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General Bounds on the Downhill Domination Number in Graphs.

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General Bounds on the Downhill Domination Number in Graphs

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by

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ABSTRACT

General Bounds on the Downhill Domination Number in Graphs

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A path $\pi = (v_1, v_2, \dots v_{k+1})$ in a graph $G = (V, E)$ is a *downhill path* if for every i, $1 \leq i \leq k$, $\deg(v_i) \geq \deg(v_{i+1})$, where $\deg(v_i)$ denotes the degree of vertex $v_i \in V$. The *downhill domination number* equals the minimum cardinality of a set $S \subseteq V$ having the property that every vertex $v \in V$ lies on a downhill path originating from some vertex in S. We investigate downhill domination numbers of graphs and give upper bounds. In particular, we show that the downhill domination number of a graph is at most half its order, and that the downhill domination number of a tree is at most one third its order. We characterize the graphs obtaining each of these bounds.

DEDICATION

I would like to dedicate this thesis to Taz, my only true friend for many years. He showed me true compassion and kindness. He has given me the strength to live and to learn. May he rest in peace with all of my love.

And to Jessie, she has given me hope for the future, and allowed me to work so very hard.

ACKNOWLEDGMENTS

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I would like to thank my parents, Dr. Mary Jane Jamieson and Robert Jamieson. They have supported me through thick and thin, been inspirational, and provided me with more love and support than anyone else through my life. Without the environment they provided me with, I would not be able to achieve anything.

I would like to thank my friends, all of them. Everyone needs people to have fun with outside of academics, may they all achieve their dreams.

Finally, I would like to thank Taz again, through the darkest times he was there.

CONTENTS

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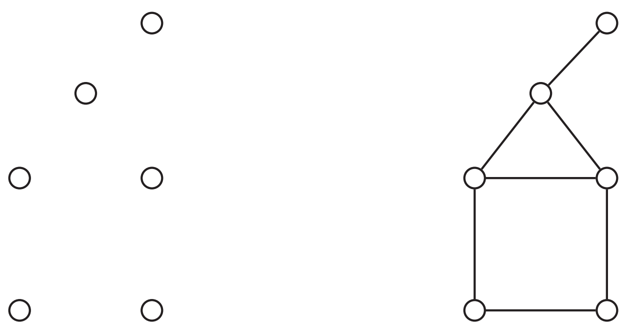
1 Introduction to Basic Graph Theory

Graph theory is an extensively studied field of mathematics, which can trace its roots to the mathematician Euler, but its more modern formulations began with the work of Ore [15]. In graph theory, our most fundamental structure is called a graph. A graph in its most general terms is a collection of objects together with relationships between the objects. In more formal terms it is a set of vertices together with a set of edges between vertices. The vertices represent the objects and the edges represent the relationships between the objects.



A set of "vertices"
(a)

A Graph
(b)

Figure 1: The House Graph

For example consider Figure 1, the so called "house graph". In panel (a), there is a collection of circles which are a pictorial representation of the vertices. In panel (b) of the figure, a full pictorial representation of the house graph is given with the vertices and lines representing edges. It is also worth noting at this point that in pictorial representations of the graph's the geometric positions of the vertices are meaningless, as illustrated in Figure 2. In each of these panels the same graph is given. The geometric positions give no additional information; the only thing which matters is the relationships between the vertices.

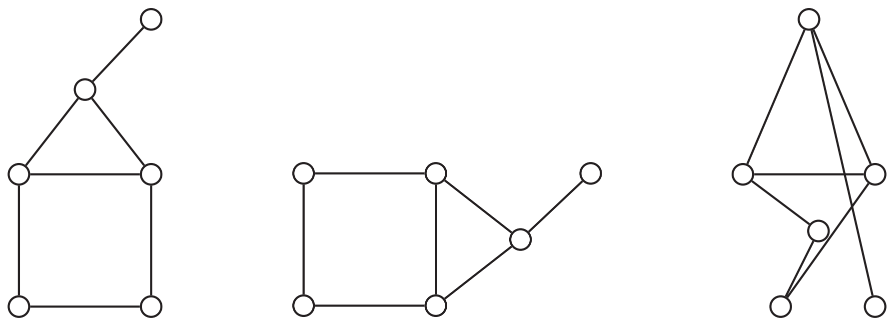


Figure 2: The House Graph Representations

As defined in [6], graph is formally denoted $G = (V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. For our purposes we will be working with simple undirected graphs. That is, there is only one edge between any pair of vertices, each edge exists between two distinct vertices (no loops), and the edge has no directionality. For example, the graph in Figure 2 meets these criteria as well as all future graphs listed.

For a more formal explanation of what an edge is first let $u, v \in V(G)$ be two distinct vertices. An edge of a simple undirected graph is a two element subset of $V(G)$ so if $e \in E(G)$ is an edge between vertices u and v, $e = \{u, v\}$. Often e would be written as uv or vu, either one is equivalent. Further our edge e is said to *join* u and v, also we say that our edge e is *incident* with u and *incident* with v. If $u, v \in V(G)$ and $uv \in E(G)$, we say that u is *adjacent* to v and v is adjacent to u. For instance, in Figure 3 edge η is incident with e and f so it joins the vertices, further due to this we can say that e is adjacent to f. Also, edge δ is incident with c and d, so it joins these vertices, and furthermore, we may say c is adjacent to d.

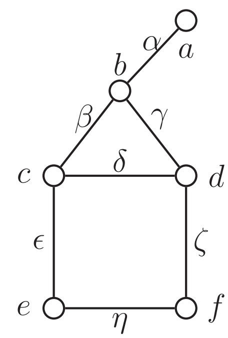


Figure 3: Labeled House Graph

Two of the most basic properties of a graph are its *order* and its *size*. The order of a graph G is the cardinality of its vertex set $V(G)$. We will typically use n to denote the order of a graph. The size of a graph G is the cardinality of its edge set $E(G)$, and we typically will use m to denote the size of a graph. For example the house graph in Figure 3 is of order 6 and size 7.

Notice that in Figure 3, each vertex and edge has been given a label. Some properties of a graph depend on the particular labeling given to the graph, while others are independent of the labeling. Properties of a graph which do not depend on a given labeling are called invariants or graph parameters. Two examples of graph invariants are the order and size of the graph, since no matter how we label the vertices and edges of a graph the number of vertices or edges does not change.

Note that for the remainder of the definitions and invariants listed in this thesis we will make use of the ones listed in [6]. At times these will be supplemented by the more extensive collection in [7], and most of the notation is that of [5, 9].

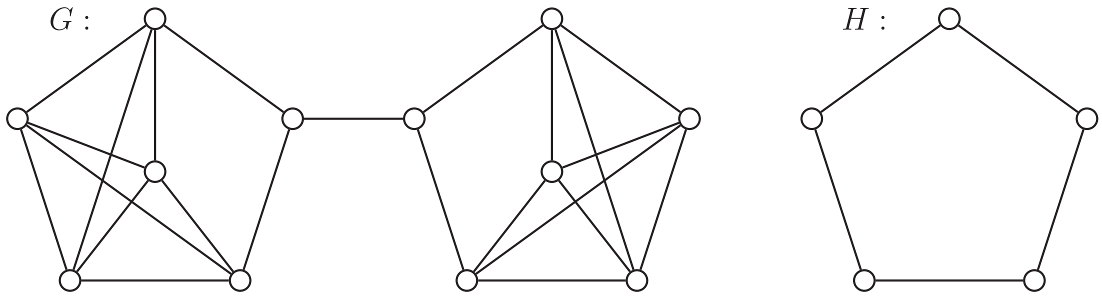


Figure 4: Subgraph Example

1.1 Neighborhoods and Subgraphs

In a graph it is sometimes useful to consider a set of vertices adjacent to something specific. That is to consider all of the vertices adjacent to a particular vertex or a given set of particular vertices. This is called a neighborhood in the graph of a vertex or set of vertices. For a graph G with vertex v, we say the *open neighborhood* of v is the set of vertices $u \in V(G)$ such that $vu \in E(G)$, we will denote this set $N(v)$ and if more than one graph is referred to then we say $N_G(v)$. Further note that if $u \in N(v)$ then u and v are called neighbors. The *closed neighborhood* of a vertex v is $N(v) \cup \{v\}$ and is denoted $N[v]$. For instance consider in Figure 3 vertex b, $N(b) = \{a, c, d\}$ and $N[b] = \{a, b, c, d\}$. For a set S of vertices the open neighborhood of the set is

$$N(S) = \left[\bigcup_{v \in S} N(v)\right] \setminus S,$$

and the closed neighborhood of the set is $N(S) \cup S = N[S]$. Now in Figure 3 consider the set $S = \{c, d\}$, $N(S) = \{b, e, f\}$ and $N[S] = \{b, c, d, e, f\}$. At this point it is worth noting that if we refer to a neighborhood without specifying whether it is open or closed, it will be assumed that the neighborhood is an open neighborhood.

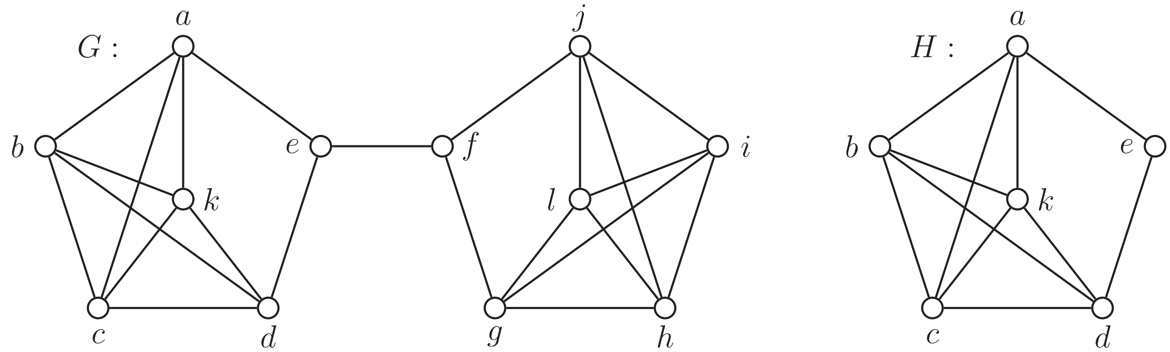


Figure 5: Vertex Induced Subgraph Example

We may also wish to address parts of a graph which also form graphs, that is a subgraph. More formally if G is a graph we say a graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In Figure 4, graph H is a subgraph of G.

It is also useful to address subgraphs which preserve certain aspects of the original graphs. For our purposes we will need subgraphs known as *induced* subgraphs. There are two major types of induced subgraphs, vertex induced and edge induced subgraphs. Let G be some graph and S be some subset of the vertices of G, the subgraph of G induced by the set S is the graph H with vertex set S and edge set $E(G) = \{e \in E(G) \mid e \cap S = e\}$, that is $E(H)$ is the set of all in edges G which are incident to only vertices in S. For example in Figure 5 if we select the vertices $S = \{a, b, c, d, e, k\}$ in G, the subgraph induced by S will be H.

Now for an edge induced subgraph, if G is a graph and L some subset of the edges of G, the subgraph of G induced by L is the graph H with edge set $E(H) = L$ and vertex set $V(H) = \{v \in V(G) \mid \exists\ e \in L \text{ such that } v \in e\}$, that is $V(H)$ is the set of vertices incident with some edge in L. For example in Figure 6, if we let

$L = \{a, b, f, g\}$, the subgraph induced by L will be H. For any subgraph of a graph G induced by some set S of vertices or edges, we denote this graph $G[S]$. So in Figures 5 and 6 the graph H can be denoted $G[S]$ and $G[L]$ respectively.

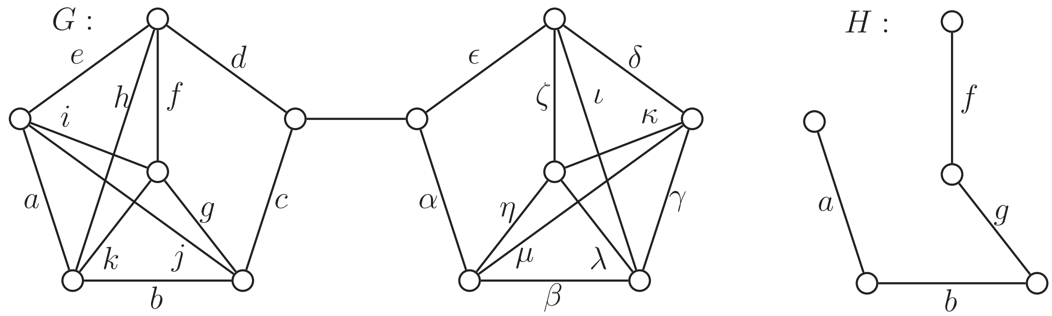


Figure 6: Edge Induced Subgraph Example

1.2 Paths and Cycles

When one considers a graph from the perspective of a set of objects together with a set of relationships, it can be useful to consider how objects are related when they are not adjacent to each other. Often we use a structure known as a path. In a graph sequence $\Pi = (v_1, v_2, \ldots, v_k)$, where for $1 \leq i \leq k$, $v_i \in V(G)$, $v_i \neq v_j$ when $i \neq j$ and $v_i v_{i+1} \in E(G)$, is called a path from vertex v_1 to vertex v_k. We may refer to path Π as a v_1-v_k path. For example in Figure 7, there is an a-d path whose edges are colored red and an f-b path whose edges are colored blue.

A graph G is said to be a *connected* graph if for every pair of vertices $u, v \in V(G)$, there is some u-v path in G. The length of a particular path is the number of edges traversed by the path. For example in Figure 7 the a-d path in red is of length 2 and the f-b path in blue is of length 3. A *cycle* in a graph can be thought of as a

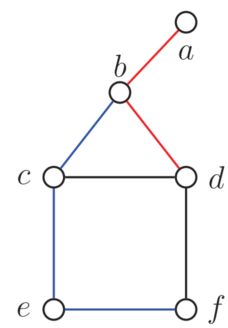


Figure 7: Paths in the House Graph

path $(v_1, v_2, \ldots, v_k)$, where $k \geq 3$ and $v_1 v_k \in E(G)$. For example in Figure 8, we will illustrate two cycle subgraphs, one in red and one in blue.

1.3 Degrees

In graphs it is useful to discuss the number of vertices adjacent to a particular vertex v, we call this the degree of the vertex denoted $deg(v)$. Another way to think of the degree of a vertex is $deg(v) = |N(v)|$. In Figure 9 each vertex of the house graph is labeled with its degree.

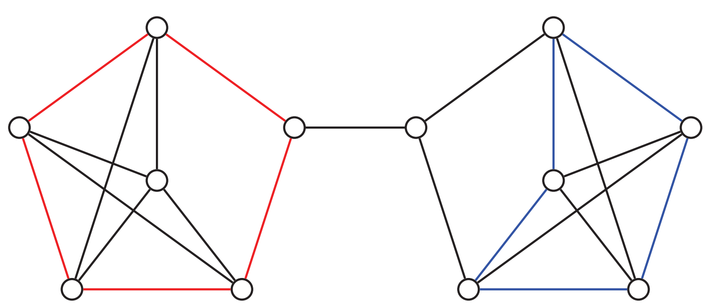


Figure 8: Example of Cycles in a Graph

We may define global graph properties pertaining to degrees in the graph. These

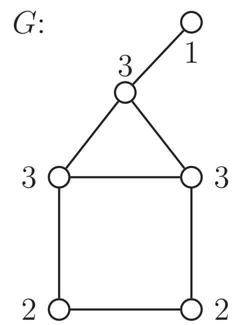


Figure 9: Degrees in the House Graph

are the maximum degree $\Delta(G)$ and the minimum degree $\delta(G)$, which are the maximum and minimum over all degrees in the graph respectively. Thus in Figure 9 it can be seen that $\Delta(G) = 3$ and $\delta(G) = 1$.

One of the simplest results in graph theory has to do with counting degrees in a graph. This result is often called the First Theorem of Graph Theory because it is usually the first proof one does in a graph theory class.

Theorem 1.1 (First Theorem of Graph Theory). *If G is a graph of size m, then*

$$\sum_{v \in V(G)} deg(v) = 2m \tag{1}$$

Proof. Let G be a graph of size m. Note that each edge of G must be adjacent to two vertices in G. Thus summing the degrees of the vertices of G must count each edge twice. Thus the left and right hand sides of equation (1) are equal. $\square$

Note that there are special names for vertices of degree 0 and vertices of degree 1. They are isolate and leaf, respectively.

1.4 Independence

For a graph G we would like to find a set S of vertices such that for any pair $u, v \in S$, $uv \notin E(G)$. The set S is called an independent set, since none of the vertices have an edge between them. We can now define the *independence number* of a graph G to be the maximum possible cardinality for an independent set in G. We denote this $\alpha(G)$. For example in Figure 10 the blue vertices form an independent set which is not as large as possible and the red vertices form an independent set of maximum size.

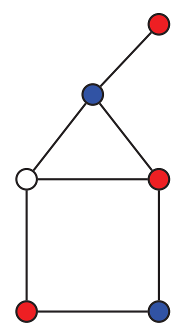


Figure 10: Independent Sets in the House Graph

It is worth noting that we may talk about maximal and maximum sets in the a graph. For independent sets, a maximal set is a set for which no additional vertex in the graph may be added and still preserve the independent property; a maximum independent set is a set of maximum possible cardinality, that is no independent set in the graph with larger cardinality. In Figure 10 the blue vertices form a maximal independent set but not a maximum independent set, and the red vertices form a maximum independent set.

Independence number and independent sets rank among some of the most well studied invariants of a graph.

1.5 Matchings

Let G be a graph. Just as with independent sets of vertices, we may wish to find *independent* sets of edges. A set $M \subseteq E(G)$ such that if $e_1, e_2 \in M$ then $e_1 \cap e_2 = \emptyset$, that is M is a vertex disjoint set of edges, is called a matching. For example in Figure 11 the red edges form one matching and the blue edges form another matching in the graph.

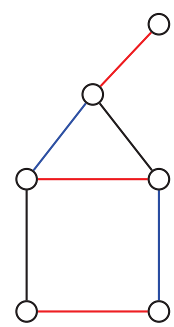


Figure 11: Matchings in the House Graph

If we have two disjoint sets of vertices U and W, it can be helpful to be able to pair the vertices in U with adjacent vertices in W. In other words we would like to match U to W. Obviously to do this we at least the condition $|U| \leq |W|$. We can match U to W if there exists a matching M such that for each vertex $u \in U$ there exists an edge $e \in M$ for which $u \in e$ and $e \cap W \neq \emptyset$. Such a matching M is said to match U to W.

This problem is sometimes called the "marriage problem". The marriage problem is as follows. *Suppose we have a set U of girls and a set W of boys. If each girl $u \in U$ likes some subset of the boys $W_u \subseteq W$, can each girl marry a boy she likes?* This question can be phrased as asking whether or not a matching exists between two sets of vertices in the graph. In fact this problem is the motivation behind one of the results used later in this thesis.

1.6 Domination

Let G be a graph, we would like to find a set S of vertices such that for all vertices $v \in V(G) \backslash S$ there exists a vertex $s \in S$ where $v \in N(s)$, that is we wish to find a set S of vertices so that each vertex of the graph is either in S or adjacent to some vertex in S. This set S is called a dominating set of the graph [10, 11]. We then can define a graph invariant called the domination number, $\gamma(G)$, where the domination number is the minimum cardinality for a dominating set in a graph G. For example in Figure 12 the set of vertices labeled in black form a minimum dominating set and the circled vertices form another dominating set.

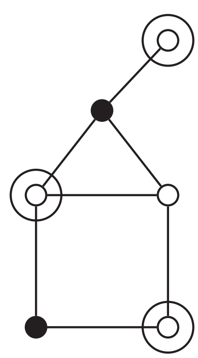


Figure 12: Dominating Sets in the House Graph

Just as with independent sets it is worth discussing that we make a distinction between a minimal dominating set a and a minimum dominating set. A minimal dominating set of a graph G is a dominating set such that the removal of any vertex

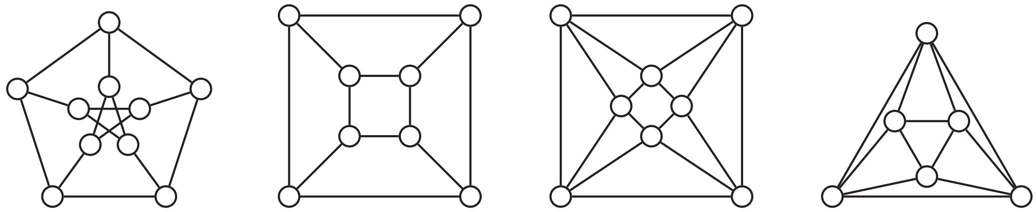


Figure 13: Examples of Regular Graphs

from the set results in a set that does not dominate G. A minimum dominating set is a dominating set of minimum possible cardinality among all possible cardinalities, note that any minimum dominating set is a minimal dominating set but the reverse is not necessarily true. Further we may refer to a minimum dominating set as a $\gamma(G)$-set, since $\gamma(G)$ is the cardinality of such a set. For example in Figure 12, the vertices in black form a minimum dominating set, and the circled vertices form a minimal dominating set.

1.7 Special Graph Families

Finally it is worth discussing three families of graphs which we will return to later: regular graphs, bipartite graphs and trees. These three families will play roles in the main results of this thesis.

1.7.1 Regular Graphs

A graph G is a regular graph if for each vertex $v \in V(G)$, $deg(v) = r$. That is each vertex of G has the same degree, if that degree is r we say G is a r-regular graph. Figure 13 has two 3-regular graphs and two 4-regular graphs.

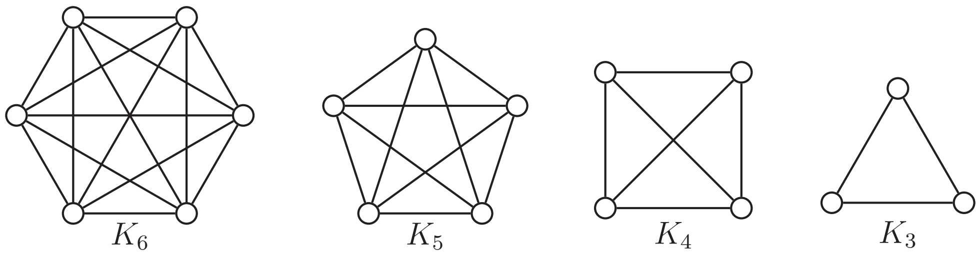


Figure 14: Examples of Complete Graphs

One special subfamily of regular graphs is called the complete graph. The complete graph G on n vertices is the graph for which every pair of vertices is joined by an edge. Thus G will be a $(n-1)$-regular graph. We denote the complete graph on n vertices as K_n. Figure 14 has a few complete graphs shown.

1.7.2 Bipartite Graphs

A graph G is a bipartite graph if $V(G)$ can be partitioned into two sets U and W such that U and W are both independent sets. Note that by partition, we mean that $U \cup W = V(G)$ and $U \cap W = \emptyset$. For example in Figure 15, the graph G is bipartite with partite sets being the red vertices and the blue vertices, but H is not bipartite

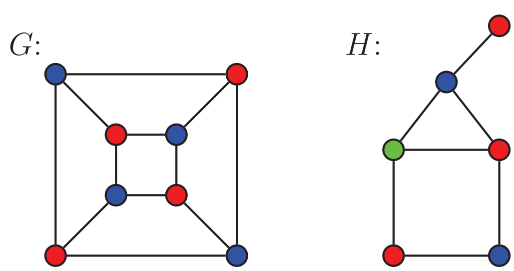


Figure 15: Bipartite vs Not Bipartite

because no matter how we partition the vertices of H we need at least three sets. To see this color the vertices of the outer 5 cycle of the graph H with two colors, say red and blue, such that no two vertices receive the same color. Since there are an odd number of vertices in this cycle, this is not possible. So we require a third color, say green as shown in the figure.

A special subfamily of bipartite graphs, is the complete bipartite graph $K_{r,s}$ with $r \leq s$. We define $K_{r,s}$ as the bipartite graph with partite sets $|U| = r$ and $|W| = s$ where $E(K_{r,s}) = \{uw \mid u \in U \text{ and } w \in W\}$ that is all possible edges exist between U and W. For example in Figure 16 all $K_{r,5}$ with $1 \leq r \leq 5$ are given.

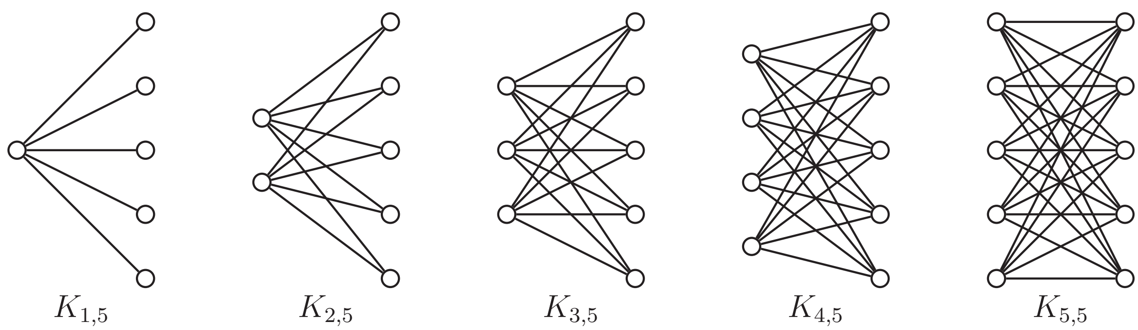


Figure 16: Complete Bipartite Graph Examples

1.7.3 Trees

Our last family of graphs is the trees. A graph T is a tree if and only if T is connected and T contains no cycles. Trees are a very important family of graphs because all trees have been counted and many properties can be found quickly for trees without being too difficult, that is in polynomial time. Figure 17 contains a few examples of some trees.

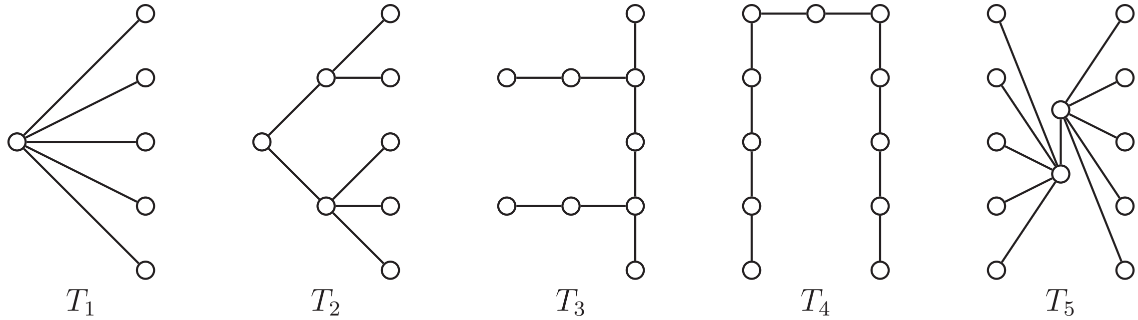


Figure 17: Examples of Trees

Note that we will refer a specific type of tree called a *caterpillar*. A caterpillar is a tree T, such that every vertex of T lies on a single central path or is adjacent to a vertex on the central path. In other words, removing all the leaves from T results in a path. For example Figure 18 contains a few caterpillars.

Note that, we will employ a special notation for caterpillars. It is called the caterpillar code. The caterpillar code is a k-tuple, where k is the length of the central path of the caterpillar, and the ordering of the k-tuple is the order of the vertices along the central path. Finally the values in each part of the k-tuple denote the

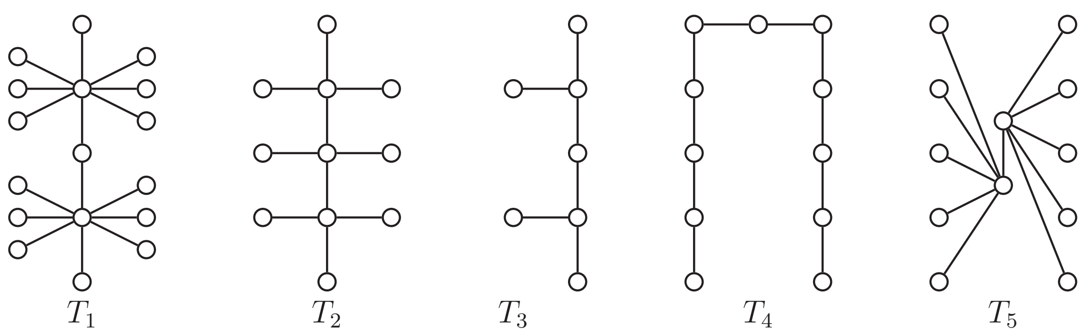


Figure 18: Examples of Caterpillars

number of leaves attached to the particular vertex. Thus in Figure 18, $T_1 = (7, 0, 7)$, $T_2 = (3, 2, 3)$, $T_3 = (2, 0, 2)$, $T_4 = (1, 0, 0, 0, 0, 0, 0, 0, 1)$, and $T_5 = (5, 5)$.

2 Downhill Domination

Recall that in Section 1.6 we defined and gave examples for a well known graph invariant called the domination number. The idea of the domination number and dominating sets will be expanded upon to produce a new type of domination, which we will term *downhill domination.* These expansions rely on the ideas in Section 1.2 and Section 1.3.

2.1 Definition of Downhill Domination

Downhill domination, relies essentially on dominating through a special type of path. We will term these paths *downhill paths.* These downhill paths are paths with some restrictions on them using the degrees of vertices.

2.1.1 Downhill Paths

We formally define a downhill path by the following.

Definition 2.1. *In a graph G. A u-v path,* $\Pi = (u = v_1, v_2, \ldots, v_k = v)$*, is a downhill path if $deg(v_i) \geq deg(v_{i+1})$ for all $1 \leq i \leq k$, that is, the degrees of the vertices of* Π *form a non-increasing sequence.*

For example in Figure 19 several downhill paths are given for the house graph. Notice that (a) contains a downhill path that is of maximum possible length, and (b) cannot be extended further.

Note that in many cases we will piece together paths like the one shown in (d) part, where each vertex on the path is of the same degree. We may refer to these

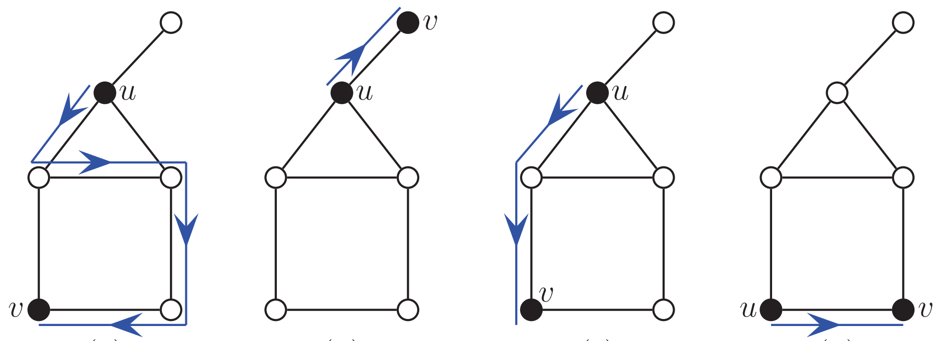


Figure 19: Example of Downhill Paths

paths as regular paths since the degree remains constant along the path, in a similar fashion to regular graphs.

We say that if there is a u-v downhill path between vertices u and v in our graph that v is *downhill* from u.

2.1.2 Downhill Domination

In examining Figure 19, one might notice that downhill paths do not originate at every vertex. So suppose that we wish to find a set of vertices such that any vertex in our graph is downhill from a vertex in the set. This type of set is called a *downhill dominating set*, we give the formal definition.

Definition 2.2. *Let G be a graph. A set $S \subseteq V(G)$ is a downhill dominating set, if $\forall v \in V(G) \backslash S$, there exists a $u \in S$ s.t. G contains a u-v downhill path.*

Just as with domination, our goal will be to find the cardinality of the smallest downhill dominating set, that is, for a graph G, we wish to find $\gamma_{dn}(G)$, which is the minimum possible cardinality of a downhill dominating set of G. We refer to downhill dominating sets of cardinality $\gamma_{dn}(G)$ as $\gamma_{dn}(G)$-sets, and a DDS is a downhill dominating set. Thus a *minimal* DDS is a downhill dominating set such that the removal of any vertex ends the downhill dominating property of this set.

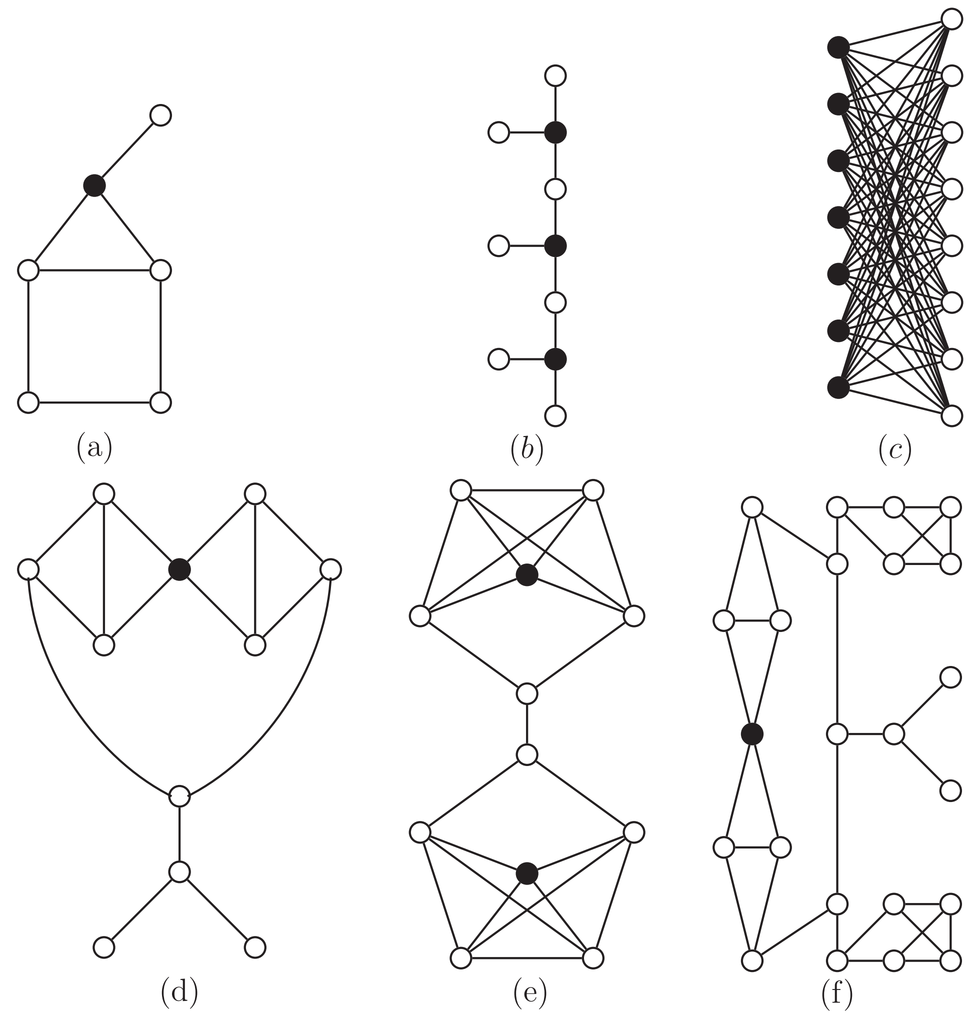


Figure 20: Examples of Downhill Domination

In Figure 20 several graphs are given with $\gamma_{dn}(G)$-sets marked by darkened vertices. If one examines the paths given by (a) and (b) in Figure 19, one can see how the darkened vertex in (a) of Figure 20 downhill dominates the house graph. We leave the discovery of the downhill paths that allow the rest of the to be downhill dominating in Figure 20 to the reader.

2.2 Preliminary Results

The preliminary results on downhill domination focus on two fronts. Firstly, we need a few results concerning specific graph families. This is so that later on when we work with the main results we will have a clear examples for graphs that achieve the general bounds. We also need a few additional results concerning downhill dominating sets, and their general properties, which will be useful in establishing our upper bounds.

2.2.1 Special Graph Families

We start with some basic results concerning some of our graph families. Beginning with regular graphs, which in many cases we will have to treat separately in our proofs.

Proposition 2.3. *If G is a connected regular graph, then $\gamma_{dn}(G) = 1$.*

Proof. Let G be a connected regular graph and v be some vertex of G. Since G is connected there exists a path between v and every vertex $u \in V(G) \backslash \{v\}$. All of these paths are downhill since the degree of each vertex of G is the same. Hence $\{v\}$ downhill dominates G, so $\gamma_{dn}(G) = 1$.

□

From this we immediately get a result concerning complete graphs.

Corollary 2.4. *If $G = K_n$, then $\gamma_{dn}(G) = 1$.*

Proof. Let $G = K_n$, so by definition G is a connected regular graph. Thus Proposition 2.3 gives that $\gamma_{dn}(G) = 1$.

□

Now we can find the downhill dominating number for all complete bipartite graphs, which we will need later on for one of the main results.

Proposition 2.5. *Let $G = K_{r,s}$. If $r < s$, then $\gamma_{dn}(G) = r$. Furthermore, if $r = s$, then $\gamma_{dn}(G) = 1$.*

Proof. Let $G = K_{r,s}$ where $r < s$. Since G is a complete bipartite graph, it can be partitioned into two independent sets $|R| = r$ and $|S| = s$. Further for $v \in R$, $deg(v) = s$ and $u \in S$, $deg(u) = r$. Since R and S are independent sets any $\gamma_{dn}(G)$-set will require every vertex of R since these vertices have no neighbors of equal or greater degree. Finally since every vertex of R is adjacent to every vertex of S, the vertices of S are downhill from those in R. Hence, R is a $\gamma_{dn}(G)$-set. So $\gamma_{dn}(G) = r$. Note that if $r = s$ then G is a regular graph so Proposition 2.3 gives $\gamma_{dn}(G) = 1$.

□

2.2.2 Introductory Results

One of the most important results, that we used in order to attack the downhill domination problem in a general way is the following lemma concerning the independence of downhill dominating sets.

Lemma 2.6. *Any minimal DDS of a graph G is an independent set of G.*

Proof. Assume S is a minimal DDS of G. If two vertices u and v in S are adjacent, then, relabeling u and v if necessary, we may assume that $\deg(u) \geq \deg(v)$. Hence, there exists a downhill path from u through v to all vertices which are downhill from v. Therefore, $S \setminus \{v\}$ is a DDS of G, contradicting that S is a minimal DDS. Hence, we conclude that S is an independent set.

□

This result is extremely important because it allows us to gain some general insight into the structure of our downhill dominating sets. Namely that they must be independent sets. Now as an aside on independence. Recall from Section 1.4 the independence number of a graph, $\alpha(G)$. Since Lemma 2.6 gives that any downhill dominating set is an independent set we obtain the following corollary.

Corollary 2.7. *For a graph G, $\gamma_{dn}(G) \leq \alpha(G)$.*

Proof. Since Lemma 2.6 gives that any DDS is an independent set, and $\gamma_{dn}(G)$ is the minimum size for a DDS in G. This is clearly less than the maximum size for an independent set of a graph, $\alpha(G)$.

□

We now give an interesting result concerting the relationship a minimal DDS and $\gamma_{dn}(G)$-sets

Theorem 2.8. *If D is a minimal DDS of G, then D is a $\gamma_{dn}(G)$-set.*

Proof. Suppose to the contrary that there exists a minimal DDS, say D, of G, such that $|D| > \gamma_{dn}(G)$. Among all γ_{dn}-sets of G, select D' to be one that has the maximum number of vertices in common with D, that is, $|D' \cap D|$ is maximized.

Since $|D'| < |D|$, there exists a vertex $u \in (D \setminus D')$. Thus u is downhill dominated by a vertex, say d' in D'. Then u and all the vertices downhill from u are downhill dominated by d'. If $d' \in D$, then $D \setminus \{u\}$ is a DDS with cardinality less than $|D|$, contradicting the minimality of D. Hence we may assume that $d' \notin D$.

Thus there exists a vertex $v \in D$ that downhill dominates d' and all of the vertices downhill from d'. Suppose $u \neq v$, then v downhill dominates u and so, again, $D \setminus \{u\}$ is a DDS, contradicting the minimality of D. If $u = v$, then since v downhill dominates d' and d' downhill dominates u, it follows that $\deg(u) = \deg(d')$. Moreover, u downhill dominates d' and the vertices downhill dominated by d'. Thus, $D'' = (D' \setminus \{d'\}) \cup \{u\}$ is a γ_{dn}-set of G such that $|D'' \cap D| > |D' \cap D|$, contradicting our choice of D'.

□

This result gives a very interesting concept, that we do not have to make a distinction between a minimal downhill dominating set and a minimum downhill dominating set. This is very different from the standard concepts of Sections 1.4 and 1.6 for independent sets and dominating sets respectively.

2.3 Introduction to Main Results

The two main results of this thesis focus first on finding an over arching upper bound on γ_{dn} for a general connected graph, and then improving this bound when exploring trees. In both cases a characterization of the graphs obtaining bound is given.

A characterization of the family of graphs for a bound is finding a family of graphs which achieve equality with the bound and then showing that if a graph achieves equality with the bound, then the graph is in the family. For our first bound our characterization involves certain types of complete bipartite graphs, the so called "almost balanced" complete bipartite graphs, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ where n is odd and two complete graphs, K_2 and K_3. For the trees bound, the characterization family is a special collection of caterpillars, $\mathcal{T}$, which follow specific caterpillar codes.

The two main results formally are as follows.

Theorem A. *If G is a graph of order $n \geq 2$, then $\gamma_{dn}(G) \leq \lfloor \frac{n}{2} \rfloor$, with equality if and only if G is one of the complete graphs K_2 or K_3, or the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ of odd order.*

Theorem B. *If T is a tree of order $n \geq 4$, then $\gamma_{dn}(T) \leq \lfloor \frac{n-1}{3} \rfloor$, with equality if and only if T is the path of order 4 or $T \in \mathcal{T}$.*

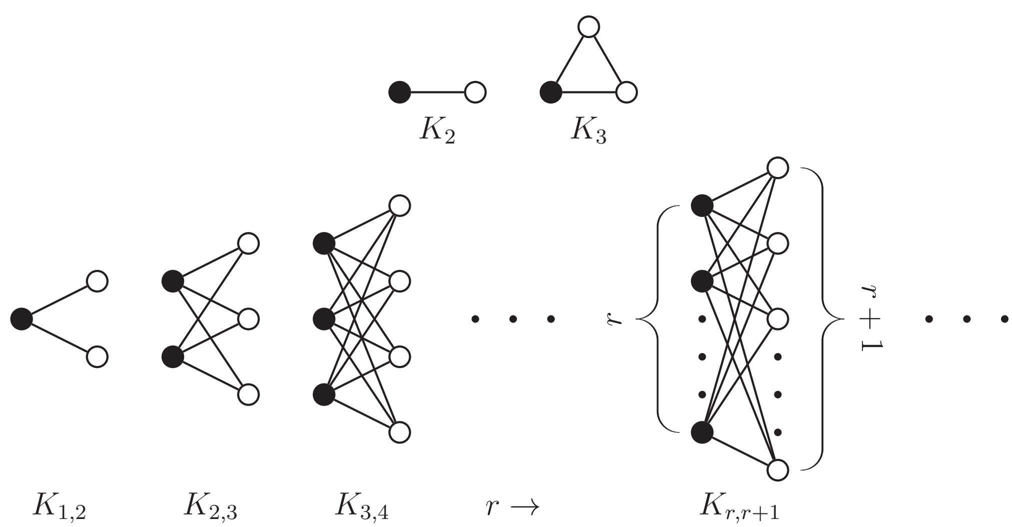


Figure 21: Characterization of Theorem A

As one can see from Theorem A, the maximum value for γ_{dn} is $\lfloor \frac{n}{2} \rfloor$ with equality in most cases when $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ when n is odd. One can easily see that the equality is true in this case from Proposition 2.5. And that the special cases for K_2 and K_3 come from Corollary 2.4. Thus our characterization family is given, in a sense, by Figure 21, where some of the initial graphs of the characterization are listed and then a progression to one of order $2r+1$ is given. The darkened vertices represent a $\gamma_{dn}(G)$-set.

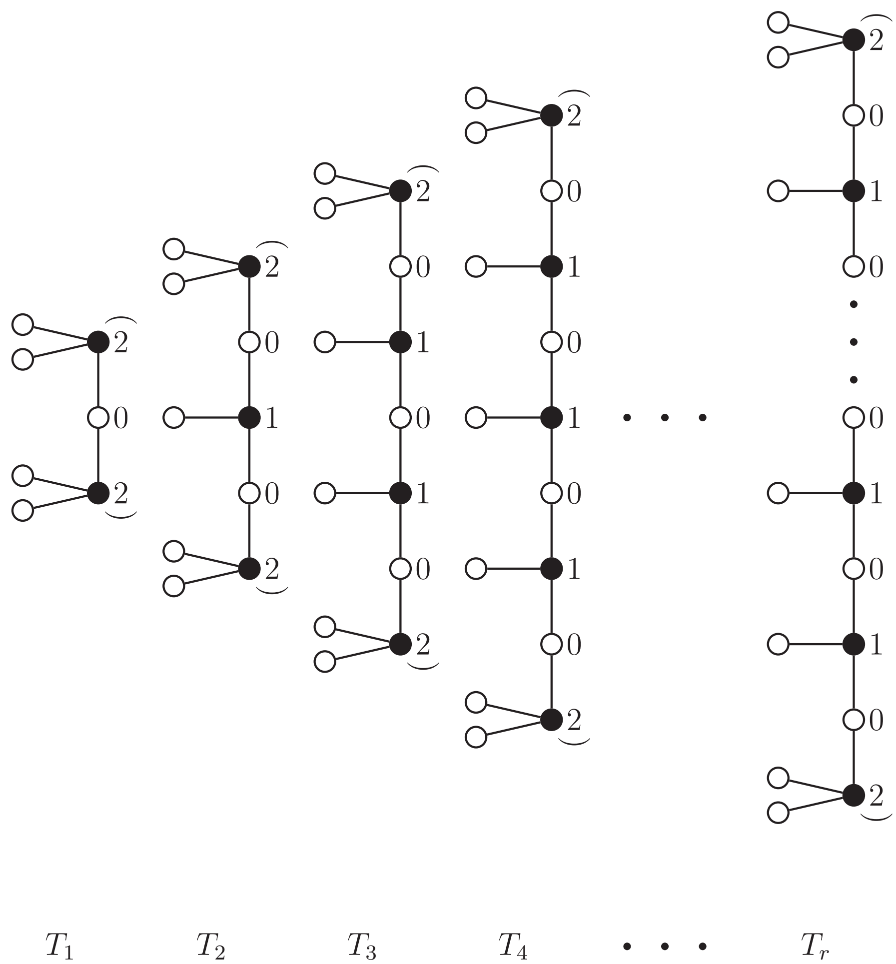


Figure 22: Some Graphs in the Characterization of Theorem B

As for Theorem B, before we fully define how to construct $\mathcal{T}$, we will give a few elements of a $\mathcal{T}$ in Figure 22 as an example.

3 General Graph Bound

We present the proof of Theorem A, which is one of the the two main results of this thesis. This result relies partially on the classification of vertices by degrees first presented by Hedetniemi, Hedetniemi, Hedetniemi, and Lewis in [14, 12, 13]. These ideas are used to prove a series of minor results which culminate in our main general bound.

3.1 The Classification of Vertices

Definition 3.1. *A vertex $u \in V(G)$ in a graph G is called:*

1. ***very strong (VS)*** *if* $\deg(u) \geq 1$ *and for every vertex* $v \in N(u)$, $\deg(u) > \deg(v)$.

2. ***strong (S)*** *if* $\deg(u) \geq 2$ *and for every vertex* $v \in N(u)$, $\deg(u) \geq \deg(v)$, *at least one neighbor* $v \in N(u)$ *has* $\deg(u) > \deg(v)$, *and at least one neighbor* $w \in N(u)$ *has* $\deg(u) = \deg(w)$.

3. ***regular (R)*** *if* $\deg(u) \geq 0$ *and for every vertex* $v \in N(u)$, $\deg(u) = \deg(v)$.

4. ***very typical (VT)*** *if* $\deg(u) \geq 2$ *and at least one neighbor* $v \in N(u)$ *has* $\deg(u) > \deg(v)$, *and at least one neighbor* $w \in N(u)$ *has* $\deg(u) < \deg(w)$.

5. ***typical (T)*** *if* $\deg(u) \geq 3$ *and there are three distinct vertices* $v, w, x \in N(u)$ *such that* $\deg(v) < \deg(u) = \deg(x) < \deg(w)$.

6. ***weak (W)*** *if* $\deg(u) \geq 2$ *and for every vertex* $v \in N(u)$, $\deg(u) \leq \deg(v)$, *at least one neighbor* $v \in N(u)$ *has* $\deg(u) < \deg(v)$, *and at least one neighbor*

$w \in N(u)$ has $\deg(u) = \deg(w)$.

7. ***very weak (VW)*** *if* $\deg(u) \geq 1$ *and for every vertex* $v \in N(u)$, $\deg(u) < \deg(v)$.

For a graph G, let $VS(G)$ be the set of very strong vertices in G, $S(G)$ be the set of strong vertices in G, and $R(G)$ be the set of regular vertices of G. Our next observations follow directly from the above definitions and the minimality of a $\gamma_{dn}(G)$-set. Since very strong, strong, and regular vertices are the only vertices with no neighbors of higher degree, and a $\gamma_{dn}(G)$-set is minimal, observe the following.

Observation 3.2. *If D is a $\gamma_{dn}(G)$-set for a graph G, then $D \subseteq \big(R(G) \cup S(G) \cup VS(G)\big)$.*

Now since very strong vertices have no neighbors of equal or greater degree, no vertex can downhill dominate a very strong vertex v, except v its self. Hence, observe the following.

Observation 3.3. *If D is a $\gamma_{dn}(G)$-set for a graph G, then $VS(G) \subseteq D$.*

Next we wish to explore the concept referred to earlier in Section 2.1.1, concerning regular paths. So we will define a special type of set in a graph.

Definition 3.4. *Let G be a graph. For a vertex $v \in V(G)$, we define the regular path neighborhood (RPN) of v, $A(v)$, to be the set of all $u \in V(G)$ such that there exists a v-u path $v = v_1, v_2, \ldots, v_k = u$ such that $deg(v_i) = deg(v)$ for all $1 \leq i \leq k$.*

We can think of this like in the following generalized graph in Figure 23. Each of the large black ellipses contains all the vertices of a given degree and each of the red ellipses contained inside the black ellipses contains a RPN of a particular vertex such as the ones marked. Clearly edges can exists between levels, these have not been drawn however. The next lemma concerns vertices in RPNs. Now for a particular degree level, there is a set of RPNs that will partition the level. As illustrated in part by the following lemma, the shown edges in blue cannot exist, since this would merge two RPNs together. This idea is used to prove the next lemma.

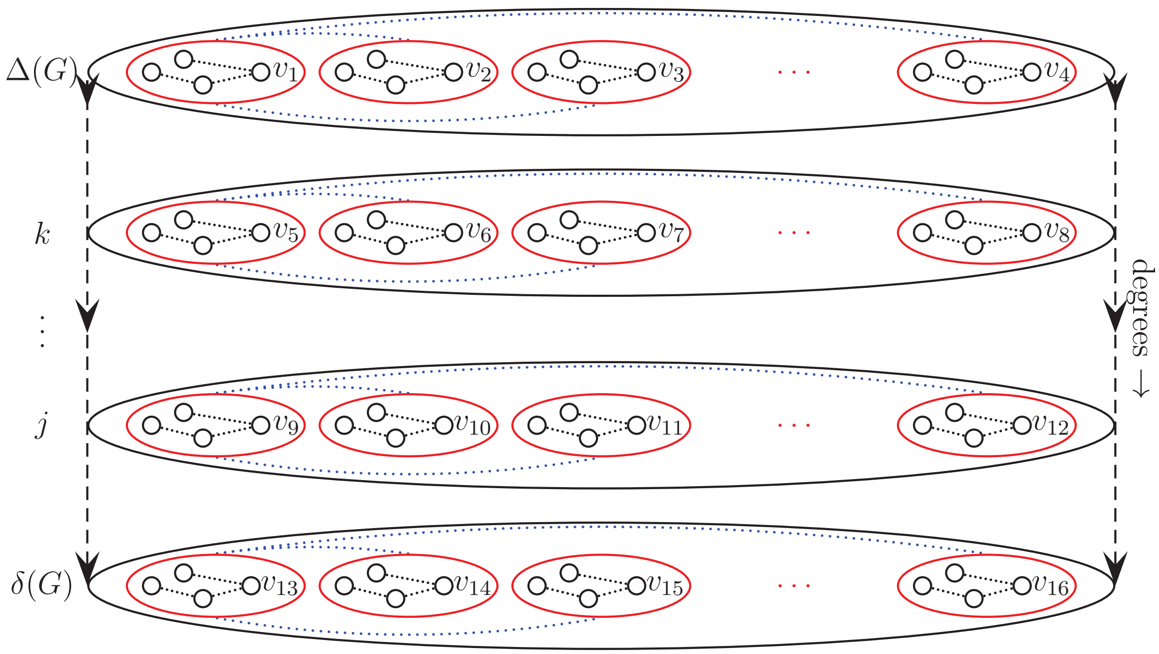


Figure 23: Downhill Path Neighborhood

Lemma 3.5. *If D is a DDS of G, then for all $u, v \in D$*

$$A(u) \cap A(v) = \emptyset. \tag{2}$$

Proof. Let D be a DDS of a graph G. Suppose to the contrary that there are two vertices $u, v \in D$ such that $A(u) \cap A(v) \neq \emptyset$. Then there exists a vertex w such that

$w \in A(u)$ and $w \in A(v)$. By definition there is a u-w regular path and a w-v regular path in G. It follows that the path from u to w which continues on the w-v path is a regular path between u and v. Thus v is downhill from u and everything downhill from v is downhill from u as well. Thus $D \backslash \{v\}$ is a DDS of G as well, a contradiction of D being a minimal downhill dominating set.

$\square$

We next we use the idea of the RPN to prove an interesting lemma concerning the composition of $\gamma_{dn}(G)$-set. It proves that we need only be concerned with two types of vertices for a $\gamma_{dn}(G)$-set, the strong and very strong vertices.

Lemma 3.6. *Let G be a connected graph. There exists a $\gamma_{dn}(G)$-set that contains no regular vertices if and only if G is not regular.*

Proof. If G has a $\gamma_{dn}(G)$-set which contains no regular vertices, then G is not regular.

Assume that G is not regular. Among all $\gamma_{dn}(G)$-sets of G, select D to minimize $|D \cap R(G)|$, that is, D contains the minimum number of regular vertices. If $D \cap R(G) = \emptyset$, then the result holds. Thus, assume that there is a regular vertex $v \in D$. By Lemma 2.6, D is independent. We now will explore the RPN of v, $A(v)$.

We first show that $A(v) \subseteq R(G) \cup S(G)$. Since $v \in R(G)$, v has a neighbor of the same degree. Hence, $A(v) \backslash \{v\} \neq \emptyset$. Let $u \in A(v)$. Since $\deg(u) = \deg(v)$, it follows from Definition 3.1 that u is either weak, typical, regular, or strong. If $u \in R(G) \cup S(G)$, then we are finished. Thus assume that u is weak or typical. Then there exists a vertex $y \in N(u)$ such that $\deg(y) > \deg(u)$. Hence, $y \notin A(v)$. Further since $\deg(u) = \deg(v)$ and there is a path between u and v consisting of vertices having degree $\deg(v)$, it follows that v does not downhill dominate y. Hence, there is some vertex $w \in D \backslash \{v\}$ such that y is downhill from w or $y = w$. But then w downhill dominates v and all the vertices downhill dominated by v, and so $D \setminus \{v\}$ is a DDS of G with cardinality less than $\gamma_{dn}(G)$, a contradiction. Hence, we may assume that every vertex in $A(v)$ is regular or strong.

We note that if $A(v) \subseteq R(G)$, then since G is connected, G must be regular, a contradiction. Therefore, there exists a strong vertex, say x, in $A(v)$. Further, from the definition of $A(v)$, v is downhill from x, implying that everything downhill from v is downhill from x. Hence, $(D \setminus \{v\}) \cup \{x\}$ is a γ_{dn}-set of G having fewer regular vertices than D, contradicting our choice of D.

$\square$

3.2 The Proof of Theorem A

We shall use the well-known theorem by Hall [8]. This theorem has to do with the ideas of Section 1.5, specifically it answers the small problem posed in this section, that is *"When can a girl marry a boy that she likes?"* The answer is when U is the set of girls and W is the set of boys, with an edge representing a girl liking the boy.

Theorem 3.7 (**Hall's Theorem**). *Let G be a bipartite graph with partite sets U and W. Then U can be matched to a subset of W if and only if for all $S \subseteq U$, $|N(S)| \geq |S|$.*

In order to prove Theorem A, we need to establish some properties of $S(G)$ and $VS(G)$. We present these properties as separate results as they are interesting in their own right.

Proposition 3.8. *Let G be a connected graph of order $n \geq 3$. If $VS(G) \neq \emptyset$, then $VS(G)$ can be matched to $N(VS(G))$.*

Proof. Let G be a connected graph of order $n \geq 3$ and $VS(G) \neq \emptyset$. Now let $X_e \subseteq E(G)$ be the set of edges having at least one endvertex in $VS(G)$. By Lemma 2.6 and Observation 3.3, $VS(G)$ is an independent set. Thus, the edge induced subgraph $G[X_e]$ is a bipartite graph with partite sets $VS(G)$ and $N(VS(G))$. We wish to show that there exists a matching from $VS(G)$ to $N(VS(G))$ in the edge induced subgraph $G[X_e]$. By Hall's Theorem, it suffices to show that for all $X \subseteq VS(G)$, $|N(X)| \geq |X|$.

To establish this, we proceed by induction on $|X|$ for a subset $X \subseteq VS(G)$. Since X is an independent set and G has no isolated vertices, every vertex in X has a neighbor in $N(VS(G))$. Hence, the result holds for $|X| = 1$. For $|X| = 2$, suppose to the contrary that $|N(X)| < |X|$. Again since G has no isolated vertices, we have that $|N(X)| \geq 1$, so $|N(X)| = 1$. But then the two vertices of X each have degree one, while their common neighbor in $N(X)$ has degree at least two, contradicting that the vertices of X are very strong. Thus, $|N(X)| \geq |X| = 2$, and so the result holds for $1 \leq |X| \leq 2$.

Assume that $|N(X)| \geq |X|$ holds for any $X \subseteq VS(G)$ such that $|X| \leq k$ for some $k \geq 2$. Let $|X| = k + 1$, and suppose to the contrary that $|N(X)| < |X|$. Let $X' = X \backslash \{v\}$ for some $v \in X$. Since $X' \subseteq VS(G)$ and $|X'| = k$, by our inductive hypothesis, $|N(X')| \geq |X'|$. Thus, we obtain the following relations

$$|N(X')| \geq |X'| \tag{3}$$

$$|N(X)| < |X| \tag{4}$$

$$|X| = |X'| + 1. \tag{5}$$

From (4) and (5), we have that $|N(X)| \leq |X| - 1 = |X'|$. Thus, by (3), $|N(X)| \leq |X'| \leq |N(X')|$. Since $N(X') \subseteq N(X)$, we have $|N(X)| \geq |N(X')|$. Thus, $|N(X)| = |N(X')|$, implying that $|N(X')| = |X'|$. Moreover, by our inductive hypothesis, $|N(X'')| \leq |X''|$ for all $X'' \subseteq X'$. Thus, by Hall's Theorem, there is a matching in $G[X_e]$ between the vertices of X' and the vertices of $N(X')$. Label the vertices of $X' = \{x_1, x_2, ..., x_k\}$ and $N(X') = \{y_1, y_2, ..., y_k\}$ such that $M = \{x_i y_i \,|\, 1 \leq i \leq k, x_i \in X'$ and $y_i \in N(X')\}$ is a perfect matching.

Now there are exactly $\sum_{i=1}^{k} \deg_G(x_i)$ edges incident to vertices in X' and vertices in $N(X')$, implying that $\sum_{i=1}^{k} \deg_G(y_i) \geq \sum_{i=1}^{k} \deg_G(x_i)$. But since $x_i \in VS(G)$, $\deg(y_i) < \deg(x_i)$ for all $1 \leq i \leq k$, and so $\sum_{i=1}^{k} \deg(y_i) < \sum_{i=1}^{k} \deg(x_i)$, a contradiction.

Thus, we conclude that $|N(X)| \geq |X|$ for every set $X \subseteq VS(G)$ such that $|X| = k + 1$. By the Principle of Mathematical Induction, $|N(X)| \geq |X|$ where $|X| \geq 1$. Therefore, by Hall's Theorem, the set $VS(G)$ can be matched to $N(VS(G))$ in the subgraph $G[X_e]$, and so $VS(G)$ can be matched to $N(VS(G))$ in G.

$\square$

Proposition 3.9. *Let G be a connected graph of order $n \geq 3$. If $VS(G) \neq \emptyset$, then $|VS(G)| < |N(VS(G))|$.*

Proof. Let G be a connected graph of order $n \geq 3$ and $X = VS(G) \neq \emptyset$. By Proposition 3.8, X can be matched to the set $N(X)$. Thus, $|X| \leq |N(X)|$. Suppose that $|X| = |N(X)| = k$ and that $X = \{x_1, x_2, \ldots, x_k\}$ and $N(VS(G)) = \{y_1, y_2, \ldots, y_k\}$, where x_i is matched to y_i for $1 \leq i \leq k$. Since X is an independent set, $\sum_{i=1}^{k} \deg(x_i) \leq \sum_{i=1} \deg(y_i)$. However, since x_i is very strong, $\deg(x_i) > \deg(y_i)$ for $1 \leq i \leq k$. Thus, $\sum_{i=1}^{k} \deg(x_i) > \sum_{i=1} \deg(y_i)$, a contradiction. Hence, $|X| < |N(X)|$.

□

For a set S, an *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted $\mathrm{epn}(v, S)$.

Proposition 3.10. *Let G be a connected graph of order $n \geq 2$. If $v \in D \cap S(G)$, then there exists a vertex $x \in \mathrm{epn}(v, D)$ such that $\deg(x) = \deg(v)$.*

Proof. Let G be a connected graph of order $n \geq 2$ and $v \in D \cap S(G)$. Since v is a strong vertex, there exists $x \in N(v)$ such that $\deg(x) = \deg(v)$. By Lemma 2.6, D is an independent set, so $x \in V \setminus D$. Suppose to the contrary that $x \notin \mathrm{epn}(v, D)$, that is, x has another neighbor, say y, in D. If $\deg(y) \geq \deg(x)$, then x, and hence, v and the vertices downhill from v are downhill from y. Thus, $D \setminus \{v\}$ is a DDS of G with cardinality less than $\gamma_{dn}(G)$, a contradiction. Assume then that $\deg(y) < \deg(x)$. But then y is downhill from v, and so $D \setminus \{y\}$ is a DDS with cardinality less than $\gamma_{dn}(G)$. Hence, we conclude that $x \in \mathrm{epn}(v, D)$, and every strong vertex in D has at least one neighbor of the same degree in its private neighborhood.

$\square$

We are now ready to prove one of our main results, Theorem A.

Theorem A. *If G is a graph of order $n \geq 2$, then $\gamma_{dn}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, with equality if and only if G is one of the complete graphs K_2 or K_3, or the complete bipartite graph $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$ of odd order.*

Proof. We first prove the upper bound. Let G be a connected graph of order $n \geq 2$. As noted in the introduction, if G is a regular graph, then $\gamma_{dn}(G) = 1$, and the result holds. Suppose now that G is not a regular graph. By Observation 3.2 and Lemma 3.6, we may choose a $\gamma_{dn}(G)$-set D such that $D \subseteq S(G) \cup VS(G)$. By Observation 3.3, $VS(G) \subseteq D$. To prove the upper bound, it suffices to show that each vertex in D can be uniquely paired with a vertex in $V(G) \setminus D$.

By Proposition 3.10, each vertex of $D \cap S(G)$ has an external private neighbor that is of the same degree. Let S' be the set of these private neighbors. Thus, $S(G)$ can be matched to S'. By Proposition 3.8, there exists a matching from $VS(G)$ to $N(VS(G))$. Further, $S' \cap N(VS(G)) = \emptyset$ and $S' \cup N(VS(G)) \subseteq V \setminus D$. It follows that $\gamma_{dn}(G) \leq |D| = |D \cap S(G)| + |D \cap VS(G)| \leq |S'| + |N(VS(G))| \leq |V \setminus D| = n - \gamma_{dn}(G)$. Hence, $\gamma_{dn}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Next we prove the characterization. Clearly, if $G \in \{K_2, K_3\}$, then $\gamma_{dn}(G) = 1 = \left\lfloor \frac{n}{2} \right\rfloor$, and if $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ with odd order n, then $\gamma_{dn}(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

Now suppose that G is a connected graph of order $n \geq 2$ with $\gamma_{dn}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Since for any r-regular graph, $\gamma_{dn}(G) = 1$, if G is regular, then $n = \{2, 3\}$, implying that $G \in \{K_2, K_3\}$. Note also that $\gamma_{dn}(P_3) = 1$ and $P_3 = K_{1,2}$.

Henceforth, we may assume that G is not a regular graph and that $n \geq 4$. Again, we may assume that G has a $\gamma_{dn}(G)$-set D, such that $D \subseteq S(G) \cup VS(G)$, D is an independent set, and $VS(G) \subseteq D$.

Let $X = VS(G) = \{x_1, x_2, ..., x_j\}$ and $Y = D \cap S(G) = \{y_1, y_2, ..., y_k\}$ for some integers j and k. Then $|D| = |X| + |Y| = j + k$.

If $k \geq 1$, then by Proposition 3.10, every vertex $y_i \in Y$ has a private neighbor $y_i' \in V \setminus D$ such that $\deg(y_i) = \deg(y_i')$. Let $Y' = \{y_i' \,|\, 1 \leq i \leq k\}$. Then $|Y'| = |Y|$ and $Y' \cap N(X) = \emptyset$.

If $X = \emptyset$, then $D = Y$, and $|V \setminus D| \leq |D| + 1 = |Y| + 1 = |Y'| + 1$. Since each $y_i \in Y$ is a strong vertex, $\deg(y_i) \geq 2$. Moreover, since each y_i has exactly one neighbor external private neighbor in Y' and D is an independent set, it follows that each y_i has at least one neighbor in $V \setminus (D \cup Y')$, that is, $V \setminus (D \cup Y') \neq \emptyset$. Hence, $|V \setminus D| \geq |Y'|+1 = |Y|+1 = |D|+1$, and so, $|V \setminus D| = |D|+1$. Then $V \setminus D = Y' \cup \{w\}$ for some vertex w. Since y_i is a strong vertex and $\deg(y_i) = \deg(y_i') \geq 2$, we have $N(y_i) = \{w, y_i'\}$ for $1 \leq i \leq k$. But then $\deg(w) \geq |Y| = |D| = \left\lfloor \frac{n}{2} \right\rfloor \geq 2 \geq \deg(y_i) = 2$, implying that no neighbor of y_i has degree less than $\deg(y_i)$, a contradiction since y_i is a strong vertex. Hence, we may assume that $X \neq \emptyset$, that is, $j \geq 1$.

Thus, we have

$$|Y'| + |N(X)| \leq |V \setminus D| = \left\lceil \frac{n}{2} \right\rceil$$

$$|Y| + |N(X)| \leq \left\lceil \frac{n}{2} \right\rceil$$

$$|Y| + |N(X)| \leq |D| + 1 = |X| + |Y| + 1$$

$$|N(X)| \leq |X| + 1 \tag{6}$$

Since $X \neq \emptyset$, by Proposition 3.9, we have $|N(X)| > |X|$, so $|N(X)| = |X|+1$. By Proposition 3.8, every vertex in X can be matched with a vertex in $N(X)$. Let $X' = N(X) = \{x'_1, x'_2, ..., x'_j\} \cup \{x\}$, such that $\{x_i x'_i \,|\, 1 \leq i \leq j\}$ is a matching from X to $N(X)$. Since D is an independent set and each vertex in Y' has exactly one neighbor in D, the number of edges incident to vertices of D is $m' = \sum_{i=1}^{k} \deg(y_i) + \sum_{i=1}^{j} \deg(x_i) \leq |Y'| + \sum_{i=1}^{j} \deg(x'_i) + \deg(x)$. However, since x_i is very strong, $\deg(x_i) > \deg(x'_i)$ for $1 \leq i \leq j$. Thus, $\sum_{i=1}^{j} \deg(x_i) \geq \sum_{i=1}^{j} \deg(x'_i) + j$. And since y_i is strong, we have that $\deg(y_i) \geq 2$ for $1 \leq i \leq k$. Hence, $2k + \sum_{i=1}^{j} \deg(x'_i) + j \leq \sum_{i=1}^{k} \deg(y_i) + \sum_{i=1}^{j} \deg(x_i) \leq |Y'| + \sum_{i=1}^{j} \deg(x'_i) + \deg(x) = k + \sum_{i=1}^{j} \deg(x'_i) + \deg(x)$. Thus, $\deg(x) \geq j + k = |X| + |Y| = |D| = \left\lfloor \frac{n}{2} \right\rfloor$. Since m' counts only the edges incident to a vertex in D and to a vertex in $V \setminus D$, it follows that x is adjacent to every vertex in D. Since $X \neq \emptyset$ and every vertex $x_i \in X$ is very strong, it follows that $\deg(x_i) > \deg(x) = \left\lfloor \frac{n}{2} \right\rfloor$ for $1 \leq i \leq j$. Since D is independent, we conclude that $N(x_i) = V \setminus D$ for each $x_i \in X$. But $V \setminus D = Y' \cup N(X)$ and $Y' \cap N(X) = \emptyset$, implying that $Y = \emptyset$. It follows that $|D| = |X| = j$ and $|V \setminus D| = |N(X)| = |X| + 1 = j + 1$. Moreover, since $\deg(x'_i) < \deg(x_i)$ for $1 \leq i \leq j$, we have that $\deg(x'_i) < |V \setminus D| = j+1$. On the other hand, every vertex in D is adjacent to every vertex in X', and so $\deg(x'_i) \geq |D| = j$, implying that $\deg(x'_i) = j$ and $N(x'_i) = D$. Thus, $V \setminus D$ is an independent set, and G is the complete bipartite graph $K_{j,j+1}$, as desired.

$\square$

As stated before Figure 21 lists the graphs in the characterization of the bound. Notice that this gives the general bound that for any connected graph G, $\gamma_{dn}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, and equality occurs if and only if the graph one of the types of graphs listed in the figure. Notice that this can be applied to disconnected graphs, by applying this to each maximal connected subgraph of the connected graph.

4 General Bounds on Trees

We have seen, $\left\lfloor \frac{n}{2} \right\rfloor$ as a characterized upper bound on $\gamma_{dn}(G)$ for connected graphs G. We now will restrict our attention to trees as described in Section 1.7.3 in hopes of improving this bound and again characterizing this bound. We achieve this with our second main theorem, Theorem B. We begin by giving the construction of the trees for the characterization, and then present the proof.

4.1 Construction of the Characterization Trees for Theorem B

We will call the subfamily of trees for which the bound given in Theorem B, $\mathcal{T}$. To construct the members of T begin with the graph $K_{1,3}$ as the base graph. To create a new graph in $\mathcal{T}$, take any graph in $\mathcal{T}$, say T. Then take the vertex labeled a in the P_3 in Figure 24 to any leaf vertex in T. Thus any graph in $\mathcal{T}$ can be formed recursively by this process from $K_{1,3}$.

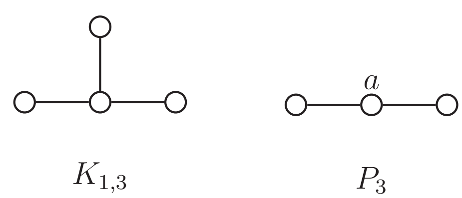


Figure 24: $K_{1,3}$ and P_3

Thus for instance we can form caterpillars such as those in Figure 22 with this process. However, there are other trees in $\mathcal{T}$, which are formed by some bifurcations of the caterpillars.

4.2 Proof of Theorem B

Using the graphs in $\mathcal{T}$ we can obtain the following improvements on the $\lfloor \frac{n}{2} \rfloor$.

Theorem B. *If T is a tree of order $n \geq 4$, then $\gamma_{dn}(T) \leq \lfloor \frac{n-1}{3} \rfloor$, with equality if and only if T is the path of order 4 or $T \in \mathcal{T}$.*

Proof. Let T be a tree of order $n \geq 4$. We first prove the upper bound. Note that if $\Delta(T) = 2$, then T is a path, and since $n \geq 4$, we have that $\gamma_{dn}(P_n) = 1 \leq \lfloor \frac{n-1}{3} \rfloor$. Hence, we may assume that $\Delta(T) \geq 3$.

Assume, for the purpose of a contradiction, that $\gamma_{dn}(T) > \lfloor \frac{n-1}{3} \rfloor$. By Lemma 3.6, T has a γ_{dn}-set D such that $D \subseteq S(T) \cup VS(T)$. To reach a contradiction, we show that T has size $m > n-1$, that is, T has a cycle. Since by Lemma 2.6, D is an independent set, every edge incident to a vertex in D is incident to a vertex in $V \setminus D$. Thus, it suffices to show that each vertex in D has degree at least 3 because this implies that $m \geq 3(\lfloor \frac{n-1}{3} \rfloor + 1) > n - 1$.

Assume to the contrary that there exists a vertex $u \in D$ with $\deg(u) \leq 2$. Then u is either strong or very strong, so $\deg(u) = 2$. Since T is connected and $n \geq 4$, it follows that $u \in S(T)$, and u is adjacent to a leaf and to a vertex, say w, of degree two. If w is downhill from a vertex in D, then so is u and its leaf neighbor, implying that $D \setminus \{u\}$ is a DDS with cardinality less than $\gamma_{dn}(T)$, a contradiction. Thus, w is not downhill from any vertex in $D \setminus \{u\}$. Let $v = w_1, w_2, ..., w_k = w$ be a v-w path for some $v \in D \setminus \{u\}$. Since the v-w path is not a downhill path, there exists a w_i such that $\deg(w_{i+1}) > \deg(w_i)$. Let i be the largest index such that $\deg(w_{i+1}) > \deg(w_i)$ on the v-w path. Since $\deg(w) = 2 < \deg(w_{i+1})$, we have that $w \neq w_{i+1}$ and w is downhill from w_{i+1}. Therefore, $w_{i+1} \notin D$, and so $w_{i+1} \in V \setminus D$. Thus, there exists a downhill path from some vertex $v' \in D$ to w_{i+1}. But then w is downhill from v', a contradiction. Hence, we may conclude that every vertex in D has degree at least 3, proving the upper bound.

Clearly, the bound is sharp for the path P_4 and the claw $K_{1,3}$. Let T be a tree in $\mathcal{T}$ with $n \geq 5$ vertices. By the construction of T, the set of $\frac{n-1}{3}$ vertices of degree 3 in T are very strong vertices of T. By Observation 3.3, we have that every very strong vertex is in every γ_{dn}-set of T, so $\gamma_{dn}(T) \geq \frac{n-1}{3}$. Hence, $\gamma_{dn}(T) = \frac{n-1}{3}$.

Next, let T be a tree with order $n \geq 4$ and $\gamma_{dn}(T) = \frac{n-1}{3}$. Then $n-1$ is divisible by 3. If $n = 4$, then $T \in \{P_4, K_{1,3}\}$, the result holds. Thus, assume that $n \geq 7$. We show that $T = T_k \in \mathcal{T}$.

Let D be a γ_{dn}-set of T. By our previous argument, every vertex in D has degree at least three. Since $|D| = \frac{n-1}{3}$ and D is independent, $3(\frac{n-1}{3}) \leq m = n-1$. It follows every vertex of D has degree 3, and the edges of T are precisely the edges incident to a vertex of D and a vertex of $V \setminus D$. In other words, both D and $V \setminus D$ are independent sets. Note that a pair of vertices in D have at most one common neighbor in $V \setminus D$, else a cycle is formed. To show that $T \in \mathcal{T}$, it suffices to show that every vertex of $V \setminus D$ has degree 1 or 2.

Assume to the contrary, that $u \in V \setminus D$ and $\deg(u) \geq 3$. Without loss of generality, let v_1, v_2, and v_3 be neighbors of u. Necessarily, $v_i \in D$, for $1 \leq i \leq 3$. But then $(D \setminus \{v_1, v_2, v_3\}) \cup \{u\}$ is a DDS of T having cardinality less than $\gamma_{dn}(T)$, a contradiction. Hence, $T \in \mathcal{T}$.

□

The way we proved the initial bound by proving that if $\gamma_{dn}(G) > \left\lfloor \frac{n-1}{3} \right\rfloor$ G must contain some sort of cycle. This proves the following result.

Corollary 4.1. *Let G be a connected graph of order n. If $\gamma_{dn}(G) > \left\lfloor \frac{n-1}{3} \right\rfloor$, then G contains a cycle.*

Note that the converse is not true, that is if $\gamma_{dn}(G) \leq \left\lfloor \frac{n-1}{3} \right\rfloor$, this does not imply that G is acyclic, for example see (a), (d), (e), and (f) in Figure 20. These graphs all have cycles, but have downhill domination numbers clearly less than $\frac{n-1}{3}$.

BIBLIOGRAPHY

[1] Claude Berge. *Theory of Graphs and its Applications*. Merthuen, London, 1962.

[2] F. T. Boesch, S. Chen, and J. A. M. McHugh. On covering the points of a graph with point disjoint paths. *Graphs and Combinatorics*, (406):202–212, 1874.

[3] Béla Bollabás. *Modern Graph Theory*. Springer, New York, 1998.

[4] Fred Buckley and Frank Harary. *Distance in Graphs*. Perseus Books, 1990.

[5] Fred Buckley and Marty Lewinter. *A Friendly Introduction to Graph Theory*. Prentice Hall, New Jersey, 2003.

[6] Gary Chartrand, Linda Lesniak, and Ping Zhang. *Graphs and Digraphs*. CRC Press, Boca Raton, FL, 5th edition, 2011.

[7] Johnathan Gross and Jay Yellen, editors. *Handbook of Graph Theory*. CRC Press, Boca Raton, FL, 2004.

[8] Philip Hall. On representatives of subsets. *Journal of London Mathematics Society*, 10(1):26–30, 1935.

[9] Frank Harary. *Graph Theory*. Addison-Wesley, Reading, MA, 1969.

[10] Teresa Haynes, Stephen Heditniemi, and Peter Slater. *Domination in Graphs, Advanced Topics*. Marcel Dekker, New York, 1998.

[11] Teresa Haynes, Stephen Heditniemi, and Peter Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, New York, 1998.

[12] Jason Hedetniemi, Sandra Hedetniemi, Stephen Hedetniemi, and Thomas Lewis. Classifying graphs by degrees. March 2012.

[13] Jason Hedetniemi, Sandra Hedetniemi, Stephen Hedetniemi, and Thomas Lewis. Classifying trees by degrees. January 2012.

[14] Jason Hedetniemi, Sandra Hedetniemi, Stephen Hedetniemi, and Thomas Lewis. Analyzing graphs by degrees. *AKCE International Journal Graphs and Combinatorics*, to appear.

[15] Oystein Ore. *Theory of Graphs*. American Mathematics Society, 1962.

[16] Peter Slater. Path coverings of the vertices of a tree. *Discrete Mathematics*, 25(1):65–74, 1979.

[17] R. Srikant, R. Sundaran, K. S. Singh, and C. P. Rangan. Optimal path cover problem on block graphs and bipartite permutation graphs. *Theoretical Computer Science*, (115):351–357, 1993.

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	K. Bennett, J. Deering, W. Jamieson, P. Johnson, and A. Purcilly, “On the Chromatic Numbers of Some 3-Uniform Hypergraphs in the Plane.” *Submitted* Feb. 2013.
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	J. Deering, T. Haynes, S. Hedetniemi, and W. Jamieson, “Downhill Domination in Graphs.” *Submitted* Apr. 2013.
	J. Deering, T. Haynes, S. Hedetniemi, and W. Jamieson, “A Polynomial Time Algorithm for Downhill and Uphill Domiation.” *Submitted* Apr. 2013.