8-2003

Placing Monitoring Devices in Electric Power Networks Modeled by Block Graphs.

David Wayne Atkins
East Tennessee State University

Placing Monitoring Devices in Electric Power Networks

Modeled by Block Graphs

A thesis

presented to the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment of the requirements for the Degree

Master of Science in Mathematical Sciences

---

by

David Wayne Atkins

August 2003

Teresa Haynes, Ph.D., Chair

Robert Gardner, Ph.D.

Debra Knisley, Ph.D

Keywords: Block Graph, Power Domination, PMU Placement

ABSTRACT

Placing Monitoring Devices in Electric Power Networks

Modeled by Block Graphs

by

David Wayne Atkins

The problem of monitoring an electric power system by placing as few measurement devices in the system as possible is closely related to the well known vertex covering and dominating set problems in graph theory. A set S of vertices is defined to be a power dominating set of a graph if every vertex and every edge in the system is monitored by the set S (following a set of rules for power system monitoring). The minimum cardinality of a power dominating set of a graph is its power domination number. In this thesis, we investigate the power domination number of a block graph.

DEDICATION

I dedicate this thesis to my family for all their support while receiving my education. I also dedicate this to my girlfriend Jamie Howard who has sacrificed to help me through school. Thank You and I Love You!!!

ACKNOWLEDGEMENTS

I would like to thank my advisors Dr. Teresa Haynes and Dr. Michael Henning for all of their useful knowledge, advice, and support during the preparation of this thesis. Thank you and I will always be grateful for the help.

Contents

ABSTRACT ii

COPYRIGHT iii

DEDICATION iv

ACKNOWLEDGEMENTS v

LIST OF FIGURES vii

1 Introduction **1**

1.1 Graph Theory 1

1.2 PMU Placement and Power Domination 7

2 Previous Results **13**

3 New Results **22**

BIBLIOGRAPHY 37

VITA 41

List of Figures

1 Example 1. 2

2 Example 2 3

3 A path on five vertices. 3

4 A complete graph on five vertices. 4

5 A complete bipartite graph. 5

6 The star, $K_{1,s}$. 5

7 A tree T. 6

8 Distance Example. 6

9 Domination Example 9

10 The graph of a spider. 15

11 A tree with $\gamma_P(T) = \frac{k+2}{3}$ where $k = 4$. 16

12 A 2× 3 grid graph. 20

13 An 8×10 grid graph with $\gamma_P(G) = 2$. 21

14 A block graph G. 22

15 A block-star. 23

16 A block-spider. 24

17 A block-spider G with $\gamma_P(G) = 1$. 25

18 A connected block graph G with $\mathrm{sp}_b(G) = 4$ and $\gamma_P(G) = 6$. 29

19 A connected block graph G with $k = 7$ vertices of block-degree at least 3 and $\gamma_P(G) = (k + 2)/3$. 32

20 A connected block graph G with $\gamma_P(G) > \mathrm{sp}_b(G)$ in which all but one block is K_2 or K_3. 36

1 Introduction

1.1 Graph Theory

Graph theory is an area of discrete mathematics. In recent decades, graph theory has developed an importance in many concentrations including computer science, biology, and chemistry. For example, graph theory can be used to model DNA as well as schedule final examinations for a university. First, a few basic definitions are needed.

In graph theory, we use a finite nonempty set of objects called **vertices** (the singular is vertex) together with a (possibly empty) set of unordered pairs of distinct vertices called **edges**. We represent a vertex by a dot, and if two vertices have a particular interest in common, there is an edge joining them. The vertex set and the edge set make up the **graph** G. For example, suppose the vertex set represents the classes offered at a university. If a student is in two classes, there is an edge joining the corresponding vertices. Consequently, the final exams for these two classes would have to be scheduled at different times. The solution to scheduling of exams so that no two with a common student are scheduled at the same time while trying to minimize time slots for the exams is a well known "coloring" problem in graph theory.

The edge $e = \{u, v\}$ is said to join the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then we say that u and v are **adjacent** vertices. Furthermore, u and e are **incident**, as are v and e. The vertex set of a graph G is denoted $V(G)$, and the edge set is denoted $E(G)$. The **order** of G, denoted n, is the cardinality of its vertex set, and the **size** of G, denoted m, is the cardinality of its edge set.

Consider the graph G of Figure 1 below. This graph has vertex set, $V(G) = \{u, v, w, x\}$ and edge set, $E(G) = \{uv, vx, wx, uw\}$. In addition this graph has order $n = 4$ and size $m = 4$.

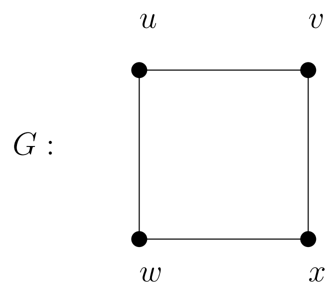


Figure 1: Example 1.

We need a few more definitions. The **degree** of a vertex v is the number of edges incident with v, denoted $deg\ v$. A vertex with degree 0 is called an isolated vertex or an **isolate**, while a vertex with degree 1 is called an **end-vertex**. The minimum degree among all the vertices of G is denoted $\delta(G)$ and the maximum degree among all the vertices of G is denoted $\Delta(G)$. Notice in Figure 2 below, that $\delta(G) = 0$ because $deg\ z = 0$ and $\Delta(G) = 3$ because $deg\ v = 3$. Also notice that w is an end-vertex and, of course, z is an isolate.

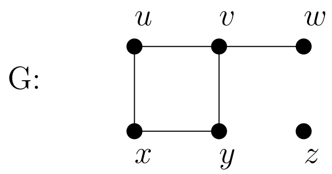


Figure 2: Example 2

In graph theory, we spend time studying properties of certain families of graphs. (Families are collections of graphs which may vary in order and size, but all have the same basic structure.) There are several different families of graphs that we consider, and we will briefly describe a few of them.

1) Let u and v be (not necessarily distinct) vertices of a graph G. A $u - v$ **path** of G is a finite, alternating sequence

$$u = u_0, e_1, u_1, e_2, ..., u_{k-1}, e_k, u_k = v$$

of vertices and edges, beginning with vertex u and ending with vertex v such that $e_i = u_{i-1}u_i$ for $i = 1, 2, ..., k$, and no vertex is repeated. The number k is called the **length** of the path, and a trivial path is one which contains no edges, that is, $k = 0$. A path is denoted P_n. An example of the path P_5 is shown in Figure 3 below.

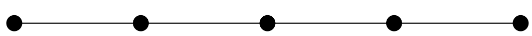


Figure 3: A path on five vertices.

2) A path that starts and ends at the same vertex is called a **cycle.** A cycle is denoted C_n. An example of the cycle C_4 is shown in Figure 1.

3) A graph is said to be **complete** if every pair of its vertices are adjacent. A complete graph is denoted K_n. An example of the complete graph K_5 is shown in Figure 4 below.

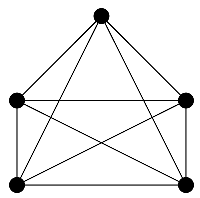


Figure 4: A complete graph on five vertices.

4) A **bipartite graph** is a graph with the property that the $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every element of $E(G)$ joins a vertex of V_1 to a vertex of V_2. A **complete bipartite graph** G is a bipartite graph having the added property that for all $u \in V_1$ and $v \in V_2$, then $uv \in E(G)$. If $|V_1| = r$ and $|V_2| = s$, then the complete bipartite graph is denoted $K_{r,s}$. An example of the complete bipartite graph $K_{2,3}$ is shown in Figure 5.

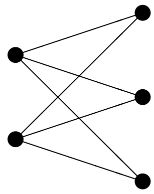


Figure 5: A complete bipartite graph.

5) A special case of a complete bipartite graph where $|V_1| = 1$ and $|V_2| = s$ is called a star and is denoted $K_{1,s}$. An example of a star is shown in Figure 6 below.

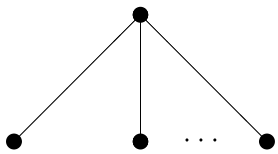


Figure 6: The star, $K_{1,s}$.

6) A vertex u is said to be **connected** to a vertex v in a graph G if there exists a $u-v$ path in G. A graph G is connected if every two of its vertices are connected. A **tree** T is a connected graph with no cycles. An example of a tree is shown in Figure 7.

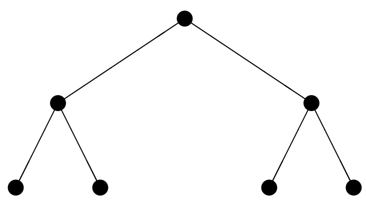


Figure 7: A tree T.

Many times it is useful to study the distance between pairs of vertices in a graph. For a connected graph G, we define the **distance** $d(u, v)$ between two vertices u and v as the minimum of the lengths of the $u - v$ paths of G. Notice that $d(u, v) = 0$ if and only if $u = v$. Consider Figure 8 below. Notice there are three $u - x$ paths, but the shortest path is simply the edge $e = \{u, x\}$. Therefore, the distance from u to x, $d(u, x) = 1$.

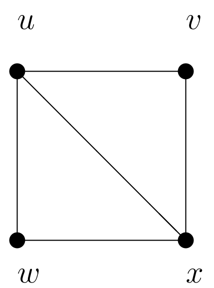


Figure 8: Distance Example.

We have merely scratched the surface of graph theory with this introduction. Nevertheless, we have explained some basic definitions and graphs that will be helpful in understanding the later material. In the next two chapters, additional definitions will be given that are essential to understanding the desired results. In general, we follow the terminology of [9].

1.2 PMU Placement and Power Domination

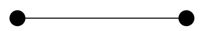
An **electrical power system** includes a set of buses and a set of lines connecting the buses. A **bus** is a substation where transmission lines are joined. A power system also includes a set of generators, which supply power and a set of loads, into which the power is directed [4]. Electric power companies need to continually monitor their system's state as defined by a set of state variables, for example, the voltage magnitude at loads and the machine phase angle at generators. A type of measurement device is placed at select locations in the system to collect information on these state variables which is sent back to the central control. One such measurement device that is used at these locations is called a *Phase Measurement Unit, PMU.* PMUs are extremely expensive, so the electric companies want to minimize the number of PMUs while making sure the whole system is monitored or *observed*. Some measurement devices are not as efficient as PMUs and require a monitoring device at each bus, which is one reason that PMUs are widely used. A system is said to be observed if all the state variables can be determined from the set of measurements, such as voltage and currents, taken at the PMUs. We will represent these systems using a graph where the vertices represent the buses and the edges represent the transmission lines. If two buses have a transmission line between them, there is an edge between the corresponding vertices in the graph. A PMU measures the state variable for the vertex (bus) at which it is placed, its incident edges, and their endvertices. Using Ohm's Law and Kirchoff's Law, we can state the other rules to show observability [4].

From this point on, a thick line represents an observed edge, while a thin line represents an edge which is unobserved. In addition, a closed circle represents an observed vertex, and an open circle represents an unobserved vertex. A vertex at which a PMU is placed is indicated by an arrow.

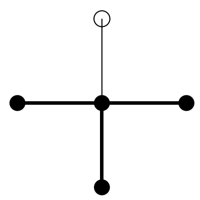
Rules:

1. *Ohm's Law*, P=IR: Any bus that is incident to an observed line connected to an observed bus is observed (the known current in the line, the known voltage at the observed bus, and the known resistance of the line determines the voltage at the bus).

2. *Ohm's Law*, I=P/R: Any line joining two observed buses is observed (the known voltage at both observed buses and the known resistance of the line determines the current on the line).



3. *Kirchoff's Law*: If all the lines incident to an observed bus are observed, except one, then all of the lines incident to that bus are observed (the net current flowing through a bus is zero).



The power system monitoring problem was first studied as a variation of the well-known dominating set problem. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a **dominating set** of G if every vertex in $V \setminus S$ has at least one neighbor in S. The cardinality of a minimum dominating set of G is the **domination number** $\gamma(G)$. Consider Figure 9.

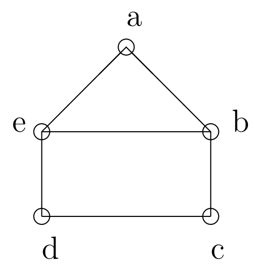
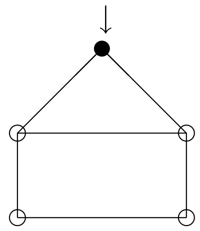


Figure 9: Domination Example

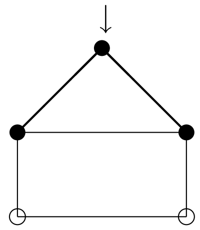
Let $S = \{a, c\}$. Notice that the remaining vertices of the graph are adjacent to at least one vertex in S. Therefore, S is a dominating set of the graph G, and so $\gamma(G) \leq 2$. If the set S consists of any one vertex in G, there is at least one vertex that is not dominated by S. Consequently, $\gamma(G) \geq 2$. Hence, $\gamma(G) = 2$.

Considering the power system monitoring problem as a variation of the dominating set problem, we define a set S to be a **power dominating set** if every vertex and every edge in G is observed by S. The **power domination number** $\gamma_P(G)$ is the minimum cardinality of a power dominating set of G. A power dominating set of cardinality $\gamma_P(G)$ we call a $\gamma_P(G)$-set or PDS. Since any dominating set is a power dominating set, $1 \leq \gamma_P(G) \leq \gamma(G)$ for all graphs G. Let us use Figure 9 again as an example.

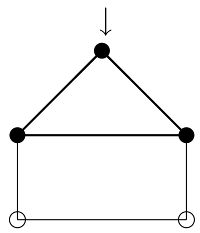
Suppose we place a PMU at the darkened vertex denoted by the arrow.



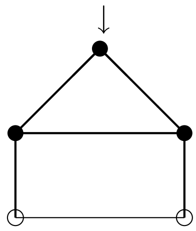
By definition, we observe the darkened vertices and edges.



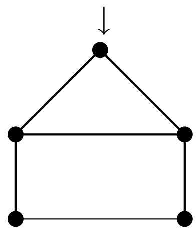
Now, applying Rule 2, we observe another edge.



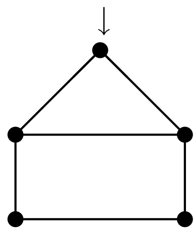
At this point, we may use Rule 3 to observe two more edges.



By using Rule 1, we observe the two remaining vertices.



Finally, using Rule 2 again, we complete the power domination of the graph.



For a given set of vertices $P \subseteq V$ representing the buses where the PMUs are placed, the following algorithm determines the sets of (observed) vertices C and edges F.

1. Initialize $C = P$ and $F = \{e \in E \mid e \text{ is incident to a vertex in } P\}$.
2. Add to C any vertex not already in C which is incident to an edge in F.
3. Add to F any edge not already in F such that
    a. both of its end-vertices are in C or
    b. it is incident to a vertex v of degree greater than one for which all the other edges incident to v are in F.
4. If steps 2 and 3 fail to locate any new edges or vertices for inclusion, stop. Otherwise, go to step 2.

Therefore, to solve the power system monitoring problem, we want $C = V$, $F = E$, and to minimize $|P|$.

We have introduced the concept of PMU placement and power domination as related to the power system monitoring problem. In the following chapters, results from previous works and new results from my work will be presented.

2 Previous Results

We first consider results from [6]. Recall that any dominating set is a power dominating set, and so we have the following observation.

Observation 1 *For any graph G, $1 \leq \gamma_P(G) \leq \gamma(G)$.*

The next observation determines the power domination number for several of the families of graphs we introduced in Chapter 1.

Observation 2 *For the graph G where $G \in \{K_n, C_n, P_n, K_{2,n}\}, \gamma_P(G) = 1$.*

In fact, for $G \in \{K_n, C_n, P_n\}$, placing a PMU at any vertex will power dominate the entire graph. For $G = K_{2,n}$, a PMU placed at a vertex in V_1 will power dominate the complete bipartite graph.

Observation 3 *There is no forbidden subgraph characterization of the graphs G for which $\gamma_P(G) = \gamma(G)$.*

The next observation notes the significance of placing PMUs at vertices of large degree.

Observation 4 *If G is a graph with maximum degree at least 3, then G contains a $\gamma_P(G)$-set in which every vertex has degree at least 3.*

A problem is said to be **NP-Complete** if it is not solvable in polynomial time. For example, consider the following question.

POWER DOMINATING SET (PDS)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k > 1$.

QUESTION: Does G have a power dominating set of size at most k?

Theorem 5 *POWER DOMINATING SET is NP-complete for bipartite graphs.*

Another graph considered is called a chordal graph. A graph G is called **chordal** if every cycle of G of length greater than 3 has a chord, that is, an edge joining two nonconsecutive vertices of the cycle [9].

Theorem 6 *POWER DOMINATING SET is NP-complete for chordal graphs.*

The next results concern power domination applied to trees. Recall that a tree is defined as a connected graph containing no cycles. A **leaf** is defined to be a vertex of degree 1, while a vertex adjacent to a leaf is called a **support vertex**. If a vertex is adjacent to two or more leaves, it is called a **strong support vertex.**

A **subdivision** of a nonempty graph G is a graph obtained from G by removing some edge $e = uv$ and adding a new vertex w and edges uw and vw [9]. A special type of tree we will consider is called a spider. Let T be a tree formed from a star by subdividing any number of its edges any number of times, that is, T has at most one vertex of degree 3 or more. We call such a tree T a **spider**. Consider Figure 10.

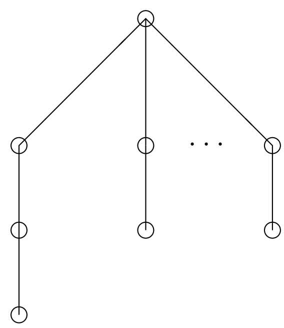


Figure 10: The graph of a spider.

Theorem 7 *For any tree T, $\gamma_P(T) = 1$ if and only if T is a spider.*

Sometimes it is useful to know what graphs have the property that their domination number and power domination number are equal. The next two results characterize these trees.

Observation 8 *If v is a strong support vertex in a graph G, then v is in every $\gamma(G)$-set and every $\gamma_P(G)$-set.*

Theorem 9 *For a tree T of order at least 3, $\gamma_P(T) = \gamma(T)$ if and only if T has a unique $\gamma(T)$-set S and every vertex in S is a strong support vertex.*

In order to bound the power domination number of a tree T, we partition the tree into spiders. The minimum number of subsets into which $V(T)$ can be partitioned so that each subset is a spider is called the **spider number**, denoted $\mathrm{sp}(T)$, of the tree T. A lower bound for the power domination number of T is given.

Lemma 10 *For any tree T, $sp(T) \leq \gamma_P(T)$.*

An upper bound for the power domination of T is also given.

Lemma 11 *For any tree T, $\gamma_P(T) \leq sp(T)$.*

Since the power domination number of T is bounded above and below by the spider number of T, they must be equal. Hence, the following theorem is proved.

Theorem 12 *For any tree T, $\gamma_P(T) = sp(T)$.*

They also determine a lower bound on the power domination number of a tree in terms of the number of vertices of degree at least 3 [6].

Theorem 13 *If T is a tree having k vertices of degree at least three, then*

$$\gamma_P(T) \geq \frac{k+2}{3},$$

and this bound is sharp.

To illustrate the sharpness of this bound, consider Figure 11.

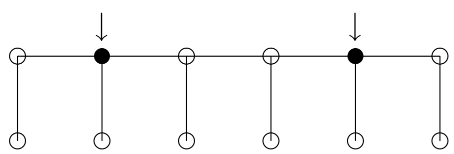


Figure 11: A tree with $\gamma_P(T) = \frac{k+2}{3}$ where $k = 4$.

The next result shows that the power domination number of a tree is at most a third the order of the tree.

Theorem 14 *For any tree of order* $n \geq 3$, $\gamma_P(T) \leq \frac{n}{3}$ *with equality if and only if* T *is the corona* $T' \circ \overline{K_2}$ *where* T' *is any tree.*

The last result presents a linear algorithm for finding a minimum power dominating set in a nontrivial tree T.

Algorithm 1 :

Input: A tree T on $n \geq 2$ vertices rooted at a vertex of maximum degree with the vertices labeled $v_1, v_2, \ldots, v_n$ so that $\ell(v_i) \leq \ell(v_j)$ for $i > j$. [Note: the root of T is labeled v_n.]

Output: A minimum power dominating set S of T and a partition of $V(T)$ into $|S|$ subsets $\{V_x \mid x \in S\}$ so that each subset induces a spider.

Begin

1. *If* T *is a spider, then* $S \leftarrow \{v_n\}$ *and* $V_{v_n} \leftarrow V(T)$, *and output* S *and* $\{V_x \mid x \in S\}$; *otherwise, continue.*

2. *Type*$(v_i) \leftarrow$ TRUE *and* $V_{v_i} \leftarrow \emptyset$ *for all* $i = 1, 2, \ldots, n$.

3. $i \leftarrow 1$, $T \leftarrow T$, $\mathcal{I} \leftarrow \{1, 2, \ldots, n\}$, *and* $S \leftarrow \emptyset$.

4. $v \leftarrow v_i$.

5. *If* $\deg_T v \leq 2$, *then*

   5.1. *if there exists a child* u *of* v (in G) *such that Type*(u) =TRUE *and* $u \in V_x$ *for some* $x \in S$, *then*

5.1.1. *if* v *is a leaf* (in T), *then*

5.1.1.1. $V_x \leftarrow V_x \cup \{v\}$,

5.1.1.2. $T \leftarrow T - v$, $\mathcal{I} \leftarrow \mathcal{I} \setminus \{i\}$ *and go to Step* 13;

5.1.2. *if* $\deg_T v = 2$, *then*

5.1.2.1. $V_x \leftarrow V_x \cup V(T_v)$,

5.1.2.2. $w \leftarrow v$ *and go to Step* 12;

5.2. *otherwise* (if no such child exists), $\mathcal{I} \leftarrow \mathcal{I} \setminus \{i\}$ *and go to Step* 13;

otherwise (if $\deg_T v \geq 3$), *then continue.*

6. $S \leftarrow S \cup \{v\}$.

7. $w \leftarrow v_m$ *where* $m \leftarrow \min\{k \mid v_k$ *is an ancestor of* v *of degree at least* 3 *in* T *or* $k = n\}$.

8. $u \leftarrow \{$*child of* w *on the* w–v *path*$\}$.

9. *If* $w = v_n$, *then*

9.1. *if the component of* $T - uw$ *containing* w *is the trivial path* w *or a path with* w *as a leaf, then* $V_v \leftarrow V(T_w)$, *and output* S *and* $\{V_x \mid x \in S\}$;

9.2. *otherwise,* $V_v \leftarrow V(T_u)$ *and go to Step* 11.

10. *If* $w \neq v_n$, *then*

10.1. $z \leftarrow parent(w)$;

10.2. *if* $\deg_T w \geq 4$ *or if* $\deg_T w = 3$ *and the component of* $T - \{uw, wz\}$ *containing* w *is a not a path, then* $V_v \leftarrow V(T_u)$ *and go to Step* 11*;*

10.3. *if* $\deg_T w = 3$ *and the component of* $T - \{uw, wz\}$ *containing* w *is a path, then* $V_v \leftarrow V(T_w)$ *and go to Step* 12*.*

11. $T \leftarrow T - V(T_u)$, $\mathcal{I} \leftarrow \mathcal{I} \setminus \{k \mid v_k \in V(T_u)\}$, *and go to Step* 13*.*

12. $T \leftarrow T - V(T_w)$, $\mathcal{I} \leftarrow \mathcal{I} \setminus \{k \mid v_k \in V(T_w)\}$, $Type(w) \leftarrow \text{FALSE}$. *Go to Step* 13*.*

13. $i \leftarrow \min\{k \mid k \in \mathcal{I}\}$.

14. *If* $i < n$, *then return to Step* 4*.*

15. *If* $i = n$, *then*

15.1. *if* (*a*) T *is the trivial path* v_n *or* T *is a path with* v_n *as a leaf, and* (*b*) *there exists a child* u *of* v (in G) *such that* $Type(u) = \text{TRUE}$ *and* $u \in V_x$ *for some* $x \in S$*, then*

15.1.1. $V_x \leftarrow V_x \cup V(T_{v_n})$,

15.1.2. *output* S *and* $\{V_x \mid x \in S\}$.

15.2. *otherwise* (if (a) or (b) in 15.1 does not hold)*, then*

15.2.1. $S \leftarrow S \cup \{v_n\}$,

15.2.2. $V_{v_n} \leftarrow V(T_{v_n})$,

15.2.3. *output* S *and* $\{V_x \mid x \in S\}$.

End

Theorem 15 *Algorithm 1 produces a* $\gamma_P(T)$*-set in a nontrivial tree* T*.*

The next results are taken from [5]. These results focus on power domination applied to grid graphs. An $n \times m$ **grid graph** is a graph consisting of a $P_n \times P_m$. A 2×3 grid graph is shown in Figure 12.

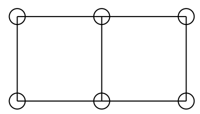


Figure 12: A 2× 3 grid graph.

The main result in [5] shows the power domination of grid graphs for $m \geq n \geq 1$. Notice if m=1 or n=1, we have a path and the power domination number of a path was shown in [6].

Theorem 16 *If G is an $n \times m$ grid graph $P_n \times P_m$ where $m \geq n \geq 1$, then*

$$\gamma_P(G) = \begin{cases} \left\lceil \frac{n+1}{4} \right\rceil & \textit{if } n \equiv 4 \pmod 8 \\ \left\lceil \frac{n}{4} \right\rceil & \text{otherwise.} \end{cases}$$

We will illustrate this result for an 8×10 grid graph G in Figure 13 . Notice that $8 \equiv 0 (\text{mod } 8)$, and so $\gamma_P(G) = 2$.

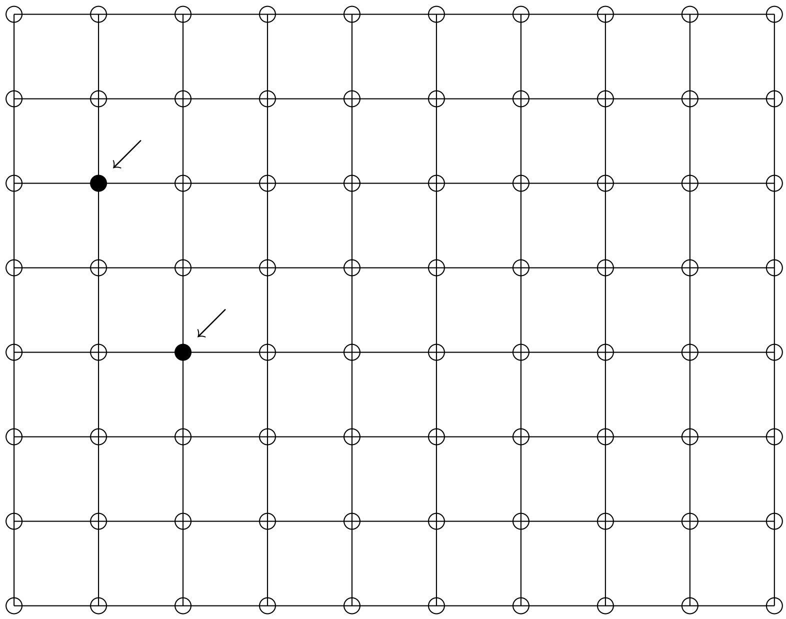


Figure 13: An 8×10 grid graph with $\gamma_P(G) = 2$.

Using the rules of PMU placement, it can be verified that these two PMUs will monitor the entire 8×10 grid. In fact, two PMUs will monitor an entire $8 \times m$ grid for $m \geq 8$.

This concludes the previous results from other works. In the next chapter, my results will be presented. These results generalize the tree theorem from [6].

3 New Results

In this chapter, I present new results that we have obtained. Our results are based on block graphs. First, we will define a block graph and some other useful terminology.

A **block** of a graph G is a maximal, 2-connected subgraph of G, where a subgraph is part of the original graph G. We call a graph G a **block graph** if and only if every block of G is a complete graph. We call a block of G that is a complete graph K_r, a K_r**-block** of G, and the number of blocks in G is denoted by $b(G)$. Consider the graph of G in Figure 14 below. Notice that $b(G) = 7$, and G has two K_4-blocks, four K_2-blocks, and one K_3-block.

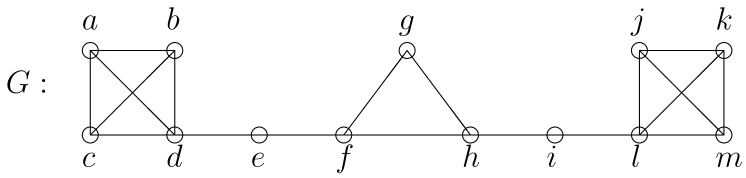


Figure 14: A block graph G.

We will also extend the notion of degree to block graphs. The **block-degree** of a vertex v in G is the number of blocks in G that contain v. In Figure 14, vertices a, b, c, g, j, k, and m have block-degree 1 while the vertices d, e, f, h, i, and l have block degree 2. A **cut-vertex** v of G is a vertex such that $G \setminus \{v\}$ disconnects the graph. An **end-block** of G is a block that contains only one cut-vertex of G. In Figure 14, the two K_4 blocks are end-blocks because they both have only one cut-vertex, namely d and l.

We also need to further the idea of the star and spider graphs. Let G be a block graph. If G itself is a block or if every block of G is an end-block, then we call G a **block-star**. Notice that a star $K_{1,n}$ where $n \geq 1$ is simply a block-star where every block is a K_2 block. An example of a block-star is given in Figure 15.

G :

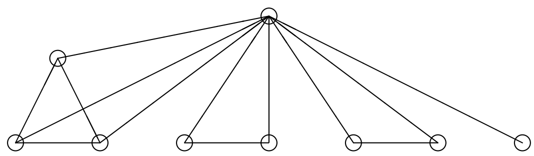


Figure 15: A block-star.

Now, if we take the graph of a block-star and attach a path to all or to some (including the possibility of none) of its vertices so that the resulting paths are vertex-disjoint, we have a **block-spider**. Again, note that if every block of a block-spider is a K_2-block, then we also have a spider. If the block-star of the block-spider has a cut-vertex, then this vertex is called the **head** of the block-spider. Notice that every vertex of a block-spider, except for possibly its head, belongs to at most two blocks in the block-spider. If a vertex other than the head belongs to two blocks, then at least one of these blocks is a K_2. An example of a block-spider is given in Figure 16.

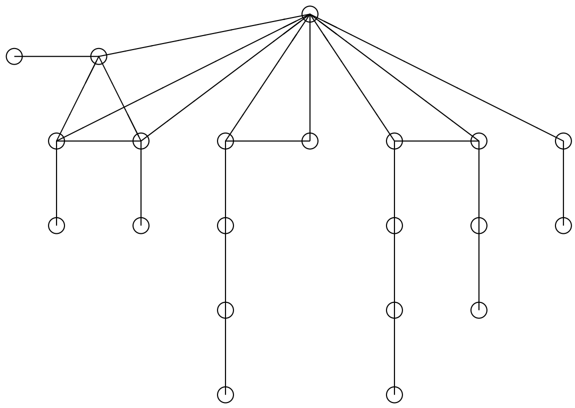


Figure 16: A block-spider.

We define the **block-spider number** of a block graph G, to be the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a block-spider. The block-spider number is denoted $\mathrm{sp}_b(G)$.

Finally, we need to discuss relationships between vertices. Given a connected block graph G, we **root** G as follows. We first identify a vertex r of G which we call the *root* of G. For each vertex $v \in V(G)$, define the *level number* of v, denoted $\ell(v)$, to be its distance $d(v, r)$ from r. If a vertex u of G is adjacent to v and $\ell(u) > \ell(v)$, then we call u a *block-child* of v, and v its *block-parent*. A vertex w is a *block-descendant* of v (and v is a *block-ancestor* of w) if the level numbers of the vertices on the v–w path are monotonically increasing. We let $D_b(v)$ denote the set of block-descendants of v in the rooted block graph G, and we define $D_b[v] = D_b(v) \cup \{v\}$. We define the *maximal block subgraph of G rooted at v* to be the block subgraph of G induced by $D_b[v]$, and we denote it G_v.

Recall, a tree is a block graph where every block is a K_2. So, our first result is a generalization of Theorem 7.

Theorem 17 *For any block graph G, $\gamma_P(G) = 1$ if and only if G is a block-spider. Furthermore, the head of a block-spider is a $\gamma_P(G)$-set.*

Proof. Suppose G is a block-spider. Then its head is a $\gamma_P(G)$-set, and so $\gamma_P(G) = 1$. To prove, the necessity, suppose that G is not a block-spider. Then G contains at least two vertices, u and v say, that both have block-degree at least 3 or that both belong to at least two blocks of order at least 3. We now root the block graph G at any vertex of G. Let S be any $\gamma_P(G)$-set. If $|S| = 1$, then, renaming u and v if necessary, we may assume that no vertex in the maximal block subgraph G_u rooted at u belongs to S. Since there are at least two edges of G_u incident with u, no edge in G_u is observed, a contradiction. Therefore, $|S| \geq 2$, and so $\gamma_P(G) \geq 2$. □

Theorem 17 is illustrated in Figure 17. It is easy to see the darkened vertex will power dominate the whole graph.

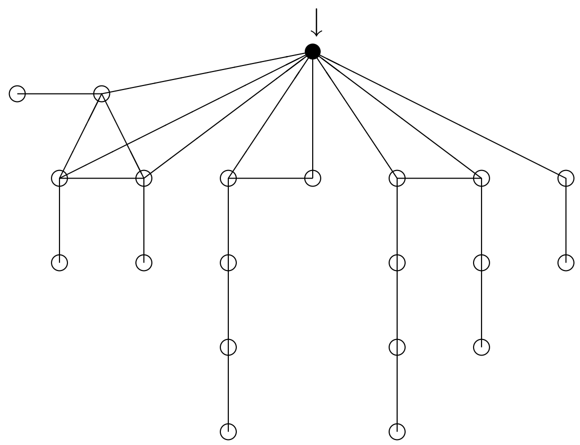


Figure 17: A block-spider G with $\gamma_P(G) = 1$.

The main goal of our research is to find a relationship between the power domination number and the block-spider number of a block graph. The next result shows this relationship.

Theorem 18 *If G is a connected block graph, then* $\mathrm{sp}_b(G) \leq \gamma_P(G) \leq 2\mathrm{sp}_b(G) - 1$.

Proof. To prove that $\mathrm{sp}_b(G) \leq \gamma_P(G)$, we proceed by induction on $m = \gamma_P(G)$. Suppose $m = 1$. Then G is a block-spider, and so, by Theorem 17, $\mathrm{sp}_b(G) = 1 = \gamma_P(G)$. Suppose, then, that for all connected block graphs G' with $\gamma_P(G') = m$, where $m \geq 1$, that $\mathrm{sp}_b(G') \leq \gamma_P(G')$. Let G be a connected block graph with $\gamma_P(G) = m+1$. By Observation 4, G contains a $\gamma_P(G)$-set $S = \{h_1, h_2, \ldots, h_{m+1}\}$ in which every vertex has degree at least 3.

We now root the block graph G at the vertex h_{m+1}. Renaming the vertices of S if necessary, we may assume that among all the vertices of S, h_1 has the largest level number, that is, among all vertices in S, h_1 is at maximum distance from h_{m+1} in G. Let w be the block-parent of h_1 and let B_1 be the block containing h_1 and w. We now consider two possibilities.

Case 1. w has block-degree two.

Let u be the block-ancestor of w of degree at least 3 that is at minimum distance from w. Then every internal vertex on the u–w path has degree 2 in G. Let v be the block-child of u on the u–w path (possibly, $v = w$). Note that no internal vertex on the u–h_1 path (including the vertex w) belongs to S.

We now define a set V_1 as follows. If S contains two vertices of B_1, let $V_1 = D_b[h_1]$. If h_1 is the only vertex of S in B_1 and if $u \in S$, let $V_1 = D_b[v]$. If h_1 is the only vertex of S in B_1 and if $u \notin S$, let $V_1' = D_b[v] \cup \{u\}$. If now $G - V_1'$ contains a path-component P that contains no vertex of S, then let $V_1 = V_1' \cup V(P)$ (notice that since S is a PDS of G, there is at most one such path-component P and u is adjacent to an end-vertex of P and to no other vertex of P). On the other hand, if every component of $G - V_1'$ contains a vertex of S, then let $V_1 = V_1'$. In all of the above cases, let $G' = G - V_1$.

By construction, $G[V_1]$ is a block-spider with head h_1, and G' is a block graph (possibly disconnected) in which $S - \{h_1\}$ is a PDS of G'. Thus, $\gamma_P(G') \leq m$. Applying the inductive hypothesis to each component of G', we have $\mathrm{sp}_b(G') \leq \gamma_P(G')$. Thus there exists a block-spider partition of $V(G')$ with m or fewer subsets. Adding the subset V_1 to the block-spider partition of $V(G')$ produces a block-spider partition of $V(G)$ with at most $m + 1$ subsets. Thus, $\mathrm{sp}_b(G) \leq \gamma_P(G)$.

Case 2. w has block-degree at least 3.

We now define a set V_1 as follows. If S contains at least two vertices of B_1, let $V_1 = D_b[h_1]$. If $h_1 \in S$ and $w \in S$, let $V_1 = D_b[h_1]$. If h_1 is the only vertex of S in B_1 and if $w \notin S$, let

$$V_1' = \left(\bigcup_{x \in N[h_1] - \{w\}} D_b[x] \right) \cup \{w\}.$$

If now $G - V_1'$ contains a path-component P that contains no vertex of S, then let $V_1 = V_1' \cup V(P)$. On the other hand, if every component of $G - V_1'$ contains a vertex of S, then let $V_1 = V_1'$. In all of the above cases, let $G' = G - V_1$. Proceeding now exactly as in paragraph three of Case 1 above, we have $\mathrm{sp}_b(G) \leq \gamma_P(G)$.

Next we prove that $\gamma_P(G) \leq 2\mathrm{sp}_b(G) - 1$. Suppose $\mathrm{sp}_b(G) = m$. If $m = 1$, then G is a block-spider, and so, by Theorem 17, its head is a PDS of G and $\gamma_P(G) = 1 =$

$\mathrm{sp}_b(G)$. Suppose, then, that $m \geq 2$. Let G be a block graph with $\mathrm{sp}_b(G) = m$. Let $\{V_1, V_2, \ldots, V_m\}$ be a block-spider partition of $V(G)$. For $i = 1, 2, ..., m$, let G_i be the block-spider induced by V_i, and so $G_i = G[V_i]$, and let h_i be the head of G_i. Then, $\{h_i\}$ is a PDS of G_i. Let F be the graph with vertex set $\{V_1, V_2, \ldots, V_m\}$ where two vertices V_i and V_j are adjacent in F if and only if there is an edge of G joining a vertex of V_i and a vertex of V_j. Since G is a block graph, so too is F. Further, every block in F corresponds to a block in G. For each block in F, we select one vertex from the corresponding block in G and we let S_F denote the resulting set of selected vertices. Since F has order m, there are at most $m-1$ blocks in F, and so $|S_F| = b(F) \leq m-1$ with equality if and only if F is a tree. Then, $S_F \cup \{h_1, h_2, \ldots, h_m\}$ is a PDS of G, and so $\gamma_P(G) \leq |S_F| + m \leq 2m - 1 = 2\mathrm{sp}_b(G) - 1$. $\Box$

The next result shows that the bounds of Theorem 18 is sharp.

Theorem 19 *Given any integers k and ℓ with $1 \leq \ell \leq k \leq 2\ell - 1$, there exists a connected block graph G satisfying $\mathrm{sp}_b(G) = \ell$ and $\gamma_P(G) = k$.*

Proof. Let $t \geq 3$ be a fixed integer. Consider two complete graphs K_t that have exactly one vertex h in common, and let u_1 and v_1 be vertices from the two different complete graphs where $u_1 \neq h$ and $v_1 \neq h$. Attach to u_1 a path u_1, u_2, u_3, u_4 and to v_1 a path v_1, v_2, v_3, v_4 (so that the resulting paths are vertex disjoint). Let F denote the resulting graph. Then, F is a block-spider with head h. Let $F_1, \ldots, F_\ell$ be ℓ disjoint copies of F. For $i = 1, \ldots, \ell$, we label the vertices u_j and v_j, $1 \leq j \leq 4$, of F by $u_{i,j}$ and $v_{i,j}$, respectively, in F_i and the vertex h of F by h_i in F_i. Let F' be the disjoint union $\cup_{i=1}^{\ell} F_i$ of the graphs F_i.

Suppose that $k = \ell + r$ where $r \in \{0, \ldots, \ell - 1\}$. If $r = 0$, let $E_1 = \emptyset$; otherwise, let $E_1 = \{v_{i,2}u_{i+1,j}, v_{i,3}u_{i+1,j} \mid 1 \leq i \leq r \text{ and } 2 \leq j \leq 3\}$. If $r = \ell - 1$, let $E_2 = \emptyset$; otherwise, let $E_2 = \{v_{i,3}u_{i+1,3} \mid i = r+1, \ldots, \ell - 1\}$. Let G be obtained from F by adding the set of edges $E_1 \cup E_2$. (The graph G when $t = 3$, $\ell = 4$ and $k = 6$ is illustrated in Figure 18 where the six darkened vertices form a $\gamma_P(G)$-set.)

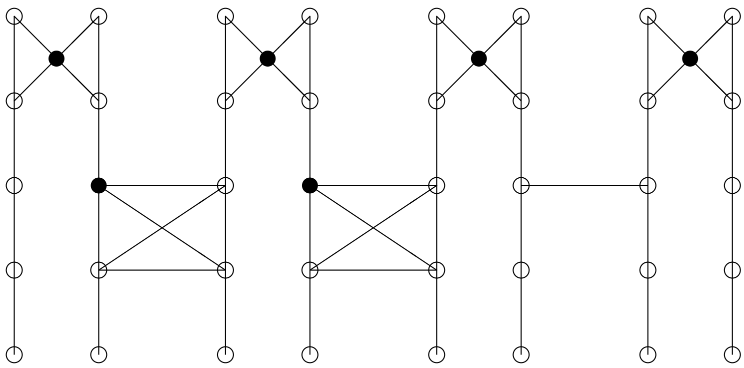


Figure 18: A connected block graph G with $\text{sp}_b(G) = 4$ and $\gamma_P(G) = 6$.

Then, G is a connected block graph in which every block is K_2, K_4, or K_t. The partition $\{V(F_1), \ldots, V(F_\ell)\}$ of $V(G)$ is a minimum block-spider partition of G, and so $\text{sp}_b(G) = \ell$. If $r = 0$, let $S = \emptyset$; otherwise, let $S = \cup_{i=1}^{r}\{v_{i,3}\}$. Then, the set $\{h_1, \ldots, h_\ell\} \cup S$ is a minimum PDS of G, and so $\gamma_P(G) = \ell + r = k$. $\square$

As a consequence of Theorem 18, we can determine a lower bound of a connected block graph in terms of the number of vertices of block-degree at least 3. The next theorem shows this bound.

Theorem 20 *If G is a connected block graph having k vertices of block-degree at least three, then*

$$\gamma_P(G) \geq \frac{k+2}{3},$$

with equality if and only if G has a block-spider partition such that the blocks of G containing vertices from different block-spiders form a disjoint union of K_2s.

Proof. Let $\mathrm{sp}_b(G) = m$. Then by Theorem 18, $\gamma_P(G) \geq m$. Let $\{V_1, V_2, \ldots, V_m\}$ be a block-spider partition of $V(G)$. For $i = 1, 2, \ldots, m$, let G_i be the block-spider induced by V_i; that is, $G_i = G[V_i]$. Further, let h_i be the head of G_i. Since each G_i is a block-spider, every vertex of G_i, except for possibly its head, belongs to at most two blocks in the block-spider.

Let F be the graph with vertex set $\{V_1, V_2, \ldots, V_m\}$ where two vertices V_i and V_j are adjacent in F if and only if there is an edge of G joining a vertex of V_i and a vertex of V_j. Since G is a block graph, so too is F. Suppose that K_t is the largest block in F. For $\ell = 2, \ldots, t$, let b_ℓ denote the number of K_ℓ-blocks in F. Let T_F be the tree obtained from F by replacing every K_ℓ-block in F where $\ell \geq 3$ by a spanning tree of the K_ℓ-block (of order ℓ and size $\ell - 1$). Then,

$$m - 1 = |E(T_F)| = \sum_{\ell=2}^{t} (\ell - 1) b_\ell \geq \sum_{\ell=2}^{t} b_\ell. \tag{1}$$

Each vertex of $V_i - \{h_i\}$ that is adjacent in G to only vertices of V_i has block-degree at most 2 in G. Let B be a block in G corresponding to a K_ℓ-block of F. Then, $|V(B) \cap V_j| \leq 2$ for all $j = 1, 2, \ldots, m$ unless $h_i \in V(B)$ in which case possibly $|V(B) \cap V_j| > 2$. Let E_B denote the set of all edges of B that do not belong to any of the block-spiders G_i. Suppose $|V(B) \cap V_i| \geq 2$ for some i. Then the block-degree of each vertex of $V(B) \cap V_i$ in G is the same as its block-degree in $G - E_B$. On the other hand, if $V(B) \cap V_i = \{v\}$ for some i, then the block-degree of v in G is one larger than its block-degree in $G - E_B$. Since $|V(B) \cap V_i| \geq 1$ for exactly ℓ values of i, at most ℓ vertices in G (each from different sets V_i) have block-degree in G one more than their block-degrees in $G - E_B$. This implies that G contains at most $m + \sum_{\ell=2}^{t} \ell b_\ell$ vertices of block-degree at least 3. Hence,

$$k \leq m + \left(\sum_{\ell=2}^{t} b_\ell \right) + \sum_{\ell=2}^{t} (\ell - 1) b_\ell. \tag{2}$$

Thus, by Equations (1) and (2), $k \leq 3m - 2$. Hence, $\gamma_P(T) \geq m \geq (k+2)/3$. Suppose $\gamma_P(T) = (k+2)/3$. Then we must have equality throughout Equations (1) and (2). In particular, F is a tree and each block B of G corresponding to an edge of F is a K_2-block the two vertices of which are from different block-spiders and are not the heads of the block-spiders. Further, no two such blocks B have any vertex in common. The desired characterization follows.

That this bound is sharp, may be seen as follows. Let $n \geq 1$ and $t \geq 2$ be integers. Let G' be the corona of a path P on $(2t-1)n$ vertices; that is, $G' = P_{(2t-1)n} \circ K_1$ (the corona of a path is also called a *comb*). Let the path be denoted by $P\colon v_1, v_2, \ldots, v_{(2t-1)n}$. For each $i = 0, \ldots, n-1$, let E_i' and E_i'' be the set of edges defined by $E_i' = \{v_{(2t-1)i+j}\, v_{(2t-1)i+\ell} \mid 1 \leq j < \ell \leq t\}$ and $E_i'' = \{v_{(2t-1)i+j}\, v_{(2t-1)i+\ell} \mid t \leq j < \ell \leq 2t-1\}$. Each of the sets E_i' and E_i'' induce a complete graph K_t and have only the vertex $v_{(2t-1)i+t}$ in common. For $i = 0, \ldots, n-1$, let $E_i = (E_i' \cup E_i'') - E(P)$. Let G be the graph obtained from G' by adding the edges $\cup_{i=1}^{n} E_i$. Then, G is a connected block graph in which every block is K_2 or K_t. (The graph G when $n = t = 3$ is illustrated in Figure 19 where the three large darkened vertices from a $\gamma_P(G)$-set.)

Let $D = \cup_{i=0}^{n-1}\{v_{(2t-1)i+t}\}$. Then, D is a PDS of G, and so $\gamma_P(G) \le |D| = n$. Let $S = \cup_{i=1}^{n-1}\{v_{(2t-1)i}, v_{(2t-1)i+1}\}$. Then the set of vertices of G of block-degree at least three is the set $D \cup S$. Hence, G has $k = |D| + |S| = 3n - 2$ vertices of block-degree at least three, and so as shown earlier, $\gamma_P(G) \ge (k+2)/3 = n$. Consequently, $\gamma_P(G) = (k+2)/3 = n$.

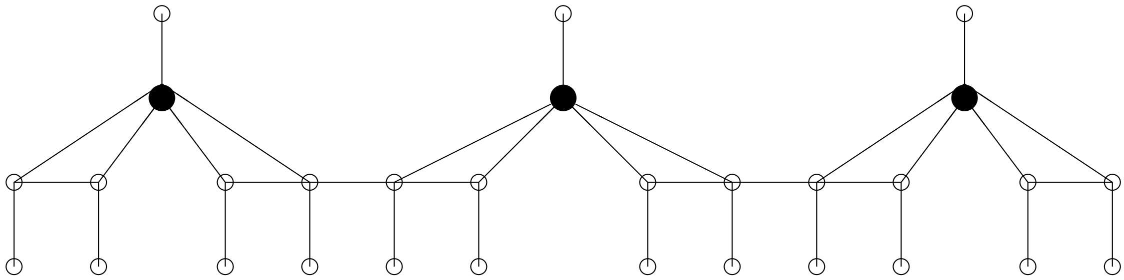


Figure 19: A connected block graph G with $k = 7$ vertices of block-degree at least 3 and $\gamma_P(G) = (k+2)/3$.

Originally, it was thought the following result would hold for any block graph. However, once a K_4-block or larger is in the graph, we no longer get equality in the power domination number and the block-spider-number. We now show that if every block of a connected block graph is K_2 or K_3, then its power domination number is precisely its block spider number. Hence, we have the following theorem.

Theorem 21 *If G is a connected block graph in which every block is K_2 or K_3, then the heads of the block-spiders induced by a block-spider partition of $V(G)$ form a PDS of G. Consequently, $\gamma_P(G) = \mathrm{sp}_b(G)$.*

Proof. We proceed by induction on $m = \mathrm{sp}_b(G)$. Suppose $m = 1$. Then G is a block-spider, and so, by Theorem 17, its head is a PDS of G and $\gamma_P(G) = 1 = \mathrm{sp}_b(G)$.

Suppose, then, that for all connected block graphs G' in which every block is K_2 or K_3 with $\mathrm{sp}_b(G') = m$, where $m \geq 1$, the heads of the block-spiders induced by a block-spider partition of $V(G')$ power dominate G'. Let G be a block graph in which every block is K_2 or K_3 with $\mathrm{sp}_b(G) = m+1$. Let $\{V_1, V_2, \dots, V_{m+1}\}$ be a block-spider partition of $V(G)$. For $i = 1, 2, ..., m+1$, let G_i be the block-spider induced by V_i, and so $G_i = G[V_i]$, and let h_i be the head of G_i.

Let F be the graph with vertex set $\{V_1, V_2, \dots, V_{m+1}\}$ where two vertices V_i and V_j are adjacent in F if and only if there is an edge of G joining a vertex of V_i and a vertex of V_j. Since G is a block graph in which every block is K_2 or K_3, so too is F. We must now consider two cases, depending on whether F contains a K_2 end-block or whether every end-block of F is a K_3.

Case 1. F contains a K_2 endblock.

Without loss of generality, we may assume that V_1 is an end-vertex of F and that $V_1V_2 \in E(F)$. The edge V_1V_2 in F corresponds to a block B in G that contains at least one vertex of each of V_1 and V_2, but no vertex in $V(G) - V_1 - V_2$. Let $G' = G - V_1$; that is, G' is the connected block graph (in which every block is K_2 or K_3) obtained from G by deleting the vertices in the subset V_1. If $\mathrm{sp}_b(G') < m$, then we can add the subset V_1 to a minimum block-spider partition of $V(G')$ to produce a block-spider partition of $V(G)$ of cardinality $\mathrm{sp}_b(G')+1 < m+1 = \mathrm{sp}_b(G)$, which is impossible. Hence, $\mathrm{sp}_b(G') \geq m$. Since $\{V_2, \dots, V_{m+1}\}$ is a block-spider partition of $V(G')$, $\mathrm{sp}_b(G') \leq m$. Consequently, $\mathrm{sp}_b(G') = m$. Applying the inductive hypothesis to G', $S' = \{h_2, \dots, h_{m+1}\}$ is a PDS of G'. Thus all vertices and edges of G' are observed by S'. Let $S = S' \cup \{h_1\}$. We show that S is a PDS of G, and so $\gamma_P(G) \le m + 1 = \mathrm{sp}_b(G)$. We consider two possibilities depending on whether $B = K_2$ or $B = K_3$.

Suppose that $B = K_2$. Let $V(B) = \{v_1, v_2\}$ where $v_i \in V_i$ for $i = 1, 2$. Thus, $v_1v_2 \in E(G)$ and this is the only edge joining a vertex in V_1 and $V(G) - V_1$. Since S' is a PDS of G', the vertex v_2 is observed by the set S' in G. The vertex v_1 is observed by the vertex h_1 in G. Hence the edge v_1v_2 is observed by the set S. It follows that S is a PDS of G.

Suppose that $B = K_3$. Suppose, first, that $V(B) = \{u_1, v_1, v_2\}$ where $\{u_1, v_1\} \subseteq V_1$ and $v_2 \in V_2$. Let G_1 be rooted at h_1. We may assume that u_1 is a block-child of v_1 (possibly, $h_1 = v_1$). The vertex v_1 and all its incident edges except for the edge u_1v_1 are observed by the vertex h_1 in G, while v_2 is observed by the set S' in G. Hence the edge v_1v_2 is observed by S in G. Thus, the edges v_1u_1 and u_1v_2 in turn become observed by S in G. It follows that S is a PDS of G.

Suppose, secondly, that $V(B) = \{v_1, v_2, u_2\}$ where $v_1 \in V_1$ and $\{u_2, v_2\} \subseteq V_2$. The vertex v_1 is observed by h_1 in G. If u_2 and v_2 are both observed by S' in G' before the edge u_2v_2 is observed, then S observes both v_1u_2 and v_1v_2 in G. Consequently, the edge u_2v_2 is observed by S in G. It follows that S is a PDS of G. On the other hand, if u_2 or v_2, say u_2, is only observed after the edge u_2v_2 is observed by S' in G', then S observes the vertices v_1 and v_2 in G, and therefore the edge v_1v_2. Thus, all edges incident with v_2 are observed by S in G, and so the vertex u_2 becomes observed by S in G. It follows that S is a PDS of G.

Case 2. Every end-block of F is K_3.

Without loss of generality, we may assume that $F[\{V_1, V_2, V_3\}]$ is an end-block of F.

The block $F[\{V_1, V_2, V_3\}]$ in F corresponds to a block $B = G[\{v_1, v_2, v_3\}] = K_3$ in G where $v_i \in V_i$ for $i = 1, 2, 3$. If $m = 3$, then the set $\{h_1, h_2, h_3\}$ observes each of v_1, v_2 and v_3 in G, and therefore observes each of the edges v_1v_2, v_1v_3 and v_2v_3. It follows that if $m = 3$, then $\{h_1, h_2, h_3\}$ is a PDS of G. Hence we may assume that $m \geq 4$ and that V_3 is a cut-vertex of F. Thus each of V_1 and V_2 has degree 2 in F, while V_3 has degree at least 3 in F.

Let $G' = G - V_1 - V_2$; that is, G' is the connected block graph (in which every block is K_2 or K_3) obtained from G by deleting the vertices in the subsets V_1 and V_2. If $\text{sp}_b(G') < m - 1$, then we can add the subsets V_1 and V_2 to a minimum block-spider partition of $V(G')$ to produce a block-spider partition of $V(G)$ of cardinality $\text{sp}_b(G') + 1 < m + 1 = \text{sp}_b(G)$, which is impossible. Hence, $\text{sp}_b(G') \geq m - 1$. Since $\{V_3, \ldots, V_{m+1}\}$ is a block-spider partition of $V(G')$, $\text{sp}_b(G') \leq m - 1$. Consequently, $\text{sp}_b(G') = m - 1$.

Applying the inductive hypothesis to G', $S' = \{h_3, \ldots, h_{m+1}\}$ is a PDS of G'. Therefore all vertices and edges of G' are observed by S'. Let $S = S' \cup \{h_1, h_2\}$. Then the vertex v_1 is observed by h_1, the vertex v_2 by h_2, and the vertex v_3 by S' in G. Thus each of the edges v_1v_2, v_1v_3 and v_2v_3 is observed by S in G. It follows that since h_i is a PDS of G_i for $i = 1, 2$ and since S' is a PDS of G', S is a PDS of G, and so $\gamma_P(G) \leq m + 1 = \text{sp}_b(G)$. However by Theorem 18, $\gamma_P(G) \geq \text{sp}_b(G)$. Consequently, $\gamma_P(G) = \text{sp}_b(G)$. $\square$

As shown, Theorem 21 is not true if we allow our block graph to contain complete blocks of order greater than 3. Our next theorem illustrates when a connected block graph, where all blocks are K_2 or K_3, has $\gamma_P(G) > \text{sp}_b(G)$.

Observation 22 *For any integer $t \geq 4$, there exists a connected block graph G with one K_t-block and all other blocks either K_2 or K_3 that satisfies $\gamma_P(G) > \mathrm{sp}_b(G)$.*

Proof. Suppose first that $t = 2k$ for some integer $k \geq 2$. For $i = 1, \ldots, k$, let G_i be the graph obtained from the path $u_{i,1}, u_{i,2}, \ldots, u_{i,6}$ by adding the edge $u_{i,4}u_{i,6}$. Then, G_i is a block-spider. Let $h_i = u_{i,4}$. Let G be obtained from the disjoint union $\cup_{i=1}^{k} G_i$ of the graphs G_i by forming a clique on the set $S = \cup_{i=1}^{k}\{u_{i,2}, u_{i,3}\}$. Then, $G[S] = K_t$ and G is a connected block graph in which every block is K_2 or K_3, except for the block induced by the set S. (The graph G when $t = 6$ is illustrated in Figure 20 where the four darkened vertices form a $\gamma_P(G)$-set.) The partition $\{V(G_1), \ldots, V(G_k)\}$ of $V(G)$ is a minimum block-spider partition of G, and so $\mathrm{sp}_b(G) = k$. On the other hand, the set $\{h_1, \ldots, h_k\} \cup \{u_{1,3}\}$ is a minimum PDS of G, and so $\gamma_P(G) = k + 1$. Thus, $\gamma_P(G) > \mathrm{sp}_b(G)$.

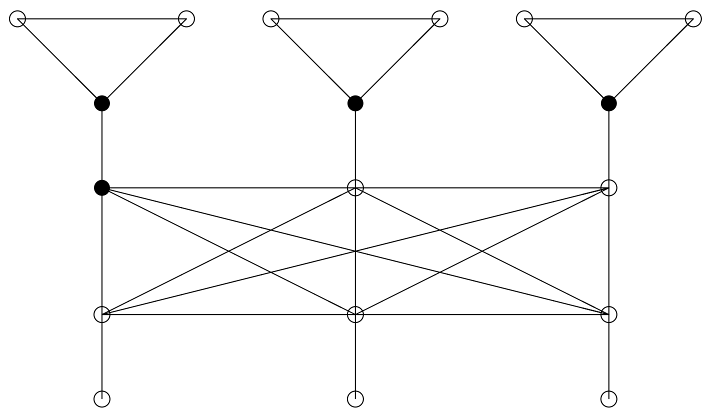


Figure 20: A connected block graph G with $\gamma_P(G) > \mathrm{sp}_b(G)$ in which all but one block is K_2 or K_3.

Suppose secondly that $t = 2k + 1$ where $k \geq 2$. Let G_{k+1} be the graph obtained from the path $u_{k+1,1}, u_{k+1,2}, \ldots, u_{k+1,6}$ and adding the edge $u_{k+1,4}u_{k+1,6}$. Then, G_i is a block-spider. Let H be the graph obtained from the disjoint union of graph G constructed earlier and the graph G_{k+1} by adding all edges joining $u_{k+1,3}$ to vertices in the set S. Then, $H[S \cup \{u_{k+1,3}\}] = K_t$ and H is a connected block graph in which every block is K_2 or K_3, except for the block induced by the set $S \cup \{u_{k+1,3}\}$, with $\gamma_P(G) = k + 2 > k + 1 = \mathrm{sp}_b(G)$. $\square$

There are several more interesting questions to be answered using power domination in block graphs. What are the block-graphs that have power domination number equal to their domination number? Is there an efficient algorithm for placing the PMUs in a block graph? There are many more. Hopefully, we can investigate more of these questions.

BIBLIOGRAPHY

References

[1] T.L. Baldwin, L. Mili, M.B. Boisen Jr., and R. Adapa, Power System Observability with Minimal Phasor Measurement Placement, *IEEE Trans. on Power Systems* 8 (1993) 707-715.

[2] M.B. Boisen Jr., T.L. Baldwin, and L. Mili, Stimulated Annealing and Graph Theory Applied to Electrical Power Networks, manuscript (2000).

[3] D.J. Brueni, *Minimal PMU Placement for Graph Observability: A Decomposition Approach*, Master Thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia (1993).

[4] D.J. Brueni and L.S. Heath, *The PMU Placement Problem*, manuscript (2002).

[5] M. Dorfling and M.A. Henning, Power Domination in Grid Graphs. Submitted for publication.

[6] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, and M.A. Henning, Power Domination in Graphs Applied to Electrical Power Networks. *SIAM J. Discrete Math.* **15**(4) (2002), 519-529.

[7] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[8] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, editors, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[9] L. Lesniak and G. Chartrand, *Graphs and Digraphs*, 3rd ed, Chapman and Hall, New York, 1996.

[10] L. Mili, T. Baldwin, and A. Phadke, Phasor Measurement Placement for Voltage and Stability Monitoring and Control, *Proceedings of the EPRI-NSF Workshop on Application of Advanced Mathematics to Power Systems*, San Francisco, California (1991).

VITA

David Wayne Atkins

303 Sycamore Street

Bean Station, TN 37708

davidatkins_tn@hotmail.com

Education

•Associates of Science Degree in Mathematics, Walters State Community College, May 1999.

•Bachelors of Science Degree in Mathematics, East Tennessee State University, May 2001.

•Masters of Science Degree in Mathematics, East Tennessee State University, August 2003.

Publications

•Placing Monitoring Devices in Electric Power Networks Modeled by Block Graphs, submitted for publication.

•Double Domination Edge Critical Graphs, submitted for publication.

Organizations and Honors

•Kappa Mu Epsilon

•Mathematical Association of America

•Outstanding Mathematics Graduate Student for 2002-2003