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Radical $p$-chains in $L_3(2)$.

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RADICAL $p$-CHAINS IN $L_3(2)$

A Thesis

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In Partial Fulfillment

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Master of Science in Mathematical Sciences

by

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May 2001

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The McKay-Alperin-Dade Conjecture, which has not been finally verified, predicts the number of complex irreducible characters in various $p$-blocks of a finite group $G$ as an alternating sum of the numbers of characters in related $p$-blocks of certain subgroups of $G$. The subgroups involved are the normalizers of representatives of conjugacy classes of radical $p$-chains of $G$. For this reason, it is of interest to study radical $p$-chains. In this thesis, we examine the group $L_3(2)$ and determine representatives of the conjugacy classes of radical $p$-subgroups and radical $p$-chains for the primes $p = 2, 3,$ and $7$. We then determine the structure of the normalizers of these subgroups and chains.
Contents

ABSTRACT ii
COPYRIGHT iii

LIST OF TABLES v

1. INTRODUCTION ................................ 1
   1.1 Definitions and Minor Results ..................... 1
   1.2 Examples ..................................... 3

2. THE GROUP $L_3(2)$ ................................ 6
   2.1 Radical 7-chains ................................ 6
   2.2 Radical 3-chains ................................ 8
   2.3 Radical 2-chains ................................ 10

3. CONCLUSIONS ..................................... 16

BIBLIOGRAPHY ..................................... 18
List of Tables

1  Radical $p$-chain Summary for $A_5$ ........................................ 5
2  Radical $p$-chain Summary for $L_3(2)$ ............................... 17
CHAPTER 1
INTRODUCTION

The McKay-Alperin-Dade Conjecture, which has not been finally verified, predicts the number of complex irreducible characters in various $p$-blocks of a finite group $G$ as an alternating sum of the numbers of characters in related $p$-blocks of certain subgroups of $G$. The subgroups involved are the normalizers of representatives of conjugacy classes of radical $p$-chains of $G$. For this reason, it is of interest to study radical $p$-chains.

We will begin by defining some terms which will be referred to throughout the thesis, with the main definitions being that of a radical $p$-subgroup and radical $p$-chain. This will lead to some minor results concerning radical $p$-subgroups. Then we will look at an example group and its radical $p$-subgroups and radical $p$-chains. Next, we examine the group $L_3(2)$ and determine representatives of the conjugacy classes of radical $p$-subgroups and radical $p$-chains for the primes $p = 2, 3, \text{ and } 7$. In addition, we will determine the structure of the normalizers of these subgroups and chains. Finally, we will summarize the results.

1.1 Definitions and Minor Results

We begin with some definitions. Let $G$ be any group and $p$ be any prime. Let $|G|$ be the order of $G$. We define $H \leq G$ as “$H$ is a subgroup of $G$”. We call $H$ a normal subgroup of $G$ if $g^{-1}hg \in H$ for all $h \in H$ and $g \in G$. From this we can say that $G$ is a normal subgroup of itself, since $g_1^{-1}g_2g_1 \in G$ for $g_1, g_2 \in G$. We will call the product $g_1^{-1}g_2g_1$ the conjugate of $g_2$ by $g_1$. If $H, K \leq G$, then we define the normalizer of $H$ in $K$ as $N_K(H) = \{k \in K|k^{-1}hk \in H, \forall h \in H\}$. We will call $G$ the semi-direct product of $H$ and $K$, denoted $H \rtimes K$, if $K$ is a normal subgroup of $G$, $G = HK = \{hk|h \in H, k \in K\}$, where
multiplication is defined by \((h_1k_1)(h_2k_2) = h_1h_2h_2^{-1}k_1h_2k_2 = h_3k_3 \in HK\), and \(H \cap K = 1\), the trivial subgroup. A \(p\)-subgroup of \(G\) is a subgroup of \(G\) with order \(p^n, n = 0, 1, 2, \ldots\), such that \(p^n\) divides \(|G|\). A **Sylow-\(p\) subgroup** of \(G\) is a \(p\)-subgroup of order \(p^m\), where \(m\) is the largest exponent such that \(p^m\) divides \(|G|\). We will use the notation \(O_p(G)\) to denote the largest normal \(p\)-subgroup of \(G\). We may now define a **radical \(p\)-subgroup** of \(G\) to be a subgroup \(P \leq G\) such that \(P = O_p(N_G(P))\). That is, \(P\) is the largest normal \(p\)-subgroup of its normalizer in \(G\). For this thesis, we will use the notation \(P_{p,n}\) to denote a radical \(p\)-subgroup of order \(p^n\), with an interesting exception we will see later. A **\(p\)-chain** \(C\) of \(G\) is any non-empty, strictly increasing chain \(C : P_0 < P_1 < P_2 < \cdots < P_n\) of \(p\)-subgroups \(P_i\) of \(G\). The **stabilizer** of \(C\) in any \(K \leq G\) is the “normalizer” \(N_K(C) = N_K(P_0) \cap N_K(P_1) \cap \cdots \cap N_K(P_n)\). A **radical \(p\)-chain** of \(G\) is a \(p\)-chain \(C : P_0 < P_1 < \cdots < P_n\) of \(G\) satisfying \(P_0 = O_p(G)\) and \(P_i = O_p(N_G(C_i))\) for \(i = 1, \ldots, n\), where \(C_i : P_0 < \cdots < P_i\).

We may now discuss some minor results. First, \(N_K(H) \leq G\). If \(K = G\), then \(H \leq N_G(H)\), since \(H \leq G\) and \(h_1^{-1}h_2h_1 \in H\) for any \(h_1, h_2 \in H\). In fact, \(H\) is a normal subgroup of \(N_G(H)\) by the definition of a normalizer. With this information we can conclude that \(H\) is a normal subgroup of \(G\) if and only if \(N_G(H) = G\). Now suppose \(H\) is a Sylow-\(p\) subgroup of \(G\). Then \(H\) is a normal subgroup of \(N_G(H)\) and, since there can be no \(p\)-subgroup larger than \(H\), \(H = O_p(N_G(H))\). Therefore, if \(H\) is a Sylow-\(p\) subgroup of \(G\), then \(H\) is a radical \(p\)-subgroup of \(G\). Finally, we note that the trivial subgroup of \(G\), denoted by \(1\), is a \(p\)-subgroup for any prime \(p\) which divides \(|G|\), since \(|1| = 1 = p^0\), while \(N_G(1) = G\).
1.2 Examples

As an example, let us examine $A_5$, the set of all even permutations of five elements. The order of this group is $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$. Note that this group is simple. That is, the only normal subgroups are $A_5$ and $1$. For this reason, we may conclude that $1$ is a radical $p$-subgroup for $p = 2, 3,$ and $5$. It must also be noted that in each case we only need to find one representative radical $p$-subgroup of each conjugacy class, for the others can then be found by conjugation. That is, radical $p$-subgroups of the same order are in the same conjugacy class for each $p$, unless otherwise noted. This is true of their normalizers as well.

For $p = 2$, the 2-subgroups of $A_5$ have orders $2^2 = 4, 2^1 = 2,$ and $2^0 = 1$. The subgroups of order 4 are the Sylow-2 subgroups. These are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. An example of such a group is $\{1, (12)(45), (14)(25), (15)(24)\}$, and we will denote these groups as $P_{2,2}$. The normalizers of these subgroups in $A_5$ are subgroups of $A_5$ isomorphic to $A_4$. For an example of this subgroup, consider all the even permutations of the set $\{1, 2, 4, 5\}$. That is, take all the even permutations of any four of the five elements of $\{1, 2, 3, 4, 5\}$, and you will have a subgroup of $A_5$ isomorphic to $A_4$. No subgroup of $A_5$ with order 4 can be isomorphic to $\mathbb{Z}_4$ because there is no element of order 4 in $A_5$. That is, no element of $A_5$ can generate a subgroup of order 4. The next subgroups we will look at have order 2, and are isomorphic to $\mathbb{Z}_2$, an example of which is $\{1, (12)(34)\}$. The normalizers of these subgroups in $A_5$ are the $P_{2,2}$ subgroups, so no subgroup of order 2 can be a radical 2-subgroup. The last subgroup we consider is the trivial subgroup $1$, whose normalizer is $N_{A_5}(1) = A_5$. No other 2-subgroups of $A_5$ are normal subgroups of $A_5$, so $1$ is a radical 2-subgroup, denoted $P_{2,0}$. The radical 2-chains are then $C_{21} : P_{2,0},$ and $C_{22} : P_{2,0} < P_{2,2}$. The stabilizers of these chains are $N_{A_5}(C_{21}) = A_5$ and $N_{A_5}(C_{22}) \cong A_4$.
The $p$-subgroups of $A_5$ for $p = 3$ have orders $3^1 = 3$ and $3^0 = 1$. The subgroups of order 3 are the Sylow-3 subgroups, which we will denote $P_{3,1}$, and are isomorphic to $\mathbb{Z}_3$. An example of such a subgroup is $< (124) >= \{1, (124), (142)\}$. The normalizers in $A_5$ of these subgroups are isomorphic to $\mathbb{Z}_2 \rtimes \mathbb{Z}_3$. As an example of this subgroup, let $\mathbb{Z}_2 = \{1, (24)(35)\}$ and let $\mathbb{Z}_3$ be as above. Then $\mathbb{Z}_2 \rtimes \mathbb{Z}_3$ would consist of the products of the elements of $\mathbb{Z}_2$ with the elements of $\mathbb{Z}_3$. As a side note, this subgroup is also isomorphic to $S_3$, the set of all permutations of three elements. Now we consider the only other 3-group, the trivial subgroup $1$, whose normalizer in $A_5$ is $A_5$. No other 3-subgroup is normal in $A_5$, so $1$ is a radical 3-subgroup, denoted $P_{3,0}$. The radical 3-chains are $C_{31} : P_{3,0}$ and $C_{32} : P_{3,0} < P_{3,1}$. The stabilizers of these chains are $N_{A_5}(C_{31}) = A_5$ and $N_{A_5}(C_{32}) \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_3$.

For $p = 5$, the 5-subgroups of $A_5$ have orders $5^1 = 5$ and $5^0 = 1$. The subgroups of order 5 are the Sylow-5 subgroups, denoted $P_{5,1}$, and are isomorphic to $\mathbb{Z}_5$. An example of this subgroup is

$$< (12345) >= \{1, (12345), (13524), (14253), (15432)\}.$$ 

The normalizer of this group in $A_5$ is isomorphic to $\mathbb{Z}_2 \rtimes \mathbb{Z}_5$. Using the above example as $\mathbb{Z}_5$, we let $\mathbb{Z}_2$ be the group $\{1, (12)(35)\}$. Then $\mathbb{Z}_2 \rtimes \mathbb{Z}_5$ will consist of the product of the elements of $\mathbb{Z}_2$ with the elements of $\mathbb{Z}_5$. The only other subgroup to consider is $1$, with $N_{A_5}(1) = A_5$. No other 5-subgroup is normal in $A_5$, so $1$ is a radical 5-subgroup, denoted $P_{5,0}$. The radical 5-chains are then $C_{51} : P_{5,0}$ and $C_{52} : P_{5,0} < P_{5,1}$. The stabilizers of these chains are $N_{A_5}(C_{51}) = A_5$ and $N_{A_5}(C_{52}) \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_5$. 

4
Table 1: Radical $p$-chain Summary for $A_5$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Radical $p$-subgroups</th>
<th>Normalizers</th>
<th>Radical $p$-chains</th>
<th>Stabilizers</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$P_{2.2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>$N_{A_5}(P_{2.2}) \cong A_4$</td>
<td>$C_{2} : P_{2,0}$</td>
<td>$N_{A_5}(C_{2}) = A_5$</td>
</tr>
<tr>
<td></td>
<td>$P_{2,0} = 1$</td>
<td>$N_{A_5}(P_{2,0}) = A_5$</td>
<td>$C_{22} : P_{2,0} &lt; P_{2.2}$</td>
<td>$N_{A_5}(C_{22}) \cong A_4$</td>
</tr>
<tr>
<td>3</td>
<td>$P_{3.1} \cong \mathbb{Z}_3$</td>
<td>$N_{A_5}(P_{3.1}) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$</td>
<td>$C_{31} : P_{3,0}$</td>
<td>$N_{A_5}(C_{31}) = A_5$</td>
</tr>
<tr>
<td></td>
<td>$P_{3,0} = 1$</td>
<td>$N_{A_5}(P_{3,0}) = A_5$</td>
<td>$C_{32} : P_{3,0} &lt; P_{3.1}$</td>
<td>$N_{A_5}(C_{32}) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$</td>
</tr>
<tr>
<td>5</td>
<td>$P_{5.1} \cong \mathbb{Z}_5$</td>
<td>$N_{A_5}(P_{5.1}) \cong \mathbb{Z}_2 \times \mathbb{Z}_5$</td>
<td>$C_{51} : P_{5,0}$</td>
<td>$N_{A_5}(C_{51}) = A_5$</td>
</tr>
<tr>
<td></td>
<td>$P_{5,0} = 1$</td>
<td>$N_{A_5}(P_{5,0}) = A_5$</td>
<td>$C_{52} : P_{5,0} &lt; P_{5.1}$</td>
<td>$N_{A_5}(C_{52}) \cong \mathbb{Z}_2 \times \mathbb{Z}_5$</td>
</tr>
</tbody>
</table>
We will begin our examination of $L_3(2)$ by discussing some properties of this group. Most of this information is provided by the *Atlas of Finite Groups* [1]. First, $L_3(2)$ is the group of invertible three by three matrices whose entries come from a field of order two. We will represent an element of this group by a matrix whose entries are either 1 or 0. For example, 
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} \in L_3(2).
\]
The order of this group is $|L_3(2)| = 168 = 2^3 \cdot 3 \cdot 7$.

$L_3(2)$ is a simple group, just as $A_5$ is. This means, of course, that the trivial subgroup $1$ is a radical $p$-subgroup for $p = 2, 3, \text{and } 7$, with normalizer $N_{L_3(2)}(1) = L_3(2)$.

There are three types of maximal subgroups in $L_3(2)$, that is, there are no subgroups of $L_3(2)$ which contain them as subgroups. One such subgroup is isomorphic to $\mathbb{Z}_3 \rtimes \mathbb{Z}_7$. The other two are isomorphic to $S_4$. However, these two subgroups are not conjugate. This means that all the elements of one of these $S_4$ subgroups cannot be found by conjugating the elements from the other $S_4$ subgroup by the same element.

### 2.1 Radical 7-chains

We will now determine the radical 7-chains of $L_3(2)$, as well as their stabilizers, by proving the following theorem.

**Theorem 2.1** The radical 7-subgroups of $L_3(2)$ are $P_{7,0} = 1$ and $P_{7,1} \cong \mathbb{Z}_7$. The radical 7-chains of $L_3(2)$ are $C_{71} : P_{7,0}$, and $C_{72} : P_{7,0} < P_{7,1}$. The stabilizers are $N_{L_3(2)}(C_{71}) = L_3(2)$ and $N_{L_3(2)}(C_{72}) \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_7$. 

Proof: The only possible radical 7-subgroups must have order seven or order one. The subgroups of order seven are the Sylow-7 subgroups, which we have already determined to be radical 7-subgroups. The only subgroup of order one is the trivial subgroup 1. We have already concluded that 1 is a radical 7-subgroup with \( N_{L_3(2)}(1) = L_3(2) \). Thus it is clear that \( C_7 : P_{7,0} = 1 \) is a radical 7-chain. The stabilizer of this chain is \( N_{L_3(2)}(C_7) = N_{L_3(2)}(1) = L_3(2) \).

It is only left to determine the structure of the Sylow-7 subgroups and their normalizers. Since the order of the Sylow-7 subgroups is seven, a prime, they must all be isomorphic to \( \mathbb{Z}_7 \). To demonstrate this we need only to find one example of such a group, and the others may be found by conjugation. Consider the matrix \( A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in L_3(2) \). The group it generates is

\[
<A_1> = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\}
\]

which has order seven, and must be isomorphic to \( \mathbb{Z}_7 \). Thus \( P_{7,1} \cong \mathbb{Z}_7 \).

We will now use \( <A_1> \) to determine \( N_{L_3(2)}(P_{7,1}) \). Consider \( A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in L_3(2) \). This element has order three, that is, this element generates a subgroup of order three. It is simple to check that \( A_2, A_2^{-1} \in N_{L_3(2)}(<A_1>) \). In particular,

\[
A_2 : A_1 \rightarrow (A_1)^4 \rightarrow (A_1)^2 \rightarrow A_1,
\]

\[
A_2 : (A_1)^3 \rightarrow (A_1)^5 \rightarrow (A_1)^6 \rightarrow (A_1)^3,
\]

and sends the identity element to itself under conjugation. By this we can tell that a subgroup isomorphic to \( \mathbb{Z}_3 \ltimes \mathbb{Z}_7 \) is contained in \( N_{L_3(2)}(<A_1>) \). Since \( \mathbb{Z}_3 \ltimes \mathbb{Z}_7 \) is a
maximal subgroup structure in \(L_3(2)\), either \(N_{L_3(2)}(<A_1>) \cong \mathbb{Z}_3 \times \mathbb{Z}_7\), or \(N_{L_3(2)}(<A_1>) = L_3(2)\). However, \(L_3(2)\) is a simple group, which means \(<A_1>\) is not a normal subgroup of \(L_3(2)\). Thus \(N_{L_3(2)}(<A_1>) \neq L_3(2)\), and \(N_{L_3(2)}(<A_1>) \cong \mathbb{Z}_3 \times \mathbb{Z}_7\). Since \(<A_1>\) is a representative of the conjugacy class of radical all 7-subgroups of \(L_3(2)\), we can say \(N_{L_3(2)}(P_{7,1}) \cong \mathbb{Z}_3 \times \mathbb{Z}_7\), for any radical 7-subgroup \(P_{7,1}\) of \(L_3(2)\).

This gives us the radical 7-chain \(C_{72} : P_{7,0} < P_{7,1}\), with stabilizer \(N_{L_3(2)}(C_{72}) = N_{L_3(2)}(1) \cap N_{L_3(2)}(P_{7,1}) = N_{L_3(2)}(P_{7,1}) \cong \mathbb{Z}_3 \times \mathbb{Z}_7\). 2

2.2 Radical 3-chains

We will prove the following theorem for the radical 3-chains of \(L_3(2)\):

**Theorem 2.2** The radical 3-subgroups of \(L_3(2)\) are \(P_{3,0} = 1\) and \(P_{3,1} \cong \mathbb{Z}_3\). The radical 3-chains of \(L_3(2)\) are \(C_{31} : P_{3,0}\) and \(C_{32} : P_{3,0} < P_{3,1}\). The stabilizers are \(N_{L_3(2)}(C_{31}) = L_3(2)\) and \(N_{L_3(2)}(C_{32}) \cong S_3\).

**Proof:** The only possible radical 3-subgroups must have order three or order one. The subgroups of order three are the Sylow-3 subgroups, which we have already determined to be radical 3-subgroups. The only subgroup of order one is the trivial subgroup 1. We have already concluded that 1 is a radical 3-subgroup with \(N_{L_3(2)}(1) = L_3(2)\). Thus it is clear that \(C_{31} : P_{3,0} = 1\) is a radical 3-chain. The stabilizer of this chain is \(N_{L_3(2)}(C_{31}) = N_{L_3(2)}(1) = L_3(2)\).
Of course, it only remains to determine the structure of the Sylow-3 subgroups and their normalizers. Since the order of the Sylow-3 subgroups is three, a prime, the subgroups must be isomorphic to \( \mathbb{Z}_3 \). As proof, consider the element \( A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \). The subgroup it generates is

\[
< A_3 > = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}
\]

which has order three and is isomorphic to \( \mathbb{Z}_3 \). Thus \( P_{3,1} \cong \mathbb{Z}_3 \).

To determine the structure of \( N_{L_{3}(2)}(P_{3,1}) \), we will use \( < A_3 > \). Consider the element

\[
A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \].
\]

This element has order two, and \( A_4 \in N_{L_{3}(2)}(< A_3 >) \) since \( A_4 : A_3 \rightarrow (A_3)^2 \rightarrow A_3 \) by conjugation. This gives us a subgroup isomorphic to \( \mathbb{Z}_2 \cong \mathbb{Z}_3 \cong S_3 \). We will show that this is, in fact, equal to \( N_{L_{3}(2)}(< A_3 >) \).

To show this, we must look at the centralizer of \( A_3 \), which is the subgroup \( C(A_3) = \{ A \in L_{3}(2) | AA_3 = A_3A \} \). According to [1], the order of its centralizer is \( |C(A_3)| = 3 \). This means \( C(A_3) = < A_3 > \).

Consider an element of order seven. In order for it to be in \( N_{L_{3}(2)}(< A_3 >) \), it must fix each element of \( < A_3 > \) by conjugation. This is because of its odd order and the fact that there are only two non-trivial elements in \( < A_3 > \). In other words, this element must be in the centralizer of \( A_3 \). Since this is not the case, no element of order seven can be in \( N_{L_{3}(2)}(< A_3 >) \). The same argument can be made for elements of order three which are not in \( < A_3 > \).
Now consider an element of order four. In order for it to be in $N_{L_3(2)}(<A_3>)$, its square must fix each element of $<A_3>$ by conjugation. The square of an order four element is an element of order two. Thus, for an element of order four to be in $N_{L_3(2)}(<A_3>)$, its square must be in $C(A_3)$. Again, this is not the case, so no element of order four can be in $N_{L_3(2)}(<A_3>)$. This case rules out the possibility of $N_{L_3(2)}(<A_3>) \cong S_4$.

Finally, we note that $S_3$ is a maximal subgroup structure of $S_4$. Since we have ruled out $S_4$ and all elements of order seven, we can conclude that $N_{L_3(2)}(<A_3>) \cong S_3$. Since $<A_3>$ is a representative of the conjugacy class of all radical 3-subgroups of $L_3(2)$, we can say $N_{L_3(2)}(P_{3,1}) \cong S_3$.

Hence we have the radical 3-chain $C_{32} : P_{3,0} < P_{3,1}$, which has stabilizer $N_{L_3(2)}(C_{32}) = N_{L_3(2)}(1) \cap N_{L_3(2)}(P_{3,1}) = N_{L_3(2)}(P_{3,1}) \cong S_3$. 2

### 2.3 Radical 2-chains

We will now determine the radical 2-chains for $L_3(2)$. In the following theorem, note that there are two conjugacy classes for the radical 2-subgroups of order 4.

Theorem 2.3 The radical 2-subgroups of $L_3(2)$ are $P_{2,0} = L_3(2)$, $P_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(1)$, $P'_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(1)$, and $P_{2,3} \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \cong D_4$. The radical 2-chains are $C_{21} : P_{2,0}$, $C_{22} : P_{2,0} < P_{2,2}$, $C_{23} : P_{2,0} < P'_{2,2}$, $C_{24} : P_{2,0} < P_{2,2} < P_{2,3}$, $C_{25} : P_{2,0} < P'_{2,2} < P_{2,3}$, and $C_{26} : P_{2,0} < P_{2,3}$. The stabilizers are $N_{L_3(2)}(C_{21}) = L_3(2)$, $N_{L_3(2)}(C_{22}) \cong S_4(1)$, $N_{L_3(2)}(C_{23}) \cong S_4(1)$, $N_{L_3(2)}(C_{24}) \cong D_4$, $N_{L_3(2)}(C_{25}) \cong D_4$, and $N_{L_3(2)}(C_{26}) \cong D_4$. We use (1) and (11) to denote the two non-conjugate elementary abelian 2-subgroups of order 4 and their respective normalizers.
Proof: The only possible radical 2-subgroups of $L_3(2)$ have order $2^3 = 8$, $2^2 = 4$, $2^1 = 2$, and $2^0 = 1$. The only subgroup of order one is the trivial subgroup $1$, which is a radical 2-subgroup. This, of course, gives us the radical 2-chain $C_{21} : P_{2,0}$ with stabilizer $N_{L_3(2)}(C_{21}) = N_{L_3(2)}(P_{2,0}) = L_3(2)$.

The radical 2-subgroups of order eight are the Sylow-2 subgroups. They are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4 \cong D_4$, a dihedral subgroup. To demonstrate such a group let us consider the elements $A_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which has order 2, and $A_6 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which has order 4. Then $< A_5 > \cong \mathbb{Z}_2$ and $< A_6 > \cong \mathbb{Z}_4$. We get $\mathbb{Z}_2 \times \mathbb{Z}_4$ by taking products of elements from these two groups. In this case, the group is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4 \cong D_4$. Thus $P_{2,3} \cong D_4$.

Before we can determine the normalizer of this group, we must first determine possible radical 2-subgroups of order 4. Using our $D_4$ subgroup as a guide, we can find two possible structures. One is isomorphic to $\mathbb{Z}_4$, an example of which is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$ 

This group is $< A_6 >$, and is a subgroup of our $D_4$ subgroup.

Any other possible subgroup is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, two examples of which are

$$H_1 = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : b, c \in F_2 \right\}$$
and

\[ H_2 = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a, b \in \mathbb{F}_2 \right\} \]

where \( \mathbb{F}_2 \) denotes a field of order two. Both are subgroups of our \( D_4 \) subgroup, and with tedious calculations it can be shown that they are not conjugate.

As an example of such a calculation, consider \( A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H_2 \) and \( D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in H_1 \). For a matrix \( \beta = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \) to send \( A \) to \( D \) under conjugation, it must be true that

\[
\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},
\]

or

\[
\begin{bmatrix} a & a+b & c \\ d & d+e & f \\ g & g+h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e+h & f+i \\ g & h & i \end{bmatrix}.
\]

Under this condition and the condition that \( \det \left( \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = 1 \), we get \( a = g = i = 0 \), \( c = d = h = 1 \), and our matrix becomes \( \beta = \begin{bmatrix} 0 & b & 1 \\ 0 & e & f \\ 0 & 1 & 0 \end{bmatrix} \).

Now consider \( B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H_2 \), and \( E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), \( F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H_1 \). In order to send \( B \) to \( E \) under conjugation by \( \beta \), it must be true that \( \beta B = E \beta \), or

\[
\begin{bmatrix} 0 & b & 1 \\ 0 & e & 1+f \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1+b & 1 \\ 0 & 1+e & f \\ 0 & 1 & 0 \end{bmatrix}.
\]
This is impossible. In order to send $B$ to $F$ under conjugation by $\beta$, it must be true that
\[ \beta B = F \beta, \]
or
\[
\begin{bmatrix}
0 & b & 1 \\
1 & e & 1 + f \\
0 & 1 & 0
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 + b & 1 \\
0 & e & f \\
0 & 1 & 0
\end{bmatrix}.
\]
This is impossible as well. In this way, we determine that it is not possible to conjugate the elements of $H_2$ by $\beta$ and get elements of $H_1$.

We now refer to Proposition 1.48(iv), page 40-41, in The Classification of Finite Simple Groups [2], which states, "If $X$ is a group with dihedral Sylow 2-subgroup $S$, then we have ... According as $|S| = 4$ or $|S| > 4$, $X$ has one or two conjugacy classes of four-subgroups." Since the Sylow-2 subgroups are dihedral with order greater than four, there are two conjugacy classes of 4-subgroups. This confirms our calculations. We can conclude that these two conjugacy classes have subgroups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, since both $H_1$ and $H_2$ are both isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, yet are not conjugate. We will denote the conjugacy class which contains $H_1$ as $\mathbb{Z}_2 \times \mathbb{Z}_2$(I) and the class which contains $H_2$ as $\mathbb{Z}_2 \times \mathbb{Z}_2$(II).

We will now look at the normalizers of these groups, beginning with $H_1$. It can be verified that $A_6$, the element of order four in $D_4$, is in $N_{L_3(2)}(H_1)$, so we can conclude that $D_4 \leq N_{L_3(2)}(H_1)$. Since $D_4$ is a maximal subgroup structure of $S_4$ and $S_4$ is maximal in $L_3(2)$, either $N_{L_3(2)}(H_1) \cong D_4$ or $N_{L_3(2)}(H_1) \cong S_4$. However, the element $A_7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin D_4$, of order three, is an element of $N_{L_3(2)}(H_1)$. Thus $N_{L_3(2)}(H_1) \cong S_4$. The same holds for $N_{L_3(2)}(H_2)$, using the element $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, also of order three. Since there are two conjugacy
classes of $S_4$ and two conjugacy classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$, we can conclude $N_{L_3(2)}(H_1) \cong S_4(I)$ and $N_{L_3(2)}(H_2) \cong S_4(II)$.

With this information, we can now determine $N_{L_3(2)}(P_{2,3})$. We know $A_7 \in N_{L_3(2)}(H_1)$, but $A_7 \notin N_{L_3(2)}(D_4)$, and also $A_2 \in N_{L_3(2)}(H_2)$, but $A_2 \notin N_{L_3(2)}(D_4)$. Hence $N_{L_3(2)}(D_4) \neq S_4$.

We must conclude that $N_{L_3(2)}(D_4) \cong D_4$. Thus $N_{L_3(2)}(P_{2,3}) = P_{2,3}$.

Now we will determine $N_{L_3(2)}(\mathbb{Z}_4)$ by finding $N_{L_3(2)}(<A_6>)$, where again $A_6$ is our element of order four in $D_4$. It can be verified that $A_5 \in N_{L_3(2)}(<A_6>)$, where $A_5$ has order two and is in $D_4$, so $D_4 \leq N_{L_3(2)}(<A_6>)$. However, no element of order three can be an element of $N_{L_3(2)}(<A_6>)$, since sending $A_6 \to (A_6)^2 \to (A_6)^3$ is impossible because $(A_6)^2$ has order two. The order of $C(A_6)$ is 4, however, which means $C(A_6) = <A_6>$, and does not have any elements of order three. Thus $N_{L_3(2)}(<A_6>) \neq S_4$, and we can conclude $N_{L_3(2)}(<A_6>) \cong D_4$. Hence $N_{L_3(2)}(\mathbb{Z}_4) = P_{2,3}$.

The last possible radical 2-subgroups have order two, and are isomorphic to $\mathbb{Z}_2$. As an example, consider the element $A_8 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The subgroup it generates is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

which is isomorphic to $\mathbb{Z}_2$. For any element $A$ to be in $N_{L_3(2)}(<A_8>)$, $A^{-1}A_8A$ must equal $A_8$, since the conjugate of the identity by any other element is itself. Thus $N_{L_3(2)}(<A_8>) = C(A_8)$. Well, $|C(A_8)| = 8 ([1])$, so $N_{L_3(2)}(<A_8>) \cong D_4$. Hence $N_{L_3(2)}(\mathbb{Z}_2) \cong D_4$. 

14
Based on this information, $\mathbb{Z}_4$ and $\mathbb{Z}_2$ cannot be radical 2-subgroups. Therefore, the radical 2-subgroups are $P_{2,3} \cong D_4$, $P_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(I)$, $P'_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(II)$, and $P_{2,0} = L_3(2)$.

The radical 2-chains are then $C_{21} : P_{2,0}, C_{22} : P_{2,0} < P_{2,2}, C_{23} : P_{2,0} < P'_{2,2}, C_{24} : P_{2,0} < P_{2,2} < P_{2,3}, C_{25} : P_{2,0} < P'_{2,2} < P_{2,3}$, and $C_{26} : P_{2,0} < P_{2,3}$, with stabilizers $N_{L_3(2)}(C_{21}) = L_3(2), N_{L_3(2)}(C_{22}) = N_{L_3(2)}(P_{2,2}) \cong S_4(I), N_{L_3(2)}(C_{23}) = N_{L_3(2)}(P'_{2,2}) \cong S_4(II), N_{L_3(2)}(C_{24}) = N_{L_3(2)}(P_{2,3}) \cong D_4, N_{L_3(2)}(C_{25}) = N_{L_3(2)}(P_{2,3}) \cong D_4,$ and $N_{L_3(2)}(C_{26}) \cong D_4$. 2
CHAPTER 3

CONCLUSIONS

The table on the next page summarizes our results.

As you can see, finding radical $p$-chains and their stabilizers can be an interesting undertaking. As future research, the information gathered here can be applied to the McKay-Alperin-Dade Conjecture to verify the claim for $L_3(2)$, or it could prove to be a counterexample. Only time will tell.
Table 2: Radical $p$-chain Summary for $L_3(2)$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Radical $p$-subgroups</th>
<th>Normalizers</th>
<th>Radical $p$-chains</th>
<th>Stabilizers</th>
</tr>
</thead>
</table>
| 2   | $P_{2,3} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_4$  
    | $P_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong L_3(2)$  
    | $P_{2,0} = 1$  
    | $N_{L_3(2)}(P_{2,3}) \cong D_4$  
    | $N_{L_3(2)}(P_{2,2}) \cong S_4(1)$  
    | $N_{L_3(2)}(P_{2,0}) = L_3(2)$  
    | $C_{21} : P_{2,0}$  
    | $C_{22} : P_{2,0} < P_{2,2}$  
    | $C_{23} : P_{2,0} < P'_{2,2}$  
    | $C_{24} : P_{2,0} < P_{2,2} < P_{2,3}$  
    | $C_{25} : P_{2,0} < P'_{2,2} < P_{2,3}$  
    | $C_{26} : P_{2,0} < P_{2,3}$ | $N_{L_3(2)}(C_{21}) = L_3(2)$  
    | $N_{L_3(2)}(C_{22}) = S_4(1)$  
    | $N_{L_3(2)}(C_{23}) = S_4(1)$  
    | $N_{L_3(2)}(C_{24}) = D_4$  
    | $N_{L_3(2)}(C_{25}) = D_4$  
    | $N_{L_3(2)}(C_{26}) = D_4$ |
| 3   | $P_{3,1} \cong \mathbb{Z}_3$  
    | $P_{3,0} = 1$  
    | $N_{L_3(2)}(P_{3,1}) \cong S_3$  
    | $N_{L_3(2)}(P_{3,0}) = L_3(2)$  
    | $C_{31} : P_{3,0}$  
    | $C_{32} : P_{3,0} < P_{3,1}$ | $N_{L_3(2)}(C_{31}) = L_3(2)$  
    | $N_{L_3(2)}(C_{32}) = S_3$ |
| 7   | $P_{7,1} \cong \mathbb{Z}_7$  
    | $P_{7,0} = 1$  
    | $N_{L_3(2)}(P_{7,1}) \cong \mathbb{Z}_3 \times \mathbb{Z}_7$  
    | $N_{L_3(2)}(P_{7,0}) = L_3(2)$  
    | $C_{71} : P_{7,0}$  
    | $C_{72} : P_{7,0} < P_{7,1}$ | $N_{L_3(2)}(C_{71}) = L_3(2)$  
    | $N_{L_3(2)}(C_{72}) = \mathbb{Z}_3 \times \mathbb{Z}_7$ |
BIBLIOGRAPHY


