of the negative part of the random variable X_1 should be finite.

The following are from [8]:

Theorem 4.1 *[8] Let $X_k, k \geq 1$ be i.i.d random variables such that $E|X_1|^r < \infty$ for some $r \geq 1$ and let $S_n = \sum_{k=1}^n X_k, n \geq 1$.Then*

$\frac{S_n}{n} \to EX_1$ *almost surely and in* L^r *as* $n \to \infty$.

Theorem 4.2 *[8] Let* $X_k, k \geq 1$ *be i.i.d random variables and let* $S_n = \sum_{k=1}^{n}, n \geq 1$.*Assume that* $EX_1 = 0$, *that Var* $X_1 = \sigma^2 < \infty$, *and that the* r*th moment of* X *is finite for some* $r \geq 2$. *Then*
$E\left|\frac{S_n}{\sqrt{n}}\right|^p \to E|Z|^p$ *as* $n \to \infty$ *for all* $p, 0 < p \leq r$, *where* Z *is a normal random variable with mean 0 and variance* σ^2

Theorem 4.3 *[8] Suppose that* $E|X_1|^r \leq \infty$ *for some* r $(1 \leq r < \infty)$. *There exist a numerical constant* K_r' *depending on* r *only such that* $E|S_\tau|^r \leq K_r'{}^{\cdot} E|X_1|^{r\cdot} E\tau^r$.
Remark: If τ *is the stopping time and* X_1, *an i.i.d increment of a random variable have finite moments of order* $r \geq 1$, *then the partial sums of the stopping times also has a finite moment. If* $EX_1 = 0$ *then a weaker condition in Theorem 4.4 suffices.*

Theorem 4.4 *[8] Suppose that* $E|X_1|^r < \infty$ *for some* r $(0 < r < \infty)$ *and that* $EX_1 = 0$ *when* $r \geq 1$. *Then*

(i) $E|S_\tau|^r \leq E|X_1|^{r\cdot} E\tau$ *for* $0 < r \leq 1$;

(ii) $E|S_\tau|^r \leq K_r'{}^{\cdot} E|X_1|^{r\cdot} E\tau$ *for* $1 \leq r \leq 2$;

(iii) $E|S_\tau|^r \leq K_r'((EX_1^2)^{\frac{r}{2}\cdot} E\tau^{\frac{r}{2}} + E|X_1|^{r\cdot} E\tau) \leq 2K_r'{}^{\cdot} E|X_1|^{r\cdot} E\tau^{\frac{r}{2}}$ *for* $r \geq 2$,

where K_r' *is a a numerical constant depending on* r *only*

The next results are the Wald Lemmas.

Theorem 4.5 *[8] If* $EX_1 = \mu$ *and* $E\tau < \infty$, *then* $ES\tau = \mu^{\cdot} E\tau$.
If $VarX_1 = \sigma^2 < \infty$, *then* $E(S_\tau - \tau\mu)^2 = \sigma^{2\cdot} E\tau$

Let $\{X_k, k \geq 1\}$ be independent, identically distributed nonnegative random variables and their partial sums to be $\{S_n, n \geq 0\}$, thus the sequence $\{S_n, n \geq 0\}$ defined is a renewal process. The renewal counting process $\{N(t), t \geq 0\}$ is defined as $N(t) = \min\{n : S_n \geq t\}$ which is in our context equal to $N(t) = \min\{n : S_n \geq S - B\}$ for the models treated in the earlier chapters. There is an inverse relationship that exist between renewal processes and counting processes, i.e $\{t > n\} = \{S_n < t\}$.

For a renewal function $U(t) = \sum_{n=1}^{\infty} F_n(t)$,

$$EN(t) = \sum_{n=1}^{\infty} P(N(t) \geq n) = \sum_{n=1}^{\infty} P(S_n \leq t) = \sum_{n=1}^{\infty} F_n(t)$$

Theorem 4.6 *[8]* **(The Elementary Renewal Theorem)** *Let* $0 \leq \mu = EX_1 \leq \infty$. *Then*

$$\frac{U(t)}{t} \to \frac{1}{\mu}$$

the limit being 0 as $t \to +\infty$.

Lattice and Non-Lattice Renewal Processes

Theorem 4.7 *[8] (i) For nonlattice renewal processes we have*

$$U(t) - U(t-h) \to \frac{h}{\mu} \text{ as } t \to \infty$$

(ii) For lattice renewal processes, we have

$$u_n = \sum_{k=1}^{\infty} P(S_k = nd) \to \frac{d}{\mu} \text{ as } n \to \infty$$

the limit being 0 as $\mu \to +\infty$.

Remarks: For (i) For a change h, in the renewal function U(t) is given as $\frac{h}{\mu}$, *where* $\mu > 0$

For (ii) A probability distribution of the renewal process S_k *for some* k *is given as* $\frac{d}{\mu}$, *where* $d > 0$ *is the span of the lattice renewal process.*

4.2 The Central Limit Theorem for Counting Processes

Theorem 4.8 *[8] Suppose that* $0 < \mu = EX_1 < \infty$ *and* $\sigma^2 = VarX_1 < \infty$. *Then*

(i)
$$\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{\sigma^2 t}{\mu^3}}} \xrightarrow{d} N(0,1) \ \ as \ \ t \to \infty$$

(ii) If the renewal process is nonlattice, then

$$EN(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(1) \ \ as \ \ t \to \infty$$

$$VarN(t) = \frac{\sigma^2 t}{\mu^3} + o(t) \ \ as \ \ t \to \infty$$

If the renewal process is lattice, then

$$EN(nd) = \frac{nd}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + \frac{d}{2\mu} + o(1) \ \ as \ \ n \to \infty$$

$$VarN(nd) = \frac{\sigma^2 nd}{\mu^3} + o(n) \ \ as \ \ n \to \infty$$

Remarks: The first term of the expectation of the renewal counting process for the nonlattice case could be derived from Wald's equation and the second term describes the overshoot, which is the main issue when $t \not\to \infty$. The nature of $o(t)$ could be described by Keener's work [4] and the variance in the nonlattice case is valid as $t \to \infty$.

4.3 Expected Overshoot of Counting Renewal Processes

$R(t) = S_{N(t)+1} - t = S_{\upsilon(t)} - t = H$ is the overshoot of the counting renewal process with the expected overshoot given as

$$EH = ES_{\upsilon(t)} - t = \mu\left(E\upsilon(t) - \frac{t}{\mu}\right),$$

where $\upsilon(t)$ is the first passage time of the renewal process. The expected overshoot for both lattice and nonlattice renewal processes is given below;

Theorem 4.9 *[8] Suppose that $VarX_1 = \sigma^2 < \infty$*

(i) If the renewal process is nonlattice, then

$$EH \rightarrow \frac{\sigma^2 + \mu^2}{2\mu} \quad as \ t \rightarrow \infty$$

(i) If the renewal process is lattice, then

$$ER(nd) \rightarrow \frac{\sigma^2 + \mu^2}{2\mu} + \frac{d}{2} \quad as \ n \rightarrow \infty$$

Remark: With reference to the models discussed $t \rightarrow \infty \Rightarrow S - B \rightarrow \infty$, which is again bringing us back to the same issue as before, namely that all the known results in the literature focus on the overshoot when the boundary to be crossed is large, which is not the case in portfolio finance situations.

4.4 Distribution of the Overshoot for a Renewal Process as $S \rightarrow \infty$

Theorem 4.10 *[8] Suppose that $0 < EX_1 = \mu < \infty$.*

(i) If the renewal process is nonlattice, then for $x > 0$, we have

$$\lim_{t\rightarrow\infty} P(R(t) \leq x) = \frac{1}{\mu}\int_0^x (1 - F(s))ds.$$

(ii) If the renewal process is lattice, then for $k = 1, 2, 3, ...,$.we have

$$\lim_{n\to\infty} P(R(nd) \le kd) = \frac{d}{\mu} \sum_{j=0}^{k-1} (1 - F(jd)).$$

or, equivalently,

$$\lim_{n\to\infty} P(R(nd) \le kd) = \frac{d}{\mu} P(X_1 \ge kd).$$

Remark: The overshoot under appropriate conditions, may converge without normalization.

4.5 Moments of Overshoot

Theorem 4.11 *[8] Suppose that $EX_1^r < \infty$ for some $r > 1$. If the renewal process is nonlattice, then*

$$E(R(t))^{r-1} \to \frac{1}{r\mu} EX_1^r \text{ as } t \to \infty$$

4.6 Condition for finite moments of overshoot and first passage time

Theorem 4.12 *[8] Let $r \ge 1$. We have, in the case of random walk with positive drift,*

(i) $$E(X_1^-)^r < \infty \Leftrightarrow E(\upsilon(t))^r = E(\tau^+)^r < \infty; (i)$$

$\infty \Leftrightarrow E(S_\upsilon(t))^r = E(H) < \infty;$

Remark: The r-th moment of the expectation of the negative part of X_1, i.e., $X_1^- < \infty$ if and only if the expectation of the first passage time is less than ∞

The r-th moment of the expectation of $X_1^+ < \infty$ if and only if the expectation of the overshoot is less than ∞.

Theorem 4.13 *[8]*

$$E\left(\frac{\upsilon(t)}{t}\right) \to \frac{1}{u}$$

$\therefore$ *for large* $t = (S - B)$

$$E(\upsilon(t)) \approx \frac{t}{u}$$

for $p = 2$

$$E\left(\frac{\upsilon^2(t)}{t^2}\right) \to \frac{1}{u^2}$$

$\therefore$

$$V(\upsilon(t)) \approx \frac{t^2}{u^2} - \frac{t^2}{u^2} = 0$$

Remark: $E(\upsilon(t)/t) \to\ 0$ for $\mu = +\infty$ for theorem to hold. Now the variance of the $\upsilon(t)$ is not really zero; the way to interpret the above is to note that second order terms force $V(\upsilon(t))$ to be of order $O(t)$.

4.7 Distribution of the Overshoot for a Random Walk with Positive Drift

Theorem 4.14 *[8]*

(i) Suppose that the random walk is nonlattice, then for $x > 0$*, we have*

$$\lim_{t\to\infty} P(R(t) \le x) = \frac{1}{\mu_H}\int_0^x (S_{T_1} > y)dy.$$

(ii) If the renewal process is lattice, then for $k = 1, 2, 3, ...,$*.we have*

$$\lim_{n\to\infty} P(R(nd) \le kd) = \frac{d}{\mu_H}\sum_{j=0}^{k-1}(S_{T_1} > jd)$$

or, equivalently,

$$\lim_{n\to\infty} P(R(nd) \le kd) = \frac{d}{\mu}P(S_{T_1} \ge kd).$$

Remark: The overshoot for a positive random walk, in fact converges without normalization, but as with all the above results, we need the level to be crossed to be going to infinity.

4.8 Expectation of the Overshoot for a Random Walk with Positive Drift

Theorem 4.15 *[8] Suppose that $E(X_1^+)^2 < \infty$, then*

(i)If the random walk is nonlattice, we have

$$\lim_{t\to\infty} ER(t) = \frac{EY_1^2}{2\mu_H}$$

(i)If the random walk is lattice, we have

$$\lim_{t\to\infty} ER(nd) = \frac{EY_1^2}{2\mu_H} + \frac{d}{2}$$

Bounds in Theorem 4.15 are not easy to calculate.

Theorem 4.16 *[8] Suppose that $E(X_1^+)^2 < \infty$, then*

(i) If the random walk is nonlattice, we have

$$\lim_{t\to\infty} ER(t) \leq \frac{E(X_1^+)^2}{2\mu} + o(1) \text{ as } t \to \infty.$$

(i) If the random walk is lattice, we have

$$\lim_{n\to\infty} ER(nd) \leq \frac{E(X_1^+)^2}{2\mu} + \frac{d}{2} + o(1) \text{ as } n \to \infty.$$

4.9 Lorden's Inequality

Let R_∞ be a *positive* random variable having the distribution

$$P\{R_\infty \in dx\} = g(x) = \frac{P(x > x)}{\mu}dx$$

and let L_∞ be positive random variable having the distribution

$$P\{L_\infty \in \ dx\} = h(x) = \frac{xP\{X \in dx\}}{\mu} dx,$$

then it can be shown that

$$E(R_\infty^p) = \frac{1}{\mu(p+1)} E(X^{p+1}) \text{ and } E(L_\infty^p) = \frac{1}{\mu} E(X^{p+1}) (p > 0)$$

Moreover, Lorden's inequality states that if $X \geq 0$ then

$$E(R_b^p) \leq (p+2) E(R_\infty^p).$$

Moreover for $p = 1$, the first moment of the overshoot is bounded as follows:

$$E(R_b) \leq 2E(R_\infty).$$

Comparing this result with the first moment of Theorem 4.16

$$E(R_b) \leq 2E(R_\infty)$$

and

$$E(R) \leq \frac{1}{2\mu} E(X_1^2) \ \textit{if} \ E(X_1^2) < \infty \ \textit{and} \ t \to \infty,$$

Lorden comes closest to giving answers for all nonnegative b (not just $b \to \infty$, but with an upper bound inequality.) Also, Lorden's work gives a bound on the tail probability distribution of R_b

$$R(R_b > x) \leq \frac{b + ER_b}{b + x} P(L_\infty > x)$$

Is this of use in better computing or bounding the quantities in Keener's work for positive random variables? We leave this question for future work.

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