5-2013

Restricted and Unrestricted Coverings of Complete Bipartite Graphs with Hexagons

Wesley M. Surber
East Tennessee State University

Restricted and Unrestricted Coverings of Complete Bipartite Graphs with Hexagons

---

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

---

by

Wesley Surber

May 2013

Robert Gardner, Ph.D., Chair

Robert Beeler, Ph.D.

Ariel Cintron-Arias, Ph.D.

Keywords: graph theory, experimental design, covering, decomposition

ABSTRACT

Restricted and Unrestricted Coverings of Complete Bipartite Graphs with Hexagons

by

Wesley Surber

A *minimal covering* of a simple graph G with isomorphic copies of a graph H is a set $\{H_1, H_2, \ldots, H_n\}$ where $H_i \cong H$, $V(H_i) \subset V(G)$, $E(G) \subset \cup_{i=1}^{n} E(H)_i$, and $|\cup_{i=1}^{n} E(H_i) \setminus E(G)|$ is minimal (the graph $\cup_{i=1}^{n} H_i$ may not be simple and $\cup_{i=1}^{n} E(H_i)$ may be a multiset). Some studies have been made of covering the complete graph, in which case an added condition of "$E(H_i) \subset E(G)$ for all i" implies no additional restrictions. However, if G is not the complete graph then this condition may have implications. We will give necessary and sufficient conditions for minimal coverings (as defined above, without the added restriction) of $K_{m,n}$ with 6-cycles, which we call minimal *unrestricted coverings*. We also give necessary and sufficient conditions for minimal coverings of $K_{m,n}$ with 6-cycles with the added condition $E(H_i) \subset E(G)$ for all i, and call these minimal *restricted coverings*.

TABLE OF CONTENTS

LIST OF FIGURES

1 GRAPH DECOMPOSITION

1.1 Definition

Graph decomposition is a part of design theory where we can take a graph and break it into isomorphic subgraphs whose union is the original graph. Formally, a *decomposition* of a graph G is a set of isomorphic subgraphs H, $\{H_1, H_2, \ldots H_n\}$, where $H_i \cong H$, $E(H) \subset E(G)$, and $V(H_i) \subset V(G)$ for all $i \in \{1, 2, \ldots, n\}$. Also, $E(H_i) \cap E(H_j) = \emptyset$ if $i \neq j$ and $\cup_{i=1}^{n} E(H_i) = E(G)$. In Figure 1, we see an example of graph decomposition.

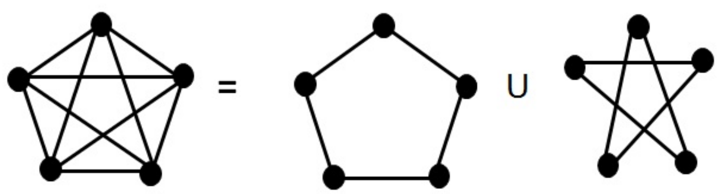

Figure 1: A K_5 and its C_5 decomposition.

1.2 History

In 1853, Jakob Steiner started to elaborate on an idea on the conditions that it would take to decompose a complete graph into smaller subgraphs which were all isomorphic [12]. His result was that if a complete graph is of order 1 or 3 (mod 6), then there exists a decomposition into C_3's [12], where C_3 denotes a 3-cycle. This research was unique during that time period since modern day graph theory was established many years later. Today, we refer to such decompositions as Steiner triple systems.

A *steiner triple system* of order n, $STS(n)$, is a decomposition of a complete graph on n vertices into C_3's.

Oddly, a similar theorem was mentioned in 1847. Thomas P. Kirkman stated the necessary and sufficient conditions for the same decompositions. He states that there is a C_3-decomposition of a complete graph if and only if the order of the complete graph is 1 or 3 (mod 6) [8]. An interesting fact is that even though Thomas P. Kirkman's publication was similar and earlier than Steiner's research, we still refer to *Steiner triple systems* when working with C_3 decompositions of complete graphs.

1.3 Motivation

Graph decomposition is part of experimental design theory because it is used to model situations where ideal grouping will lead to efficient and cost effective testing. For example, consider a company that needs to compare a certain characteristic of all samples in a test group with equipment that the business owns. However, this equipment is only able to compare three samples at a time. This example leads to the creation of a model that relates to graph decomposition. Assume we have seven samples in the test group, so the graph that would represent the samples would be a complete graph on seven vertices where the vertices represent samples and the edges represent comparisons. Assume the machine is only able to handle three samples at a time; each run of the machine would be represented by a C_3. We know that a decomposition exists in this case because the necessary and sufficient conditions for a complete graph to have a decomposition into C_3's. In Figure 2, we create a model that would represent this problem.

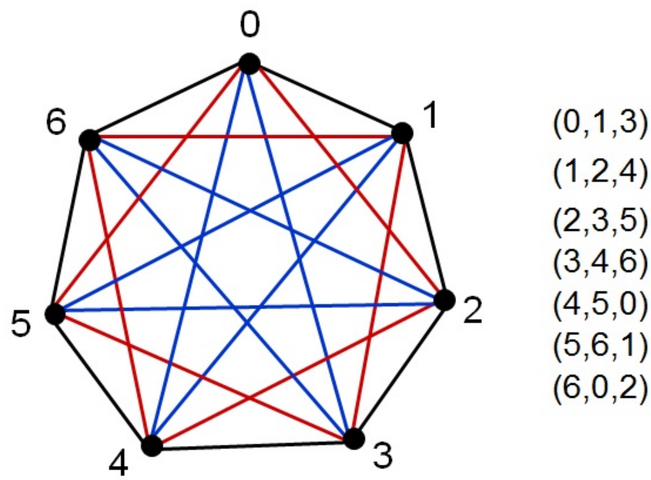

Figure 2: A C_3 decomposition of a K_7.

In Figure 2, we have a C_3 decomposition of a K_7. The triples are in the form of (a, b, c) which is a C_3 that is created by the vertices a, b, and c. We know by Jakob Steiner that this has a C_3 decomposition so we are able to make a construction. We can use a technique called difference method to easily create this construction. Let us define the difference associated with the edge (a, b) as $min\{(a-b)(\text{mod } (n-1)), (b-a)(\text{mod } (n-1))\}$. For example, the edge $(1, 3)$ has a difference of 2 and the edge $(0, 6)$ has a difference of 1. Then we want to create a triple that uses each difference only once. We can observe from Figure 2 that a black edge has associated difference of 1, a red edge has associated difference of 2, and a blue edge has associated difference of 3. We create the first C_3 as $(0, 1, 3)$ which uses each distance exactly once. Therefore, if we add 1 to each vertex label, then we will create another triple that is unique from the original, in this case, $(1, 2, 4)$. We repeat this process until all edges are used.

2 GRAPH PACKINGS AND COVERINGS

2.1 Definitions

Since graph decompositions are types of experimental designs, they are used in real world situations by creating models and solutions to given questions. However, not all models will have the conditions that is needed for a decomposition. We know the necessary and sufficient conditions for complete graphs to be decomposed into cycles of length n by Alspach, Brian, and Heather Gavlas [4] and complete bipartite graphs to be decomposed into cycles of length n by D. Sotteau [11]. However, we know there are cases where the conditions are not met. In these cases, we can define a graph covering and graph packing.

Assume we have parameters of the complete graph that do not meet the conditions for a decomposition. If we are not required to have every edge in the original graph G in the union of the subgraphs, H, then we have an alternative graph decomposition called a graph packing.

A *maximal packing* of a simple graph G with isomorphic copies of a graph H is a set $\{H_1, H_2, \ldots, H_n\}$ where $H_i \cong H$ and $V(H_i) \subset V(G)$ for all i, $E(H_i) \bigcap E(H_j) = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{n} E(H_i) \subset E(G)$, and $|E(G) \setminus \bigcup_{i=1}^{n} E(H_i)|$ is minimal. The set of edges for the *leave*, L, of the packing is $E(L) = E(G) \setminus \bigcup_{i=1}^{n} E(H_i)$.

Assume we have a graph G such as a K_4 and we want to decompose this graph into C_4's. Let $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{(1,2), (2,3), (3,4), (1,4), (1,3), (2,4)\}$. Since we know that a C_4 will take 4 edges to create, then we will have 2 edges left over. These edges will be our leave. The C_4 that is created is $\{(1, 2, 3, 4)\}$ which is the edges$\{(1,2),(1,4),(2,3),(3,4)\}$ and the leave will be, $E(L) = \{(1,3),(2,4)\}$.

In certain models, it is required that we use every edge that exists in the original graph. However, if there exists conditions where a decomposition does not exist and every edge must be used, then we will use a graph covering.

A *minimal graph covering* of a simple graph G is a set of isomorphic graphs H, $\{H_1, H_2, \ldots H_n\}$ where $E(H) \subset E(G)$, $V(H_i) \subset V(G)$, and $E(G) \subset \cup_{i=1}^{n} E(H)_i$. It is possible for $\cup_{i=1}^{n} E(H)_i$ to be a multiset since we are allowing the reuse of edges. The padding is equal to these edges that are reused, $E(P) = \cup_{i=1}^{n} E(H)_i \setminus E(G)$. So we know that the cardinality of a padding is the number of edges that must be repeated for a graph covering, $|E(P)| = |\cup_{i=1}^{n} E(H_i) \setminus E(G)|$.

Assume we have a graph G such as K_6 and we want to create a graph covering with cycles of size 3, C_3. The first observation that should be made is the argument against the degree of each vertex. We know that in a complete graph, K_n, each vertex has degree $n-1$. Every vertex of a cycle is of even degree. In the case of a K_6, we know that each vertex has degree 5. This means we need to change every vertex to an even degree by adding edges from the padding. However, we want this padding to be minimal, so we must use the most cost efficient padding. In this case, since we have six vertices, we have a minimum of 3 edges can be added from a matching, the vertices which give us an even degree at each vertex of K_6. There exists 15 edges in a K_6 so if we add 3 more edges for the padding as a matching that would bring the total edge cardinality to 18, which is divisible by 3, the number of edges in a C_3.

In Figure 3, we have a K_6. From the difference method once again, where the red edges are difference 1, green edges are difference 2, and black edges are difference

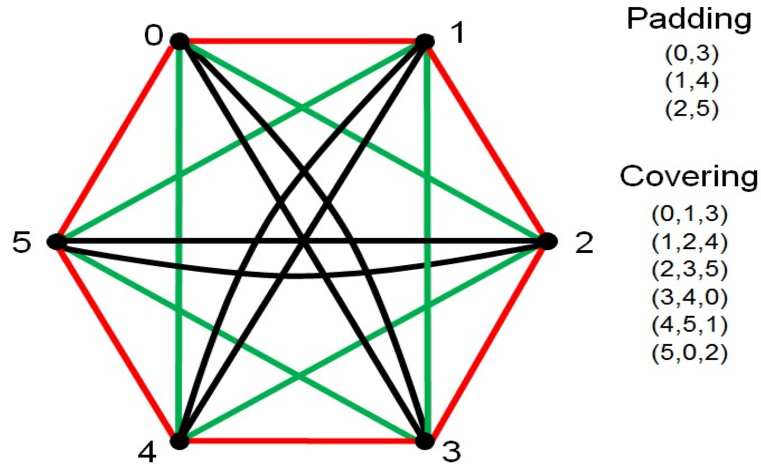

Figure 3: A C_3-covering of a K_6.

3, we can create a covering. What we notice though, is that we need to reuse the edges of distance 3, this well be represented by the curved edges. Note, that since there is an even amount of vertices and each has odd degree, we can pair each vertex with another vertex to correct the odd degree argument. In Figure 3, we can see our covering and the padding that is generated. We have the necessary and sufficient conditions for a minimal restricted covering where padding is minimal and there is a minimal covering of a K_6 with C_3.

2.2 Restricted and Unrestricted Coverings

Graph coverings have often been studied in the setting of complete graphs. What if our model should take on a different form? Let us say, a partite graph. A partite graph has n partitions of its vertices, $\{V_1, V_2, \ldots, V_n\}$, where all possible edges exist between V_i and V_j where $i \neq j$ and no edges exist between the partitions when $i = j$.

For example, a $K_{1,2,3}$ would have a total of 11 edges. If all possible connections existed, then we would have a total of 15 edges. Let us assume that we have a graph G which is a $K_{1,2,3}$ and we want to decompose it into C_3's. In this example, let us assume that the edges that are not present in G are "forbidden," then we would have what we call a *restricted covering.*

A *minimal restricted covering* of a simple graph G is a set of isomorphic graphs H, $\{H_1, H_2, \ldots H_n\}$, where $E(H) \subset E(G)$, $V(H_i) \subset V(G)$, and $E(G) \subset \cup_{i=1}^{n} E(H)_i$. It may be that $\cup_{i=1}^{n} E(H)_i$ is a multiset since we are allowing the reuse of edges. The padding is equal to these edges that are reused, $E(P) = \cup_{i=1}^{n} E(H_i)/E(G)$. So we know that the cardinality of padding is the number of edges that must be repeated for a graph covering, $|E(P)| = |\cup_{i=1}^{n} E(H_i)/E(G)|$ where $|E(P)|$ is minimal. The requirement to stress here is $E(H) \subset E(G)$ which limits us to only be able to use the edges that exist in the original graph G.

If we wanted a restricted covering of a graph G, $K_{1,2,3}$, with C_3's, then we will make a construction after predicting the minimum number of edges required for a covering. For the vertex set, we have $V(G) = \{1_1, 1_2, 2_2, 1_3, 2_3, 3_3\}$, and for the edge set we have $E(G) = \{(1_1, 1_2), (1_1, 2_2), (1_1, 1_3), (1_1, 2_3), (1_1, 3_3), (1_2, 1_3), (1_2, 2_3),$ $(1_2, 3_3)$, $(2_2, 1_3)$, $(2_2, 2_3)$, $(2_2, 3_3)\}$. Let us name the partite sets V_1, V_2, and V_3 where $E(V_1) = \{1_1\}$, $E(V_1) = \{1_2, 2_2\}$, and $E(V_1) = \{1_3, 2_3, 3_3\}$. We know that there exists four vertices with odd degree, three in V_3, and one V_1. We can predict that we can connect one edge between V_1 and V_3 to make two even degree vertices, and would need another two edges to fix the degree of the other two odd vertices. However, we also need more edges in the V_2 because we have is 3 edges going to V_3 and only one coming from V_1. So we need a minimum of four extra edges from V_1 to V_2 to create cycles that uses all the edges connected to V_2 and V_3. This in total would be an extra seven edges for a covering. Let the construction of the covering be $\{(1_1, 1_2, 1_3)$,$(1_1, 3_3, 2_2)$,$(1_1, 2_3, 2_2)$,$(1_1, 1_3, 2_2)$, $(1_1, 2_3, 1_2)$,$(1_1, 3_3, 1_2)\}$. Then the set of the padding for this example will be $E(P) = \{(1_1, 2_3), (1_1, 2_2)$,$(1_1, 1_3)$,$(1_1, 2_2)$,$(1_1, 1_2)$, $(1_1, 3_3)$, $(1_1, 1_2)\}$. There exists a construction, as seen in Figure 4, where the padding is seven. Therefore, we have found a minimal restricted covering of a $K_{1,2,3}$ with C_3's.

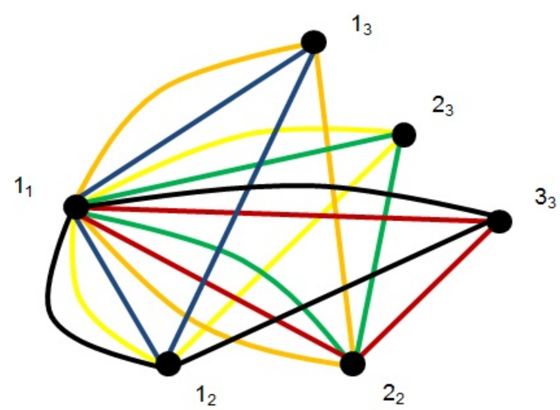

Figure 4: A C_3 restricted covering of a $K_{1,2,3}$.

Let us now assume that the restriction of a partite graph does not exist and we can now use edges that would connect vertices that exist in the same partite set. This means even though the edge $(1_2, 2_2)$ is not in the original graph, we are able to use it in the covering. This leads us to unrestricted graph coverings.

A *minimal unrestricted covering* of simple graph G with isomorphic copies of a graph H is a set $\{H_1, H_2, \ldots, H_n\}$ where $H_i \cong H$, $V(H_i) \subset V(G)$, $E(G) \subset$

$\cup_{i=1}^{n} E(H)_i$, and $| \cup_{i=1}^{n} E(H_i) \setminus E(G)|$ is minimal. The graph $\cup_{i=1}^{n} E(H)_i$ may not be simple and $\cup_{i=1}^{n} E(H_i)$ may be a multiset. The thing to note here is the fact that the $E(H) \subset E(G)$ is missing. We are now allowing these forbidden edges to be in our covering.

If we wanted an unrestricted covering of a graph G, $K_{1,2,3}$, with C_3's, then we will make a construction after predicting the minimum number of edges required for a covering. For the vertex set, we have $V(G) = \{1_1, 1_2, 2_2, 1_3, 2_3, 3_3\}$ and for the edge set we have $E(G) = \{(1_1, 1_2), (1_1, 2_2), (1_1, 1_3), (1_1, 2_3), (1_1, 3_3), (1_2, 1_3), (1_2, 2_3), (1_2, 3_3), (2_2, 1_3), (2_2, 2_3), (2_2, 3_3)\}$. Let us name the three partitions V_1, V_2, and V_3 as $E(V_1) = \{1_1\}$, $E(V_2) = \{1_2, 2_2\}$, and $E(V_3) = \{1_3, 2_3, 3_3\}$. We know that there exists four vertices with odd degree, three in V_3 and one in V_1. So, we have four odd degree vertices. Since we can have edges within the partite set, we can say we need two edges to fix all the odd degree arguments. Since we started with 11 edges and we need at least 2 more, we are at a total of 13 edges which is not divisible by three. We need the number of edges to be divisble by the order of the cycle or a covering cannot exist, so we need to add two more edges, bringing the total up to 15 edges. Let the construction of the covering be $\{(1_1, 1_2, 1_3), (1_1, 3_3, 2_2), (1_1, 1_3, 2_2), (1_1, 2_3, 2_2), (1_2, 2_3, 3_3)\}$. Then the set of the padding for this example will be $E(P) = \{(1_1, 1_3), (1_1, 2_2), (1_1, 2_2), (2_3, 3_3)\}$. Since there exists a construction where the padding is four and we predicted a padding of four, we have found an unrestricted minimal covering of a $K_{1,2,3}$ with C_3's. This is shown in Figure 5, where the curved lines are the padding.

As observed, the padding is reduced when using the unrestricted method. If we observe a covering of a complete graph, then we can assume that the restricted and

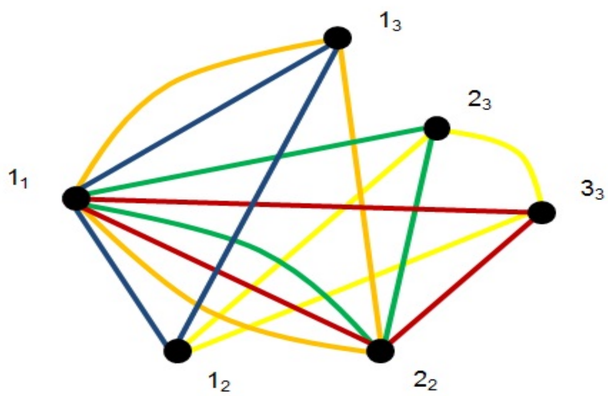

Figure 5: A C_3 unrestricted covering of a $K_{1,2,3}$.

unrestricted coverings will be equal since all edges exist. In cases where there are missing edges in the original graph, then it is possible for the unrestricted covering to be more efficient.

2.3 Previous Results

The following theorems are previous results that give us a foundation of decompositons for complete bipartite graphs.

Theorem 2.1 [2] *The complete bipartite graph $K_{m,n}$ can be decomposed into hexagons if and only if $m \equiv 0$ (mod 6) and $n \equiv 0$ (mod 2),$n \geq 4$.*

The following theorem originates from one of Dominique Sotteau's published papers [11]. In 1981, Sotteau published a paper that stated the conditions that are required for a complete bipartite graph, $K_{m,n}$ to have a decompositon of cycles of length $2k$.

Theorem 2.2 [11] *$K_{m,n}$ can be decomposed into 2k-cycles if and only if $m \geq k$, $n \geq k$, and k divides mn.*

This theorem implies the three conditions that are required: m and n must be even, m and n must be larger than k, and k must divide mn. The proof is intuitive because m and n must be even for there to exist a cycle. Think of one degree being an entrance and another degree being an exit. If either are missing, then it is impossible for there to exist a cycle. The cardinality of the partite sets must be larger than k for there to exist a cycle because if m or n was smaller than k, then we would not have enough vertices to complete the cycle. Lastly, k must divide mn because of the edge cardinality arguement; if there are not enough edges or too many edges, then there cannot exist a decomposition.

In the case for which $k = 3$, we are trying to decompose a complete bipartite graph into hexagons. Theorem 2.1 requires m to be 0 (mod 6) and n to be 0 (mod 2) and by Sotteau, m and n must be larger than three. In Theorem 2.1, since m can not be zero, it would then be six or larger and n must be four or larger because that is the first even number greater than 3. Also, since m and n are even, we have met two out of the three requirements. For the last requirement, since m will always be divisible by six, then mn will always be divisible by six. Therefore, Theorem 2.1 meets all requirements by Sotteau's conditions in Theorem 2.2 for a complete bipartite graph to be decomposed by hexagons.

Also, we have to use theorems that were proved for the use of graph packings. If we are able to generate a leave and then cover that as a seperate graph, then it may be possible to generate a minimal covering. We will be using a previous theorem thats states a decomposition exists if a matching is subtracted from the original graph.

Theorem 2.3 [2] *A hexagon decomposition of $K_{n,n} \setminus M$, where M is a perfect matching of $K_{n,n}$, exists if and only if $n \equiv 1$ or 3 (mod 6).*

2.4 Proof Technique

For a theorem which states the necessary and sufficient conditions for a restricted or unrestricted covering of the complete bipartite graph, we need lemmas that state all possible coverings of a complete bipartite graph. The most basic cases are if m and n are both odd, if m and n are both even, and if one is odd and the other is even. Then we break those cases into sub cases, such as if m and n are both odd, then we need to check all cases of the values of m and n against each other for all possible situations. We will define the larger partition of a complete bipartite graph as V_m that has the cardinality of m and the smaller partition of the same complete bipartite graph as V_n that has the cardinality of n. Let us introduce the lemmas that prove the padding for restricted and unrestriced coverings. First, let us look at all the lemmas that are required for restricted hexagon coverings.

Lemma 2.4 *A minimal restricted hexagon covering of $K_{m,n}$ where m and n are even, $m, n \geq 4$, has a padding P satisfying:*

(1) $|E(P)| = 0$ when $m \equiv 0$ (mod 6),

(2) $|E(P)| = 2$ when $m \equiv n \equiv 2$ (mod 6) or $m \equiv n \equiv 4$ (mod 6), and

(3) $|E(P)| = 4$ when $m \equiv 2$ (mod 6) and $n \equiv 4$ (mod 6).

Proof. We consider cases.

Case 1. Suppose $m \equiv 0$ (mod 6), $n \equiv 0$ (mod 2), and $n \geq 4$. Then $K_{m,n}$ can be decomposed into hexagons by Theorem 2.2 and in a minimal covering $|E(P)| = 0$.

Case 2. Suppose $m \equiv n \equiv 2 \pmod 6$, $m, n \geq 4$. Now $|E(K_{m,n})| \equiv 4 \pmod 6$, so it is necessary that a covering have a padding with $|E(P)| \geq 2$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-8} \cup K_{8,8}$ where the partite sets of $K_{m-8,n}$ are $\{9_1, 10_1, \ldots, m_1\}$ and V_n, the partite sets of $K_{8,n-8}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{9_2, 10_2, \ldots, n_2\}$, and the partite sets of $K_{8,8}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 8_2\}$. Now $K_{m-8,n}$ and $K_{8,n-8}$ can be decomposed into hexagons by Theorem 2.2. Next we note that there is a restricted hexagon covering of $K_{8,8}$, namely the set $\{$ $[2_1, 2_2, 4_1, 8_2, 1_1, 3_2]$, $[3_1, 3_2, 5_1, 1_2, 2_1, 4_2]$, $[5_1, 5_2, 7_1, 3_2, 4_1, 6_2]$, $[7_1, 7_2, 1_1, 5_2, 6_1, 8_2]$, $[8_1, 3_2, 6_1, 1_2, 4_1, 7_2]$, $[6_1, 6_2, 1_1, 4_2, 5_1, 7_2]$, $[1_1, 1_2, 3_1, 8_2, 5_1, 3_2]$, $[2_1, 5_2, 8_1, 6_2, 3_1, 7_2]$, $[8_1, 2_2, 7_1, 6_2, 2_1, 8_2]$, $[8_1, 2_2, 6_1, 4_2, 7_1, 1_2]$, $[8_1, 2_2, 3_1, 5_2, 4_1, 4_2]\}$. This is a minimal restricted covering of $K_{m,n}$ with padding P where $E(P) = \{(8_1, 2_2), (8_1, 2_2)\}$ and so $|E(P)| = 2$.

Case 3. Suppose $m \equiv n \equiv 4 \pmod 6$. As in Case 2, a packing with padding P satisfies $|E(P)| \geq 2$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-4} \cup K_{4,4}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-4}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$, and the partite sets of $K_{4,4}$ are $\{1_1, 2_2, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2, 4_2\}$. Now $K_{m-4,n}$ and $K_{4,n-4}$ can be decomposed into hexagons by Theorem 2.2. Next we note that there is a restricted hexagon covering of $K_{4,4}$, namely the set $\{[1_1, 1_2, 3_1, 4_2, 4_1, 2_2]$, $[1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, $[1_1, 1_2, 4_1, 3_2, 2_1, 4_2]\}$. This is a minimal restricted covering of $K_{m,n}$ with padding P where $E(P) = \{2 \times (1_1, 1_2)\}$ and so $|E(P)| = 2$.

Case 4. Suppose $m \equiv 2 \pmod 6$, $m \geq 8$, and $n \equiv 4 \pmod 6$. Now $|E(K_{m,n}| \equiv 2 \pmod 6$, so it is necessary that a covering have a padding with $|E(P)| \geq 4$. Now $K_{m,n} = K_{8,n-4} \cup K_{m-8,n} \cup K_{8,4}$ where the partite sets of $K_{8,n-4}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and

$V_n \backslash \{1_2, 2_2, 3_2, 4_2\}$, the partite sets of $K_{m-8,n}$ are $\{9_1, 10_1, \ldots, m_1\}$ and V_n, and the partite sets of $K_{8,4}$ are $\{1_1, 2_2, \ldots, 8_1\}$ and $\{1_2, 2_2, 3_2, 4_2\}$. Now $K_{8,n-4}$ and $K_{m-8,n}$ can be decomposed into hexagons by Theorem 2.2. Next we note that there is a restricted hexagon covering of $K_{8,4}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, $[4_1, 1_2, 3_1, 4_2, 2_1, 3_2]$, $[5_1, 1_2, 6_1, 2_2, 7_1, 3_2]$, $[8_1, 1_2, 7_1, 4_2, 6_1, 3_2]$, $[1_1, 4_2, 8_1, 1_2, 5_1, 2_2]$, $[4_1, 4_2, 5_1, 1_2, 8_1, 2_2]\}$. This is a minimal restricted covering of $K_{m,n}$ with padding P where $E(P) = \{(5_1, 1_2)$, $(5_1, 1_2)$, $(8_1, 1_2)$, $(8_1, 1_2)\}$ and so $|E(P)| = 4$. $\square$

Lemma 2.5 *A minimal restricted hexagon covering of $K_{m,n}$ where m is even and n is odd ($m \geq 4, n \geq 3$) has a padding P satisfying $|E(P)| = m + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m + k) \equiv 0 \pmod 6$.*

Proof. Since each vertex of V_m is of odd degree in $K_{m,n}$, in the padding of a covering each of these vertices will be of odd degree. Therefore, in a restricted covering of $K_{m,n}$ with padding P, it is necessary that $|E(P)| \geq m$. Since a covering yields a decomposition of $K_{m,n} \cup P$, then it is necessary that $|E(K_{m,n})| + |E(P)| \equiv 0$ (mod 6).

Case 1. First, suppose $m \equiv 0 \pmod 6$ and $n = 5$. Consider the set of hexagons $\{[(1+6i)_1, 1_2, (2+6i)_1, 2_2, (3+6i)_1, 3_2], [(4+6i)_1, 3_2, (5+6i)_1, 4_2, (6+6i)_1, 5_2], [(3+6i)_1, 1_2, (5+6i)_1, 2_2, (4+6i)_1, 4_2], [(1+6i)_1, 2_2, (6+6i)_1, 3_2, (2+6i)_1, 4_2], [(3+6i)_1, 1_2, (6+6i)_1, 3_2, (5+6i)_1, 5_2], [(1+6i)_1, 1_2, (4+6i)_1, 2_2, (2+6i)_1, 5_2] \mid i = 0, 1, \ldots, m/6 - 1\}$. This is a restricted hexagon covering of $K_{m,n}$ with padding P satisfying $E(P) = \{((1+6i)_1, 1_2), ((2+6i)_1, 2_2), ((3+6i)_1, 1_2), ((4+6i)_1, 2_2), ((5+6i)_1, 3_2), ((6+6i)_1, 3_2) \mid i = 0, 1, \ldots, m/6 - 1\}$, and so $|E(P)| = m$ and the restricted covering is minimal.

Next, suppose $m \equiv 0 \pmod 6$, $n \equiv 1 \pmod 2$, and $n \neq 5$. Now $K_{m,n} = K_{m,n-3} \cup \frac{m}{6} \times K_{6,3}$ where the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$, and $E(\frac{m}{6} \times K_{6,3}) = \{[(1+6i)_1, 1_2, (4+6i)_1, 2_2, (2+6i)_1, 3_2], [(3+6i)_1, 2_2, (5+6i)_1, 1_2, (6+6i)_1, 3_2], [(3+6i)_1, 1_2, (5+6i)_1, 3_2, (4+6i)_1, 2_2], [(1+6i)_1, 2_2, (6+6i)_1, 1_2, (2+6i)_1, 3_2] \mid i = 0, 1, \ldots, \frac{m}{6} - 1\}$. Now $K_{m,n-3}$ can be decomposed into hexagons by Theorem 2.2. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding P where $E(P) = \{((1+6i)_1, 3_2), ((2+6i)_1, 3_2), ((3+6i)_1, 2_2), ((4+6i)_1, 2_2), ((5+6i)_1, 1_2), ((6+6i)_1, 1_2) \mid i = 0, 1, \ldots, \frac{m}{6} - 1\}$ and so $|E(P)| = m$ and the restricted covering is minimal.

Case 2. Suppose $m \equiv 2 \pmod 6$, $m \geq 8$, $n \equiv 1 \pmod 6$, and $n \geq 7$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-7} \cup K_{8,7}$ where the partite sets of $K_{m-8,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and V_n, the partite sets of $K_{8,n-7}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{8,7}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$. Now $K_{m-8,n}$ has a restricted hexagon covering with padding P where $|E(P)| = m - 8$ (by Case 1) and $K_{8,n-7}$ can be decomposed into hexagons by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{8,7}$, namely the set $\{[2_1, 4_2, 3_1, 5_2, 4_1, 6_2]$, $[6_1, 4_2, 7_1, 5_2, 8_1, 6_2]$, $[1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, $[5_1, 1_2, 6_1, 2_2, 7_1, 3_2]$, $[1_1, 2_2, 3_1, 1_2, 4_1, 7_2]$, $[5_1, 2_2, 7_1, 1_2, 8_1, 7_2]$, $[1_1, 4_2, 4_1, 3_2, 2_1, 5_2]$, $[5_1, 4_2, 8_1, 3_2, 6_1, 5_2]$, $[1_1, 6_2, 3_1, 7_2, 2_1, 1_2]$, $[5_1, 6_2, 7_1, 7_2, 6_1, 1_2]$, $[4_1, 2_2, 8_1, 3_2, 3_1, 1_2]\}$ with padding P_2 satisfying $E(P_2) = \{(3_1, 2_2), (7_1, 2_2), (1_1, 1_2), (2_1, 1_2), (5_1, 1_2), (6_1, 1_2), (8_1, 3_2), (3_1, 3_2), (3_1, 1_2), (4_1, 1_2)\}$ and so $|E(P_2)| = 10$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 2$ and the restricted covering is minimal.

Case 3. Suppose $m \equiv 2 \pmod 6$, $n \equiv 3 \pmod 6$, and $m \geq 8$. Now $K_{m,n} =$

$K_{m-8,n} \cup K_{8,n-3} \cup K_{8,3}$ where the partite sets of $K_{m-8,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and V_n, the partite sets of $K_{8,n-3}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2\}$, and the partite sets of $K_{8,3}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, 3_2\}$. Now $K_{m-8,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m - 8$ (by Case 1), and there is a hexagon decomposition of $K_{8,n-3}$ by Theorem 2.2.

Next, we note that there is a restricted hexagon covering of $K_{8,3}$, namely the set $\{[1_1, 3_2, 3_1, 2_2, 2_1, 1_2], [4_1, 1_2, 5_1, 2_2, 6_1, 3_2], [6_1, 1_2, 7_1, 3_2, 8_1, 2_2], [2_1, 1_2, 7_1, 2_2, 5_1, 3_2], [1_1, 1_2, 8_1, 3_2, 4_1, 2_2], [3_1, 1_2, 4_1, 2_2, 5_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(6_1, 2_2), (2_1, 1_2), (7_1, 1_2), (5_1, 2_2), (1_1, 1_2), (4_1, 3_2), (8_1, 3_2), (3_1, 3_2), (5_1, 3_2), (5_1, 2_2), (4_1, 2_2), (4_1, 1_2)\}$ and so $|E(P_2)| = 12$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 4$ and the restricted covering is minimal.

Case 4. Suppose $m \equiv 2 \pmod 6$, $n \equiv 5 \pmod 6$, and $m \geq 8$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-5} \cup K_{8,5}$ where the partite sets of $K_{m-8,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and V_n, the partite sets of $K_{8,n-5}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 5_2\}$, and the partite sets of $K_{8,5}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$. Now $K_{m-8,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m - 8$ (by Case 1), and there is a hexagon decomposition of $K_{8,n-5}$ by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{8,5}$, namely the set $\{[1_1, 1_2, 2_1, 3_2, 3_1, 2_2], [5_1, 1_2, 6_1, 3_2, 7_1, 2_2], [1_1, 1_2, 3_1, 4_2, 4_1, 5_2,] \; [5_1, 1_2, 7_1, 4_2, 8_1, 5_2], [2_1, 2_2, 4_1, 3_2, 3_1, 5_2], [6_1, 2_2, 8_1, 3_2, 7_1, 5_2], [5_1, 3_2, 8_1, 1_2, 6_1, 4_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 1_2), (3_1, 3_2), (5_1, 1_2), (7_1, 3_2), (6_1, 1_2), (8_1, 3_2), (2_1, 1_2), (4_1, 3_2)\}$ and so $|E(P_2)| = 8$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 5. Suppose $m \equiv 4 \pmod 6$, $n \equiv 1 \pmod 6$, and $n \geq 7$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-7} \cup K_{4,7}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{4,7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, \ldots 7_2\}$. Now $K_{m-4,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m - 4$ (by Case 1), and there is a hexagon decomposition of $K_{4,n-7}$ by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{4,7}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [2_1, 5_2, 3_1, 2_1, 4_2, 4_1, 5_2], [1_1, 3_2, 2_1, 4_2, 4_1, 3_1, 7_2], [1_1, 2_2, 4_1, 3_2, 2_1, 4_2], [2_1, 5_2, 3_1, 1_2, 4_1, 6_2]\}$, with padding P_2 satisfying $E(P_2) = \{(1_1, 3_2), (2_1, 4_2), (2_1, 3_2), (2_1, 4_2), (2_1, 5_2), (2_1, 6_2), (3_1, 5_2), (4_1, 6_2)\}$ and so $|E(P_2)| = 8$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 4$.

Case 6. Suppose $m \equiv 4 \pmod 6$ and $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-3} \cup K_{4,3}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-3}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2\}$, and the partite sets of $K_{4,3}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2\}$. Now $K_{m-4,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m - 4$ (by Case 1) and there is a hexagon decomposition of $K_{4,n-3}$ by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{4,3}$, namely the set $\{[1_1, 1_1, 2_1, 2_2, 3_1, 3_2], [1_1, 1_2, 2_1, 3_2, 4_1, 2_2], [2_1, 2_2, 3_1, 1_2, 4_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 1_2), (2_1, 1_2), (2_1, 2_2), (2_1, 3_2), (3_1, 2_2), (4_1, 3_2)\}$ and so $|E(P_2)| = 6$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 2$.

Case 7. Suppose $m \equiv 4 \pmod 6$, $n \equiv 5 \pmod 6$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-5} \cup K_{4,5}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of

$K_{4,n-5}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 5_2\}$, and the partite sets of $K_{4,5}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, \ldots 5_2\}$. Now $K_{m-4,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m-4$ (by Case 1) and there is a hexagon decomposition of $K_{4,n-5}$ by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{4,5}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [1_1, 2_2, 4_1, 5_2, 3_1, 4_2], [2_1, 3_2, 3_1, 1_2, 4_1, 4_2], [1_1, 1_2, 4_1, 3_2, 2_1, 5_2]\}$ with padding P_2 satisfying $E(P_2) = \{(3_1, 3_2), (1_1, 1_2), (2_1, 3_2), (4_1, 1_2), \}$ and so $|E(P_2)| = 4$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$. $\Box$

Lemma 2.6 *A minimal restricted hexagon covering of $K_{m,n}$ where $m = n \equiv 5$ (mod 6) has padding P satisfying $|E(P)| = m$.*

Proof. Each vertex of V_m is of odd degree, so in a minimal covering, as in Lemma 2.5, it is necessary that $|E(P)| \geq m$. In the constructions for this case, we assume that $V_m = \{0_1, 1_1, \ldots, (m-1)_1\}$ and $V_n = \{0_2, 1_2, \ldots, (n-1)_2\}$. In each of the following two cases, we reduce the vertex labels modulo m.

Case 1. Suppose $m = n \equiv 5$ (mod 12). Consider the set of hexagons $\{[j_1, (4+j)_2, (1+j)_1, (1+j)_2, (m-1+j)_1, j_2]\} \cup \{[j_1, (11+12i+j)_2, (2+j)_1, (9+12i+j)_2, (1+j)_1, (6+12i+j)_2], [j_1, (16+12i+j)_2, (1+j)_1, (15+12i+j)_2, (2+j)_1, (12+12i+j)_2] \mid i = 0, 1, \ldots, (n-17)/12; j = 0, 1, 2, \ldots, m-1\}$. This is a minimal hexagon covering of $K_{m,n}$ with padding P where $E(P) = \{(i_1, i_2) \mid i = 0, 1, \ldots, m-1\}$, and so $|E(P)| = m$.

Case 2. Suppose $m = n \equiv 11$ (mod 12). Consider the set of hexagons $\{[j_1, (6+12i+j)_2, (2+j)_1, (4+12i+j)_2, (1+j)_1, (1+12i+j)_2], [j_1, (11+12i+j)_2, (1+j)_1, (10+12i+j)_2, (2+j)_1, (7+12i+j)_2] \mid i = 0, 1, \ldots, (n-11)/12; j = 0, 1, 2, \ldots, m-1\}$ This is a minimal hexagon covering of $K_{m,n}$ with padding P where $E(P) = \{(i_1, i_2) \mid i = 0, 1, \ldots, m-1\}$, and so $|E(P)| = m$. □

Lemma 2.7 *A minimal restricted hexagon covering of $K_{m,n}$ where m and n are both odd, $m \geq n \geq 3$, has a padding P satisfying $|E(P)| = m + k$, where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m + k) \equiv 0 \pmod 6$.*

Proof. The necessary conditions follow as in Lemma 2.5. We now establish sufficiency.

Case 1. Suppose $m \equiv n \equiv 1 \pmod 6$ and $m \geq n \geq 7$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1 \ldots, m_1\}$ and V_n, and the partite sets of $K_{n,n}$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n. By Theorem 2.3, there is a decomposition of $K_{n,n} \setminus M$ where M is a perfect matching which referes to the edge (a_1, a_2) where $a \in \{1, 2, 3 \ldots m\}$ of $K_{n,n}$, say $E(M) = \{(i_1, 1_2) \mid i = 1, 2, \ldots, n\}$. We take the collection of hexagons for such a decomposition along with the set of hexagons $\{[(1+3i)_1, (3+3i)_2, (3+3i)_1, (2+3i)_2, (2+3i)_1, (1+3i)_2] \mid i = 0, 1, \ldots, \frac{n-4}{3}\}$. We see that $K_{n,n} \setminus \{(n_1, n_2)\}$ has a hexagon covering with padding P_1 where $E(P_1) = \{((1+3i)_1, (3+3i)_2), ((2+3i)_1, (1+3i)_2), (3+3i)_1, (2+3i)_2) \mid i = 0, 1, \ldots, \frac{n-4}{3}\}$. So $|E(P_1)| = n - 1$. We cover edge (n_1, n_2) with hexagon $[n_1, n_2, (n-1)_1, (n-1)_2, (n-2)_1, (n-2)_2]$ and add to the padding the edges in $E(P_2) = \{(n_2, (n-1)_1), ((n-1)_1, (n-1)_2), ((n-1)_2, (n-2)_1), ((n-2)_1, (n-2)_2), (n_1, (n-2)_2)\}$ and so $|E(P_2)| = 5$. By Lemma 2.5 Case 1, there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_3 where $|E(P_3)| = m - n$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2 \cup P_3$ where $|E(P)| = m + 4$.

Case 2. Suppose $m \equiv 1 \pmod 6$, $n \equiv 3 \pmod 6$, and $m \geq n$. Now $K_{m,n} =$

$K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{(n+1)_1, (n+2)_1 \ldots, m_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n. By Theorem 2.3, there is a decomposition of $K_{n,n} \setminus M$ where M is a perfect matching of $K_{n,n}$, say $E(M) = \{(i_1, i_2) \mid i = 1, 2, \ldots, n\}$. We take the collection of hexagons for such a decomposition along with the set of hexagons $\{[(1+3i)_1, (3+3i)_2, (3+3i)_1, (2+3i)_2, (2+3i)_1, (1+3i)_2] \mid i = 0, 1, \ldots, \frac{n-4}{3}\}$. We see that $K_{n,n}$ has a hexagon covering with padding P_1 where $E(P_1) = \{((1+3i)_1, (3+3i)_2), ((2+3i)_1, (1+3i)_2), ((3+3i)_1, (2+3i)_2) \mid i = 0, 1, \ldots, \frac{n-4}{3}\}$. So $|E(P_1)| = n$. By Lemma 2.5, Case 6, there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2, where $|E(P_2)| = m - n + 2$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 2$.

Case 3. First, suppose $m = n + 2 \equiv 1 \pmod 6$. Then the set of hexagons $\{[1_1, 2_2, 3_1, 1_2, 2_1, 3_2],\ [4_1, 4_2, 6_1, 3_2, 5_1, 5_2],\ [3_1, 3_2, 4_1, 2_2, 7_1, 5_2],\ [1_1, 1_2, 6_1, 2_2, 5_1, 4_2],\ [1_1, 1_2, 7_1, 3_2, 6_1, 5_2],\ [2_1, 2_2, 4_1, 1_2, 7_1, 4_2],\ [2_1, 4_2, 3_1, 1_2, 5_1, 5_2]\} \cup \{[1_1, (6+6i)_2, (8+6i)_1, 1_2, (9+6i)_1, 7_2],\ [2_1, (8+6i)_2, (10+6i)_1, 2_2, (11+6i)_1, 9_2],\ [1_1, (8+6i)_2, (12+6i)_1, 5_2, (9+6i)_1, 9_2],\ [3_1, (7+6i)_2, (10+6i)_1, 3_2, (9+6i)_1, 8_2],\ [3_1, (10+6i)_2, (12+6i)_1, 3_2, (13+6i)_1, 11_2],\ [4_1, (6+6i)_2, (9+6i)_1, 4_2, (8+6i)_1, 7_2],\ [4_1, (9+6i)_2, (13+6i)_1, 1_2, (11+6i)_1, 11_2],\ [5_1, (8+6i)_2, (11+6i)_1, 5_2, (10+6i)_1, 9_2],\ [6_1, (10+6i)_2, (13+6i)_1, 2_2, (12+6i)_1, 11_2],\ [7_1, (6+6i)_2, (13+6i)_1, 5_2, (8+6i)_1, 8_2],\ [7_1, (9+6i)_2, (12+6i)_1, 4_2, (11+6i)_1, 10_2],\ [1_1, (10+6i)_2, 5_1, (7+6i)_2, 2_1, (11+6i)_2],\ [2_1, (6+6i)_2, 6_1, (8+6i)_2, 4_1, (10+6i)_2],\ [(6+6i)_2, (10+6i)_1, 1_2, (12+6i)_1, (7+6i)_2, (11+6i)_1],\ [(10+6i)_2, (8+6i)_1, 2_2, (9+6i)_1, (11+6i)_2, (10+6i)_1],\ [3_1, (6+6i)_2, 5_1, (11+6i)_2, (8+6i)_1, (9+6i)_2]$ $[6_1, (7+6i)_2, (11+6i)_1, 3_2, (8+6i)_1, (9+6i)_2],\ [7_1, (7+6i)_2, (13+6i)_1, 4_2, (10+$

$6i)_1, (11+6i)_2]$, $[(6+6i)_2, (9+6i)_1, (10+6i)_2, (13+6i)_1, (8+6i)_2, (12+6i)_1] \mid i = 0, 1, \ldots, (m-7)/6 - 1\}$ is an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{(1_1, 1_2), (2_1, 4_2), (3_1, 1_2), (4_1, 2_2), (5_1, 5_2), (6_1, 3_2), (7_1, 1_2)\} \cup \{((8+6i)_1, (9+6i)_2), ((9+6i)_1, (6+6i)_2), ((10+6i)_1, (11+6i)_2), ((11+6i)_1, (7+6i)_2), ((12+6i)_1, (8+6i)_2), ((13+6i)_1, (10+6i)_2) \mid i = 0, 1, \ldots, (m-7)/6 - 1\}$ and so $|E(P)| = m$. Next, suppose $m \equiv 1 \pmod 6$, $n \equiv 5 \pmod 6$, and $m \geq n+8$. Now, $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1 \ldots, n_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. Now $K_{n,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = n$ (by Lemma 2.5), and there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 satisfying $|E(P_2)| = m-n$ (by Lemma 2.4, Case 4). Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 4. Suppose $m \equiv 3 \pmod 6$, $n \equiv 1 \pmod 6$, and $m > n \geq 7$. Now $K_{m,n} = K_{n-2,n} \cup K_{m-n+2,n}$ where the partite sets of $K_{n-2,n}$ are $\{1_1, 2_1 \ldots, (n-2)_1\}$ and V_n, and the partite sets of $K_{m-n+2,n}$ are $\{(n-1)_1, n_1, \ldots, m_1\}$ and V_n. By Case 3, there is a restricted covering of $K_{n-2,n}$ with padding P_1 where $|E(P_1)| = n-2$, and there is a restricted covering of $K_{m-n+2,n}$ with padding P_2 satisfying $|E(P_2)| = m-n+6$ (by Lemma 2.5, Case 5). Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m+4$.

Case 5. Suppose $m \equiv n \equiv 3 \pmod 6$ and $m \geq n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1 \ldots, m_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n. By Theorem 2.3, there is a decomposition of $K_{n,n} \setminus M$ where M is a perfect matching of $K_{n,n}$, say $E(M) = \{(i_1, i_2) \mid$

$i = 1, 2, \ldots, n\}$. Taking the collection of hexagons for such a decomposition along with the set of hexagons $\{[(1+3i)_1, (3+3i)_2, (3+3i)_1, (2+3i)_2, (2+3i)_1, (1+3i)_2] \mid i = 0, 1, \ldots, \frac{n}{3} - 1\}$, we see that $K_{n,n}$ has a hexagon covering with padding P_1 where $E(P_1) = \{((1+3i)_1, (3+3i)_2), ((2+3i)_1, (1_3 i)_2), (3+3i)_1, (2+3i)) \mid i = 0, 1, \ldots, \frac{n}{3} - 1\}$ and so $|E(P_1)| = n$. By Lemma 2.5, Case 1, there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 where $|E(P_1)| = m - n$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 6. Suppose $m \equiv 3 \pmod 6$, $n \equiv 5 \pmod 6$, and $m > n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1 \ldots, n_1\}$ and V_n and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. Now $K_{n,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = n$ (by Lemma 3.3), and there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 satisfying $|E(P_2)| = m-n$ (by Lemma 2.5, Case 7). Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 7. Suppose $m \equiv 5 \pmod 6$, $n \equiv 1 \pmod 6$, and $m > n$. Now $K_{m,n} = K_{n-2,n} \cup K_{m-n+2,n}$ where the partite sets of $K_{n-2,n}$ are $\{1_1, 2_1 \ldots, (n-2)_1\}$ and V_n and the partite sets of $K_{m-n+2,n}$ are $\{(n-1)_1, n_1, \ldots, m_1\}$ and V_n. By Case 3, there is a restricted covering of $K_{n-2,n}$ with padding P_1 where $|E(P_1)| = n-2$ and there is a restricted covering of $K_{m-n+2,n}$ with padding P_2 satisfying $|E(P_2)| = m-n+2$ (by Lemma 2.5 Case 1). Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 8. First, suppose $m = n + 2 \equiv 5 \pmod 6$. Then the set of hexagons $\{[1_1, 1_2, 2_1, 3_2, 3_1, 2_2], [1_1, 1_2, 2_1, 2_2, 5_1, 3_2], [3_1, 1_2, 5_1, 3_2, 4_1, 2_2], [3_1, 1_2, 4_1, 3_2, 5_1, 2_2]\} \cup \{[1_1, (4+$

$6i)_2, (6+6i)_1, 1_2, (7+6i)_1, (5+6i)_2]$, $[2_1, (4+6i)_2, (7+6i)_1, 2_2, (6+6i)_1, (5+6i)_2]$, $[2_1, (6{+}6i)_2, (8{+}6i)_1, 2_2, (9{+}6i)_1, (7{+}6i)_2]$, $[4_1, (6{+}6i)_2, (9{+}6i)_1, 1_2, (8{+}6i)_1, (7{+}6i)_2]$, $[3_1, (8{+}6i)_2, (10{+}6i)_1, 3_2, (11{+}6i)_1, (9{+}6i)_2]$, $[5_1, (8{+}6i)_2, (11{+}6i)_1, 2_2, (10{+}6i)_1, (9{+}6i)_2]$, $[3_1, (5{+}6i)_2, (8{+}6i)_1, 3_2, (7{+}6i)_1, (6{+}6i)_2]$, $[4_1, (8{+}6i)_2, (9{+}6i)_1, (4{+}6i)_2, (8{+}6i)_2, (9{+}6i)_2]$, $[3_1, (4{+}6i)_2, 4_1, (5{+}6i)_2, 5_1, (7{+}6i)_2]$, $[(11{+}6i)_1, (4{+}6i)_2, (10{+}6i)_1, (6{+}6i)_2, (6{+}6i)_1, (7{+}6i)_2]$, $[1_1, (8{+}6i)_2, (6{+}6i)_1, 3_2, (9{+}6i)_1, (9{+}6i)_2]$, $[1_1, (6{+}6i)_2, (11{+}6i)_1, 1_2, (10+6i)_1, (7+6i)_2]$, $[2_1, (8+6i)_2, (7+6i)_1, (7+6i)_2, (6+6i)_1, (9+6i)_2]$, $[5_1, (4{+}6i)_2, (9{+}6i)_1, (5{+}6i)_2, (10{+}6i)_1, (6{+}6i)_2]$, $[(11{+}6i)_1, (5{+}6i)_2, (8{+}6i)_1, (8{+}6i)_2, (7+6i)_1, (9+6i)_2] \mid i = 0, 1, \ldots, (m-5)/6-1\}$ is an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{(3_1, 2_2), (1_1, 1_2), (2_1, 1_2), (3_1, 1_2), (3_1, 2_2), (4_1, 3_2), (5_1, 2_2), (5_1, 3_2), (5_1, 3_2)\} \cup \{((6+6i)_1, (7+6i)_2), ((7+6i)_1, (8+6i)_2), ((8+6i)_1, (5+6i)_2), ((9+6i)_1, (4+6i)_2), ((10+6i)_1, (6+6i)_2), ((11+6i)_1, (9+6i)_2) \mid i = 0, 1, \ldots, (m-5)/6-1\}$ and so $|E(P)| = m+4$. Next, suppose $m \equiv 5 \pmod 6$, $n \equiv 3 \pmod 6$, $m \geq n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1 \ldots, n_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. By Theorem 2.3, there is a decomposition of $K_{n,n} \setminus M$ where M is a perfect matching of $K_{n,n}$, say $E(M) = \{(i_1, 1_2) \mid i = 1, 2, \ldots, n\}$. We take the collection of hexagons for such a decomposition along with the set of hexagons $\{[(1+3i)_1, (3+3i)_2, (3+3i)_1, (2+3i)_2, (2+3i)_1, (1+3i)_2] \mid i = 0, 1, \ldots, \frac{n}{3}-1\}$. We see that $K_{n,n}$ has a hexagon covering with padding P_1 where $E(P_1) = \{((1+3i)_1, (3+3i)_2), ((2+3i)_1, (1+3i)_2), (3+3i)_1, (2+3i)) \mid i = 0, 1, \ldots, \frac{n}{3}-1\}$. So $|E(P_1)| = n$. By Lemma 2.5, Case 3, there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 where $|E(P_1)| = m-n+4$. Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 4$.

Case 9. Suppose $m \equiv n \equiv 5 \pmod 6$, and $m > n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1 \ldots, n_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. Now $K_{n,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = n$ (by Lemma 3.3). There is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 satisfying $|E(P_2)| = m - n$ (by Lemma 2.5 Case 1). Therefore, there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$. □

For these cases, we take some value of m and n that we are able to break into smaller cases that has previously been solved or we are able to create a covering. For example, in case 7 of Lemma 2.4, $m \equiv 4 \pmod 6$ and $n \equiv 5 \pmod 6$. In this case, we can say $K_{m,n} = K_{m-4,n} \cup K_{4,n-5} \cup K_{4,5}$ which means we are breaking the original bipartite graph into three subgraphs. If we look at $K_{m-4,n}$, then we would have $m - 4 \equiv 0 \pmod 6$ and $n \equiv 5 \pmod 6$ which are the conditions for Case 1. We know the padding is equal to $m - 4$, which for this specific subgraph would be $|E(P_1)| = m - 4$. If we look at $K_{4,n-5}$, then the partite sets are size 4 and 0 (mod 6) which are the conditions for a decomposition by a previous theorem. Then all we have left is $K_{4,5}$ which we can predict would need four extra edges to create a covering. We are able to do it with a padding of four, so $|E(P_2)| = 4$. This means that the padding for $K_{m,n}$ is $P = P_1 \cup P_2$ which is $P = m - 4 + 4 = m$. Next, let us look at a lemma that is used in the unrestricted coverings of complete bipartite graphs.

Lemma 2.8 *A minimal unrestricted covering of $K_{1,n}$, $n \geq 5$, has a padding P where $|E(P)| = 2n$ when n is even and $|E(P)| = 2n + 3$ when n is odd.*

Proof. For n even, $n \geq 6$, we have V_1 and V_n as the partite sets of $K_{2,n}$. If a hexagon in such a covering contains no vertices of V_1, then it must contain 6 edges in the padding. If a hexagon in a covering contains 1 vertex of V_1, then it must contain at least 4 edges in P and at most 2 edges in $K_{2,n}$. Since $K1, n$ contains n edges, then an unrestricted covering with padding P must satisfy $|E(P)| \geq 2n$. The set of hexagons $\{[1_1, 1_2, 5_2, 4_2, 3_2, 2_2], [1_1, 3_2, 2_2, 1_2, 5_2, 4_2], [1_1, 5_2, 4_2, 3_2, 2_2, 6_2]\} \cup \{[1_1, (5+2i)_2, 3_2, 2_2, 1_2, (6+2i)_2] \mid i = 1, 2, \ldots, (n-6)/2$ forms an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{(1_2, 5_2),\ (4_2, 5_2), (3_2, 4_2), (2_2, 3_2),\ (2_2, 3_2), (1_2, 3_2), (1_2, 5_2), (4_2, 5_2),\ (4_2, 5_2), (3_2, 4_2), (2_2, 3_2), (2_2, 6_2)\} \cup \{(3_2, (5 + 2i)_2),\ (2_2, 3_2), (1_2, 2_2), (1_2, (6 + 2i)_2) \mid i = 1, 2, \ldots, (n - 6)/2\}$ and so $|E(P)| = 2n$.

For n odd, as when n is even, each hexagon of an unrestricted covering contains at least 4 edges in the padding, so an unrestricted with padding P must satisfy $|E(P)| \geq 2n$. Since $|E(K_{m,n})| + |E(P)| \equiv 0 \pmod 6$, it follows that $|E(P)| \geq 2n + 3$. The set of hexagons $\{[1_1, 1_2, 5_2, 4_2, 3_2, 2_2], [1_1, 3_2, 2_2, 1_2, 5_2, 4_2], [1_1, 4_2, 1_2, 2_2, 3_2, 5_2]\} \cup \{[(1_1, (4 + 2i)_2, 3_2, 2_2, 1_2, (5 + 2i)_2] \mid i = 1, 2, \ldots, (n - 5)/2\}$ forms an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{(1_2, 5_2), (4_2, 5_2), (3_2, 4_2),\ (2_2, 3_2), (2_2, 3_2),\ (1_2, 3_2),\ (1_2, 5_2),\ (4_2, 5_2), (1_1, 4_2), (1_2, 4_2),\ (1_2, 2_2),\ (2_2, 3_2), (3_2, 5_2)\} \cup \{(3_2, (4+2i)_2), (2_2, 3_2), (1_2, 2_2), (1_2, (5 + 2i)_2) \mid i = 1, 2, \ldots, (n - 5)/2\}$ and so $|E(P)| = 2n + 3$.

□

Lemma 2.9 *A minimal unrestricted covering of $K_{2,n}$ where $n \equiv 0 \pmod 2$, $n \geq 4$, has a padding P where $|E(P)| = n$.*

Proof. We have V_2 and V_n as the partite sets of $K_{2,n}$. If a hexagon in such a covering contains no vertices of V_2, then it must contain 6 edges in the padding. If a hexagon in a covering contains 1 vertex of V_2, then it must contain 4 edges in P and 2 edges in $K_{2,n}$. If a hexagon in a covering contains 2 vertices of V_2, then it must contain either (1) 2 edges in P and 4 edges in $K_{2,n}$, or (2) 4 edges in P and 2 edges in $K_{2,n}$. Since $K_{2,n}$ contains $2n$ edges, then an unrestricted covering with padding P must satisfy $|E(P)| \geq n$. Since $n \geq 4$ is even, then $n = 4n_1 + 6n_2$ for some $n_1, n_2 \in \mathbb{N}$. Then the set of hexagons: $\{[1_1, (1+4i)_2, (4+4i)_2, (3+4i)_2, 2_1, (2+4i)_2], [1_1, (3+4i)_2, (2+4i)_2, (1+4i)_2, 2_1, (4+4i)_2] \mid i = 0, 1, \ldots, n_1 - 1\} \cup \{[1_1, (6+6j)_2, (2+6j)_2, (1+6j)_2, 2_1, (5+6j)_2], [1_1, (1+6j)_2, (4+6j)_2, (3+6j)_2, 2_1, (2+6j)_2], [1_1, (3+6j)_2, (2+6j)_2, (6+6j)_2, 2_1, (4+6j)_2] \mid j = 0, 1, \ldots, n_2 - 1\}$ is an unrestricted covering of $K_{2,n}$ with padding $P = \{((1+4i)_2, (2+4i)_2), ((2+4i)_2, (3+4i)_2), ((3+4i)_2, (4+4i)_2), ((1+4i)_2, (4+4i)_2) \mid i = 0, 1, \ldots, n_1 - 1\} \cup \{((1+6j)_2, (2+6j)_2), ((2+6j)_2, (3+6j)_2), ((3+6j)_2, (4+6j)_2), ((1+6j)_2, (4+6j)_2), 2 \times ((2+6j)_2, (6+6j)_2) \mid j = 0, 1, \ldots, n_2 - 1\}$ and so $|E(P)| = 4n_1 + 6n_2 = n$ and this unrestricted covering is minimal. $\Box$

Lemma 2.10 *A minimal unrestricted covering of $K_{2,n}$ where $n \equiv 1 \pmod 2$ and $n \geq 5$, has a padding P where $|E(P)| = n + 3$.*

Proof. As in Lemma 2.8, each hexagon of a covering of $K_{2,n}$ contains at least 2 edges of the padding and at most 4 edges of $K_{2,n}$. Since $|E(K_{2,n})| = 2n$, then the number of hexagons in a covering must be at least $\lceil 2n/4 \rceil = \lceil n/2 \rceil = (n+1)/2$ since n is odd. Since each hexagon contains at least 2 edges of the padding P, we have $|E(P)| \geq n+1$. Now we need $|E(K_{2,n})| + |E(P)| \equiv 0 \pmod 6$ and $|E(K_{2,n})| + |E(P)| \geq$

$3n+1$, so $|E(P)| \geq n+3$. Since n is odd and $n \geq 5$, then either (1) $n = 4\ell+5$ where $\ell = (n-5)/4 \in \mathbb{N}$, or (2) $n = 4\ell+7$ where $\ell = (n-7)/4 \in \mathbb{N}$.

Define $A = \{[1_1, (1+4i)_2, (4+4i)_2, (3+4i)_2, 2_1, (2+4i)_2], [1_1, (3+4i)_2, (2+4i)_2, (1+4i)_2, 2_1, (4+4i)_2] \mid i = 0, 1, \ldots, \ell-1\}$. For $n = 4\ell+5$, consider the set of blocks $A \cup \{[1_1, (n-4)_2, 2_1, n_2, (n-1)_2, (n-2)_2], [1_1, (n-4)_2, 2_1, (n-1)_2, (n-2)_2, (n-3)_2], [1_1, (n-1)_2, (n-2)_2, 2_1, (n-3)_2, n_2]\}$. This is an unrestricted covering of $K_{2,n}$ with padding P where $E(P) = \{((1+4i)_2, (2+4i)_2), ((2+4i)_2, (3+4i)_2), ((3+4i)_2, (4+4i)_2), ((1+4i)_2, (4+4i)_2) \mid i = 0, 1, \ldots, \ell-1\} \cup \{(1_1, (n-4)_2), (2_1, (n-4)_2), ((n-3)_2, (n-2)_2), ((n-3)_2, n_2), 3 \times ((n-2)_2, (n-1)_2), ((n-1)_2, n_2)\}$. Since $|E(P)| = 4\ell+8 = n+3$, the covering is minimal. For $n = 4\ell+7$, consider the set of blocks $A \cup \{[1_1, (n-6)_2, 2_1, (n-2)_2, (n-3)_2, (n-4)_2], [1_1, (n-6)_2, 2_1, (n-3)_2, (n-4)_2, (n-5)_2], [1_1, (n-1)_2, (n-4)_2, 2_1, (n-5)_2, n_2], [1_1, (n-3)_2, n_2, 2_1, (n-1)_2, (n-2)_2]\}$. This is an unrestricted covering of $K_{2,n}$ with padding $P = \{((1+4i)_2, (2+4i)_2), ((2+4i)_2, (3+4i)_2), ((3+4i)_2, (4+4i)_2), ((1+4i)_2, (4+4i)_2) \mid i = 0, 1, \ldots, \ell-1\} \cup \{(1_1, (n-6)_2), (2_1, (n-6)_2), ((n-5)_2, (n-4)_2), ((n-5)_2, n_2), 2 \times ((n-4)_2, (n-3)_2), ((n-4)_2, (n-1)_2), ((n-3)_2, n_2), ((n-3)_2, (n-2)_2), ((n-2)_2, (n-1)_2)\}$. Since $|E(P)| = 4\ell+10 = n+3$, the covering is minimal. □

Lemma 2.11 *A minimal unrestricted hexagon covering of $K_{m,n}$ where m and n are even, $m, n \geq 4$, has a padding P satisfying:*

(1) $|E(P)| = 0$ *when* $m \equiv 0 \pmod 6$,

(2) $|E(P)| = 2$ *when* $m \equiv n \equiv 2 \pmod 6$ *or* $m \equiv n \equiv 4 \pmod 6$, *and*

(3) $|E(P)| = 4$ *when* $m \equiv 2 \pmod 6$ *and* $n \equiv 4 \pmod 6$.

Lemma 2.12 *A minimal unrestricted hexagon covering of $K_{m,n}$ where m is even and n is odd ($m \geq 4, n \geq 3$) has a padding P satisfying $|E(P)| = m/2 + k$, where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m/2 + k) \equiv 0 \pmod 6$.*

Proof. Since each vertex of V_m is of odd degree in $K_{m,n}$, in the padding of a covering each of these vertices will be of odd degree. Therefore, in a restricted covering of $K_{m,n}$ with padding P, it is necessary that $|E(P)| \geq m/2$. Since a covering yields a decomposition of $K_{m,n} \cup P$, then it is necessary that $|E(K_{m,n})| + |E(P)| \equiv 0$ (mod 6).

Case 1. First, suppose $m \equiv 0$ (mod 12) and $n = 3$. Then the set of hexagons $\{[(1+12i)_1, 2_2, (3+12i)_1, 1_2, (2+12i)_1, 3_2], [(4+12i)_1, 2_2, (6+12i)_1, 1_2, (5+12i)_1, 3_2], [(7+12i)_1, 2_2, (9+12i)_1, 1_2, (8+12i)_1, 3_2], [(10+12i)_1, 2_2, (12+12i)_1, 1_2, (11+12i)_1, 3_2], [(1+12i)_1, 1_2, (4+12i)_1, (5+12i)_1, 2_2, (2+12i)_1], [(6+12i)_1, 3_2, (9+12i)_1, (10+12i)_1, 1_2, (7+12i)_1], [(3+12i)_1, 3_2, (12+12i)_1, (11+12i)_1, 2_2, (8+12i)_1] \mid i = 0, 1, \ldots, m/12-1\}$ is an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{((1+12i)_1, (2+12i)_1), ((3+12i)_1, (8+12i)_1), ((4+12i)_1, (5+12i)_1), ((6+12i)_1, (7+12i)_1), ((9+12i)_1, (10+12i)_1), ((11+12i)_1, (12+12i)_1) \mid= 0, 1, \ldots, m/12-1\}$ and so $|E(P)| = m/2$.

Second, suppose $m \equiv 0$ (mod 12) and $n = 5$. Then the set of hexagons $\{[(1+12i)_1, 1_2, (2+12i)_1, 2_2, (3+12i)_1, 3_2], [(1+12i)_1, 4_2, (3+12i)_1, 1_2, (5+12i)_1, 5_2], [(2+12i)_1, 3_2, (6+12i)_1, 2_2, (4+12i)_1, 4_2], [(4+12i)_1, 3_2, (5+12i)_1, 4_2, (6+12i)_1, 5_2], [(7+12i)_1, 4_2, (8+12i)_1, 5_2, (9+12i)_1, 1_2], [(7+12i)_1, 2_2, (9+12i)_1, 4_2, (11+12i)_1, 3_2], [(8+12i)_1, 1_2, (12+12i)_1, 5_2, (10+12i)_1, 2_2], [(10+12i)_1, 1_2, (11_1+12i)_1, 2_2, (12+12i)_1, 3_2], [(1+12i)_1, 2_2, (5+12i)_1, (7+12i)_1, 5_2, (11+12i)_1], [(2+12i)_1, 5_2, (3+12i)_1, (10+12i)_1, 4_2, (12+12i)_1], [(4+12i)_1, 1_2, (6+12i)_1, (8+12i)_1, 3_2, (9+12i)_1] \mid$

$i = 0, 1, \ldots, m/12 - 1\}$ is an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{((1+12i)_1, (11+12i)_1), ((2+12i)_1, (12+12i)_1), ((3+12i)_1, (10+12i)_1), ((4+12i)_1, (9+12i)_1), ((5+12i)_1, (7+12i)_1), ((6+12i)_1, (8+12i)_1) \mid i = 0, 1, \ldots, m/12-1\}$ and so $|E(P)| = m/2$.

Finally, suppose $m \equiv 0 \pmod{12}$, $n \equiv 1 \pmod 2$, and $n > 5$. Now $K_{m,n} \subset K_{m,n-3} \cup \frac{m}{3} \times C_6 \cup \frac{m}{4} \times C_6$ where the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$, $\frac{m}{3} \times C_6 = \{[(1+3i)_1, 2_2, (3+3i)_1, 1_2, (2+3i)_1, 3_2] \mid i = 0, 1, \ldots, \frac{m}{3} - 1\}$, and $\frac{m}{4} \times C_6 = \{[(1+12i)_1, 1_2, (4+12i)_1, (2+12i)_1, 2_2, (5+12i)_1], [(3+12i)_1, 3_2, (6+12i)_1, (7+12i)_1, 1_2, (10+12i)_1], [(8+12i)_1, 2_2, (11+12i)_1, (12+12i)_1, 3_2, (9+12i)_1] \mid i = 0, 1, \ldots, \frac{m}{4}-1\}$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding P where $E(P) = \{((1+12i)_1, (5+12i)_1), ((2+12i)_1, (4+12i)_1), ((3+12i)_1, (10+12i)_1), ((6+12i)_1, (7+12i)_1), ((8+12i)_1, (9+12i)_1), ((11+12i)_1, (12+12i)_1) \mid i = 0, 1, \ldots, \frac{m}{12} - 1\}$. So $|E(P)| = \frac{m}{2}$ and the unrestricted covering is minimal.

Case 2. Suppose $m \equiv 2 \pmod{12}$ and $m \geq 14$, and $n \equiv 1 \pmod 6$, $n \geq 7$. Now $K_{m,n} = K_{14,7} \cup K_{m-14,7} \cup K_{m,n-7}$ where the partite sets of $K_{14,7}$ are $\{1_1, 2_1, \ldots, 14_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$, the partite sets of $K_{m-14,7}$ are $\{15_1, 16_1, \ldots, m_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{m,n-7}$ are V_m and $\{8_2, 9_2, \ldots, n_2\}$. There exists an unrestricted covering of $K_{m-14,7}$ with padding P_1 where $|E(P_1)| = (m-14)/2$ by Case 1 and there exists a hexagon decomposition of $K_{m,n-7}$ by Theorem 2.2. Next, we note that $K_{14,7} = K_{7,7} \cup K_{7,7}$ where the partite sets of the first copy of $K_{7,7}$ are $\{1_1, 2_1, \ldots, 7_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of the second copy of $K_{7,7}$ are $\{8_1, 9_1, \ldots, 14_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$. By Theorem 2.3, there is a hexagon decomposition of $K_{7,7} \setminus M$ where M is a matching of $K_{7,7}$. So there is a hexagon decomposition of $K_{14,7} \setminus M_1$ where $E(M_1) = \{(i_1, i_2), ((i+7)_1, i_2) \mid i = 1, 2, \ldots, 7\}$. This decomposition along with the set $\{[1_1, 1_2, 8_1, 9_1, 2_2, 2_1]$, $[3_1, 3_2, 10_1, 11_1, 4_2, 4_1]$, $[5_1, 5_2, 12_1, 13_1, 6_2, 6_1], [6_1, 6_2, 13_1, 14_1, 7_2, 7_1]\}$ forms an unrestricted covering of $K_{14,7}$ with padding P_2 where $E(P_2) = \{$ $(1_1, 2_1)$, $(3_1, 4_1)$, $(5_1, 6_1)$, $(6_1, 7_1)$, $(6_1, 6_2)$, $(8_1, 9_1)$, $(10_1, 11_1)$, $(12_1, 13_1)$, $(13_1, 14_1)$, $(13_1, 6_2)\}$ and so $|E(P_2)| = 10$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 3$.

Case 3. Suppose $m \equiv 2 \pmod{12}$, $m \geq 14$, and $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{14,3} \cup K_{m-14,3} \cup K_{m,n-3}$, where the partite sets of $K_{14,3}$ are $\{1_1, 2_1, \ldots, 14_1\}$ and $\{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-14,3}$ are $V_m \setminus \{1_1, 2_1, \ldots, 14_1\}$ and $\{1_2, 2_2, 3_2\}$, and the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$. There exists an unrestricted covering of $K_{m-14,3}$ with padding P_1 where $|E(P_1)| = (m-14)/2$ by Case 1 and $K_{m,n-3}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{14,3}$, namely the set $\{[1_1, 2_2, 2_1, 11_1, 3_2, 14_1]$, $[3_1, 1_2, 6_1, 7_1, 2_2, 4_1]$, $[9_1, 1_2, 12_1, 13_1, 2_2, 10_1,]$ $[3_1, 2_2, 5_1, 1_2, 4_1, 3_2]$, $[6_1, 2_2, 8_1, 1_2, 7_1, 3_2]$, $[9_1, 2_2, 11_1, 1_2, 10_1, 3_2], [12_1, 2_2, 14_1, 1_2, 13_1, 3_2], [1_1, 1_2, 2_1, 5_1, 3_2, 8_1], [1_1, 1_2, 3_1, 2_2, 2_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 14_1)$, $(2_1, 11_1)$, $(3_1, 4_1)$, $(6_1, 7_1)$, $(9_1, 10_1)$, $(12_1, 13_1)$, $(1_1, 1_2)$, $(1_1, 8_1)$, $(2_1, 2_2)$, $(2_1, 5_1)$, $(3_1, 1_2)$, $(3_1, 2_2)\}$ and so $|E(P_2)| = 12$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$, where $|E(P)| = m/2 + 5$.

Case 4. Suppose $m \equiv 2 \pmod{12}$ and $m \geq 14$, and $n \equiv 5 \pmod 6$, $n \geq 5$. Now $K_{m,n} = K_{14,5} \cup K_{m-14,5} \cup K_{m,n-5}$ where the partite sets of $K_{14,5}$ are $\{1_1, 2_1, \ldots, 14_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$, the partite sets of $K_{m-14,5}$ are $\{15_1, 16_1, \ldots, m_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$,

and the partite sets of $K_{m,n-5}$ are V_m and $\{6_2, 7_2, \ldots, n_2\}$. There exists an unrestricted covering of $K_{m-14,5}$ with padding P_1 where $|E(P_1)| = (m-14)/2$ by Case 1 and there exists a hexagon decomposition of $K_{m,n-5}$ by Theorem 2.2. Next, we note that $K_{14,5} = K_{8,5} \cup K_{6,5}$ where the partite sets of the first copy of $K_{8,5}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$, and the partite sets of the second copy of $K_{6,5}$ are $\{9_1, 10_1, \ldots, 14_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$. By a theorem in a previous paper by Brown, Coker, Gardner, and Kennedy [2] , there is a hexagon decomposition of $K_{8,5} \backslash M_1$ where $E(M_1) = \{(1_1, 5_2), (2_1, 5_2), (3_1, 1_2), (4_1, 4_2), (5_1, 2_2), (5_1, 4_2), (5_1, 5_2), (6_1, 5_2), (7_1, 1_2), (8_1, 2_2)\}$. Also, by the same paper [2], there exists a decomposition of a $K_{6,5} \setminus M_2$ where $E(M_2) = \{(9_1, 1_2), (10_1, 2_2), (11_1, 3_2), (12_1, 1_2), (13_1, 2_2), (14_1, 3_2)\}$ So there is a hexagon decomposition of $K_{14,7} \setminus M_1 \cup M_2$. This decomposition along with the set $\{[1_1, 5_2, 2_1, 9_1, 1_2, 12_1], [3_1, 1_2, 7_1, 5_1, 2_2, 8_1], [10_1, 2_2, 13_1, 5_1, 5_2, 6_1],$
$[11_1, 3_2, 14_1, 4_1, 4_2, 5_1]\}$ forms an unrestricted covering of $K_{14,5}$ with padding P_2 where $E(P_2) = \{ (1_1, 12_1), (2_1, 9_1), (3_1, 8_1), (4_1, 14_1), (5_1, 7_2), (5_1, 11_1), (5_1, 13_1), (6_1, 10_1)\}$ and so $|E(P_2)| = 8$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 1$.

Case 5. Suppose $m \equiv 4 \pmod{12}$, $n \equiv 1 \pmod 6$, and $n \neq 7$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-7} \cup K_{4,7}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{4,7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$. There exists an unrestricted covering of $K_{m-4,n}$ with padding P_1 where $|E(P_1)| = (m-4)/2$ by Case 1 and $K_{4,n-7}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{4,7}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [2_1, 5_2, 4_1, 7_2, 3_1, 6_2],$

$[1_1, 6_2, 4_1, 4_2, 2_1, 7_2]$, $[1_1, 2_2, 4_1, 1_2, 3_1, 5_2]$, $[1_1, 4_2, 3_1, 4_1, 3_2, 2_1]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (3_1, 4_1)\}$ So $|E(P_2)| = 2$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$, where $|E(P)| = m/2$.

Case 6. Suppose $m \equiv 4 \pmod{12}$ and $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-3} \cup K_{4,3}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-3}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2\}$, and the partite sets of $K_{4,3}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2\}$. There exists an unrestricted covering of $K_{m-4,n}$ with padding P_1 where $|E(P_1)| = (m-4)/2$ by Case 1 and $K_{4,n-3}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{4,3}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, $[1_1, 2_2, 4_1, 1_2, 3_1, 2_1]$, $[1_1, 1_2, 2_1, 3_2, 4_1, 2_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (2_1, 3_1), (1_1, 1_2), (1_1, 2_2), (2_1, 1_2), (4_1, 2_2)\}$. So $|E(P_2)| = 6$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$, where $|E(P)| = m/2 + 4$.

Case 7. Suppose $m \equiv 4 \pmod{12}$, $n \equiv 5 \pmod 6$, and $n \geq 17$. Now $K_{m,n} = K_{m,n-4} \cup K_{m,4}$ where the partite sets of $K_{m,n-4}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$ and the partite sets of $K_{m,4}$ are V_m and $\{1_2, 2_2, 3_2, 4_2\}$. There exists an unrestricted hexagon covering of $K_{m,n-4}$ with padding P_1 where $|E(P_1)| = m/2$ by Case 5 and there is a restricted hexagon covering of $K_{m,4}$ with padding P_2 where $|E(P_2)| = 2$ by Lemma 2.11 Case 3. Therefore, there is an unrestricted hexagon covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 2$.

Case 8. First, suppose $m \equiv 6 \pmod{12}$ and $n = 3$. Now $K_{m,n} = K_{6,3} \cup K_{m-6,3}$ where the partite sets of $K_{6,3}$ are $\{1_1, 2_1, \ldots, 6_1\}$ and V_n and the partite sets of $K_{m-6,3}$ are $V_m \setminus \{1_1, 2_1, \ldots, 6_1\}$ and V_n. There exists an unrestricted covering of

$K_{m-6,3}$ with padding P_1 where $|E(P_1)| = (m-6)/2$ by Case 1. Next, we note that there is an unrestricted hexagon covering of $K_{6,3}$, namely the set $\{[1_1, 2_2, 3_1, 1_2, 2_1, 3_2]$, $[4_1, 2_2, 6_1, 1_2, 5_1, 3_2]$, $[1_1, 1_2, 4_1, 5_1, 2_2, 2_1]$, $[3_1, 1_2, 5_1, 2_2, 6_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1)$, $(4_1, 5_1)$, $(3_1, 1_2)$, $(5_1, 1_2)$, $(5_1, 2_2)$, $(6_1, 2_2)\}$ and so $|E(P_2)| = 6$. Therefore, there is an unrestricted covering of $K_{m,n}$ with heagons with padding $P + P_1 \cup P_2$, where $|E(P)| = m/2 + 3$. Second, suppose $m \equiv 6 \pmod{12}$ and $n = 5$. Now $K_{m,n} = K_{6,5} \cup K_{m-6,5}$ where the partite sets of $K_{6,5}$ are $\{1_1, 2_1, \ldots, 6_1\}$ and V_n and the partite sets of $K_{m-6,5}$ are $V_m \setminus \{1_1, 2_1, \ldots, 6_1\}$ and V_n. There exists an unrestricted covering of $K_{m-6,n}$ with padding P_1 where $|E(P_1)| = (m - 6)/2$ by Case 1. Next, we note that there is an unrestricted hexagon covering of $K_{6,5}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, $[1_1, 4_2, 3_1, 1_2, 5_1, 5_2]$, $[2_1, 3_2, 6_1, 2_2, 4_1, 4_2]$, $[4_1, 3_2, 5_1, 4_2, 6_1, 5_2]$, $[1_1, 2_2, 5_1, 3_1, 5_2, 2_1]$, $[4_1, 1_2, 6_1, 5_2, 5_1, 4_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (3_1, 5_1), (4_1, 4_2), ((5_1, 4_2), (5_1, 5_2), (6_1, 5_2)\}$ and so $|E(P_2)| = 6$. Therefore, there is an unresticted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$, where $|E(P)| = m/2 + 3$. Finally, suppose $m \equiv 6 \pmod{12}$, $n \equiv 1 \pmod{2}$, and $n > 5$. Now $K_{m,n} = K_{6,n} \cup K_{m-6,n}$ where the partite sets of $K_{6,n}$ are $\{1_1, 2_1 \ldots, 6_1\}$ and V_n and the partite sets of $K_{m-6,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 6_1\}$ and V_n. There exists a restricted covering of $K_{6,n}$ with padding P_1 where $|E(P_1)| = 6$ by Lemma 3.2 Case 1 and there exists an unrestricted covering of $K_{m-6,n}$ with padding P_2 where $|E(P_2)| = (m - 6)/2$ by Case 1. Therefore, there is a unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 3$ and the unrestricted covering is minimal.

Case 9. Suppose $m \equiv 8 \pmod{12}$, $n \equiv 1 \pmod{6}$, and $n \geq 7$. Now $K_{m,n} = K_{8,7} \cup$

$K_{m-8,7} \cup K_{m,n-7}$ where the partite sets of $K_{8,7}$ are $\{1_1, 2_1, \dots, 8_1\}$ and $\{1_2, 2_2, \dots, 7_2\}$, the partite sets of $K_{m-8,7}$ are $V_m \setminus \{1_1, 2_1, \dots, 8_1\}$ and $\{1_2, 2_2, \dots, 7_2\}$, and the partite sets of $K_{m,n-7}$ are V_m and $V_n \setminus \{1_2, 2_2, \dots, 7_2\}$. There exists an unrestricted covering of $K_{m-8,7}$ with padding P_1 where $|E(P_1)| = (m-8)/2$ by Case 1 and $K_{m,n-7}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{8,7}$, namely the set $\{[1_1, 2_2, 6_1, 5_2, 2_1, 3_2],\ [1_1, 4_2, 5_1, 1_2, 4_1, 6_2],\ [3_1, 3_2, 5_1, 2_2, 4_1, 4_2],\ [3_1, 6_2, 5_1, 5_2, 4_1, 7_2],\ [6_1, 3_2, 8_1, 2_2, 7_1, 4_2],\ [6_1, 6_2, 8_1, 5_2, 7_1, 7_2],\ [1_1, 1_2, 8_1, 7_2, 3_1, 5_2],\ [2_1, 1_2, 7_1, 6_2, 3_1, 2_2],\ [1_1, 7_2, 5_1, 8_1, 4_2, 2_1],\ [3_1, 1_2, 6_1, 7_1, 3_2, 4_1]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (3_1, 4_1), (5_1, 8_1), (6_1, 7_1)\}$ and so $|E(P_2)| = 4$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2$.

Case 10. Suppose $m \equiv 8 \pmod{12}$ and $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{8,3} \cup K_{m-8,n} \cup K_{8,n-3}$ where the partite sets of $K_{8,3}$ are $\{1_1, 2_1, \dots, 8_1\}$ and $\{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-8,n}$ are $V_m \setminus \{1_1, 2_1, \dots, 8_1\}$ and V_n, and the partite sets of $K_{8,n-3}$ are $\{1_1, 2_1, \dots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2\}$. There exists an unrestricted covering of $K_{m-8,n}$ with padding P_1 where $|E(P_1)| = (m-8)/2$ by Case 1 and $K_{8,n-3}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{8,3}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 4_1, 3_2],\ [1_1, 2_2, 7_1, 5_1, 3_2, 2_1],\ [3_1, 1_2, 6_1, 8_1, 3_2, 4_1],\ [3_1, 2_2, 5_1, 1_2, 4_1, 3_2],\ [6_1, 2_2, 8_1, 1_2, 7_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (3_1, 4_1), (5_1, 7_1), (6_1, 8_1), 2 \times (4_1, 3_2)\}$ and so $|E(P_2)| = 6$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 2$.

Case 11. Suppose $m \equiv 8 \pmod{12}$, $n \equiv 5 \pmod 6$. Now $K_{m,n} = K_{8,5} \cup K_{m-8,5} \cup$

$K_{m,n-5}$ where the partite sets of $K_{8,5}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$, the partite sets of $K_{m-8,5}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$, and the partite sets of $K_{m,n-5}$ are V_m and $V_n \setminus \{1_2, 2_2, \ldots, 5_2\}$. There exists an unrestricted covering of $K_{m-8,5}$ with padding P_1 where $|E(P_1)| = (m-8)/2$ by Case 1 and $K_{m,n-5}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{8,5}$, namely the set $\{[1_1, 1_2, 3_1, 3_2, 6_1, 2_2]$, $[2_1, 1_2, 4_1, 5_1, 2_2, 3_1]$, $[3_1, 4_2, 5_1, 3_2, 4_1, 5_2]$, $[6_1, 4_2, 8_1, 3_2, 7_1, 5_2]$, $[1_1, 3_2, 2_1, 2_2, 8_1, 5_2]$, $[2_1, 4_2, 7_1, 1_2, 5_1, 5_2]$, $[4_1, 2_2, 7_1, 8_1, 1_2, 6_1]$, $[1_1, 1_2, 2_1, 2_2, 4_1, 4_2]\}$ with padding P_2 satisfying $E(P_2) = \{(2_1, 3_1)$, $(4_1, 5_1)$, $(1_1, 1_2)$, $(2_1, 1_2)$, $(2_1, 2_2)$, $(4_1, 2_2)$, $(4_1, 6_1)$, $(7_1, 8_1)\}$ and so $|E(P_2)| = 8$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$, where $|E(P)| = m/2 + 4$.

Case 12. Suppose $m \equiv 10 \pmod{12}$, $n \equiv 1 \pmod 6$, and $n \neq 7$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-7} \cup K_{4,7}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{4,7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$. There exists an unrestricted covering of $K_{m-4,n}$ with padding P_1 where $|E(P_1)| = m/2 + 1$ by Case 8 and $K_{4,n-7}$ can be decomposed by Theorem 2.2. Then Case 5 gives an unrestricted hexagon covering of $K_{4,7}$ with padding P_2 where $|E(P_2)| = 2$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$, where $|E(P)| = m/2 + 3$.

Case 13. Suppose $m \equiv 10 \pmod{12}$ and $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{10,3} \cup K_{m-10,3} \cup K_{m,n-3}$ where the partite sets of $K_{10,3}$ are $\{1_1, 2_1, \ldots, 10_1\}$ and $\{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-10,3}$ are $V_m \setminus \{1_1, 2_1, \ldots, 10_1\}$ and $\{1_2, 2_2, 3_2\}$, and the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$. There exists an unrestricted hexagon covering of $K_{m-10,3}$ with padding P_1 where $|E(P_1)| = (m-10)/2$ by Case 2 and $K_{m,n-3}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{10,3}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 10_1, 3_2], [1_1, 2_2, 3_1, 5_1, 1_2, 8_1], [4_1, 3_2, 7_1, 9_1, 2_2, 6_1], [2_1, 2_2, 4_1, 1_2, 3_1, 3_2], [5_1, 2_2, 7_1, 1_2, 6_1, 3_2], [8_1, 2_2, 10_1, 1_2, 9_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 8_1), (2_1, 2_2), (3_1, 5_1), (4_1, 6_1), (7_1, 9_1), (10_1, 2_2)\}$ and so $|E(P_2)| = 6$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$, where $|E(P)| = m/2 + 1$.

Case 14. Suppose $m \equiv 10 \pmod{12}$ and $n \equiv 5 \pmod 6$. Now $K_{m,n} = K_{m,n-4} \cup K_{m,4}$ where the partite sets of $K_{m,n-4}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$ and the partite sets of $K_{m,4}$ are V_m and $\{1_2, 2_2, 3_2, 4_2\}$. There exists an unrestricted hexagon covering of $K_{m,n-4}$ with padding P_1 where $|E(P_1)| = m/2+3$ by Case 12 and there is a restricted hexagon covering of $K_{m,4}$ with padding P_2 where $|E(P_2)| = 2$ by Lemma 3.1 Case 2. Therefore, there is an unrestricted hexagon covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$, where $|E(P)| = m/2 + 5$. □

Lemma 2.13 *A minimal unrestricted hexagon covering of $K_{m,n}$ where m and n are both odd, $m \geq n \geq 3$, has a padding P satisfying $|E(P)| = (m+n)/2 + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m+n)/2 + k \equiv 0 \pmod 6$.*

Proof. Since each vertex of $K_{m,n}$ is of odd degree, in the padding of a covering each of these vertices will be of odd degree. Therefore, in an unrestricted covering of $K_{m,n}$ with padding P, it is necessary that $|E(P)| \geq (m+n)/2$. Since a covering yields a decomposition of $K_{m,n} \cup P$, it is necessary that $|E(K_{m,n})| + |E(P)| \equiv 0 \pmod 6$.

Case 1. Suppose $m \equiv 1 \pmod{12}$, $m \geq 13$, and $n \equiv 7 \pmod{12}$. We have $K_{m,n} =$

$K_{n,n} \cup K_{m-n,n-3} \cup K_{6,3} \cup K_{m-n-6,3}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_2, \ldots, n_1\}$ and V_n, the partite sets of $K_{m-n,n-3}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and $\{4_2, 5_2, \ldots, n_2\}$, the partite sets of $K_{6,3}$ are $\{(n+1)_1, (n+2)_1, \ldots, (n+6)_1\}$ and $\{1_2, 2_2, 3_3\}$, and the partite sets of $K_{m-n-6,3}$ are $\{(n+7)_1, (n+8)_1, \ldots, m_1\}$ and $\{1_2, 2_2, 3_2\}$. There is a hexagon decomposition of $K_{n,n} \backslash M$ by Theorem 2.3, where (without loss of generality) $E(M) = \{(i_1, i_2) \mid i = 1, 2, \ldots, n\}$. There is a hexagon decomposition of $K_{m-n,n-3}$ by Theorem 2.2. There is an unrestricted covering of $K_{m-n-6,3}$ with padding P_1 where $|E(P_1)| = (m-n-6)/2$ by Lemma 2.11 Case 1. Taking these decompositions, the covering, and $\{((3i-2)_1, (3i-2)_2, (3i)_1, (3i)_2, (3i-1)_1, (3i-1)_2) \mid i = 1, 2, \ldots, (n-1)/3\} \cup \{((n+1)_1, 1_2, (n+4)_1, (n+5)_1, 2_2, (n+2)_1), (n_1, n_2, (n+6)_1, 3_2, (n+3)_1, (n+2)_1)\}$ yields an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = E(P_1) \cup \{((3i-2)_1, (3i-1)_2), ((3i-1)_1, (3i)_2), ((3i)_1, (3i-2)_2) \mid i = 1, 2, \ldots, (n-1)/3\} \cup \{((n+1)_1, (n+2)_1), ((n+2)_1, (n+3)_1), ((n+4)_1, (n+5)_1), (n_1, (n+2)_1), ((n+6)_1, n_2)\}$ and so $|E(P)| = (m+n)/2 + 1$.

Case 2. First, suppose $m = 3$ and $n \equiv 1 \pmod{12}$, $n \geq 13$. We have $K_{m,n} = K_{3,13} \cup K_{3,n-13}$ where the partite sets of $K_{3,13}$ are $\{1_1, 2_1, 3_1\}$ and $\{1_2, 2_2, \ldots, 13_2\}$, and the partite sets of $K_{3,n-13}$ are $\{1_1, 2_1, 3_1\}$ and $\{14_2, 15_2, \ldots, n_2\}$. Now $K_{3,n-13}$ has an unrestricted covering with padding P_1 where $|E(P_1)| = (n-13)/2$ by Lemma 2.14 Case 1. Next, we note that there is an unrestricted hexagon covering of $K_{3,13}$, namely the set $\{[1_1, 3_2, 3_1, 2_2, 2_1, 1_2], [1_1, 12_2, 10_2, 11_2, 3_1, 13_2], [3_1, 8_2, 10_2, 2_1, 13_2, 5_2], [1_1, 6_2, 4_2, 2_1, 7_2, 9_2]\}$ with padding $P_2 = \{((2_1, 1_2), (3_1, 2_2), (1_1, 3_2), (10_2, 11_2), (10_2, 12_2), (5_2, 13_2), (8_2, 10_2), (4_2, 6_2), (7_2, 9_2)\}$ and so $|E(P_2)| = 9$. Therefore, there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where

$|E(P)| = (m+n)/2+1$.

Now suppose $m \equiv 3 \pmod{12}$, $m \geq 15$, and $n \equiv 1 \pmod{12}$, $n \geq 13$. We have $K_{m,n} = K_{7,13} \cup K_{8,13} \cup K_{m-15,13} \cup K_{15,n-13} \cup K_{m-15,n-13}$ where the partite sets of $K_{7,13}$ are $\{1_1, 2_1, \ldots, 7_1\}$ and $\{1_2, 2_2, \ldots, 13_2\}$, the partite sets of $K_{8,13}$ are $\{8_1, 9_1, \ldots, 15_1\}$ and $\{1_2, 2_2, \ldots, 13_2\}$, the partite sets of $K_{m-15,13}$ are $\{16_1, 17_1, \ldots, m_1\}$
and $\{1_2, 2_2, \ldots, 13_2\}$, the partite sets of $K_{m-15,n-13}$ are $\{16_1, 17_1, \ldots, m_1\}$ and
$\{14_2, 15_2, \ldots, n_2\}$. Now $K_{7,13}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = 11$ by Case 1, $K_{8,13}$ has an unrestricted covering with a padding P_2 where $|E(P_2)| = 4$ by Lemma 2.12 Case 9, $K_{m-15,13}$ has an unrestricted covering with a padding P_3 where $|E(P)| = (m-15)/2$, $K_{15,n-13}$ has an unrestricted covering with a padding P_4 where $|E(P_4)| = (n-13)/2$, and there is a hexagon decomposition of $K_{m-15,n-13}$ by Theorem 2.2. Taking these coverings and the decomposition yields an unrestricted covering of $K_{m,n}$ with padding $P = P_1 \cup P_2 \cup P_3 \cup P_4$ where $|E(P)| = (m+n)/2+1$.

Case 3. Suppose $m \equiv 5 \pmod{12}$ and $n \equiv 7 \pmod{12}$. We have $K_{m,n} = K_{m-4,n} \cup K_{4,n}$ where that partite sets of $K_{m-4,n}$ are $V_m \setminus \{(m-3)_1, (m-2)_1, (m-1)_1, m_1\}$ and V_n, and the partite sets of $K_{4,n}$ are $\{(m-3)_1, (m-2)_1, (m-1)_1, m_1\}$ and V_n. Now $K_{m-4,n}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = (m+n-2)/2$ by Case 1, and $K_{4,n}$ has an unrestricted hexagon covering with padding P_2 where $|E(P_2)| = 2$ by Lemma 2.14 Case 5. Taking these two coverings together gives a covering of $K_{m,n}$ with padding P where $|E(P)| = (m+n)/2+1$.

Case 4. Suppose $m \equiv 7 \pmod{12}$ and $n \equiv 1 \pmod{12}$, $n \geq 13$. We have $K_{m,n} = K_{m-4,n} \cup K_{4,n}$ where that partite sets of $K_{m-4,n}$ are $V_m \setminus \{(m-3)_1, (m-2)_1, (m-$

$1)_1, m_1\}$ and V_n, and the partite sets of $K_{4,n}$ are $\{(m-3)_1, (m-2)_1, (m-1)_1, m_1\}$ and V_n. Now $K_{m-4,n}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = (m+n-2)/2$ by Case 2, and $K_{4,n}$ has an unrestricted hexagon covering with padding P_2 where $|E(P_2)| = 2$ by Lemma 2.14 Case 5. Taking these two coverings together gives a covering of $K_{m,n}$ with padding P where $|E(P)| = (m+n)/2 + 1$.

Case 5. Suppose $m \equiv 9 \pmod{12}$ and $n \equiv 7 \pmod{12}$. We have $K_{m,n} = K_{m-8,n} \cup K_{8,n}$ where that partite sets of $K_{m-8,n}$ are $V_m \setminus \{(m-7)_1, (m-6)_1, \ldots, m_1\}$ and V_n, and the partite sets of $K_{8,n}$ are $\{(m-7)_1, (m-6)_1, \ldots, m_1\}$ and V_n. Now $K_{m-8,n}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = (m+n-6)/2$ by Case 1, and $K_{8,n}$ has an unrestricted hexagon covering with padding P_2 where $|E(P_2)| = 4$ by Lemma 2.14 Case 9. Taking these two coverings together gives a covering of $K_{m,n}$ with padding P where $|E(P)| = (m+n)/2 + 1$.

Case 6. Suppose $m \equiv 11 \pmod{12}$ and $n \equiv 1 \pmod{12}$. We have $K_{m,n} = K_{m-8,n} \cup K_{8,n}$ where that partite sets of $K_{m-8,n}$ are $V_m \setminus \{(m-7)_1, (m-6)_1, \ldots, m_1\}$ and V_n, and the partite sets of $K_{8,n}$ are $\{(m-7)_1, (m-6)_1, \ldots, m_1\}$ and V_n. Now $K_{m-8,n}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = (m+n-6)/2$ by Case 2, and $K_{8,n}$ has an unrestricted hexagon covering with padding P_2 where $|E(P_2)| = 4$ by Lemma 2.14 Case 9. Taking these two coverings together gives a covering of $K_{m,n}$ with padding P where $|E(P)| = (m+n)/2 + 1$.

For the remaining cases, $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. There exists a restricted hexagon covering of $K_{n,n}$ with padding P_1 and an unrestricted hexagon covering of $K_{m-n,n}$ with padding P_2, by previous results.

These allow us to cover $K_{m,n}$ with padding $P = P_1 \cup P_2$ which satisfies the required conditions. We present the results in a table which covers these 30 cases.

m (mod 12)	n (mod 12)	$m-n$ (mod 12)	$\|E(P_1)\|$	Lemma/ Case	$\|E(P_2)\|$	Lemma/ Case
1	1	0	$n+4$	2.6/1	$(m-n)/2$	2.10/1
1	3	10	n	2.6/5	$(m-n)/2+1$	2.10/13
1	5	8	n	2.5	$(m-n)/2+4$	2.10/11
1	9	4	n	2.6/5	$(m-n)/2+4$	2.10/6
1	11	2	n	2.5	$(m-n)/2+1$	2.10/4
3	3	0	n	2.6/5	$(m-n)/2$	2.10/1
3	5	10	n	2.5	$(m-n)/2+5$	2.10/14
3	7	8	$n+4$	2.6/1	$(m-n)/2$	2.10/9
3	9	6	n	2.6/5	$(m-n)/2+3$	2.10/8
3	11	4	n	2.5	$(m-n)/2+2$	2.10/7
5	1	4	$n+4$	2.6/1	$(m-n)/2$	2.10/5
5	3	2	n	2.6/5	$(m-n)/2+5$	2.10/3
5	5	0	n	2.5	$(m-n)/2$	2.10/1
5	9	8	n	2.6/5	$(m-n)/2+2$	2.10/10
5	11	6	n	2.5	$(m-n)/2+3$	2.10/8
7	3	4	n	2.6/5	$(m-n)/2+4$	2.10/6
7	5	2	n	2.5	$(m-n)/2+1$	2.10/3
7	7	0	$n+4$	2.6/1	$(m-n)/2$	2.10/1
7	9	10	n	2.6/5	$(m-n)/2+1$	2.10/13
7	11	8	n	2.5	$(m-n)/2+4$	2.10/11
9	1	0	$n+4$	2.6/1	$(m-n)/2$	2.10/9
9	3	10	n	2.6/5	$(m-n)/2+3$	2.10/8
9	5	8	n	2.5	$(m-n)/2+2$	2.10/7
9	9	4	n	2.6/5	$(m-n)/2$	2.10/1
9	11	2	n	2.5	$(m-n)/2+5$	2.10/14
11	3	8	n	2.6/5	$(m-n)/2+1$	2.10/10
11	5	6	n	2.5	$(m-n)/2+4$	2.10/8
11	7	4	$n+4$	2.6/1	$(m-n)/2+3$	2.10/5
11	9	2	n	2.6/5	$(m-n)/2+4$	2.10/3
11	11	0	n	2.5	$(m-n)/2+1$	2.10/1

□

3 RESTRICTED AND UNRESTRICTED COVERINGS OF COMPLETE BIPARTITE GRAPHS WITH HEXAGONS

3.1 Restricted Coverings of $K_{m,n}$ with Hexagons

We can now construct the following theorem from all previous lemmas that were created for restricted hexagon coverings of compelte bipartite graphs.

Theorem 3.1 *A minimal restricted hexagon covering of $K_{m,n}$ (where $m \geq 3$ and $n \geq 3$) with padding P satisfies:*

(1) when $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, $|E(P)| = m + k$, where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m + k) \equiv 0 \pmod 6$,

(2) when $m \equiv n \equiv 1 \pmod 2$ and $m \geq n$, $|E(P)| = m + k$, where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m + k) \equiv 0 \pmod 6$,

(3) when $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 2$, $|E(P)| = 0$,

(4) when $m \equiv n \equiv 2 \pmod 6$ or $m \equiv n \equiv 4 \pmod 6$, then $|E(P)| = 2$, and

(5) when $m \equiv 2 \pmod 6$ and $n \equiv 4 \pmod 6$, then $|E(P)| = 4$.

As shown with the theorem, we know the necessary and sufficient conditions for a restricted covering of a complete bipartite graph with hexagons given that $m \geq 3$ and $n \geq 3$. We have a $K_{6,4}$ by the third condition of Theorem 3.1, we know there exists a decomposition of this bipartite graph since there are no edges in the padding of the covering.

A more complex example would be a $K_{10,5}$. We know we need to use condition one of the theorem. We have a V_m that is equivilent to 0 (mod 2) and an V_n that is equivilent to 1 (mod 2). This means that we have an even amount of vertices on the V_m set and an odd amount of vertices on the V_n set. By the theorem, we know that the padding is 12 which is $50 + 10 \equiv 0 \pmod 6$. This makes sense because we have a total of 50 edges in the $K_{10,5}$. We know that the vertices in the V_m set has odd degrees. This forces us to add an edge for every vertex in V_m to create an even degree since we have a restricted covering and cannot generate an even degree any other way, and since there exists an even number of vertices in V_m, we know that all the degrees in V_n will remain even with these additional edges from V_m to V_n. Then we have a total of 60 edges. This number is equivilent to $0 \pmod 6$, so we know there exists enough edges for a covering, and there exists a construction for the covering. Therefore, we have a minimal covering where the $|E(P)| = 10$

Now, because of all the previous lemmas associated with unrestricted hexagon coverings of complete bipartite graphs we are able to construct a theorem that covers all possible scenarios. A major difference between the theorems is that fact that the theorem for unrestricted hexagon coverings are able to cover the scenarios where one of the vertex sets is of order one or two.

3.2 Unrestricted Coverings of $K_{m,n}$ with Hexagons

Theorem 3.2 *A minimal unrestricted hexagon covering of $K_{m,n}$ with padding P satisfies:*

(1) *when $m = 1$ and $n \geq 4$, $|E(P)| = 2n$ for n even and $|E(P)| = 2n + 3$ for n odd,*

(2) *when $m = 2$ and $n \geq 4$, $|E(P)| = n$ for n even and $|E(P)| = n + 3$ for n odd,*

(3) *when $m \equiv 0 \pmod 2$, $m \geq 4$, and $n \equiv 1 \pmod 2$, $n \geq 3$, $|E(P)| = m/2 + k$, where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m/2 + k) \equiv 0$ (mod*

6),

(4) *when* $m \equiv n \equiv 1$ *(mod* 2*) and* $m \geq n \geq 3$, $|E(P)| = (m+n)/2 + k$, *where* k *is the smallest nonnegative integer such that* $|E(K_{m,n})| + (m+n)/2 + k \equiv 0$ *(mod* 6*),*

(5) *when* $m \equiv 0$ *(mod* 6*) and* $n \equiv 0$ *(mod* 2*),* $n \geq 4$, $|E(L)| = 0$,

(6) *when* $m \equiv n \equiv 2$ *(mod* 6*),* $n \geq 4$, *or* $m \equiv n \equiv 4$ *(mod* 6*),* $m \geq 4$, *then* $|E(P)| = 2$, *and*

(7) *when* $m \equiv 2$ *(mod* 6*),* $m \geq 8$, *and* $n \equiv 4$ *(mod* 6*), then* $|E(P)| = 4$.

Let us apply this theorem to an example of a $K_{6,7}$. We see that the first condition applies to this example where m is even and n is odd. Since we are working with an unrestricted covering, we connect odd degree vertices together. This means it takes roughly half the number of edges in the padding compared to the restricted covering. So we know $|E(P)| = m/2 + k$, where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m/2 + k) \equiv 0$ (mod 6). We know m is six in this example and k would be zero. This means the cardinality of the padding is three.

4 CONCLUSION

We see a difference between restricted and unrestricted coverings. It is important to note though that the difference is only observed when one of the sets in a complete bipartite graph is odd. When they are even, the theorems are equivalent. This means that the difference between restricted and unrestricted mostly helps correct the odd degree argument. Let us compare the first condition of each theorem which involves one set being even and the other being odd.

For the restricted covering, we have when $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, $|E(P)| = m + k$, where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m + k) \equiv 0 \pmod 6$. For the unrestricted covering we have when $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, $|E(P)| = m/2 + k$, where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m/2 + k) \equiv 0 \pmod 6$. In both theorems, n is the odd set. Therefore, every vertex in m has odd degree. We notice that the difference is in cardinality of the padding, where in the restricted $|E(P)| = m + k$ which m fixes the odd degree argument in the m set and k is to fix the cardinality argument and in the unrestricted $|E(P)| = m/2 + k$ which m/2 fixes the odd degree argument of the set m and k fixes the cardinality argument. Since m always has an even number of vertices, $m/2$ will be a whole number and $m/2$ fixed the odd degree argument in unrestricted coverings because we can allow one edge to fix two vertices.

From our observations, we can say that unrestricted coverings are more efficient than restricted coverings. This concept can be spread into other types of graph coverings and is not unique to bipartite graphs. This concept is to be explored in the future with the covering of complete graphs with a whole and other types of complete graphs that will help model real situations.

BIBLIOGRAPHY

[1] A. Brouwer. Optimal Packings of K_4's into a K_n. *Journal of Combinatorial Theory, Series A*. 26(3):278-297, 1979.

[2] L. Brown, G. Coker, R. Gardner, and J. Kennedy. Packing the Complete Bipartite Graph with Hexagons. *Congressus Numerantium*. 174:97-106, 2005.

[3] M. Fort and G. Hedlund. Minimal Coverings of Pairs by Triples. *Pacific Journal of Mathematics*. 8:709-719, 1958.

[4] B. Alspach and H. Gavlas. Graph Decompositions of K_n and K_{n-1}.*Jounal of Combinatorial Theory B*. 81:77-99, 2001.

[5] J. Kennedy. Maximum Packings of K_n with Hexagons. *Australasian Journal of Combinatorics*. 7:101-110, 1993.

[6] J. Kennedy. Maximum Packings of K_n with Hexagons: Corrigendum. *Australasian Journal of Combinatorics*. 10:293, 1994.

[7] J. Kennedy. Minimum Coverings of K_n with Hexagons. *Autstralasian Journal of Combinatorics*. 16: 295–303, 1997.

[8] T. Kirkman. On a problem in combinations. *Cambridge and Dublin Mathematics Journal*. 2:191-204, 1847.

[9] J. Schönheim. On Maximal Systems of k-Tuples. *Studia Sci. Math. Hungarica*. 363-368, 1966.

[10] J. Schönheim and A. Bialostocki. Packing and Covering of the Complete Graph with 4-Cycles. *Canadian Mathematics Bulletin*. 18(5):703–708, 1975.

[11] D. Sotteau. Decompositions of $K_{m,n}$ ($K^*_{m,n}$) into Cycles (Circuits) of Length $2k$. *Journal of Combinatorial Theory, Series B*. 30:75-81, 1981.

[12] K. Steiner. Combinatorische Aufgabe. *Journal fur die Reine und angewandte Mathematik (Crelle's Journal)*. 45; 181-182, 1853.

VITA

WESLEY SURBER

Education:	B.S. Mathematics, East Tennessee State University, Johnson City, Tennessee 2011 M.S. Mathematical Sciences, East Tennessee State University Johnson City, Tennessee 2013
Professional Experience:	Graduate Assistant, East Tennessee State University Johnson City, Tennessee, 2011–2013
Publications:	R. Gardner and W. Surber, "Restricted and Unrestricted Coverings of Complete Bipartite Graphs with Hexagons," In Preparation.