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Roots of Quaternionic Polynomials and Automorphisms of Roots

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Roots of Quaternionic Polynomials and Automorphisms of Roots

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ABSTRACT

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The quaternions are an extension of the complex numbers which were first described by Sir William Rowan Hamilton in 1843. In his description, he gave the equation of the multiplication of the imaginary component similar to that of complex numbers. Many mathematicians have studied the zeros of quaternionic polynomials. Prominent of these, Ivan Niven pioneered a root-finding algorithm in 1941, Gentili and Struppa proved the Fundamental Theorem of Algebra (FTA) for quaternions in 2007. This thesis finds the zeros of quaternionic polynomials using the Fundamental Theorem of Algebra. There are isolated zeros and spheres of zeros. In this thesis, we also find the automorphisms of the zeros of the polynomials and the automorphism group.

TABLE OF CONTENTS

1 INTRODUCTION

Quaternions are a mathematical concept that extend the idea of complex numbers. While complex numbers are expressed in terms of a real number and an imaginary number, quaternions have three imaginary components in addition to the real component. Quaternions were first introduced by the Irish mathematician Sir William Rowan Hamilton on 16th of October 1843. He was inspired by the complex numbers and sought to find a way to extend their properties to higher dimensions. Hamilton realized that he could define a set of numbers with three imaginary components by introducing a set of rules for their multiplication. They were discovered as a way to describe three-dimensional rotations in a mathematically elegant way.

Like complex numbers, quaternions consist of a real part and three imaginary parts. They can be written in the form $a + bi + cj + dk$, where a, b, c, and d are real numbers and i, j, and k satisfy $i^2 = j^2 = k^2 = -1$ and $ijk = -1$.

A quaternionic polynomial is a polynomial with coefficients in the quaternions. Quaternions are not commutative with multiplication, therefore quaternionic polynomials are usually separated into left and right polynomials. A left polynomial is an expression of the form $P(x) = a_n x^n + ... + a_1 x + a_0$, $a_i \in \mathbb{H}$, $a_n \neq 0$, $n \geq 1$ and a right polynomial is of the form $P(x) = \sum_{i=0}^{n} x^i a_i$, $a_i \in \mathbb{H}$ [8]. The study of the zeros of quaternionic polynomials is a complex area of mathematics that involves both algebraic and geometric techniques. One of the fundamental tools for studying quaternionic polynomials is the concept of the characteristic polynomial, which is the polynomial obtained by replacing each quaternion variable in the polynomial with a scalar variable.

Ivan Niven pioneered root-finding for a quaternion polynomial by proposing an algorithm [6], Gentili and Struppa proved the Fundamental Theorem of Algebra [3] . By applying the FTA, in [3] it is proven that any quaternionic polynomial has two types of zeros which are either isolated or spherical zeros. Since then, many studies have been conducted for quaternionic polynomial root finding. In 2013, Kalantari gave an algorithm for the root-finding in [5].

This thesis explores recent work in finding the zeros of quaternionic polynomials and also the automorphisms of the zeros. The automorphisms of the zeros of quaternionic polynomials acts on the zeros of the polynomial thereby preserving their algebraic properties. The automorphism group is the group of all bijective functions that map the zeros of the polynomial to themselves and preserve the algebraic relations between the zeros.

The following chapters will explore quaternions and the automorphisms of the zeros of quaternionic polynomials. Chapter 2 will introduce quaternions and its polynomials such as quadratic and cyclotomic polynomials. Some known results will be used in finding the zeros of polynomials. Chapter 3 will extend these concepts to automorphisms of the zeros. Chapter 4 will discuss possible future directions for the study of the automorphisms of roots.

2 BACKGROUND

This chapter introduces the quaternions through the idea of complex numbers. Unless otherwise noted, all material from this chapter section will reference [1]. A *complex number* is a number of the form $z = a + ib$ where a and b are real numbers and $i^2 = -1$. The set of complex numbers $\mathbb{C} = \{a+ib|a, b \in \mathbb{R}\}$, form a 2-dimensional vector space over $\mathbb{R}$. On $\mathbb{R}^2$, the *norm* of a vector is defined by $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$ which means that if $z = a + ib \in \mathbb{C}$, then $|z| = \sqrt{a^2 + b^2}$.

A *unit vector* is a vector with magnitude one. For example, the complex number $\frac{\sqrt{2}}{\sqrt{3}} + i\frac{1}{\sqrt{3}}$ is a unit complex number since $|\frac{\sqrt{2}}{\sqrt{3}} + i\frac{1}{\sqrt{3}}| = \sqrt{\frac{2}{3} + \frac{1}{3}} = 1$.

All complex numbers $z = a + ib$ can be written in the form $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and θ is an argument of z, denoted $\theta = arg(z)$, the angle between the positive real axis and the line joining 0 and z with $z \neq 0$.

If $|z| = r$ and $\theta = arg(z)$, then we have the picture in figure 1.

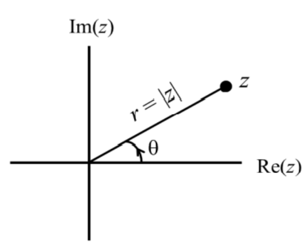

Figure 1: Complex number [1]

Any complex number can be expressed in terms of the norm r and argument θ. For any complex number $z = a + bi$ with $r = \sqrt{a^2 + b^2}$ and $\tan(\theta) = \frac{b}{a}$, then we can write $z = re^{i\theta}$.

With De Moivre's formula, $z^n = r^n(\cos n\theta + i \sin n\theta)$, we can compute roots of complex polynomials. Let $w \in \mathbb{C}$, $w \neq 0$, then we want to find all $z \in \mathbb{C}$ such that $z^n = w$ (for a given $n \in \mathbb{N}$). For such a z, we need $|z| = |w|^{\frac{1}{n}}$ and $arg(z) = \frac{arg(w)}{n}$. Let $\alpha = arg(w)$, then one such z is $z = |w|^{\frac{1}{n}}(\cos(\frac{\alpha}{n}) + i \sin(\frac{\alpha}{n}))$. However there are several choices for α. We find that the z are given by $|w|^{\frac{1}{n}}(\cos(\frac{\alpha+2k\pi}{n}) + i \sin(\frac{\alpha+2k\pi}{n}))$ for $k = 0, 1, 2, ...n-1$.

Example 2.1. *Find the sixth roots of unity*

For $|z|^6 = a = 1$, *we have* $a = |a|(\cos(\alpha) + i \sin(\alpha))$. *So the sixth roots of unity are* $|a|^{\frac{1}{6}}(\cos(\frac{\alpha+2k\pi}{6}) + i \sin(\frac{\alpha+2k\pi}{6}))$ *for* $0 \leq k \leq n-1$, *that is* $\cos(\frac{\pi k}{3}) + i \sin(\frac{\pi k}{3})$ *for* $0 \leq k \leq 5$,

$z_0 = \cos(0) + i \sin(0) = 1,$

$z_1 = \cos(\frac{\pi}{3}) + i \sin(\frac{\pi}{3}) = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1+i\sqrt{3}}{2},$

$z_2 = \cos(\frac{2\pi}{3}) + i \sin(\frac{2\pi}{3}) = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = \frac{-1+i\sqrt{3}}{2},$

$z_3 = \cos(\pi) + i \sin(\pi) = -1 + 0 = -1,$

$z_4 = \cos(\frac{4\pi}{3}) + i \sin(\frac{4\pi}{3}) = \frac{-1}{2} + i\frac{-\sqrt{3}}{2} = \frac{-1-i\sqrt{3}}{2},$

$z_5 = \cos(\frac{5\pi}{3}) + i \sin(\frac{5\pi}{3}) = \frac{1}{2} + i\frac{-\sqrt{3}}{2} = \frac{1-i\sqrt{3}}{2}.$

If a polynomial $P(x)$ *is defined on* $\mathbb{C}$, *then we can define an automorphism* $\varphi : \mathbb{C} \to \mathbb{C}$ *on the roots such that* $\varphi(z^*) = \overline{z^*}$ *where* z^* *is a root of the polynomial* $P(x)$ *in* $\mathbb{C}$ *and* $\overline{z^*}$ *is the conjugate of* z^*.

In the sixth roots of unity, with roots $z_o = 1$, $z_1 = \frac{1+i\sqrt{3}}{2}$, $z_2 = \frac{-1+i\sqrt{3}}{2}$, $z_3 = -1$, $z_4 = \frac{-1-i\sqrt{3}}{2}$, $z_5 = \frac{1-i\sqrt{3}}{2}$, *then* $\varphi(z_o) = z_o$, $\varphi(z_1) = z_5$, $\varphi(z_2) = z_4$, $\varphi(z_3) = z_3$.

2.1 Quaternions

A quaternion is a number of the form $q = a + bi + cj + dk$ such that a, b, c, d are real numbers and i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1$. The set of quaternions $\mathbb{H} = \{q = a + bi + cj + dk \quad | \quad a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$ is an extension of the complex field which was first described by Hamilton in 1843.

The quaternions i, j and k relate with each other satisfying; $ij = k$, $jk = i$ and $ki = j$. Unlike the case of complex numbers, quaternion multiplication is not commutative because $ij = -ji$, $ik = -ki$ and $jk = -kj$. In fact, the quaternions form a division ring which is similar to a field but multiplication is not commutative. and are a real vector space of dimension four with basis $(1, i, j, k)$ [2]. Every quaternion $q = a + bi + cj + dk$ can be written as $q = a + \vec{v}$ where $\vec{v} = bi + cj + dk$, called the *vector part* and a is called the *real part* (sometimes denoted as Re(q)).

Definition 2.2. *A quaternion $q = a+bi+cj+dk$ has a conjugate $\overline{q} = a-bi-cj-dk$. The* trace *of q is $Tr(q) = q + \overline{q} = 2Re(q) = 2a$. The* norm *of q denoted $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$.*

For any quaternion q such that $\overline{q} = -q$, then q is called a **pure quaternion**, this means the trace of $q = 0$.

Example 2.3. *Suppose $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$ are two quaternions, then*

$$\begin{aligned} q_1 + q_2 &= a_1 + b_1 i + c_1 j + d_1 k \quad + \quad a_2 + b_2 i + c_2 j + d_2 k \\ &= a_1 + a_2 + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k \\ &= a_2 + a_1 + (b_2 + b_1)i + (c_2 + c_1)j + (d_2 + d_1)k \end{aligned}$$

$$
\begin{aligned}
&= a_2 + b_2 i + c_2 j + d_2 k + a_1 + b_1 i + c_1 j + d_1 k \\
&= q_2 + q_1.
\end{aligned}
$$

Quaternion addition is commutative.

Example 2.4. *Given a quaternion $q = a + bi + cj + dk$, then the inverse of q denoted q^{-1} such that $qq^{-1} = q^{-1}q = 1$ and can be calculated as follows:*
Let $\overline{q} = a - bi - cj - dk$ and $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$,
then $q^{-1} = \frac{\overline{q}}{|q|^2} = \frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}$. This implies that

$$
\begin{aligned}
qq^{-1} &= (a + bi + cj + dk)\frac{(a - bi - cj - dk)}{a^2 + b^2 + c^2 + d^2} \\
&= \frac{a(a - bi - cj - dk) + bi(a - bi - cj - dk) + cj(a - bi - cj - dk)}{a^2 + b^2 + c^2 + d^2} \\
&+ \frac{dk(a - bi - cj - dk)}{a^2 + b^2 + c^2 + d^2} \\
&= \frac{a^2 - abi - acj - adk + bai + b^2 - bck - bdj + caj + bck + c^2 - cdi}{a^2 + b^2 + c^2 + d^2} \\
&+ \frac{adk + dbj - dci - d^2}{a^2 + b^2 + c^2 + d^2} \\
&= \frac{a^2 + b^2 + c^2 + d^2}{a^2 + b^2 + c^2 + d^2} \\
&= 1.
\end{aligned}
$$

This means that the inverse of a quaternion q is $q^{-1} = \frac{\overline{q}}{|q|^2}$ where $q \neq 0$.

The division of a quaternion q_1 by $q_2 \neq 0$ is specified either as $q_1 q_2^{-1}$ or $q_2^{-1} q_1$, if q_1, q_2 are non zero then $(q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}$. Similarly it can also be show that $\overline{q_1 + q_2} = \overline{q_1} + \overline{q_2}$.

Two quaternions q_1 and q_2 are said to be *congruent*, denoted by $q_2 \sim q_1$, if for some quaternion $w \neq 0$, then we have $q_2 = w q_1 w^{-1}$. The *congruence class* of $q =$

$a + bi + cj + dk$, denoted by $[q]$ is the set

$$[q_2] = \{q_1 \in \mathbb{H} \quad | \quad q_1 \sim q_2\}.$$

On the other hand, any quaternion is congruent to a complex number with the same real part and norm.

Example 2.5. *The complex number* $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ *has many quaternions congruent to it;* $q = \frac{1}{2} + i\frac{1}{2} + j\frac{1}{2} + k\frac{1}{2}$ *is one of the quaternions .*

If q is congruent to a complex number z where $z = a + i\sqrt{b^2 + c^2 + d^2}$, then

$$[q] = \{a + x_2 i + x_3 j + x_4 k | x_2^2 + x_3^2 + x_4^2 = b^2 + c^2 + d^2\}.$$

It follows $[q]$ has a singleton element if and only if q is a real number. If q is not real, its congruent class is the three dimensional sphere in the coordinate space of x_2, x_3, x_4 centered at the point $(a, 0, 0, 0)$ having radius equal to $\sqrt{b^2 + c^2 + d^2}$. [5]

2.2 Polynomials

For quaternions $q = a + bi + cj + dk \in \mathbb{H}$, the characteristic polynomial of q, denoted $P_q(x)$, is $P_q(x) = x^2 - Tr(q)x + (|q|)^2$, where $Tr(q)$ and $|q|$ are the trace and norm of q respectively. The characteristic equation is $P_q(x) = 0$.

Example 2.6. *For the quaternions* $\pm i$, $\pm j$, $\pm k$, *the characteristic polynomial is the polynomial* $x^2 + 1$.

We define a left polynomial over the quaternions as $P(x) = a_n x^n + ... + a_1 x + a_0$, $a_i \in \mathbb{H}$, $a_n \neq 0$, $n \geq 1$. The quaternion conjugate of $P(x)$ is $\overline{P(x)} = \overline{a_n} x^n +$

$... + \overline{a_1}x + \overline{a_0}$. The evaluation of the polynomial $P(x)$ for a given quaternion q is $P(q) = a_n q^n + ... + a_1 q + a_0$.

The multiplication of quaternionic polynomials is defined in terms of the regular product. Let $f(x) = \sum_{i=0}^{n} x^i a_i$ and $g(x) = \sum_{j=0}^{m} x^i b_j$ be two polynomials, then the regular product of f and g is defined as the polynomial $f * g(x) = \sum_{k=0}^{mn} x^k c_k$ where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$ for all k [3].

2.3 Cyclotomic Polynomials

A *cyclotomic polynomial* is a polynomial with coefficients in the integers that is defined in terms of roots of unity. Specifically, the $n^{\mathbf{th}}$ cyclotomic polynomial, denoted by $\Phi_n(x)$, is the polynomial whose roots are all $n^{\mathbf{th}}$ primitive roots of unity , that is, the numbers of the form $e^{\frac{2\pi ik}{n}}$, where k is an integer relatively prime to n.

For any positive integer n, a complex number z is an $n^{\mathbf{th}}$ root of unity if $z^n = 1$. There are n distinct such roots of unity. Applying the De Moivre formula as discussed above, such roots are uniquely determined as $e^{\frac{2\pi ik}{n}}$ for $k = 0, 1, ...n-1$. For some root z, we say z is a primitive $n^{\mathbf{th}}$ root of unity if $z^k \neq 1$ for all $k < n$. This is equivalent to $z = e^{\frac{2\pi ik}{n}}$, with $gcd(k, n) = 1$.

Example 2.7. *The fourth roots of unity are the solutions of* $z^4 - 1 = 0$*, which are;* $1, -1, \pm i$*. Now 1 is a primitive first root of unity, -1 is a primitive second root of unity, and* $\pm i$ *are the primitive fourth roots of unity.*

Example 2.8. *The sixth roots of unity are the solutions of* $z^6 - 1 = 0$ *which are* $z_0 = 1,\ z_1 = e^{\frac{\pi i}{3}} = \frac{1+i\sqrt{3}}{2},\ z_2 = e^{\frac{2\pi i}{3}} = \frac{-1+i\sqrt{3}}{2},\ z_3 = e^{\pi i} = -1,\ z_4 = e^{\frac{4\pi i}{3}} = \frac{-11-i\sqrt{3}}{2}$ *and* $z_5 = e^{\frac{5\pi i}{3}} = \frac{1-i\sqrt{3}}{2}$.

$z_0 = 1$ *is a primitive first root of unity,*

$z_3 = -1$ *is a primitive second root of unity,*

$z_2 = \frac{-1+i\sqrt{3}}{2}$ *and* $z_4 == \frac{-1-i\sqrt{3}}{2}$ *are primitive third roots of unity,*

$z_1 = \frac{1+i\sqrt{3}}{2}$ *and* $z_5 = \frac{1-i\sqrt{3}}{2}$ *are primitive sixth roots of unity.*

Then with the above understanding, we define the $n^{\mathbf{th}}$ cyclotomic polynomial for any positive integer as the unique irreducible polynomial (a polynomial that cannot be expressed as the product of two non-constant polynomials) with integer coefficient that is a divisor of $x^n - 1$ and not a divisor of $x^k - 1$ for any $k < n$. Its roots are all primitive roots of unity $e^{\frac{2i\pi k}{n}}$ where k $\in \mathbb{N}$, $1 \leq k \leq n$, $gcd(k, n) = 1$.

For any positive integer n the $n^{\mathbf{th}}$ cyclotomic polynomial, $\Phi_n(x)$, is given by

$$\Phi_n(x) = (x - z_1)(x - z_2)...(x - z_s),$$

where z_1, z_2, ... , z_s are the primitive $n^{\mathbf{th}}$ roots of unity. The $n^{\mathbf{th}}$ cyclotomic polynomial can be written as

$$\Phi_n(x) = \prod_{\substack{k=1 \\ gcd(k,n)=1}}^{n} (x - e^{\frac{2i\pi k}{n}}).$$

If n is a positive integer, then $\Phi_n(x)$ is monic and its degree is $\phi(n)$, where $\phi(n)$ is the Euler $\phi-$ function, that is $\phi(n)$ is defined as the number of non negative integers less than n that are relatively prime to n.

Example 2.9. *The following are examples of cyclotomic polynomials*

$$\begin{aligned} \Phi_1(x) &= x - 1, \\ \Phi_2(x) &= x + 1, \end{aligned}$$

$$
\begin{aligned}
\Phi_3(x) &= x^2 + x + 1, \\
\Phi_4(x) &= x^2 + 1, \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1, \\
\Phi_6(x) &= x^2 - x + 1, \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \\
\Phi_8(x) &= x^4 + 1, \\
\Phi_9(x) &= x^6 + x^3 + 1 \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1, \\
\Phi_{11}(x) &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.
\end{aligned}
$$

2.4 Roots of Polynomials

A quaternion q is a *root* of $P(x)$ if and only if $P(x) = Q(x) * (x - q)$ for some $Q(x) = \sum_{i=0}^{n-1} q_i x^i$. In line with this, we define various types of roots of quaternionic polynomials [3].

Definition 2.10. *Let $P(x)$ be a quaternionic polynomial with q a root.*

(i) q is a root of multiplicity k if $P(x) = Q(x)*(x-q)^k$, where $Q(q) \neq 0$. In particular, if $k = 1$, we say q is a simple root.

(ii) q is an isolated root if there exists a neighborhood of q that contains no other root of $P(x)$.

(iii) q is a spherical root if $[q]$ is contained in Z_p (the set of zeros of $P(x)$). $[q]$ is called sphere of zeros for $P(x)$.

Theorem 2.11. *Fundamental Theorem of Algebra [3]*

Let $P(x)$ be a quaternionic polynomial of degree n. Then the number of isolated zeros, plus twice the number of spheres of zeros, counted with multiplicity, is n.

In other words, this theorem can be stated as; if $P(x)$ has r real roots, such that $m = deg(P(x)) - r$, then $P(x)$ has $\frac{m}{2}$ spheres of roots.

Example 2.12. *Find the roots of the polynomial $P(x) = x^2 + 1$.*

The above polynomial is the characteristic polynomial of the quaternions $\pm i$, $\pm j$, $\pm k$.

By the above theorem, that if $P(x)$ has r real roots, with $m = deg(P(x)) - r$, then $P(x)$ has $\frac{m}{2}$ spheres of roots.

In this case, this polynomial has zero real roots which means $r = 0$, $deg(P(x)) = 2$, and hence $m = 2$ and so $P(x)$ has one sphere of roots.

The zeros constitute the 3-D unit sphere centered at the origin (0, 0, 0, 0) with zero real part, $\{bi + cj + dk \in \mathbb{H} \quad | \quad b^2 + c^2 + d^2 = 1\}$. Examples of such roots are $\pm i$, $\pm j$, $\pm k$, $-\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$, $\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$ and infinitely many others on the sphere.

We test one of the roots of the polynomial $x^2 + 1$, say when $q = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$.

$$
\begin{aligned}
P(q) = q^2 + 1 &= \left(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k\right)^2 + 1 \\
&= \left(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k\right)\left(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k\right) + 1 \\
&= \frac{1}{3}i^2 + \frac{1}{3}ij + \frac{1}{3}ik + \frac{1}{3}ji + \frac{1}{3}j^2 + \frac{1}{3}jk + \frac{1}{3}ki + \frac{1}{3}kj + \frac{1}{3}k^2 + 1 \\
&= -\frac{1}{3} + \frac{1}{3}k - \frac{1}{3}j - \frac{1}{3}k - \frac{1}{3} + \frac{1}{3}i + \frac{1}{3}j - \frac{1}{3}i - \frac{1}{3} + 1 \\
&= -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} + 1 \\
&= 0.
\end{aligned}
$$

Hence $q = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k \in Z_p$ (zeroes of the polynomial $P(x) = x^2 + 1$).

Example 2.13. *Find the roots of the polynomial $P(x) = x^2 - c$ with c a positive real.*

P(x) has two real roots; $\sqrt{c}$ and $-\sqrt{c}$. Therefore, r = 2 which means m = deg(P(x)) - r = 2 - 2 = 0.

$P(x) = x^2 - c$ has $\frac{0}{2}$ (zero) spheres of roots.

The only roots of $P(x) = x^2 - c$ with c is positive are $\sqrt{c}$ and $-\sqrt{c}$.

Example 2.14. *Find the roots of the polynomial $P(x) = x^2 + f$ such that f is a positive real.*

The above polynomial is the characteristic polynomial of the quaternions $\pm\sqrt{f}i$, $\pm\sqrt{f}j$, $\pm\sqrt{f}k$.

By the FTA for $\mathbb{H}$, if P(x) has r real roots, with m = deg(P(x)) - r, P(x) has $\frac{m}{2}$ spheres of roots.

In this case it can be seen that the polynomial has no real roots, therefore $r = 0$, $deg(P(x)) = 2$ and $m = 2$ which means the polynomial has one sphere of zeros.

The zeros constitute the radius $\sqrt{f}$ 3-sphere centered at the origin (0, 0, 0, 0) with zero real part, such that $\{bi + cj + dk \in \mathbb{H} \quad | \quad b^2 + c^2 + d^2 = f\}$.

Examples of such roots are $\pm\sqrt{f}i$, $\pm\sqrt{f}j$, $\pm\sqrt{f}k$, $-\frac{\sqrt{f}}{\sqrt{2}}i + \frac{\sqrt{f}}{\sqrt{2}}j$, $\frac{\sqrt{f}}{\sqrt{3}}i + \frac{\sqrt{f}}{\sqrt{3}}j + \frac{\sqrt{f}}{\sqrt{3}}k$ and infinitely many others on the sphere.

Proposition 2.15. *If q and a distinct conjugate q' are both roots of P(x), then so is any element of the conjugacy class $[q]$. In particular, $P(x) = Q(x) * P_q(x)$ for some quaternion polynomial $Q(x)$.*

Proof. For any quaternion q, the characteristic polynomial $P_q(x)$ is defined as $P_q(x) = x^2 - Tr(q)x + |q|^2$. Since q is a root of the polynomial, then it means $P_q(x)$ is a factor of $P(x)$. Given also that the distinct conjugate is a root of $P(x)$ and by conjugacy, $q' \in [q]$ then the conjugacy class $[q]$ is a subset of the zeros of $P(x)$.

Suppose $P_q(x)$ is not a factor of $P(x)$. Then applying Niven's division [6] it follows that both q and $q^{'}$ are solutions of the corresponding equation $fx + g = 0$, with $f, g \neq 0$. But there is a unique solution which is a contradiction. $\square$

Example 2.16. *Find the roots of the polynomial $P(x) = x^3 - 1$*

The polynomial can be expressed as $P(x) = x^3 - 1 = (x^2 + x + 1) * (x - 1)$. Using FTA, it can be seen that 1 is an isolated root of $P(x)$ with deg(P(x)) = 3. Hence $m = 3 - 1 = 2$ which means that $P(x)$ has $\frac{2}{2} = 1$ sphere of zeroes.

Solving for the root of $x^2 + x + 1$ using the quadratic formula, $\frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$ is a complex root of $x^2 + x + 1$.

Every quaternion is congruent to a complex number with the same real part, then $q \in \{\frac{-1}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{3}{4}\}$ is a root of the polynomial.

The sphere of zeros is centered at $(\frac{-1}{2}, 0, 0, 0)$ and is given by $\{\frac{-1}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{3}{4}\}$.

Some examples of roots of $P(x) = x^3 - 1$ are $\frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$, $\frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$, $\frac{-1}{2} \pm k\frac{\sqrt{3}}{2}$, $-\frac{1}{2} + i\frac{\sqrt{3}}{\sqrt{8}} + j\frac{\sqrt{3}}{\sqrt{8}}$ and infinitely many others on the sphere.

Example 2.17. *Find the quaternion roots of the cyclotomic polynomial $\Phi_{12}(x) = x^4 - x^2 + 1$.*

$\Phi_{12}(x)$ has no isolated roots, hence by the FTA, $m = \phi_{12} = 4$. This means that $\Phi_{12}(x)$ has two spheres of zeros.

The complex primitive roots of $\Phi_{12}(x)$ are $\{e^{\frac{2\pi i h}{12}} \quad | \quad 1 \leq h \leq 12, \quad gcd(h, 12) = 1\}$. Note, $\phi(12) = [1, 5, 7, 11]$. So the roots are

$e^{\frac{2\pi i}{12}} = e^{\frac{\pi i}{6}} = \cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + \frac{i}{2}$,

$e^{\frac{10\pi i}{12}} = e^{\frac{5\pi i}{6}} = \cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$,

$e^{\frac{14\pi i}{12}} = e^{\frac{7\pi i}{6}} = \cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} - \frac{i}{2}$,

$e^{\frac{22\pi i}{12}} = e^{\frac{11\pi i}{6}} = \cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{i}{2}$.

Applying conjugacy of complex number and quaternions, we can easily generate the roots of $\Phi_{12}(x)$.

From the complex roots $\frac{\sqrt{3}}{2} + \frac{i}{2}$ *and* $\frac{\sqrt{3}}{2} - \frac{i}{2}$, *the quaternions on the sphere centered at* $(\frac{\sqrt{3}}{2}, 0, 0, 0)$ *where* $q \in \{\frac{\sqrt{3}}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{1}{4}\}$ *are quaternion roots of* $\Phi_{12}(x)$. *Examples are* $\frac{\sqrt{3}}{2} + \frac{i}{2}$, $\frac{\sqrt{3}}{2} + \frac{j}{2}$, $\frac{\sqrt{3}}{2} + \frac{k}{2}$, $\frac{\sqrt{3}}{2} + \frac{i}{\sqrt{8}} + \frac{j}{\sqrt{8}}$, $\frac{\sqrt{3}}{2} + \frac{i}{\sqrt{12}} + \frac{j}{\sqrt{12}} + \frac{k}{\sqrt{12}}$

Similarly, from the complex roots $-\frac{\sqrt{3}}{2} + \frac{i}{2}$ *and* $-\frac{\sqrt{3}}{2} - \frac{i}{2}$, *the quaternions on the sphere centered at* $(-\frac{\sqrt{3}}{2}, 0, 0, 0)$ *where* $q \in \{-\frac{\sqrt{3}}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{1}{4}\}$ *are quaternion roots of* $\Phi_{12}(x)$. *Examples are* $-\frac{\sqrt{3}}{2} + \frac{i}{2}$, $-\frac{\sqrt{3}}{2} + \frac{j}{2}$, $-\frac{\sqrt{3}}{2} + \frac{k}{2}$, $-\frac{\sqrt{3}}{2} + \frac{i}{\sqrt{8}} + \frac{j}{\sqrt{8}}$, $-\frac{\sqrt{3}}{2} + \frac{i}{\sqrt{12}} + \frac{j}{\sqrt{12}} + \frac{k}{\sqrt{12}}$.

Example 2.18. *Find the quaternion roots of the cyclotomic polynomial*

$$\Phi_{13}(x) = x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x^1 + 1.$$

$\Phi_{13}(x)$ *has no isolated roots, hence by FTA,* $m = \phi(13) = 12$. *This means that* $\Phi_{13}(x)$ *has six spheres of zeros.*

The complex primitive roots of $\Phi_{13}(x)$ *are* $\{e^{\frac{2\pi ih}{13}} \quad | \quad 1 \leq h \leq 13, \quad gcd(h, 13) = 1\}$,

So, the complex roots are, $e^{\frac{2\pi i}{13}} = \cos(\frac{2\pi}{13}) + i\sin(\frac{2\pi}{13})$, $e^{\frac{4\pi i}{13}} = \cos(\frac{4\pi}{13}) + i\sin(\frac{4\pi}{13})$

$e^{\frac{6\pi i}{13}} = \cos(\frac{6\pi}{13}) + i\sin(\frac{6\pi}{13})$, $e^{\frac{8\pi i}{13}} = \cos(\frac{8\pi}{13}) + i\sin(\frac{8\pi}{13})$, $e^{\frac{10\pi i}{13}} = \cos(\frac{10\pi}{13}) + i\sin(\frac{10\pi}{13})$,

$e^{\frac{12\pi i}{13}} = \cos(\frac{12\pi}{13}) + i\sin(\frac{12\pi}{13})$, $e^{\frac{14\pi i}{13}} = \cos(\frac{14\pi}{13}) + i\sin(\frac{14\pi}{13})$, $e^{\frac{16\pi i}{13}} = \cos(\frac{16\pi}{13}) + i\sin(\frac{16\pi}{13})$

$e^{\frac{18\pi i}{13}} = \cos(\frac{18\pi}{13}) + i\sin(\frac{18\pi}{13})$, $e^{\frac{20\pi i}{13}} = \cos(\frac{20\pi}{13}) + i\sin(\frac{20\pi}{13})$, $e^{\frac{22\pi i}{13}} = \cos(\frac{22\pi}{13}) + i\sin(\frac{22\pi}{13})$

$e^{\frac{24\pi i}{13}} = \cos(\frac{24\pi}{13}) + i\sin(\frac{24\pi}{13})$.

Applying conjugacy of complex number and quaternions, we can easily generate each sphere of zeros.

From the complex roots $e^{\frac{2\pi i}{13}}$ *and* $e^{\frac{24\pi i}{13}}$, *the quaternions on the sphere centered at* $(\cos(\frac{2\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{2\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{2\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $e^{\frac{4\pi i}{13}}$ *and* $e^{\frac{22\pi i}{13}}$, *the quaternions on the sphere centered at* $(\cos(\frac{4\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{4\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{4\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $e^{\frac{6\pi i}{13}}$ *and* $e^{\frac{20\pi i}{13}}$, *the quaternions on the sphere centered at* $(\cos(\frac{6\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{6\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{6\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $e^{\frac{8\pi i}{13}}$ *and* $e^{\frac{18\pi i}{13}}$, *the quaternions on the sphere centered at* $(\cos(\frac{8\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{8\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{8\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $e^{\frac{10\pi i}{13}}$ *and* $e^{\frac{16\pi i}{13}}$, *the quaternions on the sphere centered at* $(\cos(\frac{10\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{10\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{10\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $e^{\frac{12\pi i}{13}}$ *and* $e^{\frac{14\pi i}{13}}$, *the quaternions on the sphere centered at* $(\cos(\frac{12\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{12\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{12\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

3 AUTOMORPHISM OF ROOTS

If G is a nonempty set, then a binary operation on G is a function from $G \times G$ to G. If the binary operation is denoted $*$, then we use the notation $a * b = c$ if $(a, b) \in G \times G$ is mapped to $c \in G$ under the binary operation. A *semigroup* is a nonempty set G with an associative binary operation. A *monoid* is a semigroup with an identity. A *group* is a monoid such that each $a \in G$ has an inverse $a^{-1} \in G$. A semigroup G is abelian or commutative if $a * b = b * a$ for all $a, b \in G$.

Let G and H be semigroups. A function $f : G \longrightarrow H$ is a homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in$ G. A one to one (injective) homomorphism is a monomorphism. An onto (surjective) homomorphism is an epimorphism. A one to one and onto (bijective) homomorphism is an isomorphism. If there is an isomorphism from G to H, we say that G and H are isomorphic, denoted $G \cong H$. A homomorphism $f : G \longrightarrow G$ is an endomorphism of G. An isomorphism $f : G \longrightarrow G$ is an automorphism of G.[4]

Example 3.1. *If a polynomial $P(x)$ is defined on $\mathbb{C}$, then we can define an automorphism $\varphi : \mathbb{C} \to \mathbb{C}$ on the roots such that $\varphi(z^*) = \overline{z^*}$ where z^* is a root of the polynomial $P(x)$ in $\mathbb{C}$ and $\overline{z^*}$ is the conjugate of z^*.*

In the sixth roots of unity, with roots $z_0 = 1$, $z_1 = \frac{1+i\sqrt{3}}{2}$, $z_2 = \frac{-1+i\sqrt{3}}{2}$, $z_3 = -1$, $z_4 = \frac{-1-i\sqrt{3}}{2}$, $z_5 = \frac{1-i\sqrt{3}}{2}$, then

$$\varphi(z_o) = z_o,$$

$$\varphi(z_1) = z_5,$$

$$\varphi(z_2) = z_4,$$

$$\varphi(z_3) = z_3.$$

3.1 Orthogonal/Special Orthogonal groups

The plane $\mathbb{R}^2$ can be identified with the complex plane $\mathbb{C}$ where $z = a + ib \in \mathbb{C}$ is the same as $(x, y) \in \mathbb{R}^2$. Every rotation in the two dimensional plane ρ by angle θ can be represented by multiplication with the complex numbers $e^{i\theta} = \cos\theta + \sin\theta$. If we let $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

Then $a1 + bi + cj + dk$ in matrix form is of the form A $= \begin{bmatrix} x & y \\ -\overline{y} & \overline{x} \end{bmatrix}$ where $x = a + ib$ and $y = c + id$, $a, b, c, d \in \mathbb{R}$. An orthogonal matrix is a square matrix with real entries whose columns and rows are orthonormal vectors, meaning that they are unit vectors (vectors of length 1) and are mutually perpendicular to each other. An orthogonal matrix has the property that its transpose is also its inverse. More formally, let A be an $n \times n$ matrix, then A is an orthogonal matrix if $AA^T = A^TA = I$, where I is the $n \times n$ identity matrix. The set of orthogonal matrices denoted $O(n) = \{A \in GL_n(\mathbb{R}) : A^{-1} = A^T\}$ forms the orthogonal group of $n \times n$ matrices with matrix multiplication as the group operation.

Theorem 3.2. *The set O(n) is a group under matrix multiplication.*

Proof. It is obvious that I (the identity) $\in O(n)$. By definition $A^T = A^{-1}$, then each $A \in O(n)$ has an inverse and since $A^T \in O(n)$, then $A^{-1} \in O(n)$. Matrix multiplication is associative, then O(n) is associative under matrix multiplication. Let $A, B \in O(n)$, consider $(AB)(AB)^T = ABB^TA^T = AIA^T = AA^T = I$.

This means that $O(n)$ is closed under matrix multiplication. $\square$

Similarly, the orthogonal group can be thought of as a linear symmetry distance-preserving map of a Euclidean space of dimension n that preserves a fixed point with the group operation being composition of transformations. We define an isometry f of $\mathbb{R}^n$ as a function $f : \mathbb{R}^n \to \mathbb{R}^n$ which for any vector $x, y \in \mathbb{R}^n$ we have $|f(x) - f(y)| = |x - y|$, i.e., f preserves the distance between two points in $\mathbb{R}^n$. Any isometry function $f : \mathbb{R}^n \to \mathbb{R}^n$ that fixes the origin preserves the length of all vectors in $\mathbb{R}^n$.

Lemma 3.3. *[9] Let A be an element of O(n). The transformation associated with A preserves dot products .*

Proof. For any vector $x, y \in \mathbb{R}^n$, the dot product of x and y becomes $x \cdot y = x^T y$. For any transformation $A \in O(n)$, then $(Ax) \cdot (Ay) = (Ax)^T (Ay) = x^T A^T A y = x^T y = x \cdot y$ □

Lemma 3.4. *[9] Suppose the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry that moves the origin. Then the function $g : \mathbb{R}^n \to \mathbb{R}^n$ given by g(x) = f(x) - f(0) is a linear isometry.*

Proof. For any two vectors $x, y \in \mathbb{R}^n$,

$|g(x) - g(y)| = |f(x) - f(0) - f(y) + f(0)| = |f(x) - f(y)| = |x - y|$. Then g is an isometry on $\mathbb{R}^n$. Similarly, $g(0) = f(0) - f(0) = 0$ and g fixes the origin. □

Lemma 3.5. *[9] If $A \in O(n)$ then A is a linear isometry.*

Proof. Let A be an element of $O(n)$. Since A preserves dot products, this means it must also preserve lengths in $\mathbb{R}^n$, since the length of a vector $v \in \mathbb{R}^n$ may be defined as $|v| = \sqrt{v \cdot v}$ Furthermore, it is clear that the origin is fixed since $A0 = 0$. Thus, A is a linear isometry. □

For all $n \times n$ matrices A and B, we have that $det(A) = det(A^T)$ and $det(AB) = det(A)det(B)$. This implies that for all $A \in O(n)$,

$$det(A^2) = det(A)det(A) = det(A)det(A^T) = det(AA^T) = det(I) = 1.$$

This implies all orthogonal matrices must have determinant of ± 1. By this, the orthogonal group of dimension n has the orthogonal matrices of determinant 1 and the orthogonal matrices of determinant -1.

The orthogonal matrices of determinant 1 forms the special orthogonal group, denoted $SO(n)$. This is also called the rotation group.

Theorem 3.6. *The subset $SO(n) = \{A \in O(n) : det(A) = 1\}$ is a subgroup of O(n).*

Proof. The identity $I \in SO(n)$ and all $A \in SO(n)$ has an inverse by definition. Matrix multiplication is associative, hence $SO(n)$ is associative under matrix multiplication. For closure, let $A, B \in SO(n)$, then it means $\det(A) = 1 = \det(B)$. Consider $\det(AB) = \det(A)\det(B) = 1$, hence $AB \in SO(n)$. $SO(n)$ is closed under matrix multiplication. Thus, $SO(n)$ is a subgroup of $O(n)$. $\square$

Theorem 3.7. *The group SO(n) is a normal subgroup of O(n).*
In addition, O(n)/SO(n) $\cong \mathbb{Z}_2$.

Proof. The set $\{-1, 1\}$ is a group under multiplication. We define $f : O(n) \to \{-1, 1\}$ by $f(A) = \det(A)$ for all $A \in O(n)$.
Let A, $B \in O(n)$, $f(AB) = det(AB) = det(A)det(B) = f(A)f(B)$. Thus, f is a homomorphism. Clearly f is an epimorphism with kernel of f equals all $A \in$ O(n) such that $\det(A) = 1$. Thus, $\mathrm{Ker}(f) = SO(n)$.

Since every kernel of a homomorphism f is a normal subgroup of $O(n)$, then by the first isomorphism theorem, there is an isomorphism from $O(n)/SO(n)$ with the image on f. Hence $O(n)/SO(n) \cong \mathbb{Z}_2$. $\square$

The orthogonal matrices of determinant -1 do not form a subgroup of $O(n)$. A quick check is by picking matrices A and B. It means $\det(A)$ = -1 and $\det(B)$ = -1. Consider $\det(AB) = \det(A)\det(B)$ = -1 $\times$ -1 = 1. Hence AB does not belong to the second component and the component is not closed under matrix multiplication.

3.2 Properties of SO(n)

The orthonormal basis $\{\langle 1,0\rangle, \langle 0,1\rangle\}$ for $\mathbb{R}^2$ can be used to define rotation in the plane. The group of rotations in the plane is $SO(2)$. The rotation by angle θ transfers the basis to $\rho_\theta\langle 1,0\rangle = \langle \cos\theta, \sin\theta\rangle$ and $\rho_\theta\langle 0,1\rangle = \langle -sin\theta, cos\theta\rangle$ with matrix representation $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. It can be seen that $\det(A) = +1$ and that

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The vector $\langle \cos\theta, \sin\theta\rangle$ parametrizes the unit circle centered at the origin and any element of $SO(2)$. In the unit circle two vectors $\langle -\sin\theta, \cos\theta\rangle$ and $\langle \sin\theta, -\cos\theta\rangle$ are orthogonal to $\langle \cos\theta, \sin\theta\rangle$. It is obvious that

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \notin SO(2) \quad since \quad |B| = -(\cos^2\theta + \sin^2\theta) = -1.$$

We now show $SO(2)$ is an abelian group. Let

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

and consider

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\beta - \sin\theta\sin\beta & -\cos\theta\sin\beta - \sin\theta\cos\beta \\ \sin\theta\cos\beta + \cos\theta\sin\beta & -\sin\theta\sin\beta + \cos\theta\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\beta) & -\sin(\theta+\beta) \\ \sin(\theta+\beta) & \cos(\theta+\beta) \end{bmatrix}.$$

Similarly,

$$BA = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\beta) & -\sin(\theta+\beta) \\ \sin(\theta+\beta) & \cos(\theta+\beta) \end{bmatrix}.$$

Thus, $SO(2)$ is abelian. We can also check if $SO(3)$ is abelian.

Consider the matrices $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

It can be shown that $AA^T = I$ and $BB^T = I$ and $\det(A) = 1 = \det(B)$. This means that $A, B \in SO(3)$. Consider

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta\cos\theta & \sin^2\theta \\ \sin\theta & \cos^2\theta & -\sin\theta\cos\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, BA = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta\cos\theta & \cos^2\theta & -\sin\theta \\ \sin^2\theta & \sin\theta\cos\theta & \cos\theta. \end{bmatrix}.$$

Thus $SO(3)$ is not abelian and in turn implies that $SO(4)$ is not since it contains $SO(3)$ as a subgroup.

Lemma 3.8. *[9] The one-sphere is an abelian group under complex multiplication.*

Proof. $S^{'} = \{x \in \mathbb{C} : |x| = 1\}$.

The set of non zero complex numbers forms a group under multiplication. To show that $S^{'}$ is a group, then we can show that it is a subgroup of $\mathbb{C} \setminus \{0\}$.

Note $1 \in S^{'}$. Similarly, for all $x \in S^{'}$ it means that $xx^{-1} = 1$, then $|xx^{-1}| = |1| = 1$ and $|x| = 1$, $|x^{-1}| = 1$. So every element of $S^{'}$ has an inverse in $S^{'}$. $S^{'}$ is associative since $\mathbb{C}$ is associative.

Let $x, y \in S^{'}$, then $|xy| = |x||y| = |1||1| = 1$. Hence, $S^{'}$ is closed under multiplication.

$\square$

Theorem 3.9. *The group SO(2) is isomorphic to S^1.*

Proof. Complex numbers can be written in the form $e^{i\theta} = \cos\theta + i\sin\theta$ and every rotation in $SO(2)$ can be written as $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where θ is the angle of rotation and $\theta \in [0, 2\pi)$.

We define a mapping $f : S^{'} \to SO(2)$ by $f(e^{i\theta}) = A(\theta)$. Then

$$f(e^{i\theta})f(e^{i\beta}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\beta - \sin\theta\sin\beta & -(\cos\theta\sin\beta + \sin\theta\cos\beta) \\ \sin\theta\cos\beta + \cos\theta\sin\beta & \cos\theta\cos\beta - \sin\theta\sin\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\beta) & -\sin(\theta+\beta) \\ \sin(\theta+\beta) & \cos(\theta+\beta) \end{bmatrix}.$$

Similarly,

$$\begin{aligned} f(e^{i\theta})f(e^{i\beta}) &= f(e^{i(\theta+\beta)}) \\ &= \begin{bmatrix} \cos(\theta+\beta) & -\sin(\theta+\beta) \\ \sin(\theta+\beta) & \cos(\theta+\beta) \end{bmatrix} \\ &= f(e^{i\theta})f(e^{i\beta}). \end{aligned}$$

Thus f is a homomorphism. Similarly, f is surjective and since θ is uniquely determined, then f is injective which means f is an isomorphism. $\square$

3.3 Quaternions and the Special Orthogonal group

Given any two quaternions, $X = a + bi + cj + dk$ and $Y = w + xi + yj + zk$, the maps $Y \to XY$ and $X \to XY$ are linear maps. By quaternion multiplication, the map $Y \to XY$ becomes

$$\begin{aligned} XY &= (a + bi + cj + dk)(w + xi + yj + zk) \\ &= a(w + xi + yj + zk) + bi(w + xi + yj + zk) + cj(w + xi + yj + zk) \\ &+ dk(w + xi + yj + zk) \\ &= aw + axi + ayj + azk + bwi + bxi^2 + byij + bzik + cwj + cxji + cyj^2 \\ &+ czjk + dwk + dxki + dykj + dzk^2 \\ &= aw - bx - cy - dz + (ax + bw + cz - dy)i + (ay + cw - bz + dx)j \\ &+ (az + dw + by - cx)k. \end{aligned}$$

From the above, we can write XY in matrix form as $XY = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

We say $XY = A_X Y$ where $A_X = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$.

Similarly the map $X \to XY$ can be done the same way as above and we have

$XY = \begin{bmatrix} w & -x & -y & -z \\ x & w & z & -y \\ y & -z & w & x \\ z & y & -x & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, such that $XY = B_Y X$ where

$B_Y = \begin{bmatrix} w & -x & -y & -z \\ x & w & z & -y \\ y & -z & w & x \\ z & y & -x & w \end{bmatrix}$.

Note; Let $A_X A_X^T = L_A$

$$\begin{aligned} L_A &= \begin{bmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2+c^2+d^2 & 0 & 0 \\ 0 & 0 & a^2+b^2+c^2+d^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2+d^2 \end{bmatrix} \\ &= (|X|)^2 I, \end{aligned}$$

and $det(A_X^2) = det(A_X)det(A_X^T) = det(A_X A_X^T) = det((|X|^2)I) = (|X|^2)^4 det(I) = |X|^8$. This means that $det(A_x) = |X|^4$.

If X is a unit quaternion, then A_X is an orthogonal matrix and hence A_X is a rotation matrix. Similarly, $B_Y B_Y^T = (|Y|)^2 I$ and Y a unit quaternion, means B_Y is an orthogonal matrix and a rotation matrix.

The following lemmas and proofs adopted from [2] gives the rotation and reflection maps for quaternions.

Lemma 3.10. *For every quaternion $Z \neq 0$, the map $\rho_Z : X \to ZXZ^{-1}$ (where $X \in \mathbb{H}$) is a rotation in $SO(\mathbb{H}) = SO(4)$ whose restriction to the space $\mathbb{H}_p$ of pure quaternions is a rotation in $SO(\mathbb{H}_p) = SO(3)$. Conversely, every rotation in SO(3) is of the form $\rho_Z : X \to ZXZ^{-1}$ for some quaternion $Z \neq 0$ and for all $X \in \mathbb{H}_p$. Furthermore, if two nonnull quaternions Z and Z' represent the same rotation, then $Z' = \lambda Z$ for some $\lambda \neq 0$ in $\mathbb{R}$.*

Proof. For any non-zero quaternions X, the maps $Y \to XY$ and $Y \to YX$ are linear maps and when $|X| = 1$, the maps are in $SO(4)$. The map $\rho_{Y,Z} : \mathbb{H} \to \mathbb{H}$ where for $X \in \mathbb{H}$ is defined as $\rho_{Y,Z}(X) = YXZ$. The map $\rho_{Y,Z}$ is an isometry and $\rho_{Y,Z} = \rho_{Y,1} \circ \rho_{1,Z}$.

$\rho_{Y,1}$ is the map $X \to YX$ and $\rho_{1,Z}$ is the $X :\to XZ$, which are both rotations. The composition of rotations is a rotation, hence, $\rho_{Y,Z} = \rho_{Y,1}$ o $\rho_{1,Z}$ is a rotation in $SO(4)$. If $Z = Y^{-1}$, then the map $\rho_{Y,Y^{-1}}$ is denoted by ρ_Y. It can be observed that for any $X, Y \in \mathbb{H}$, $\rho_z(X + Y) = \rho_Z(X) + \rho_Z(Y)$ and also $\rho_Z(\overline{X}) = \overline{\rho_Z(X)}$.

Then we have $\rho_Z(X + \overline{X}) = \rho_Z(X) + \rho_Z(\overline{X}) = \rho_Z(X) + \overline{\rho_Z(X)}$. If $X \in \mathbb{H}_p$, then $X + \overline{X} = 0$, then $\rho_Z(X) + \rho_Z(\overline{X}) = 0$ which means $\rho_Z(X)$ is a pure quaternion. Thus, $\rho_z \in SO(3)$.

Every rotation that is not the identity is the composition of an even number of reflections. Then we want to show that for every reflection σ of $\mathbb{H}_p$, about a hyperplane, there is some pure quaternion $Z \neq 0$ such that $\sigma(X) = -ZXZ^{-1}$ for all $X \in \mathbb{H}_p$. If

Z is a pure quaternion orthogonal to the plane, we know that

$$\sigma(X) = X - 2\frac{(X \cdot Z)}{(Z \cdot Z)}Z$$

for all $X \in \mathbb{H}_p$. However, for pure quaternions $Y, Z \in \mathbb{H}_p$, we have

$$2(Y \cdot Z) = -(YZ + ZY).$$

Then $(Z \cdot Z) = -Z^2$, and we have

$$\sigma(X) = X - 2\frac{(X \cdot Z)}{(Z \cdot Z)}Z = X + 2(X \cdot Z)Z^{-1},$$

$$\sigma(x) = X - (XZ + ZX)Z^{-1} = -ZXZ^{-1},$$

for all $X \in \mathbb{H}_p$.

If $\sigma(Z_1) = \sigma(Z_2)$, then

$$Z_1 X Z_1^{-1} = Z_2 X Z_2^{-1}$$

for all $X \in \mathbb{H}$, which is equivalent to

$$Z_2^{-1} Z_1 X = X Z_2^{-1} Z_1,$$

where $Z_2^{-1} Z_1 = a1$ for some $a \in \mathbb{R}$ and since Z_1 and Z_2 are non zero, we get $Z_2 = (\frac{1}{a})Z_1$ where $a \neq 0$. □

Lemma 3.11. *For every quaternion $Z = a + t$ where t is a nonnull pure quaternion, the axis of the rotation ρ_z associated with Z is determined by the vector in $\mathbb{R}^3$ corresponding to t, and the angle of rotation θ is equal to π when $a = 0$, or when $a \neq 0$, given a suitable orientation of the plane orthogonal to the axis of the rotation, the angle is given by*

$$\tan\frac{\theta}{2} = \frac{\sqrt{N(t)}}{|a|},$$

with $0 < \theta \leq \pi$.

Proof. A simple calculation shows that the line of direction t is invariant under the rotation ρ_Z and thus the axis of rotation. For any two non zero vectors $X, Y \in \mathbb{R}^3$ such that $|X| = |Y|$, there is some rotation ρ such that $\rho(X) = Y$. If $X = Y$, then the identity will work and if $X \neq Y$, we use the rotation of axis determined by $X \times Y$ rotating X to Y in the plane containing X and Y. Thus given any two nonnull pure quaternions X, Y such that $|X| = |Y|$, there is some nonnull quaternions W such that $Y = WXW^{-1}$.

For any non zero quaternions Z, W, the angle of rotation ρ_z is the same as the angle of the rotation $\rho_{WXW^{-1}}$. $Z = a + t$ where t is a pure nonnull quaternion, the axis of rotation $\rho_{WXW^{-1}}$ is $WtW^{-1} = \rho_w t$. WtW^{-1} is pure, and

$$WZW^{-1} = W(a1 + t)W^{-1} = Wa1W^{-1} + WtW^{-1} = a1 + WtW^{-1}.$$

Also given any pure non zero quaternion X orthogonal to t, the angle of the rotation Z is the angle between X and $\rho_z(X)$. Since rotations preserve orientation (since they preserve the cross product), the angle θ between two vectors X and Y is preserved under rotation. Since rotations preserve the inner product, if $Xt = 0$, we have $\rho_W(X) \cdot \rho_W(t) = 0$ and the angle of rotation $\rho_{WXW^{-1}} = \rho_W \circ \rho_Z \circ (\rho_W)^{-1}$ is the angle between the two vectors $\rho_w(X)$ and $\rho_{WXW^{-1}}(\rho_W(X))$. Since

$$\rho_{WXW^{-1}}(\rho_W(X)) = (\rho_W \circ \rho_Z \circ (\rho_W)^{-1} \circ \rho_W)(X) = (\rho_W \circ \rho_Z)(X) = \rho_W(\rho_Z(X)),$$

the angle of rotation of $\rho_{WXW^{-1}}$ is the angle between the two vectors $\rho_W(X)$ and $\rho_W(\rho_Z(X))$. Since the rotation preserves angles, this is also the angle between the two vectors X and $\rho_Z(X)$ which is the angle of the rotation ρ_Z.

Thus, given any quaternion $Z = a + t$, where t is a non zero pure quaternion, since there is some non zero quaternion W such that $WtW^{-1} = |t|i$ and $WZW^{-1} = a1+|t|i$, we can figure out the angle of rotation for a quaternion Z of the form $a+bi$ (a rotation of axis i). We can find the angle between j and $\rho_Z(j)$

$$\rho_Z(j) = (a+bi)j(a+bi)^{-1},$$

we get

$$\rho_Z(j) = \frac{1}{a^2+b^2}(a+bi)j(a-bi) = \frac{a^2-b^2}{a^2+b^2}j + \frac{2ab}{a^2+b^2}k.$$

Then if $a \neq 0$, we must have

$$tan\theta = \frac{2ab}{a^2-b^2} = \frac{2(b/a)}{1-(b/a)^2},$$

and since

$$tan\theta = \frac{2tan(\frac{\theta}{2})}{1-tan^2(\frac{\theta}{2})},$$

with a suitable orientation we have

$$tan\frac{\theta}{2} = \frac{b}{|a|} = \frac{|t|}{|a|}.$$

If $a = 0$, we get

$$\rho_Z(j) = -j,$$

and $\theta = \pi$. $\square$

We define the map $\rho : \mathbb{H} \to \mathbb{H}$ by $\rho_Z(X) = ZXZ^{-1}$, for $Z, X \in \mathbb{H}$ where $|Z| = 1$. Then let $Z = a+bi+cj+dk$ and $X = w+xi+yj+zk$. Since Z is a unit quaternion, then $Z^{-1} = \overline{Z} = a-bi-cj-dk$.

Evaluating $\rho_Z(X) = ZXZ^{-1} = ZX\overline{Z}$. Then

$$ZX \quad = \quad (a+bi+cj+dk)(w+xi+yj+zk)$$

$$
\begin{aligned}
&= (aw - bx - cy - dz) + (ax + bw + cz - dy)i + (ay + cw - bz + dx)j \\
&+ (az + dw + by - cx)k
\end{aligned}
$$

$$
= \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}.
$$

Let $\alpha = aw - bx - cy - dz, \qquad \beta = ax + bw + cz - dy,$

$\gamma = ay + cw - bz + dx, \qquad \sigma = az + dw + by - cx,$

$ZX = \alpha + \beta i + \gamma j + \sigma k$. Then,

$$
\begin{aligned}
ZX\overline{Z} &= (\alpha + \beta i + \gamma j + \sigma k)(a - bi - cj - dk) \\
&= (\alpha a + \beta b + \gamma c + \sigma d) + (-\alpha b + \beta a - \gamma d + \sigma c)i + (-\alpha c + \beta d + \gamma a - \sigma b)j \\
&+ (-\alpha d - \beta c + \gamma b + \sigma a)k
\end{aligned}
$$

$$
= \begin{bmatrix} \alpha & \beta & \gamma & \sigma \\ \beta & -\alpha & \sigma & -\gamma \\ \gamma & -\sigma & -\alpha & \beta \\ \sigma & \gamma & -\beta & -\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.
$$

With substitution and simplification we have

$$
ZX\overline{Z} = L_Z \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}, \text{ where}
$$

$$
L_Z = \begin{bmatrix} a^2 + b^2 + c^2 + d^2 & 0 & 0 & 0 \\ 0 & a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2ac + 2cd \\ 0 & 2ad + 2bc & a^2 - b^2 + c^2 - d^2 & -2ab + 2cd \\ 0 & -2ac + 2bd & 2ab + 2cd & a^2 - b^2 - c^2 + d^2 \end{bmatrix}
$$

This can be seen as the composition of transformations since

$$ZX = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \text{ and } X\overline{Z} = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}.$$

which implies that

$$\begin{aligned} \rho_Z(X) &= ZX\overline{Z} \\ &= \rho_{Z,1}(X) o \rho_{1,\overline{Z}}(X) \\ &= \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \\ &= L_Z \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}, where \end{aligned}$$

$$L_Z = \begin{bmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2cd \\ 0 & 2ad+2bc & a^2-b^2+c^2-d^2 & -2ab+2cd \\ 0 & -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2 \end{bmatrix}.$$

By definition

$$\begin{aligned} \rho_Z(XY) &= Z(XY)\overline{Z} \\ &= Z(X\overline{Z}ZY)\overline{Z} \\ &= ZX\overline{Z}ZY\overline{Z} \\ &= \rho_Z(X)\rho_Z(Y). \end{aligned}$$

Thus, ρ_Z is a bijective homomorphism. Similarly,

$$\begin{aligned}
\rho_Z(X+Y) &= L_Z \begin{bmatrix} w+p \\ x+q \\ y+u \\ z+v \end{bmatrix} \\
&= L_Z \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} + L_Z \begin{bmatrix} p \\ q \\ u \\ v \end{bmatrix} \\
&= \rho_z(X) + \rho_z(Y),
\end{aligned}$$

and

$$\begin{aligned}
\rho_z(\lambda x) &= L_Z \begin{bmatrix} \lambda w \\ \lambda x \\ \lambda y \\ \lambda z \end{bmatrix} \\
&= \lambda \left(L_Z \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \right) \\
&= \lambda \rho_z(X).
\end{aligned}$$

From the above results, ρ_z is a linear tranformation [2].

3.4 Automorphisms of Cyclotomic and Quadratic Polynomial roots

At this point, we can define the different automorphisms on the sphere of zeros of a quaternionic polynomial. Let Δ be a sphere of zeros of a quaternionic polynomial;

then we define the following automorphisms:

$\iota : \Delta \to \Delta$ such that $\iota(a) = a$ for all $a \in \Delta$ (Identity),

$\tau : \Delta \to \Delta$ such that $\tau(a) = \overline{a}$ for all $a \in \Delta$ (Conjugation),

$\sigma_z : \Delta \to \Delta$ such that $\sigma_z(a) = -zaz^{-1}$ for all $a \in \Delta$, $z \in \mathbb{H}_p$ (Reflection),

$\rho_z : \Delta \to \Delta$ such that $\rho_z(a) = zaz^{-1}$ for all $a \in \Delta$, $z \in \mathbb{H}$ (Rotation).

We look at general and concrete examples of quaternionic polynomials in quadratic and cyclotomic forms.

For the quadratic polynomial $P(x) = ax^2 + bx + c$ with discriminant $D = b^2 - 4ac$;

- If $D = 0$, the quadratic polynomial has one real root.
- If $D > 0$, the quadratic polynomial has two real (without any sphere of roots).
- If $D < 0$, then the quadratic polynomial has no real roots. By the FTA, the polynomial has $\frac{m}{2}$ spheres of zeros where $m = deg(P(x)) - r$.

In order to examine the automorphisms of the roots of the quadratic polynomial, we consider the case where $D < 0$.

Example 3.12. *Consider the general form of quadratic polynomial, $P(x) = ax^2 + bx + c$ with discriminant $D = b^2 - 4ac < 0$.*

The solution of the polynomial becomes $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$. The polynomial has no real roots, it means it has one sphere of zeros centered at $(\frac{-b}{2a}, 0, 0, 0)$. The quaternion roots of the polynomial on the sphere satisfies $q \in \{\frac{-b}{2a} + ui + vj + wk \quad | \quad u^2 + v^2 + w^2 = \frac{b^2-4ac}{4a^2}\}$.

Example 3.13. *Consider the quadratic polynomial $P(x) = x^2 + 1$*

Solution: The above polynomial is the characteristic polynomial of the quaternions

$\pm i$, $\pm j$, $\pm k$.

In this case, this polynomial has zero real roots which means $r = 0$, $deg(P(x)) = 2$, and hence $m = 2$.

The zeros constitute the unit sphere in the space b, c, d, centered at the origin (0, 0, 0, 0): $\{bi + cj + dk \in \mathbb{H} \quad | \quad b^2 + c^2 + d^2 = 1\}$. Example of such roots are $\pm i$, $\pm j$, $\pm k$, $-\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$, $\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$ and infinitely many others on the sphere.

We can define the automorphisms on this sphere of zeros. $\iota : \Delta \to \Delta$ such that $\iota(a) = a$ for all $a \in \Delta$ (Identity),

$\tau : \Delta \to \Delta$ such that $\tau(a) = \overline{a}$ for all $a \in \Delta$ (Conjugation),

For reflection, let $z = i$ and $a = xi + yj$.

Then $\sigma_z(xi+yj) = -i(xi+yj)(-i) = -xi+yj$. If $a = i\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}$, then $\sigma_i(i\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}) = -i(i\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})(-i) = -i\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$.

Rotating such zeros through the same axis space $z = i$ ($\theta = \pi$, z is pure quaternion), we have

$\rho_i(i\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}) = i(i\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})(-i) = i\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$.

Example 3.14. *Consider the roots of the polynomial $P(x) = x^2 + x + 1$.*

Solution: The discriminant $D = b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = -2$, $D < 0$. The polynomial has no real roots and hence one sphere of zeros.

Solving for the roots of $x^2 + x + 1$ using the quadratic formula, $\frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$ is a complex root of $x^2 + x + 1$.

If a quaternion is congruent to a complex number with the same real part, then $q \in \{\frac{-1}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{3}{4}\}$ is a root of the polynomial.

The sphere of zeros is centered at $(\frac{-1}{2}, 0, 0, 0)$ such that $\{\frac{-1}{2} + bi + cj + dk \quad | \quad b^2 +$

$c^2+d^2=\frac{3}{4}\}$.

Hence the roots of $P(x)=x^2+x+1$ *are on the sphere of zeros such as* $\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}$, $\frac{-1}{2}\pm j\frac{\sqrt{3}}{2}$, $\frac{-1}{2}\pm k\frac{\sqrt{3}}{2}$, $-\frac{1}{2}+i\sqrt{\frac{3}{8}}+j\sqrt{\frac{3}{8}}$ *and infinitely many others on the sphere.*

We define the first automorphism on the roots as the conjugate of the roots.

Let Δ *be the set of roots of the polynomial* $P(x)=x^2+x+1$ *on the sphere, we define the automorphism* $\tau:\Delta\to\Delta$ *as* $\tau(a)=\overline{a}$.

It can be verified that the conjugate of each root is on the sphere and also a root of the polynomial,

$-\frac{1}{2}+i\sqrt{\frac{3}{8}}+j\sqrt{\frac{3}{8}}$ *is a root; then* $\tau(-\frac{1}{2}+i\sqrt{\frac{3}{8}}+j\sqrt{\frac{3}{8}})=-\frac{1}{2}-i\sqrt{\frac{3}{8}}-j\sqrt{\frac{3}{8}}$ *is on the sphere and also a root. Confirming this we have;*

$$\begin{aligned}
x^2+x+1 &= (-\frac{1}{2}-i\frac{\sqrt{3}}{\sqrt{8}}-j\frac{\sqrt{3}}{\sqrt{8}})^2+(-\frac{1}{2}-i\frac{\sqrt{3}}{\sqrt{8}}-j\frac{\sqrt{3}}{\sqrt{8}})+1\\
\\
&= (\frac{1}{4}+i\frac{\sqrt{3}}{2\sqrt{8}}+j\frac{\sqrt{3}}{2\sqrt{8}}+i\frac{\sqrt{3}}{2\sqrt{8}}-\frac{3}{8}+k\frac{3}{8}+j\frac{\sqrt{3}}{2\sqrt{8}}-k\frac{3}{8}-\frac{3}{8})\\
&+ (-\frac{1}{2}-i\frac{\sqrt{3}}{\sqrt{8}}-j\frac{\sqrt{3}}{\sqrt{8}})+1\\
\\
&= (-\frac{1}{2}-i\frac{\sqrt{3}}{\sqrt{8}}-j\frac{\sqrt{3}}{\sqrt{8}})+(-\frac{1}{2}-i\frac{\sqrt{3}}{\sqrt{8}}-j\frac{\sqrt{3}}{\sqrt{8}})+1\\
&= 0.
\end{aligned}$$

We can define a rotation of the roots of the quadratic polynomial $P(x)=x^2+x+1$. *First we need to choose the axis of rotation* z *which is represented by a unit pure quaternion and angle of rotation* θ. *For this example, let* $z=\frac{\sqrt{3}}{2}+i\frac{1}{4}+j\frac{\sqrt{3}}{4}$ *be the axis of rotation through which we want to rotate the roots. From the previous lemma, we can determine the angle of rotation which is* $tan\frac{\theta}{2}=\frac{|t|}{|a|}$ *where* $t=i\frac{1}{4}+j\frac{\sqrt{3}}{4}$ *and*

$a = \frac{\sqrt{3}}{2}$. *Then* $\theta = \frac{\pi}{3}$.

As define above, $\rho_z(X) = ZXZ^{-1}$*, we rotate the root* $x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$*, then*

$$\begin{aligned}
ZXZ^{-1} &= \left(\frac{\sqrt{3}}{2} + i\frac{1}{4} + j\frac{\sqrt{3}}{4}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2} - i\frac{1}{4} - j\frac{\sqrt{3}}{4}\right) \\
&= \left(-\frac{3\sqrt{3}}{8} + i\frac{5}{8} - j\frac{\sqrt{3}}{8} - k\frac{3}{8}\right)\left(\frac{\sqrt{3}}{2} - i\frac{1}{4} - j\frac{\sqrt{3}}{4}\right) \\
&= -\frac{16}{32} + i\frac{10\sqrt{3}}{32} + j\frac{6}{32} - k\frac{12\sqrt{3}}{32}.
\end{aligned}$$

A quick check shows that $(\frac{10\sqrt{3}}{32})^2 + (\frac{6}{32})^2 + (\frac{12\sqrt{3}}{32})^2 = \frac{300}{1024} + \frac{36}{1024} + \frac{432}{1024} = \frac{768}{1024} = \frac{3}{4}$.

We can confirm that $-\frac{16}{32} + i\frac{10\sqrt{3}}{32} + j\frac{6}{32} - k\frac{12\sqrt{3}}{32}$ *is on the sphere and also a root of the polynomial:*

$$\begin{aligned}
x^2 + x + 1 = &\left(-\frac{16}{32} + i\frac{10\sqrt{3}}{32} + j\frac{6}{32} - k\frac{12\sqrt{3}}{32}\right)^2 \\
&+ \left(-\frac{16}{32} + i\frac{10\sqrt{3}}{32} + j\frac{6}{32} - k\frac{12\sqrt{3}}{32}\right) + 1 \\
&= \left(\frac{256}{1024} - i\frac{160\sqrt{3}}{1024} - j\frac{96}{1024} + k\frac{192\sqrt{3}}{1024} - i\frac{160\sqrt{3}}{1024} - \frac{300}{1024} + k\frac{60\sqrt{3}}{1024}\right. \\
&\quad + j\frac{360}{1024} - j\frac{96}{1024} - k\frac{60\sqrt{3}}{1024} - \frac{36}{1024} - i\frac{72\sqrt{3}}{1024} + k\frac{192\sqrt{3}}{1024} - j\frac{360}{1024} \\
&\quad \left. + i\frac{72\sqrt{3}}{1024} - \frac{432}{1024}\right) + \left(-\frac{16}{32} + i\frac{10\sqrt{3}}{32} + j\frac{6}{32} - k\frac{12\sqrt{3}}{32}\right) + 1 \\
&= \left(-\frac{512}{1024} - i\frac{320\sqrt{3}}{1024} - j\frac{192}{1024} + k\frac{384}{1024}\right) \\
&+ \left(-\frac{16}{32} + i\frac{10\sqrt{3}}{32} + j\frac{6}{32} - k\frac{12\sqrt{3}}{32}\right) + 1
\end{aligned}$$

$= 0.$

Now we consider the cyclotomic polynomials. The cyclotomic polynomials have no real roots, hence using the congruence of complex numbers and quaternions we know that the number of spheres of zeros is $\frac{|\phi(n)|}{2}$. On each sphere of zeros, we can define automorphisms of roots in terms of conjugation and rotation.

Example 3.15. *For the cyclotomic polynomial $\Phi_{12}(x) = x^4 - x^2 + 1$.*

We apply the FTA, $\Phi_{12}(x)$ has no isolated roots, hence $m = \phi_{12} = 4$. This means that $\Phi_{12}(x)$ has two spheres of zeros.

The complex primitive roots of $\Phi_{12}(x) = \{e^{\frac{2\pi ih}{12}} \quad | \quad 1 \leq h \leq 12, \quad gcd(h, 12) = 1\}$, $\phi_{12} = [1, 5, 7, 11]$ which are (for h = 1, 5, 7, 11)

$e^{\frac{2\pi i}{12}} = e^{\frac{\pi i}{6}} = \cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + \frac{i}{2}$,

$e^{\frac{10\pi i}{12}} = e^{\frac{5\pi i}{6}} = \cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$,

$e^{\frac{14\pi i}{12}} = e^{\frac{7\pi i}{6}} = \cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} - \frac{i}{2}$,

$e^{\frac{22\pi i}{12}} = e^{\frac{11\pi i}{6}} = \cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{i}{2}$.

Applying conjugacy of complex numbers and quaternions, we can easily generate the quaternion roots of $\Phi_{12}(x)$.

From the complex roots $\frac{\sqrt{3}}{2} + \frac{i}{2}$ and $\frac{\sqrt{3}}{2} - \frac{i}{2}$, the quaternions on the sphere centered at $(\frac{\sqrt{3}}{2}, 0, 0, 0)$ where $q \in \{\frac{\sqrt{3}}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{1}{4}\}$ are quaternion roots of $\Phi_{12}(x)$. Such as $\frac{\sqrt{3}}{2} + \frac{i}{2}$, $\frac{\sqrt{3}}{2} + \frac{j}{2}$, $\frac{\sqrt{3}}{2} + \frac{k}{2}$, $\frac{\sqrt{3}}{2} + \frac{i}{\sqrt{8}} + \frac{j}{\sqrt{8}}$, $\frac{\sqrt{3}}{2} + \frac{i}{\sqrt{12}} + \frac{j}{\sqrt{12}} + \frac{k}{\sqrt{12}}$

Similarly, from the complex roots $-\frac{\sqrt{3}}{2} + \frac{i}{2}$ and $-\frac{\sqrt{3}}{2} - \frac{i}{2}$, the quaternions on the sphere centered at $(-\frac{\sqrt{3}}{2}, 0, 0, 0)$ where $q \in \{-\frac{\sqrt{3}}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{1}{4}\}$ are quaternion roots of $\Phi_{12}(x)$. Such as $-\frac{\sqrt{3}}{2} + \frac{i}{2}$, $-\frac{\sqrt{3}}{2} + \frac{j}{2}$, $-\frac{\sqrt{3}}{2} + \frac{k}{2}$, $-\frac{\sqrt{3}}{2} + \frac{i}{\sqrt{8}} + \frac{j}{\sqrt{8}}$,

$-\frac{\sqrt{3}}{2} + \frac{i}{\sqrt{12}} + \frac{j}{\sqrt{12}} + \frac{k}{\sqrt{12}}$.

For each of the spheres, the conjugate of each root is also a root in the sphere. As defined above, we can also rotate the roots in each sphere to get another root in the sphere.

We choose the axis of rotation z which is represented by a unit pure quaternion and angle of rotation θ. For this example, let $Z = \frac{1}{2} + k\frac{\sqrt{3}}{2}$ be the axis of rotation through which we want to rotate the roots from the sphere centered at $(-\frac{\sqrt{3}}{2}, 0, 0, 0)$ where $q \in \{-\frac{\sqrt{3}}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{1}{4}\}$. From the previous lemma, we can determine the angle of rotation which is $tan\frac{\theta}{2} = \frac{|t|}{|a|}$ where $t = k\frac{\sqrt{3}}{2}$ and $a = \frac{1}{2}$. Then $\theta = \frac{2\pi}{3}$.

As define above, $\rho_z(X) = ZXZ^{-1}$, we rotate the root $x = -\frac{\sqrt{3}}{2} + \frac{i}{2}$, then

$$\begin{aligned} ZXZ^{-1} &= \left(\frac{1}{2} + k\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\left(\frac{1}{2} - k\frac{\sqrt{3}}{2}\right) \\ &= \left(-\frac{\sqrt{3}}{4} + \frac{i}{4} - k\frac{3}{4} + j\frac{\sqrt{3}}{4}\right)\left(\frac{1}{2} - k\frac{\sqrt{3}}{2}\right) \\ &= -\frac{\sqrt{3}}{2} - \frac{i}{4} + j\frac{\sqrt{3}}{4}. \end{aligned}$$

Which is a root on the sphere. We can also check that $\Phi_{12}(-\frac{\sqrt{3}}{2} - \frac{i}{4} + j\frac{\sqrt{3}}{4}) = 0$

Example 3.16. *For the cyclotomic polynomial $\Phi_{13}(x) = x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x^1 + 1$.*

Solution:

Applying the FTA, $\Phi_{13}(x)$ has no isolated roots, hence $m = \phi(13) = 12$. This means that $\Phi_{13}(x)$ has six spheres of zeros.

The complex primitive roots of $\Phi_{13}(x) = \{e^{\frac{2\pi ih}{13}} \quad | \quad 1 \leq h \leq 13, \quad gcd(h, 13) = 1\}$,

$\phi_{13} = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$ *which are (for h = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)*

$e^{\frac{2\pi i}{13}} = \cos(\frac{2\pi}{13}) + i\sin(\frac{2\pi}{13})$, $e^{\frac{4\pi i}{13}} = \cos(\frac{4\pi}{13}) + i\sin(\frac{4\pi}{13})$, $e^{\frac{6\pi i}{13}} = \cos(\frac{6\pi}{13}) + i\sin(\frac{6\pi}{13})$,

$e^{\frac{8\pi i}{13}} = \cos(\frac{8\pi}{13}) + i\sin(\frac{8\pi}{13})$, $e^{\frac{10\pi i}{13}} = \cos(\frac{10\pi}{13}) + i\sin(\frac{10\pi}{13})$, $e^{\frac{12\pi i}{13}} = \cos(\frac{12\pi}{13}) + i\sin(\frac{12\pi}{13})$,

$e^{\frac{14\pi i}{13}} = \cos(\frac{14\pi}{13}) + i\sin(\frac{14\pi}{13})$, $e^{\frac{16\pi i}{13}} = \cos(\frac{16\pi}{13}) + i\sin(\frac{16\pi}{13})$, $e^{\frac{18\pi i}{13}} = \cos(\frac{18\pi}{13}) + i\sin(\frac{18\pi}{13})$,

$e^{\frac{20\pi i}{13}} = \cos(\frac{20\pi}{13}) + i\sin(\frac{20\pi}{13})$, $e^{\frac{22\pi i}{13}} = \cos(\frac{22\pi}{13}) + i\sin(\frac{22\pi}{13})$, $e^{\frac{24\pi i}{13}} = \cos(\frac{24\pi}{13}) + i\sin(\frac{24\pi}{13})$.

Applying conjugacy of complex numbers and quaternions, we can easily generate the quaternion roots of $\Phi_{13}(x)$ *for each sphere of roots.*

From the complex roots $\cos(\frac{2\pi}{13}) + i\sin(\frac{2\pi}{13})$ *and* $\cos(\frac{24\pi}{13}) + i\sin(\frac{24\pi}{13})$, *the quaternions on the sphere centered at* $(\cos(\frac{2\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{2\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{2\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $\cos(\frac{4\pi}{13}) + i\sin(\frac{4\pi}{13})$ *and* $\cos(\frac{22\pi}{13}) + i\sin(\frac{22\pi}{13})$, *the quaternions on the sphere centered at* $(\cos(\frac{4\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{4\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{4\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $\cos(\frac{6\pi}{13}) + i\sin(\frac{6\pi}{13})$ *and* $\cos(\frac{20\pi}{13}) + i\sin(\frac{20\pi}{13})$, *the quaternions on the sphere centered at* $(\cos(\frac{6\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{6\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{6\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $\cos(\frac{8\pi}{13}) + i\sin(\frac{8\pi}{13})$ *and* $\cos(\frac{18\pi}{13}) + i\sin(\frac{18\pi}{13})$, *the quaternions on the sphere centered at* $(\cos(\frac{8\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{8\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{8\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots cos$(\frac{10\pi}{13}) + i\sin(\frac{10\pi}{13})$ *and* $\cos(\frac{16\pi}{13}) + i\sin(\frac{16\pi}{13})$, *the quaternions on the sphere centered at* $(\cos(\frac{10\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{10\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{10\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$.

From the complex roots $\cos(\frac{12\pi}{13}) + i\sin(\frac{12\pi}{13})$ *and* $\cos(\frac{14\pi}{13}) + i\sin(\frac{14\pi}{13})$*, the quaternions on the sphere centered at* $(\cos(\frac{12\pi}{13}), 0, 0, 0)$ *where* $q \in \{\cos(\frac{12\pi}{13}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (\sin(\frac{12\pi}{13}))^2\}$ *are quaternion roots of* $\Phi_{13}(x)$*.*
Similarly, for each of the spheres, the the conjugate of each root is also a root in the sphere. As defined above, we can also rotate the roots in each sphere to get another root in the sphere.

Also, we can define a mapping from one sphere of zeros to another sphere of zeros. Let Υ be the set of spheres of zeros of a quaternionic polynomial. Suppose there are m spheres of zeros; $S_1, S_2, \ldots S_m$. We can define the mapping $\varphi : \Upsilon \to \Upsilon$ by $\varphi(S_i) = S_i \times w_n^k = S_j$, where $S_i, S_j \in \Upsilon$, $0 \leq k \leq m-1$, w_n is an $n^{\mathbf{th}}$ root of unity. When $k = 0$, the sphere of zeros is mapped to itself.

Example 3.17. *Consider the cyclotomic polynomial* $\Phi_{12}(x) = x^4 - x^2 + 1 =$
$\Phi_{12}(x)$ *has two spheres of zeros,* S_1 *centered at* $(\frac{\sqrt{3}}{2}, 0, 0, 0)$ *with the zeros* $q \in \{\frac{\sqrt{3}}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{1}{4}\}$ *and* S_2 *centered at* $(-\frac{\sqrt{3}}{2}, 0, 0, 0)$ *with the zeros* $q \in \{-\frac{\sqrt{3}}{2} + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = \frac{1}{4}\}$*.*
With appropriate k *and* w_n*, when* $k = 1$*,* $w_n = e^{\pi i} = -1$*, then* $\varphi(S_1) = S_1 \times -1 = S_2$*.*
Similarly $\varphi(S_2) = S_2 \times -1 = S_1$

Example 3.18. *Consider the cyclotomic polynomial* $\Phi_{20}(x) = x^8 - x^6 + x^4 - x^2 + 1$
We determine the zeros of the polynomial using the primitive roots approach.
$\phi(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$*. The primitive roots of* $\Phi_{20}(x)$ *are* $e^{\frac{\pi i}{10}}$*,* $e^{\frac{3\pi i}{10}}$*,* $e^{\frac{7\pi i}{10}}$*,* $e^{\frac{9\pi i}{10}}$*,* $e^{\frac{11\pi i}{10}}$*,* $e^{\frac{13\pi i}{10}}$*,* $e^{\frac{17\pi i}{10}}$ *and* $e^{\frac{19\pi i}{10}}$*. This polynomial has four spheres of zeros;* S_1 *determined by* $e^{\frac{\pi i}{10}}$ *and* $e^{\frac{19\pi i}{10}}$ *centered at* $(cos(\frac{\pi}{10}), 0, 0, 0)$ *with* $q \in \{cos(\frac{\pi}{10}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (sin\frac{\pi}{10})^2\}$*,*

S_2 *determined by* $e^{\frac{3\pi i}{10}}$ *and* $e^{\frac{17\pi i}{10}}$ *centered at* $(cos(\frac{3\pi}{10}), 0, 0, 0)$ *with* $q \in \{cos(\frac{3\pi}{10}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (sin\frac{3\pi}{10})^2\}$,

S_3 *determined by* $e^{\frac{7\pi i}{10}}$ *and* $e^{\frac{13\pi i}{10}}$ *centered at* $(cos(\frac{7\pi}{10}), 0, 0, 0)$ *with* $q \in \{cos(\frac{7\pi}{10}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (sin\frac{7\pi}{10})^2\}$,

S_4 *determined by* $e^{\frac{9\pi i}{10}}$ *and* $e^{\frac{11\pi i}{10}}$ *centered at* $(cos(\frac{9\pi}{10}), 0, 0, 0)$ *with* $q \in \{cos(\frac{9\pi}{10}) + bi + cj + dk \quad | \quad b^2 + c^2 + d^2 = (sin\frac{9\pi}{10})^2\}$,

For $k = 1$ *and* $w_n = e^{\frac{2\pi i}{10}}$, $\varphi(S_1) = S_1 \times e^{\frac{2\pi i}{10}} = S_2$. *For* $k = 2$, $w_n = e^{\frac{4\pi i}{10}}$, $\varphi(S_1) = S_1 \times (e^{\frac{4\pi i}{10}})^2 = S_4$.

4 FUTURE WORK

Automorphisms of roots of quaternionic polynomials are an important topic of research in the field of algebra and its applications. There is much more research that can be done in the area of root automorphisms of quaternionic polynomials.

One possibility for future work would be to prove the structure of automorphisms of roots of quaternionic polynomials. This involves studying the properties and characteristics of automorphisms of roots of quaternionic polynomials which could lead to a better understanding of the structure of these automorphisms and their behavior.

Also one can explore the relationship between automorphisms of roots of quaternionic polynomials and the geometry of quaternions. The geometry of quaternions is a rich and interesting topic, and understanding the relationship between automorphisms of roots of quaternionic polynomials and quaternionic geometry could lead to new insights.

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