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Enestr¨om-Kakeya Type Results for Complex and Quaternionic Polynomials

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ABSTRACT

Eneström-Kakeya Type Results for Complex and Quaternionic Polynomials

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The well known Eneström-Kakeya Theorem states that: for $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$, a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, all zeros of $P(z)$ lie in $|z| \leq 1$ in the complex plane. In this thesis, we will find inner and outer bounds in which the zeros of complex and quaternionic polynomials lie. We will do this by imposing restrictions on the real and imaginary parts, and on the moduli, of the complex and quaternionic coefficients. We also apply similar restrictions on complex polynomials with complex coefficients to give a bound on the number of zeros in a disk centered at the origin. For each result, we will consider lacunary polynomials, that is polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$, as well as a new class of polynomials $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$.

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1 INTRODUCTION

Polynomials are intricately associated with many fields of study in mathematics and other STEM fields. Many real world applications, as well as many theoretical concepts, are studied through the use of polynomials. Recently there is larger interest in the study of the complex plane, but we are unable to find the exact locations of zeros of most complex polynomials. However, by imposing restrictions we are able to approximate their solutions.

1.1 Locations of Zeros

In 1816, Gauss attempted to prove the Fundamental Theorem of Algebra and in the process proved [13]:

Theorem 1.1. *If $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ is a polynomial of degree n with real coefficients, then all zeros of p lie in*

$$|z| \leq R = \max_{1 \leq k \leq n} \left\{ (n\sqrt{2}|a_k|)^{1/k} \right\}.$$

Later Cauchy [4, 22] took Gauss's result and improved it by showing:

Theorem 1.2. *If $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is polynomial of degree n with complex coefficients, then all zeros of p lie in*

$$|z| \leq 1 + \max_{0 \leq \ell \leq n-1} \left| \frac{a_\ell}{a_n} \right|.$$

Notice there are no restrictions in Gauss's or in Cauchy's theorems. A more useful form of Theorem 1.2 was proved by Eneström who showed [9]:

Theorem 1.3. *Let* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$*,* $n \geq 1$*, be a polynomial of degree* n *with positive real coefficients* $a_\ell > 0$ *for all* $0 \leq \ell \leq n$*. If*

$$\alpha = \min_{0 \leq \ell < n} \left\{ \frac{a_\ell}{a_{\ell+1}} \right\}, \qquad \beta = \max_{0 \leq \ell < n} \left\{ \frac{a_\ell}{a_{\ell+1}} \right\}$$

then all zeros of $p(z)$ *are contained in the annulus* $\alpha \leq |z| \leq \beta$*.*

Eneström first gave a result while attempting to solve a problem in the theory of pension funds. In 1893 he published in the journal *Översigt af Vetenskaps-Akademiens Förhandlingar*. In 1912, Kakeya published a result in the *Tôhoku Mathematical Journal*. By applying a monotonicity condition to the coefficients of complex polynomials, Eneström, and Kakeya, were independently able to prove [9, 20]:

Theorem 1.4 (Eneström-Kakeya)**.** *Let* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *be a polynomial of degree* n *whose coefficients satisfy* $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$*. Then* $P(z)$ *has all its zeros in the closed unit disk* $|z| \leq 1$*.*

In this thesis we will relax the conditions of Theorem 1.4. To do this we will first look at Gardner and Govil's [18] proof of Theorem 1.4 which we will use as a base for the new conditions we will apply. We will start by multiplying the polynomial by $(1 - z)$. In this thesis, we will use this technique in order to get a difference of coefficients which allows us to use the monotonicity feature as we see here:

Proof of Theorem 1.4. Define f by the equation

$$\begin{aligned} P(z)(1-z) &= a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1} \\ &= f(z) - a_n z^{n+1} \end{aligned}$$

Then for $|z| = 1$, we have

$$\begin{aligned} |f(z)| &\leq |a_0| + |a_1 - a_0| + |a_2 - a_1| + \cdots + |a_n - a_{n-1}| \\ &= a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) \\ &= a_n \end{aligned}$$

Notice that the function $z^n f\left(\frac{1}{z}\right) = \sum_{j=0}^{n}(a_j - a_{j-1})z^{n-j}$, $a_{-1} = 0$ has the same bound on $|z| = 1$ as f. Namely, $\left|z^n f\left(\frac{1}{z}\right)\right| \leq a_n$ for $|z| = 1$. Since $z^n f\left(\frac{1}{z}\right)$ is analytic in $|z| \leq 1$, we have $\left|z^n f\left(\frac{1}{z}\right)\right| \leq a_n$ for $|z| \leq 1$ by the Maximum Modulus Theorem. Hence, $\left|f\left(\frac{1}{z}\right)\right| \leq \frac{a_n}{|z|^n}$ for $|z| \leq 1$. Replace z with $\frac{1}{z}$, we see that $|f(z)| \leq a_n|z|^n$ for $|z| \geq 1$, and making use of this we get

$$\begin{aligned} |(1-z)P(z)| &= |f(z) - a_n z^{n+1}| \\ &\geq a_n|z|^{n+1} - |f(z)| \\ &\geq a_n|z|^{n+1} - a_n|z|^n \\ &= a_n|z|^n(|z| - 1). \end{aligned}$$

So if $|z| > 1$ then $(1-z)P(z) \neq 0$. Therefore, all zeros of P lie in $|z| \leq 1$. □

Many have worked to create more generalizations of Theorem 1.4 including Joyal, Labelle, Rahman who proved [19]:

Theorem 1.5. *If* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *with real coefficients satisfying* $a_0 \leq a_1 \leq \cdots \leq a_n$, *then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Notice that Theorem 1.5 reduces to Theorem 1.4 when $a_0 \geq 0$. Unfortunately Theorem 1.5 is only applicable to polynomials with real coefficients. Govil and Rahman demonstrated [16]:

Theorem 1.6. *If* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *with complex coefficients satisfying* $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2$ *for some* α *and* β *and* $\ell = 0, 1, 2, \ldots, n$ *and* $|a_0| \leq |a_1| \leq \cdots \leq |a_n|$*, then all zeros of* $P(z)$ *lie in*

$$|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{\ell=0}^{n-1} |a_\ell|.$$

Notice that even Theorem 1.6 reduces to Theorem 1.4 when $\alpha = \beta = 0$. So as we work through these new hypotheses, we can use particular values and reduce many of our results to Theorem 1.4. Govil and Rahman, in the same paper, used monotonicity on the modulus of the coefficients in Theorem 1.6. In Theorem 1.7 they proposed monotonicity on the real or imaginary parts of the coefficients of the polynomials [16]:

Theorem 1.7. *If* $P(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$. *is a polynomial of degree* n *with complex coefficients where* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$ *for* $\ell = 0, 1, 2, \ldots, n$*, satisfying* $0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$*,* $\alpha_n \neq 0$*, then all zeros of* $P(z)$ *lie in*

$$|z| \leq 1 + \frac{2}{a_n} \sum_{\ell=0}^{n} |\beta_\ell|.$$

Theorem 1.7 reduces to Theorem 1.4 by setting $\beta_k = 0$.

Similar to what Eneström did in Theorem 1.3 to prove solutions are found within an annulus, Govil and Jain [17] refined Theorem 1.7 by describing an annulus which contained the solutions by using monotonicity of moduli of complex coefficients:

Theorem 1.8. *If* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *with complex coefficients satisfying* $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for some* α *and* β *and for* $j = 0, 1, 2, \ldots, n$ *and* $|a_0| \leq |a_1| \leq \cdots \leq |a_n|$*, then all the zeros of* P *lie in*

$$\frac{1}{2M_2^2}\left[-R^2|b|(M_2 - |a_0|) + \left\{4|a_0|R^2M_2^3 + R^4|b|^2(M_2 - |a_0|)^2\right\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2}$$

and $M_1 = |a_n|r$*,* $M_2 = |a_n|R^n\left[r + R - \frac{|a_0|}{|a_n|}(\cos\alpha + \sin\alpha)\right]$*,* $c = |a_n - a_{n-1}|$*,* $b = a_1 - a_0$*, and* $r = \cos\alpha + \sin\alpha + \frac{2\sin\alpha|}{|a_n|}\sum_{\ell=0}^{n-1}|a_\ell|$.

We can also find the annulus with the conditions of Theorem 1.7 which produces:

Theorem 1.9. *If* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *with complex coefficients where* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$ *for* $\ell = 0, 1, 2, \ldots, n$*, satisfying* $0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$*,* $\alpha_n \neq 0$*, then all the zeros of* P *lie in*

$$\frac{1}{2M_4^2}\left[-R^2|b|(M_4 - |a_0|) + \left\{4|a_0|R^2M_4^3 + R^4|b|^2(M_4 - |a_0|)^2\right\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{\alpha_n} - \frac{1}{M^3}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{\alpha_n} - \frac{1}{M^3}\right)^2 + \frac{M_3}{\alpha_n}\right\}^{1/2}$$

and $M_3 = \alpha_n r$*,* $M_4 = R^n[\alpha_n + |\beta_n|)R + \alpha_n r - (\alpha_0 + |\beta_0|)]$*,* $c = |a - n - a_{n-1}|$*,* $b = a_1 - a_0$*, and* $r = 1 + \frac{1}{\alpha_n}\left(2\sum_{\ell=0}^{n-1}|\beta_\ell| + |\beta_n|\right)$.

Using Theorem 1.9 with the coefficient restrictions found in Theorem 1.5 where $a_0 \leq a_1 \leq \cdots \leq a_n$, Dewan and Govil [8] show that $R \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$ and the inner radius of the zero containing region is less than 1 which overall improves Joyal, Rabelle, Rahman's theorem. While Theorem 1.7 and Theorem 1.9 used monotonicity on either the real or the imaginary components of the coefficients, Gardner and Govil [11] placed a monotonicity condition on both the real and imaginary components which produces:

Theorem 1.10. *If* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *with complex coefficients where* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$ *for* $\ell = 0, 1, 2, \ldots, n$*, satisfying*

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$$

then all zeros of P *lie in*

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.$$

Aziz and Zargar [2] relaxed the monotonicity to show:

Theorem 1.11. *If* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *be a polynomial of degree* n *such that for some* $k \geq 1$ *and* $0 < \rho \leq 1$,

$$ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq \rho a_0 > 0,$$

then $P(z)$ *has all its zeros in the disk*

$$|z + k - 1| \leq k + \frac{2a_0}{a_n}(1 - \rho).$$

A more recent result published by Shah, Swroop, Sof, and Nisar in 2021 [27] used monotonicity of a portion of the coefficients, thus proving:

Theorem 1.12. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + a_{p-1} z^{p-1} + \cdots + a_q z^q + a_{q-1} z^{q-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree n *satisfying*

$$a_p \geq a_{p-1} \geq \cdots \geq a_q, p \geq q.$$

$$M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| \text{ and } M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$$

then all the zeros of $P(z)$ *lie in the disk*

$$|z| \leq \frac{|a_0| + M_q - a_q + a_p + M_p}{|a_n|}.$$

In the same paper, they discussed an inner disk in which there are no zeros, thus proving:

Theorem 1.13. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + a_{p-1} z^{p-1} + \cdots + a_q z^q + a_{q-1} z^{q-1} + \cdots a_1 z + a_0$$

be a polynomial of degree n *satisfying*

$$a_q \leq a_{q+1} \leq \cdots \leq a_p, q \leq p.$$

$$M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| \text{ and } M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$$

then $P(z)$ *does not vanish in*

$$|z| < \min\left\{1, \frac{|a_0|}{|M_q - a_q + a_p + M_p + |a_n||}\right\}.$$

1.2 Numbers of Zeros

Not only are the locations of zeros important when discussing polynomials, we also want to discuss the number of zeros within a disk found in the complex plane. Titchmarsh proves using Jensen's Formula which uses integration of complex functions [28]:

Theorem 1.14. *Let $F(z)$ be analytic in $|z| \leq R$. Let $|F(z)| \leq M$ in the disk $|z| \leq R$ and suppose $(f) \neq 0$. Then for $0 < \delta < 1$ then number of zeros of $f(z)$ in the disk $|z| \leq \delta R$ is less than*

$$\frac{1}{\log {}^1\!/\!_\delta} \log \frac{M}{|f(0)|}.$$

While Theorem 1.14 applies to all analytic functions, we need to note a formula for quaternionic polynomials does not exist, so, we will only find the number of zeros for complex polynomials. Now we can relax the conditions which we apply to the coefficients of Theorem 1.4 where Mohammad [24] showed:

Theorem 1.15. *Let $P(z) = \sum_{j=0}^{n}$ be such that $0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq a_n$. Then the number of zeros in $|z| \leq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \left(\frac{a_n}{a_0} \right).$$

Dewan's dissertation weakens the hypothesis of Theorem 1.15 to include complex coefficients by using the monotonicity of the moduli of the coefficients and proves [7, 23]:

Theorem 1.16. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be such that $|\arg(a_j) - \beta| \leq \alpha \leq {}^\pi\!/\!_2$ for all $1 \leq j \leq n$ and for some real α and β and $0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{n-1}| \leq |a_n|$.*

Then the number of zeros of P lie in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

By applying monotonicity to the real, non-negative components of complex coefficients Dewan, in the same paper, also proves:

Theorem 1.17. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\alpha_j = Re(a_j)$ and $\beta_j = Im(a_j)$ for all j and $0 < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n$, then the number of zeros of P in $|z| \leq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|\alpha_0|}.$$

Pukhta [25] generalizes Theorem 1.16 and Theorem 1.17 by counting the number of zeros of a function within a disk of radius δ. So, the number of zeros for polynomials where the moduli of the complex coefficient has a monotone behavior is given by:

Theorem 1.18. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be such that $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for all $1 \leq j \leq n$ and some real α and β, and $0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{n-1}| \leq |a_n|$. Then the number of zeros of P in $|z| \leq \delta$, $0 < \delta < 1$, does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Similarly to what was done above, Pukhta placed the monotonicity condition on the real components of the polynomial, thus proving:

Theorem 1.19. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be such that* $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ *for all* $1 \leq j \leq n$ *and for some real* α *and* β, *and* $0 < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n$. *Then the number of zeros of* P *in* $|z| \leq \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{2\left(\alpha_n + \sum_{j=0}^{n} |\beta_j|\right)}{|a_0|}.$$

Gardner and Shields [12] further refined Theorem 1.19 by applying the monotonicity condition on the real and imaginary coefficients thus showing:

Theorem 1.20. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $\alpha_j = Re(a_j)$ *and* $\beta_j = Im(a_j)$ *for* $0 \leq j \leq n$. *Suppose that we have*

$$0 \neq \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{1/\delta} \log \frac{(|\alpha_0| - \alpha_0) + (|\alpha_n| + \alpha_n) + (|\beta_0| - \beta_0) + (|\beta_n| + \beta_n)}{|a_0|}.$$

In this thesis, we generalize the hypotheses for each theorem to give bounds on the location of zeros as well as give a number of zeros within a bound for complex polynomials with complex coefficients. We also give related results on the location of zeros for quaternionic polynomials.

2 LOCATIONS OF ZEROS FOR COMPLEX POLYNOMIALS

In this thesis, we will introduce new parameters in order to generalize the hypotheses imposed on the coefficients of complex polynomials. We will start by imposing the parameters k and ρ from Aziz and Zargar found in Theorem 1.11, and Shah's p and q condition found in Theorem 1.12 onto the hypothesis which results in the real and imaginary parts of the coefficients of a polynomial. In this chapter we will also introduce dual gap polynomials; a new class of polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$. Results within this chapter were published in MDPI *AppliedMath* Journal.

2.1 Necessary Lemma

In proving Theorem 1.6, Govil and Rahman used the following [16, Equation (6)] and thus we will use it in several proofs within this thesis.

Lemma 2.1. *Let $\{a_\ell\}_{\ell=1}^{n}$ be a set of complex numbers which satisfy $|arg(a_\ell) - \beta| \leq \alpha \leq \pi/2$ for $0 \leq \ell \leq n$ and for some real β. Suppose $|a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_n|$. Then for $\ell \in \{1, 2, \ldots, n\}$ we have*

$$|a_\ell - a_{\ell-1}| \leq (|a_\ell| - |a_{\ell-1}|)\cos\alpha + (|a_\ell| + |a_{\ell-1}|)\sin\alpha.$$

2.2 Results

By considering a polynomial which contains the parameters, we obtain:

Theorem 2.2. *Let $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$ be a polynomial of degree n, $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$, for some real k_r, k_i, ρ_r, ρ_i where $k_r \geq 1$,*

$k_i \geq 1$, $0 < \rho_r \leq 1$, $0 < \rho_i \leq 1$,

$$\rho_r \alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r \alpha_p,$$

$$\rho_i \beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i \beta_p.$$

Then all zeros of $P(z)$ *lie in the disk*

$$\begin{aligned} |z| \;\; &\leq \;\; \frac{1}{|a_n|}\bigg(|a_0| + M_q - \rho_r \alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r - 1) + k_r \alpha_p \\ &\quad -\rho_i \beta_q + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i \beta_p + M_p \bigg), \end{aligned}$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 2.2. Let $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$ be a polynomial of degree n such that for some $k_i \geq 1$, $k_r \geq 1$, $0 < \rho_i \leq 1$, $0 < \rho_r \leq 1$, $\rho_r \alpha_q \leq \alpha_{q+1} \leq \cdots \leq k_r \alpha_p$, and $\rho_i \beta_q \leq \beta_{q+1} \leq \cdots \leq k_i \beta_p$, $q \leq p$. Define f to be the equation

$$\begin{aligned} P(z)(1-z) \;\; &= \;\; a_0 + (a_1 - a_0)z + \cdots + (a_q - a_{q-1})z^q + \cdots + (a_p - a_{p-1})z^p + \cdots \\ &\quad +(a_n - a_{n-1})z^n - a_n z^{n+1} \\ &= \;\; f(z) - a_n z^{n+1}. \end{aligned}$$

Let $|z| = 1$, then

$$\begin{aligned} |f(z)| \;\; &= \;\; |a_0 + (a_1 - a_0)z + \cdots + (a_q - a_{q-1})z^q + \cdots + (a_p - a_{p-1})z^p \\ &\quad + \cdots + (a_n - a_{n-1})z^n| \\ &\leq \;\; |a_0| + |a_1 - a_0||z| + \cdots + |a_{q-1} - a_{q-2}||z|^{q-1} + |a_q - a_{q-1}||z|^q \\ &\quad +|a_{q+1} - a_q||z|^{q+1} + \cdots + |a_p - a_{p-1}||z|^p + |a_{p+1} - a_p||z|^{p+1} \end{aligned}$$

$$
\begin{aligned}
& +\cdots+|a_n+a_{n-1}||z|^n \\
= \ & |a_0|+|a_1-a_0|+\cdots+|a_{q-1}-a_{q-2}|+|a_q-a_{q-1}|+|a_{q+1}-a_q| \\
& +\cdots+|a_p-a_{p-1}|+|a_{p+1}-a_p|+\cdots+|a_n+a_{n-1}|.
\end{aligned}
$$

Let $\alpha_\ell = \mathrm{Re}(a_\ell)$ and $\beta_\ell = \mathrm{Im}(a_\ell)$ for $q \le \ell \le p$. Thus

$$
\begin{aligned}
|f(z)| \ \le \ & |a_0|+\sum_{\ell=1}^{q}|a_\ell-a_{\ell-1}|+|\alpha_{q+1}+i\beta_{q+1}-\alpha_q-i\beta_q|+\cdots \\
& +|\alpha_p+i\beta_p-\alpha_{p-1}-i\beta_{p-1}|+\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|.
\end{aligned}
$$

Let $M_q = \sum_{\ell=1}^{q}|a_\ell-a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$. Hence

$$
\begin{aligned}
|f(z)| \ \le \ & |a_0|+M_q+|\alpha_{q+1}-\alpha_q|+|i||\beta_{q+1}-\beta_q|+\cdots+|\alpha_p-\alpha_{p-1}| \\
& +|i||\beta_p-\beta_{p-1}|+M_p \\
= \ & |a_0|+M_q+|\alpha_{q+1}-\alpha_q|+\cdots+|\alpha_p-\alpha_{p-1}|+|\beta_{q+1}-\beta_q| \\
& +\cdots+|\beta_p-\beta_{p-1}|+M_p \\
= \ & |a_0|+M_q+|\alpha_{q+1}-\rho_r\alpha_q+\rho_r\alpha_q-\alpha_q|+|\alpha_{q+2}-\alpha_{q+1}| \\
& +\cdots+|\alpha_{p-1}-\alpha_{p-2}|+|\alpha_p-k_r\alpha_p+k_r\alpha_p-\alpha_{p-1}| \\
& +|\beta_{q+1}-\rho_i\beta_q+\rho_i\beta_q-\beta_q|+|\beta_{q+2}-\beta_{q+1}|+\cdots+|\beta_{p-1}-\beta_{p-2}| \\
& +|\beta_p-k_i\beta_p+k_i\beta_p-\beta_{p-1}|+M_p.
\end{aligned}
$$

Since $\rho_r\alpha_q \le \alpha_{q+1} \le \cdots \le k_r\alpha_p$, $\rho_i\beta_q \le \beta_{q+1} \le \cdots \le k_i\beta_p$ where $q \le p$, then

$$
\begin{aligned}
|f(z)| \ \le \ & |a_0|+M_q+|\alpha_{q+1}-\rho_r\alpha_q+\rho_r\alpha_q-\alpha_q|-\alpha_{q+1}+\alpha_{p-1} \\
& +|\alpha_p-k_r\alpha_p+k_r\alpha_p-\alpha_{p-1}|+|\beta_{q+1}-\rho_i\beta_q+\rho_i\beta_q-\beta_q|-\beta_{q+1}+\beta_{p-1} \\
& +|\beta_p-k_i\beta_p+k_i\beta_p-\beta_{p-1}|+M_p
\end{aligned}
$$

$$
\begin{aligned}
&\leq |a_0| + M_q + |\alpha_{q+1} - \rho_r\alpha_q| + |\rho_r\alpha_q - \alpha_q| - \alpha_{q+1} + \alpha_{p-1} \\
&\quad +|\alpha_p - k_r\alpha_p| + |k_r\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_i\beta_q| + |\rho_i\beta_q - \beta_q| - \beta_{q+1} + \beta_{p-1} \\
&\quad +|\beta_p - k_i\beta_p| + |k_i\beta_p - \beta_{p-1}| + M_p \\
&= |a_0| + M_q + (\alpha_{q+1} - \rho_r\alpha_q) + |\alpha_q||\rho_r - 1| - \alpha_{q+1} + \alpha_{p-1} \\
&\quad +|\alpha_p||1 - k_r| + (k_r\alpha_p - \alpha_{p-1}) + (\beta_{q+1} - \rho_i\beta_q) + |\beta_q||\rho_i - 1| - \beta_{q+1} + \beta_{p-1} \\
&\quad +|\beta_p||1 - k_i| + (k_i\beta_p - \beta_{p-1}) + M_p \\
&= |a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1 - \rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p - \rho_i\beta_q \\
&\quad +|\beta_q|(1 - \rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p.
\end{aligned}
$$

We can notice $z^n f\left(\frac{1}{z}\right) = \sum_{\ell=0}^{n}(a_\ell - a_{\ell-1})z^{n-\ell}$ where $a_{-1} = 0$ has the same bound on $|z| = 1$ as $f(z)$. Namely $\left|z^n f\left(\frac{1}{z}\right)\right| \leq |a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1 - \rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p - \rho_i\beta_q + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p$ is analytic in $|z| \leq 1$ where we consider this function to have the value $a_n - a_{n-1}$ at $z = 0$ we have $\left|z^n f\left(\frac{1}{z}\right)\right| \leq |a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p - \rho_i\beta_q + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p$ for $|z| \leq 1$ by the Maximum Modulus Theorem. Thus

$$
\begin{aligned}
\left|f\left(\frac{1}{z}\right)\right| &= \frac{1}{|z|^n}\Big(|a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1 - \rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p \\
&\quad -\rho_i\beta_q + |\beta_q|(1 - \rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p\Big)
\end{aligned}
$$

for $|z| \leq 1$. Replacing z with $\frac{1}{z}$ we have

$$
\begin{aligned}
|f(z)| &\leq \Big(|a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1 - \rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p - \rho_i\beta_q \\
&\quad +|\beta_q|(1 - \rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p\Big)|z^n|
\end{aligned}
$$

for $|z| \geq 1$. We have

$$\begin{aligned}
|(1-z)P(z)| &= |f(z) - a_n z^{n+1}| \\
&\geq |a_n||z^{n+1}| - |f(z)| \\
&\geq |a_n||z^{n+1}| - \Bigg(|a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) \\
&\quad +|\alpha_p|(k_r - 1) + k_r\alpha_p - \rho_i\beta_q) + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) \\
&\quad +k_i\beta_p + M_p\Bigg)|z^n| \\
&= |z^n|\Bigg[|a_n||z| - \Bigg(|a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) \\
&\quad +|\alpha_p|(k_r - 1) + k_r\alpha_p - \rho_i\beta_q + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) \\
&\quad +k_i\beta_p + M_p\Bigg)\Bigg].
\end{aligned}$$

So if

$$\begin{aligned}
|z| &> \frac{1}{|a_n|}\Bigg(|a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p - \rho_i\beta_q \\
&\quad +|\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p\Bigg)
\end{aligned}$$

then

$$\begin{aligned}
0 &\neq |z^n|\Bigg[|a_n||z| - \Bigg(|a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p \\
&\quad -\rho_i\beta_q + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p\Bigg)\Bigg].
\end{aligned}$$

Therefore all zeros of P lie in

$$\begin{aligned}
|z| &\leq \frac{1}{|a_n|}\Bigg(|a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p \\
&\quad -\rho_i\beta_q + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p\Bigg).
\end{aligned}$$

□

Notice for β_ℓ for $0 \leq \ell \leq n$ and $k_R = \rho_R = 1$, then Theorem 2.2 reduces to Theorem 1.12. When β_ℓ for $0 \leq \ell \leq n$, $k_R = \rho_R = 1$, $q = 0$, and $p = n$, then Theorem 2.2 reduces to a result of Theorem 1.5. With $k_R = k_I = \rho_R = \rho_I = 1$, $q = 0$, and $p = n$, then Theorem 2.2 reduces to half of Theorem 1.10. If $a_0 \geq 0$, then it further reduces to Theorem 1.4, Eneström-Kakeya Theorem.

There is a connection with Bernstein inequalities, Chan and Malik [5] (independently Qazi [26]), considered the class of polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$. We can now apply the conditions from Theorem 2.2. We obtain the following:

Corollary 2.3. *Let $P(z)$ be a lacunary polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients where $a_0 \neq 0$, $m \geq 1$. Suppose that for some positive numbers k_r, k_i, ρ_r, ρ_i, p and q with $k_r \geq 1$, $k_i \geq 1$, $0 < \rho_r \leq 1$, $0 < \rho_i \leq 1$, and $m \leq q \leq p \leq n$, the coefficients satisfy*

$$\rho_r \alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r \alpha_p$$

and

$$\rho_i \beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i \beta_p.$$

Then all zeros of $P(z)$ lie in the radius

$$\begin{aligned} |z| \quad \leq \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q - \rho_r \alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r - 1) + k_r \alpha_p \\ & -\rho_i \beta_q) + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i \beta_p + M_p \Big), \end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$ and $a_{m-1} = 0$.

Notice when $m = 0$, then Corollary 2.3 reduces to Theorem 2.2. We can further the reduce Corollary 2.3 to Theorem 1.4 under the parameters discussed previously.

With similar behavior to lacunary polynomials, Gardner and Gladin [10] considered a class of polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{m'} + a_n$. Thus we obtain:

Corollary 2.4. *Let $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$ be a polynomial of degree n with complex coefficients where $a_0 \neq 0$, $a_n \neq 0$. Suppose that for some positive numbers k_r, k_i, ρ_r, ρ_i, p and q with $k_r \geq 1$, $k_i \geq 1$, $0 < \rho_r \leq 1$, $0 < \rho_i \leq 1$, and $1 \leq m \leq q \leq p \leq m' \leq n-1$, the coefficients satisfy*

$$\rho_r \alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r \alpha_p$$

and

$$\rho_i \beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i \beta_p.$$

Then all zeros of $P(z)$ lie in the radius

$$\begin{aligned} |z| \quad \leq \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q - \rho_r \alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r - 1) + k_r \alpha_p \\ & -\rho_i \beta_q) + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i \beta_p + M_p\Big), \end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

Corollary 2.4 reduces to Corollary 2.3 when $m' = n-1$. Corollary 2.4 further reduces to Theorem 2.2 when $m = 0$ and $m' = n-1$. We will now apply the parameters from Theorem 2.2 to obtain an inner radius for which no zeros exist. Hence we have:

Theorem 2.5. *Let $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$ be a polynomial of degree n such that $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$ when $q \leq \ell \leq p$. Suppose for some* *real* k_r, k_i, ρ_r, ρ_i *where* $k_r \geq 1$, $k_i \geq 1$, $0 < \rho_r \leq 1$, *and* $0 < \rho_i \leq 1$,

$$\rho_r\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r\alpha_p,$$

$$\rho_i\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i\beta_p.$$

Then $P(z)$ *does not vanish in*

$$\begin{aligned} |z| \;\; < \;\; & \min\Big\{1, |a_0|/\big(M_q + |\alpha_q|(1-\rho_r) - \rho_r\alpha_q + k_r\alpha_p + |\alpha_p|(k_r - 1) \\ & +|\beta_q|(1-\rho_i) - \rho_i\beta_q + k_i\beta_p + |\beta_p|(k_i - 1) + M_p + |a_n|\big)\Big\}, \end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 2.5. Consider the reciprocal polynomial

$$S(z) = z^n P\left(\frac{1}{z}\right) = a_0z^n + a_1z^{n-1} + \cdots + a_qz^{n-q} + \cdots + a_pz^{n-p} + \cdots + a_{n-1}z + a_n.$$

Let

$$\begin{aligned} H(z) &= (1-z)S(z) \\ &= -a_0z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_q - a_{q+1})z^{n-q} \\ &\quad + \cdots + (a_p - a_{p+1})z^{n-p} + \cdots + (a_{n-2} - a_{n-1})z^2 + (a_{n-1} - a_n)z + a_n \end{aligned}$$

This gives

$$\begin{aligned} |H(z)| &\geq |a_0||z|^{n+1} - \Big[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_q - a_{q+1}||z|^{n-q} \\ &\quad + \cdots + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| \\ &\quad + |a_n|\Big] \\ &= |a_0||z|^{n+1} - \Big[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \end{aligned}$$

$$+|\alpha_q + i\beta_q - \alpha_{q+1} - i\beta_{q+1}||z|^{n-q} + \cdots$$
$$+|\alpha_{p-1} + i\beta_{p-1} - \alpha_p - i\beta_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p}$$
$$+ \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n|\Big],$$

where $\alpha_\ell = \text{Re}(a_\ell)$ and $\beta_\ell = \text{Im}(a_\ell)$ for $q \le \ell \le p$. This gives

$$\begin{aligned}
|H(z)| \ \geq\ & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
& + |\alpha_q - \alpha_{q+1}||z|^{n-q} + |i\beta_q - i\beta_{q+1}||z|^{n-q} + \cdots + |\alpha_{p-1} - \alpha_p||z|^{n-p+1} \\
& + |i\beta_{p-1} - i\beta_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-2} - a_{n-1}||z|^2 \\
& + |a_{n-1} - a_n||z| + |a_n|\Big) \\
=\ & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
& + |\alpha_q - \rho_r\alpha_q + \rho_r\alpha_q - \alpha_{q+1}||z|^{n-q} + \cdots \\
& + |\alpha_{p-1} - k_r\alpha_p + k_r\alpha_p - \alpha_p||z|^{n-p+1} + |i||\beta_q - \rho_i\beta_q + \rho_i\beta_q - \beta_{q+1}||z|^{n-q} \\
& + \cdots + |i||\beta_{p-1} - k_i\beta_p + k_i\beta_p - \beta_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} \\
& + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n|\Big) \\
\geq\ & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
& + |\alpha_q - \rho_r\alpha_q||z|^{n-q} + |\rho_r\alpha_q - \alpha_{q+1}||z|^{n-q} + \cdots + |\alpha_{p-1} - k_r\alpha_p||z|^{n-p+1} \\
& + |k_r\alpha_p - \alpha_p||z|^{n-p+1} + |\beta_q - \rho_i\beta_q||z|^{n-q} + |\rho_i\beta_q - \beta_{q+1}||z|^{n-q} \\
& + \cdots + |\beta_{p-1} - k_i\beta_p||z|^{n-p+1} + |k_i\beta_p - \beta_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} \\
& + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n|\Big) \\
=\ & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
& + |\alpha_q||1 - \rho_r||z|^{n-q} + |\rho_r\alpha_q - \alpha_{q+1}||z|^{n-q} + \cdots + |\alpha_{p-1} - k_r\alpha_p||z|^{n-p+1}
\end{aligned}$$

$$
\begin{aligned}
&+|\alpha_p||k_r-1||z|^{n-p+1}+|\beta_q||1-\rho_i||z|^{n-q}+|\rho_i\beta_q-\beta_{q+1}||z|^{n-q}\\
&+\cdots+|\beta_{p-1}-k_i\beta_p||z|^{n-p+1}+|\beta_p||k_i-1||z|^{n-p+1}+|a_p-a_{p+1}||z|^{n-p}\\
&+\cdots+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big).
\end{aligned}
$$

Since $0<\rho_r\leq 1$, $0<\rho_i\leq 1$, $k_r\geq 1$, $k_i\geq 1$, $\rho_r\alpha_q\leq\alpha_{q+1}\leq\cdots\leq k_r\alpha_p$ and $\rho_i\beta_q\leq\beta_{q+1}\leq\cdots\leq k_i\beta_p$, then

$$
\begin{aligned}
|H(z)| \;\geq\; & |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots+|a_{q-1}-a_q||z|^{n-q+1}\\
&+|\alpha_q|(1-\rho_r)|z|^{n-q}+(\alpha_{q+1}-\rho_r\alpha_q)|z|^{n-q}+\cdots+(k_r\alpha_p-\alpha_{p-1})|z|^{n-p+1}\\
&+|\alpha_p|(k_r-1)|z|^{n-p+1}+|\beta_q|(1-\rho_i)|z|^{n-q}+(\beta_{q+1}-\rho_i\beta_q)|z|^{n-q}\\
&+\cdots+(k_i\beta_p-\beta_{p-1})|z|^{n-p+1}+|\beta_p|(k_i-1)||z|^{n-p+1}+|a_p-a_{p+1}||z|^{n-p}\\
&+\cdots+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big)\\
=\; & |z|^n\Bigg[|a_0||z|-\Bigg(|a_0-a_1|+\frac{|a_1-a_2|}{|z|}+\cdots+\frac{|a_{q-1}-a_q|}{|z|^{q-1}}+\frac{|\alpha_q|(1-\rho_r)}{|z|^q}\\
&+\frac{\alpha_{q+1}-\rho_r\alpha_q}{|z|^q}+\cdots+\frac{k_r\alpha_p-\alpha_{p-1}}{|z|^{p-1}}+\frac{|\alpha_p|(k_r-1)}{|z|^{p-1}}+\frac{|\beta_q|(1-\rho_i)}{|z|^q}\\
&+\frac{\beta_{q+1}-\rho_i\beta_q}{|z|^q}+\cdots+\frac{k_i\beta_p-\beta_{p-1}}{|z|^{p-1}}+\frac{|\beta_p|(k_i-1)|}{|z|^{p-1}}+\frac{|a_p-a_{p+1}|}{|z|^p}\\
&+\cdots+\frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}}+\frac{|a_{n-1}-a_n|}{|z|^{n-1}}+\frac{|a_n|}{|z|^n}\Bigg)\Bigg].
\end{aligned}
$$

Now for $|z|>1$ so that $\dfrac{1}{|z|^{n-\ell}}<1$, for $0\leq\ell<n$ we have

$$
\begin{aligned}
|H(z)| \;\geq\; & |z|^n\Big[|a_0||z|-\Big(|a_0-a_1|+|a_1-a_2|+\cdots+|a_{q-1}-a_q|+|\alpha_q|(1-\rho_r)\\
&+(\alpha_{q+1}-\rho_r\alpha_q)+\cdots+(k_r\alpha_p-\alpha_{p-1})+|\alpha_p|(k_r-1)+|\beta_q|(1-\rho_i)\\
&+(\beta_{q+1}-\rho_i\beta_q)+\cdots+(k_i\beta_p-\beta_{p-1})+|\beta_p|(k_i-1)|+|a_p-a_{p+1}|\\
&+\cdots+|a_{n-2}-a_{n-1}|+|a_{n-1}-a_n|+|a_n|\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= |z|^n\Bigg[|a_0||z| - \Bigg(\sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}| + |\alpha_q|(1-\rho_r) - \rho_r\alpha_q + k_r\alpha_p \\
&\quad +|\alpha_p|(k_r-1) + |\beta_q|(1-\rho_i) - \rho_i\beta_q + k_i\beta_p + |\beta_p|(k_i-1) \\
&\quad + \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}| + |a_n|\Bigg)\Bigg] \\
&> 0
\end{aligned}
$$

if

$$
\begin{aligned}
|z| &> \frac{1}{|a_0|}\Big(M_q + |\alpha_q|(1-\rho_r) - \rho_r\alpha_q + k_r\alpha_p + |\alpha_p|(k_r-1) + |\beta_q|(1-\rho_i) \\
&\quad -\rho_i\beta_q + k_i\beta_p + |\beta_p|(k_i-1) + M_p + |a_n|\Big)
\end{aligned}
$$

where

$$
M_q = \sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}| \text{ and } M_p = \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}|.
$$

Thus all zeros of $H(z)$ whose modulus is greater than 1 lie in

$$
\begin{aligned}
|z| &\leq \frac{1}{|a_0|}\Big(M_q + |\alpha_q|(1-\rho_r) - \rho_r\alpha_q + k_r\alpha_p + |\alpha_p|(k_r-1) + |\beta_q|(1-\rho_i) \\
&\quad -\rho_i\beta_q + k_i\beta_p + |\beta_p|(k_i-1) + M_p + |a_n|\Big).
\end{aligned}
$$

Hence all zeros of $H(z)$ and hence of $S(z)$ lie in

$$
\begin{aligned}
|z| &\leq \max\Big\{1, |a_0|/\big(M_q + |\alpha_q|(1-\rho_r) - \rho_r\alpha_q + k_r\alpha_p + |\alpha_p|(k_r-1) \\
&\quad +|\beta_q|(1-\rho_i) - \rho_i\beta_q + k_i\beta_p + |\beta_p|(k_i-1) + M_p + |a_n|\big)\Big\}.
\end{aligned}
$$

Therefore all the zeros $P(z)$ lie in

$$
\begin{aligned}
|z| &\geq \min\Big\{1, |a_0|/\big(M_q + |\alpha_q|(1-\rho_r) - \rho_r\alpha_q + k_r\alpha_p + |\alpha_p|(k_r-1) \\
&\quad +|\beta_q|(1-\rho_i) - \rho_i\beta_q + k_i\beta_p + |\beta_p|(k_i-1) + M_p + |a_n|\big)\Big\}.
\end{aligned}
$$

Thus the polynomial $P(z)$ does not vanish in

$$|z| < \min\Big\{1, |a_0|/\big(M_q + |\alpha_q|(1-\rho_r) - \rho_r\alpha_q + k_r\alpha_p + |\alpha_p|(k_r-1) + |\beta_q|(1-\rho_i) - \rho_i\beta_q + k_i\beta_p + |\beta_p|(k_i-1) + M_p + |a_n|\big)\Big\}.$$

$\square$

With $q = 0$, $p = n$, and $\rho = k = 1$, Theorem 2.5 becomes a slight improvement to Theorem 1.6. Additionally when $\alpha = \beta = 0$, then Theorem 2.2 reduces to Theorem 1.4. We will extend Theorem 2.5 for the class of polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$. For which we have:

Corollary 2.6. *Let $P(z)$ be a lacunary polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$, $m \geq 1$, $a_0 \neq 0$ be a polynomial of degree n such that $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$ when $q \leq \ell \leq p$ for some real k_r, k_i, ρ_r, ρ_i where $k_r \geq 1$, $k_i \geq 1$, $0 < \rho_r \leq 1$, $0 < \rho_i \leq 1$ and $m \leq q \leq p \leq n$,*

$$\rho_r\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r\alpha_p,$$

$$\rho_i\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i\beta_p.$$

Then $P(z)$ does not vanish in

$$|z| < \min\Big\{1, |a_0|/\big(M_q + |\alpha_q|(1-\rho_r) - \rho_r\alpha_q + k_r\alpha_p + |\alpha_p|(k_r-1) + |\beta_q|(1-\rho_i) - \rho_i\beta_q + k_i\beta_p + |\beta_p|(k_i-1) + M_p + |a_n|\big)\Big\},$$

where $a_{m-1} = 0$, $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

We can note when $m = 0$, Corollary 2.6 reduces to Theorem 2.5. We can also extend this to the class of polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n$. Thus we get the following corollary:

Corollary 2.7. *Let* $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$, $a_0 \neq 0, a_n \neq 0$ *be a polynomial of degree* n *such that* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$ *when* $q \leq \ell \leq p$ *for some real* k_r, k_i, ρ_r, ρ_i *where* $k_r \geq 1$, $k_i \geq 1$, $0 < \rho_r \leq 1$, $0 < \rho_i \leq 1$ *and* $1 \leq m \leq q \leq p \leq m' \leq n-1$,

$$\rho_r \alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r \alpha_p$$

$$\rho_i \beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i \beta_p.$$

Then $P(z)$ *does not vanish in*

$$\begin{aligned} |z| \quad < \quad \min\Big\{ & 1, |a_0| / \big(M_q + |\alpha_q|(1-\rho_r) - \rho_r \alpha_q + k_r \alpha_p + |\alpha_p|(k_r - 1) \\ & + |\beta_q|(1-\rho_i) - \rho_i \beta_q + k_i \beta_p + |\beta_p|(k_i - 1) + M_p + |a_n| \big) \Big\}, \end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

Now that we have considered the k and ρ parameters from Aziz and Zargar as well as Shah et al. p and q parameter onto the real and imaginary parts, we will now impose these parameters onto the modulus of the coefficients for a polynomial with complex coefficients. Thus we obtain:

Theorem 2.8. *Let* $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$ *be a polynomial of degree* n *with complex coefficients such that* $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2, \ell = q, q+1, \ldots, p$ *for some real* β, $k \geq 1$, $0 < \rho \leq 1$,

$$\rho |a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{p-1}| \leq k |a_p|$$

$$M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| \text{ and } M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|.$$

Then all zeros of $P(z)$ lie in the disk

$$\begin{aligned} |z| \leq \; & \frac{1}{|a_n|}\Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\ & +2\sum_{\ell=q+1}^{p-1} |a_\ell|\sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p\Bigg). \end{aligned}$$

Proof of Theorem 2.8. Let $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$ be a polynomial of degree n with complex coefficients such that $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2, \ell = q, q+1, \ldots, p$ for some real β, and $\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq k|a_p|$ for $0 < \rho \leq 1$ and $k \geq 1$. Without loss of generality assume $\beta = 0$. By Lemma 2.1

$$|a_\ell - a_{\ell-1}| \leq (|a_\ell| - |a_{\ell-1}|)\cos\alpha + (|a_\ell| + |a_{\ell-1}|)\sin\alpha.$$

Consider

$P(z)(1-z) = a_0 + (a_1 - a_0)z + \cdots + (a_q - a_{q-1})z^q + \cdots + (a_p - a_{p-1})z^p + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1} = f(z) - a_n z^{n+1}$. Then if $|z| = 1$, then

$$\begin{aligned} |f(z)| &= |a_0 + (a_1 - a_0)z + \cdots + (a_q - a_{q-1})z^q + \cdots + (a_p - a_{p-1})z^p \\ &\quad + \cdots + (a_n - a_{n-1})z^n| \\ &\leq |a_0| + |a_1 - a_0||z| + \cdots + |a_q - a_{q-1}||z|^q + \cdots + |a_p - a_{p-1}||z|^p \\ &\quad + \cdots + |a_n - a_{n-1}||z|^n \\ &= |a_0| + |a_1 - a_0| + \cdots + |a_q - a_{q-1}| + \cdots + |a_p - a_{p-1}| + \cdots + |a_n - a_{n-1}| \\ &= |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |a_{q+1} - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| \\ &\quad + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| \end{aligned}$$

$$= \ |a_0| + M_q + |a_{q+1} - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| + M_p,$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. Thus for $|z| = 1$

$$\begin{aligned}
|f(z)| &\leq |a_0| + M_q + |a_{q+1} - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| + M_p \\
&= |a_0| + M_q + |a_{q+1} - \rho a_q + \rho a_q - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| \\
&\quad + |a_p - k a_p + k a_p - a_{p-1}| + M_p \\
&\leq |a_0| + M_q + |a_{q+1} - \rho a_q| + |\rho a_q - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| \\
&\quad + |a_p - k a_p| + |k a_p - a_{p-1}| + M_p \\
&\leq |a_0| + M_q + (|a_{q+1}| - |\rho a_q|)\cos\alpha + (|a_{q+1}| + |\rho a_q|)\sin\alpha \\
&\quad + |a_q||\rho - 1| + \sum_{\ell=q+2}^{p-1} (|a_\ell| - |a_{\ell-1}|)\cos\alpha + \sum_{\ell=q+2}^{p-1} (|a_\ell| + |a_{\ell-1}|)\sin\alpha \\
&\quad + |a_p||1 - k| + (|k a_p| - |a_{p-1}|)\cos\alpha + (|k a_p| + |a_{p-1}|)\sin\alpha + M_p \\
&= |a_0| + M_q + (|a_{q+1}| - |\rho a_q|)\cos\alpha + (|a_{q+1}| + |\rho a_q|)\sin\alpha \\
&\quad + |a_q|(1 - \rho) + |a_{p-1}|(\cos\alpha + \sin\alpha) + 2\sum_{\ell=q+1}^{p-2} |a_\ell|\sin\alpha \\
&\quad - |a_{q+1}|(\cos\alpha + \sin\alpha) + |a_p|(k - 1) + (|k a_p| - |a_{p-1}|)\cos\alpha \\
&\quad + (|k a_p| + |a_{p-1}|)\sin\alpha + M_p \\
&= |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell|\sin\alpha \\
&\quad + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p.
\end{aligned}$$

Hence also,

$$\left| z^n f\left(\frac{1}{z}\right) \right| \ \leq \ |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1)$$

$$+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha+\sin\alpha+1) - |a_p| + M_p$$

for $|z| = 1$. By the Maximum Modulus Theorem

$$\begin{aligned}\left|z^n f\left(\frac{1}{z}\right)\right| &\leq |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1)\\ &\quad +2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha+\sin\alpha+1) - |a_p| + M_p\end{aligned}$$

holds inside the unit circle $|z| \leq 1$ as well. If $R > 1$, then $\frac{1}{R}e^{-i\alpha}$ lies inside the unit circle for every real α. Thus it follows

$$\begin{aligned}\left|P(Re^{i\alpha})\right| &\leq \Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2|a_{p-1}|\sin\alpha\\ &\quad +2\sum_{\ell=q+1}^{p-2}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha+\sin\alpha+1) - |a_p| + M_p\Bigg)R^n\end{aligned}$$

for every $R \geq 1$ and α real. Thus for every $|z| = R > 1$,

$$\begin{aligned}|P(z)(1-z)| &= |-a_n z^{n+1} + f(z)|\\ &\geq |a_n||R|^{n+1} - |f(z)|\\ &\geq |a_n|R^{n+1} - \Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1)\\ &\quad +2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha+\sin\alpha+1) - |a_p| + M_p\Bigg)R^n\\ &= R^n\Bigg[|a_n|R - \Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1)\\ &\quad +2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha+\sin\alpha+1) - |a_p| + M_p\Bigg)\Bigg]\\ &> 0\end{aligned}$$

if

$$R > \frac{1}{|a_n|}\Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1)$$

$$+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha+k|a_p|(\cos\alpha+\sin\alpha+1)-|a_p|+M_p\Big).$$

Therefore all zeros lie within

$$\begin{aligned}|z| &\leq \frac{1}{|a_n|}\Big(|a_0|+M_q+|a_q|+\rho|a_q|(\sin\alpha-\cos\alpha-1)\\ &\quad +2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha+k|a_p|(\cos\alpha+\sin\alpha+1)-|a_p|+M_p\Big).\end{aligned}$$

□

Notice when $\rho = k = 1$, $q = 0$ and $p = 1$, then Theorem 2.8 reduces to Theorem 1.6. Furthermore when $\beta_i = 0$ for all $0 \leq i \leq n$, then Theorem 2.8 reduces to Theorem 1.4. Now we will consider a polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$. Thus we have:

Corollary 2.9. *Let $P(z)$ be a lacunary polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$. Suppose $P(z)$ is a polynomial of degree n with complex coefficients such that $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2, \ell = q, q+1, \ldots, p$ for some real β, $k \geq 1$, $0 < \rho \leq 1$, where $1 \leq m \leq q \leq p \leq n$,*

$$\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{p-1}| \leq k|a_p|.$$

Then all zeros of $P(z)$ lie in the disk

$$\begin{aligned}|z| &\leq \frac{1}{|a_n|}\Big(|a_0|+M_q+|a_q|+\rho|a_q|(\sin\alpha-\cos\alpha-1)\\ &\quad +2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha+k|a_p|(\cos\alpha+\sin\alpha+1)+M_p-|a_p|\Big)\end{aligned}$$

where $M_q = \sum_{\ell=m}^{q}|a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}|$.

We notice when $m = 0$, Corollary 2.9 reduces to Theorem 2.8. This corollary reduces further to the Eneström-Kakeya Theorem under the conditions mentioned previously. We will now consider the hypotheses from Theorem 2.8 on polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n$. Thus we show:

Corollary 2.10. *Let* $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$, $a_0 \neq 0$, $a_n \neq 0$ *be a polynomial of degree* n *with complex coefficients such that* $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2, \ell = q, q+1, \ldots, p$ *for some real* β, $k \geq 1$, $0 < \rho \leq 1$, *where* $1 \leq m \leq q \leq p \leq m' \leq n-1$,

$$\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{p-1}| \leq k|a_p|.$$

Then all zeros of $P(z)$ *lie in the disk*

$$\begin{aligned} |z| \quad \leq \quad & \frac{1}{|a_n|}\Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\ & +2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) + M_p - |a_p|\Bigg), \end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

We notice Corollary 2.10 reduces to Corollary 2.9 when $m' = n-1$, and under the conditions mentioned previously will also reduce to Theorem 1.4. We will now apply the hypotheses applied to Theorem 2.8 to determine a disk in which no zeros exist. Thus we have:

Theorem 2.11. *Let* $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$. *Suppose* $P(z)$ *is a polynomial of degree* n *with complex coefficients such that* $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2, \ell = q, q+1, \ldots, p$ *for some real* β, $k \geq 1$, $0 < \rho \leq 1$ *where*

$$\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{p-1}| \leq k|a_p|.$$

Then $P(z)$ does not vanish in

$$|z| \;<\; \min\left\{1, |a_0| \Big/ \left(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha \right.\right.$$
$$\left.\left. +k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|\right)\right\},$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 2.11. Without loss of generality, assume $\beta = 0$. By Lemma 2.1

$$|a_\ell - a_{\ell-1}| \leq (|a_\ell| - |a_{\ell-1}|)\cos\alpha + (|a_\ell| + |a_{\ell-1}|)\sin\alpha.$$

Consider the reciprocal polynomial

$$S(z) = z^n P\left(\frac{1}{z}\right) = a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + \cdots + a_p z^{n-p} + \cdots + a_{n-1} z + a_n.$$

Let

$$\begin{aligned} H(z) &= (1-z)S(z) \\ &= -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_q - a_{q+1})z^{n-q} \\ &\quad + \cdots + (a_{p-1} - a_p)z^{n-p+1} + \cdots + (a_{n-2} - a_{n-1})z^2 + (a_{n-1} - a_n)z + a_n. \end{aligned}$$

This gives

$$\begin{aligned} |H(z)| &\geq |a_0||z|^{n+1} - \Big[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_q - a_{q+1}||z|^{n-q} \\ &\quad + \cdots + |a_{p-1} - a_p||z|^{n-p+1} + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| \\ &\quad + |a_n|\Big] \\ &= |a_0||z|^{n+1} - \Big[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots \\ &\quad + |a_q - \rho a_q + \rho a_q - a_{q+1}||z|^{n-q} + \cdots + |a_{p-1} - k a_p + k a_p - a_p||z|^{n-p+1} \end{aligned}$$

$$
\begin{aligned}
&+\cdots+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big]\\
\geq\ & |a_0||z|^{n+1}-\Big[|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots+|a_q-\rho a_q||z|^{n-q}\\
&+|\rho a_q-a_{q+1}||z|^{n-q}+\cdots+|a_{p-1}-ka_p||z|^{n-p+1}+|ka_p-a_p||z|^{n-p+1}\\
&+\cdots+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big].
\end{aligned}
$$

Since $k \geq 1$ and $0 < \rho \leq 1$, $\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq k|a_p|$,

$$
\begin{aligned}
|H(z)| \ \geq\ & |a_0||z|^{n+1}-\Big[|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}\\
&+\cdots+|a_q-\rho a_q||z|^{n-q}+|\rho a_q-a_{q+1}||z|^{n-q}+\cdots+|a_{p-1}-ka_p||z|^{n-p+1}\\
&+|ka_p-a_p||z|^{n-p+1}+\cdots+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big]\\
=\ & |z|^n\Bigg[|a_0||z|-\bigg(|a_0-a_1|+\frac{|a_1-a_2|}{|z|}+\cdots+\frac{|a_{q-1}-a_q|}{z^{q-1}}+\frac{|a_q||1-\rho|}{|z|^q}\\
&+\frac{|\rho a_q-a_{q+1}|}{|z|^q}+\cdots+\frac{|a_{p-1}-ka_p|}{|z|^{p-1}}+\frac{|a_p||k-1|}{|z|^{p-1}}+\cdots+\frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}}\\
&+\frac{|a_{n-1}-a_n|}{|z|^{n-1}}+\frac{|a_n|}{|z|^n}\bigg)\Bigg].
\end{aligned}
$$

Now for $|z| > 1$ so that $\frac{1}{|z|^{n-\ell}} < 1$, where $0 \leq \ell < n$ we have

$$
\begin{aligned}
|H(z)| \ \geq\ & |z|^n\Bigg[|a_0||z|-\bigg(|a_0-a_1|+|a_1-a_2|+\cdots+|a_{q-1}-a_q|+|a_q|(1-\rho)\\
&+|\rho a_q-a_{q+1}|+\cdots+|a_{p-1}-ka_p|+|a_p|(k-1)+\cdots+|a_{n-2}-a_{n-1}|\\
&+|a_{n-1}-a_n|+|a_n|\bigg)\Bigg]\\
=\ & |z|^n\Bigg[|a_0||z|-\bigg(\sum_{\ell=1}^{q}|a_\ell-a_{\ell-1}|+|a_q|(1-\rho)+|a_{q+1}-\rho a_q|\\
&+\sum_{\ell=q+2}^{p-1}|a_\ell-a_{\ell-1}|+|ka_p-a_{p-1}|+|a_p|(k-1)+\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|\\
&+|a_n|\bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\geq |z|^n\Bigg[|a_0||z| - \Bigg(\sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}| + |a_q|(1-\rho) + (|a_{q+1}| - |\rho a_q|)\cos\alpha \\
&\quad +(|a_{q+1}| + |\rho a_q|)\sin\alpha + \sum_{\ell=q+2}^{p-1}(|a_\ell| - |a_{\ell-1}|)\cos\alpha \\
&\quad + \sum_{\ell=q+2}^{p-1}(|a_\ell| + |a_{\ell-1}|)\sin\alpha + (|ka_p| - |a_{p-1}|)\cos\alpha + (|ka_p| + |a_{p-1}|)\sin\alpha \\
&\quad +|a_p|(k-1) + \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}| + |a_n|\Bigg)\Bigg] \\
&\geq |z|^n\Bigg[|a_0||z| - \Bigg(M_q + |a_q|(1-\rho) + (|a_{q+1}| - \rho|a_q|)\cos\alpha \\
&\quad +(|a_{q+1}| + \rho|a_q|)\sin\alpha + \sum_{\ell=q+2}^{p-1}(|a_\ell| - |a_{\ell-1}|)\cos\alpha \\
&\quad + \sum_{\ell=q+2}^{p-1}(|a_\ell| + |a_{\ell-1}|)\sin\alpha + (k|a_p| - |a_{p-1}|)\cos\alpha + (k|a_p| + |a_{p-1}|)\sin\alpha \\
&\quad +|a_p|(k-1) + M_p + |a_n|\Bigg)\Bigg],
\end{aligned}
$$

where $M_q = \sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}|$. Hence

$$
\begin{aligned}
|H(z)| &\geq |z|^n\Bigg[|a_0||z| - \Bigg(M_q + |a_q|(1-\rho) + (|a_{q+1}| - \rho|a_q|)\cos\alpha \\
&\quad +(|a_{q+1}| + \rho|a_q|)\sin\alpha + |a_{p-1}|(\cos\alpha + \sin\alpha) + 2\sum_{\ell=q+1}^{p-2}|a_\ell|\sin\alpha \\
&\quad -|a_{q+1}|(\cos\alpha + \sin\alpha) + (k|a_p| - |a_{p-1}|)\cos\alpha + (k|a_p| + |a_{p-1}|)\sin\alpha \\
&\quad +|a_p|(k-1) + M_p + |a_n|\Bigg)\Bigg] \\
&= |z|^n\Bigg[|a_0||z| - \Bigg(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha \\
&\quad +k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|\Bigg)\Bigg] \\
&> 0
\end{aligned}
$$

if

$$|z| > \frac{1}{|a_0|}\Bigg(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha$$
$$+k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|\Bigg).$$

Thus all zeros of $H(z)$ whose modulus is greater than 1 lie in

$$|z| \leq \frac{1}{|a_0|}\Bigg(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha$$
$$+k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|\Bigg).$$

Hence all zeros of $H(z)$ and hence of $S(z)$ lie in

$$|z| \leq \max\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha$$
$$+k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|\Bigg)\Bigg\}.$$

Therefore all the zeros of $P(z)$ lie in

$$|z| \geq \min\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha$$
$$+k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|\Bigg)\Bigg\}.$$

Thus the polynomial $P(z)$ does not vanish in

$$|z| < \min\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha$$
$$+k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|\Bigg)\Bigg\}.$$

□

Notice that if $\beta_\ell = 0$ for $0 \le \ell \le n$, $\rho = k = 1$, then Theorem 2.11 reduces to Theorem 1.13. We will now expand these hypotheses to the polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$. Thus we have:

Corollary 2.12. *Let $P(z)$ be a lacunary polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$, $a_0 \neq 0$. Suppose $P(z)$ is a polynomial of degree n with complex coefficients such that $|\arg a_\ell - \beta| \le \alpha \le \pi/2, \ell = q, q+1, \dots, p$ for some real β, $k \ge 1$ and $0 < \rho \le 1$ where $1 \le m \le q \le p \le n$,*

$$\rho|a_q| \le |a_{q+1}| \le \cdots \le |a_{p-1}| \le k|a_p|.$$

Then $P(z)$ does not vanish in

$$\begin{aligned} |z| \;&<\; \min\left\{1, |a_0| \Big/ \left(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha \right.\right. \\ &\left.\left. + k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| \right)\right\}, \end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Notice this reduces to Theorem 2.11 when $m = 0$. We will take this further to polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$. That is:

Corollary 2.13. *Let $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$, $a_0 \neq 0$, $a_n \neq 0$. Suppose $P(z)$ is a polynomial of degree n with complex coefficients such that $|\arg a_\ell - \beta| \le \alpha \le \pi/2, \ell = q, q+1, \dots, p$ for some real β, $k \ge 1$, $0 < \rho \le 1$, and $1 \le m \le q \le p \le m' \le n-1$ where*

$$\rho|a_q| \le |a_{q+1}| \le \cdots \le |a_{p-1}| \le k|a_p|.$$

Then $P(z)$ *does not vanish in*

$$|z| < \min\left\{1, |a_0| \Big/ \left(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha \right.\right.$$
$$\left.\left. +k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p|\right)\right\},$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

Notice this reduces to Theorem 2.11 when $m = 0$ and $m' = n - 1$. We can note under the conditions mentioned previously, this will reduce down to Theorem 1.13.

Throughout this chapter we were able to reduce the outer bound to resemble Theorem 1.4 and the inner bounds reduce to the inner bounds found in Chapter 1. As we continue throughout this thesis to determine inner and outer bounds of location of zeros we will use this idea in order to examine the validity of our results.

3 COUNTING ROOTS WITHIN A BOUND

Along with finding inner and outer bounds we also want to explore the number of zeros found within a bound of radius δR. We will apply the k and ρ parameters from Aziz and Zargar found in Theorem 1.11, Shah's p and q parameter found in Theorem 1.12 to the *real* and *imaginary* components to which we have:

Theorem 3.1. *Let $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$ be a polynomial of degree n, $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$, for some real k_r, k_i, ρ_r, ρ_i where $k_r \geq 1$, $k_i \geq 1$, $0 < \rho_r \leq 1$ and $0 < \rho_i \leq 1$,*

$$\rho_r \alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r \alpha_p,$$

$$\rho_i \beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i \beta_p.$$

Then the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$,

$$\begin{aligned} M &= |a_0| + M_q - \rho_r \alpha_q + |\alpha_q|(1 - \rho_r) + |\alpha_p|(k_r - 1) + k_r \alpha_p - \rho_i \beta_q \\ &\quad + |\beta_q|(1 - \rho_i) + |\beta_p|(k_i - 1) + k_i \beta_p + M_p + |a_n|, \end{aligned}$$

$M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 3.1. Consider

$$\begin{aligned} F(z) &= (1 - z)P(z) \\ &= a_0 + (a_1 - a_0)z + \cdots + (a_{q+1} - a_q)z^q + \cdots + (a_p - a_{p-1})z^p \end{aligned}$$

$$+ \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}.$$

For $|z| = 1$ we have,

$$\begin{aligned}
|F(z)| &= |a_0 + (a_1 - a_0)z + \cdots + (a_{q+1} - a_q)z^q + \cdots + (a_p - a_{p-1})z^p \\
&\quad + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}| \\
&\leq |a_0| + |a_1 - a_0||z| + \cdots + |a_{q+1} - a_q||z|^q + \cdots + |a_p - a_{p-1}||z|^p \\
&\quad + \cdots + |a_n - a_{n-1}||z|^n + |-a_n||z|^{n+1} \\
&= |a_0| + |a_1 - a_0| + \cdots + |a_{q+1} - a_q| + \cdots + |a_p - a_{p-1}| + \cdots \\
&\quad +|a_n - a_{n-1}| + |a_n| \\
&= |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |\alpha_{q+1} + i\beta_{q+1} - \alpha_q - i\beta_q| + \cdots \\
&\quad +|\alpha_p + i\beta_p - \alpha_{p-1} - i\beta_{p-1}| + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| + |a_n| \\
&\leq |a_0| + M_q + |\alpha_{q+1} - \alpha_q| + \cdots + |\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \beta_q| + \cdots \\
&\quad +|\beta_p - \beta_{p-1}| + M_p + |a_n| \\
&= |a_0| + M_q + |\alpha_{q+1} - \rho_r\alpha_q + \rho_r\alpha_q - \alpha_q| + \cdots \\
&\quad +|\alpha_p - k_r\alpha_p + k_r\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_i\beta_q + \rho_i\beta_q - \beta_q| + \cdots \\
&\quad +|\beta_p - k_i\beta_p + k_i\beta_p - \beta_{p-1}| + M_p + |a_n| \\
&\leq |a_0| + M_q + |\alpha_{q+1} - \rho_r\alpha_q| + |\rho_r\alpha_q - \alpha_q| + \cdots + |\alpha_p - k_r\alpha_p| \\
&\quad +|k_r\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_i\beta_q| + |\rho_i\beta_q - \beta_q| + \cdots + |\beta_p - k_i\beta_p| \\
&\quad +|k_i\beta_p - \beta_{p-1}| + M_p + |a_n| \\
&= |a_0| + M_q + (\alpha_{q+1} - \rho_r\alpha_q) + |\alpha_q||\rho_r - 1| + \cdots + |\alpha_p||1 - k_r| \\
&\quad +(k_r\alpha_p - \alpha_{p-1}) + (\beta_{q+1} - \rho_i\beta_q) + |\beta_q||\rho_i - 1| + \cdots + |\beta_p||1 - k_i|
\end{aligned}$$

$$
\begin{aligned}
&+(k_i\beta_p - \beta_{p-1}) + M_p + |a_n| \\
= \;& |a_0| + M_q + (\alpha_{q+1} - \rho_r\alpha_q) + |\alpha_q|(1-\rho_r) + \cdots + |\alpha_p|(k_r - 1) \\
&+(k_r\alpha_p - \alpha_{p-1}) + (\beta_{q+1} - \rho_i\beta_q) + |\beta_q|(1-\rho_i) + \cdots + |\beta_p|(k_i - 1) \\
&+(k_i\beta_p - \beta_{p-1}) + M_p + |a_n| \\
= \;& |a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r-1) + k_r\alpha_p - \rho_i\beta_q \\
&+|\beta_q|(1-\rho_i) + |\beta_p|(k_i-1) + k_i\beta_p + M_p + |a_n| \\
= \;& M.
\end{aligned}
$$

Since $F(z)$ is analytic in $|z| \le 1$, and $|F(z)| \le M$ for $|z| = 1$ so by Theorem 1.14 and the Maximum Modulus Theorem, the number of zeros of F (and hence of P) in $|z| \le \delta$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$ and

$$
\begin{aligned}
M \;=\;& |a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r-1) + k_r\alpha_p - \rho_i\beta_q \\
&+|\beta_q|(1-\rho_i) + |\beta_p|(k_i-1) + k_i\beta_p + M_p + |a_n|.
\end{aligned}
$$

□

Notice that when $q = 0$, $p = n$, and $k = \rho = 1$, then Theorem 3.1 reduces to Theorem 1.20. We can further reduce this to Theorem 1.15 by having β_ℓ for $0 \le \ell \le n$, and $a_0 > 0$. These parameters applied to a lacunary polynomial gives:

Corollary 3.2. *Let $P(z)$ be an nth degree lacunary polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$, $a_0 \neq 0$. Suppose for $1 \le m \le q \le p \le n$, $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$,* *for some real* k_r*,* k_i*,* ρ_r*,* ρ_i *where* $k_r \geq 1$*,* $k_i \geq 1$*,* $0 < \rho_r \leq 1$*,* $0 < \rho_i \leq 1$ *satisfying*

$$\rho_r\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r\alpha_p,$$

$$\rho_i\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i\beta_p.$$

Then the number of zeros of $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$*,*

$$\begin{aligned} M &= |a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1-\rho_r) + |\alpha_p|(k_r - 1) + k_r\alpha_p - \rho_i\beta_q \\ &\quad + |\beta_q|(1-\rho_i) + |\beta_p|(k_i - 1) + k_i\beta_p + M_p + |a_n|, \end{aligned}$$

$M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Notice when $m = q$, then Corollary 3.2 reduces to Theorem 3.1. We can now consider these parameters applied to a dual gap polynomial, which gives:

Corollary 3.3. *Let* $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$*,* $a_0 \neq 0$*,* $a_n \neq 0$ *be a polynomial of degree* n*,* $1 \leq m \leq q \leq p \leq m' \leq n-1$*,* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$*, for some real* k_r*,* k_i*,* ρ_r*,* ρ_i *where* $k_r \geq 1$*,* $k_i \geq 1$*,* $0 < \rho_r \leq 1$*,* $0 < \rho_i \leq 1$*,*

$$\rho_r\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{p-1} \leq k_r\alpha_p,$$

$$\rho_i\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{p-1} \leq k_i\beta_p.$$

Then the number of zeros of $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$,

$$\begin{aligned} M &= |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha \\ &\quad + k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|, \end{aligned}$$

$M_q = \displaystyle\sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \displaystyle\sum_{\ell=p+1}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

Notice when $m' = n-1$, then Corollary 3.3 reduces to Corollary 3.2. We applied Aziz and Zargar's k and ρ parameters, as well as Shah's p and q parameter to the *real* and *imaginary* parts. We will now apply these conditions to the moduli of these complex coefficients.

Theorem 3.4. *Let* $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$ *be a polynomial of degree* n *such that for some* $k \geq 1$ *and* $0 < \rho \leq 1$,

$$\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{p-1}| \leq k|a_p|.$$

Then the number of zeros of $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$,

$$\begin{aligned} M &= |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha \\ &\quad + k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|, \end{aligned}$$

$M_q = \displaystyle\sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \displaystyle\sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 3.4. Consider

$$F(z) = (1-z)P(z)$$

$$
\begin{aligned}
&= a_0 + (a_1 - a_0)z + \cdots + (a_{q+1} - a_q)z^q + \cdots + (a_p - a_{p-1})z^p \\
&\quad + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}.
\end{aligned}
$$

For $|z| = 1$ we have,

$$
\begin{aligned}
|F(z)| &= a_0 + (a_1 - a_0)z + \cdots + (a_{q+1} - a_q)z^q + \cdots + (a_p - a_{p-1})z^p + \cdots \\
&\quad +(a_n - a_{n-1})z^n - a_n z^{n+1} \\
&\leq |a_0| + |a_1 - a_0||z| + \cdots + |a_{q+1} - a_q||z|^q + \cdots + |a_p - a_{p-1}||z|^p \\
&\quad + \cdots |a_n - a_{n-1}||z|^n + |-a_n||z|^{n+1} \\
&= |a_0| + |a_1 - a_0| + \cdots + |a_{q+1} - a_q| + \cdots + |a_p - a_{p-1}| \\
&\quad |a_n - a_{n-1}| + |a_n| \\
&= |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |a_{q+1} - \rho a_q + \rho a_q - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| \\
&\quad +|a_p - k a_p + k a_p - a_{p-1}| + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| + |a_n| \\
&\leq |a_0| + M_q + |a_{q+1} - \rho a_q| + |\rho a_q - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - k a_p| \\
&\quad +|k a_p - a_{p-1}| + M_p + |a_n|,
\end{aligned}
$$

where $\sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $\sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

$$
\begin{aligned}
|F(z)| &\leq |a_0| + M_q + (|a_{q+1}| - |\rho a_q|)\cos\alpha + (|a_{q+1}| + |\rho a_q|)\sin\alpha + |a_q|(1-\rho) \\
&\quad + \sum_{\ell=q+2}^{p-1} (|a_\ell| - |a_{\ell-1}|)\cos\alpha + \sum_{\ell=q+2}^{p-1} (|a_\ell| + |a_{\ell-1}|)\sin\alpha + |a_p|(k-1) \\
&\quad +(|k a_p| - |a_{p-1}|)\cos\alpha + (|k a_p| + |a_{p-1}|)\sin\alpha + M_p + |a_n| \\
&= |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho|a_q|\sin\alpha \\
&\quad +|a_q|(1-\rho) + \sum_{\ell=q+2}^{p-1} |a_\ell|\cos\alpha - \sum_{\ell=q+2}^{p-1} |a_{\ell-1}|\cos\alpha + \sum_{\ell=q+2}^{p-1} |a_\ell|\sin\alpha
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{\ell=q+2}^{p-1} |a_{\ell-1}| \sin\alpha + |a_p|(k-1) + k|a_p|\cos\alpha - |a_{p-1}|\cos\alpha + k|a_p|\sin\alpha \\
& + |a_{p-1}|\sin\alpha + M_p + |a_n| \\
= \;& |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho|a_q|\sin\alpha \\
& + |a_q|(1-\rho) + |a_{p-1}|\cos\alpha + \sum_{\ell=q+2}^{p-2} |a_\ell|\cos\alpha - |a_{q+1}|\cos\alpha \\
& + \sum_{\ell=q+2}^{p-2} |a_\ell|\cos\alpha + |a_{p-1}|\sin\alpha + \sum_{\ell=q+2}^{p-2} |a_\ell|\sin\alpha + |a_{q+1}|\sin\alpha \\
& + \sum_{\ell=q+2}^{p-2} |a_\ell|\sin\alpha + |a_p|(k-1) + k|a_p|\cos\alpha - |a_{p-1}|\cos\alpha + k|a_p|\sin\alpha \\
& + |a_{p-1}|\sin\alpha + M_p + |a_n| \\
= \;& |a_0| + M_q - \rho|a_q|\cos\alpha + 2|a_{q+1}|\sin\alpha + \rho|a_q|\sin\alpha + |a_q|(1-\rho) \\
& + 2|a_{p-1}|\sin\alpha + 2\sum_{\ell=q+2}^{p-2} |a_\ell|\sin\alpha + |a_p|(k-1) + k|a_p|\cos\alpha + k|a_p|\sin\alpha \\
& + M_p + |a_n| \\
= \;& |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell|\sin\alpha \\
& + k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n| \\
= \;& M.
\end{aligned}
$$

Since $F(z)$ is analytic in $|z| \leq 1$, and $|F(z)| \leq M$ for $|z| = 1$ so by Theorem 1.14 and the Maximum Modulus Theorem, the number of zeros of F (and hence of P) in $|z| \leq \delta$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$, and

$$M = |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell|\sin\alpha$$

$$+k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|.$$

□

Notice when $\rho = k = 1$, $q = 0$, and $p = n$, then Theorem 3.4 reduces to Theorem 1.18. Now with similar parameters applied to lacunary polynomials, we get:

Corollary 3.5. *Let $P(z)$ be a lacunary polynomial of the form $P(z) = a_0 + \sum_{\ell=1}^{n} a_\ell z^\ell$, $a_0 \neq 0$ be a polynomial of degree n where $1 \leq m \leq q \leq p \leq n$ and for some $k \geq 1$, $0 < \rho \leq 1$,*

$$\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{p-1}| \leq k|a_p|.$$

Then the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$ and

$$\begin{aligned} M &= |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha \\ &\quad + k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n|, \end{aligned}$$

$M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

When $m = 1$, then Corollary 3.5 reduces to Theorem 3.4. We will now consider dual gap polynomials:

Corollary 3.6. *Let $P(z) = a_0 + \sum_{\ell=1}^{m'} a_\ell z^\ell + a_n z^n$, $a_0 \neq 0$, $a_n \neq 0$ be a polynomial of degree n where $1 \leq m \leq q \leq p \leq m' \leq n-1$ and for some $k \geq 1$, $0 < \rho \leq 1$,*

$$\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{p-1}| \leq k|a_p|.$$

Then the number of zeros of $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log {}^{1}\!/\!_{\delta}} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$ *and*

$$\begin{aligned} M &= |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha \\ &\quad + k|a_p|(\sin\alpha + \cos\alpha + 1) - |a_p| + M_p + |a_n| \end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

Notice when $m' = n - 1$, then Corollary 3.6 reduces to Corollary 3.5.

Throughout this chapter we considered polynomials which satisfied hypotheses also found in Chapter 2. As we continue to generalize these hypotheses, we will further relax the monotone conditions to continue counting the numbers of zeros with a bound.

4 LOCATIONS OF ZEROS, MONOTONE WITH A REVERSAL

In Chapter 2, we applied a ρ, k condition, along with the q, p condition to find inner and outer bounds for locations of zeros for complex polynomials with complex coefficients by applying the parameters to the *real* and *imaginary* parts of the coefficients as well as by applying these parameters to the moduli of the coefficients. We applied these parameters to a set of strictly monotonically increasing coefficients. We will now apply these parameters to coefficients with a monotone behavior, however, we will apply a reversal to the monotone behavior and we get:

Theorem 4.1. *Let* $P(z) = \sum_{\ell=1}^{n} a_\ell z^\ell$ *be an nth degree polynomial with complex coefficients such that for* $Re(a_\ell) = \alpha_\ell$, $Im(a_\ell) = \beta_\ell$, $0 < \rho_{r_1} \leq 1$, $0 < \rho_{r_2} \leq 1$, $0 < \rho_{i_1} \leq 1$, $0 < \rho_{i_2} \leq 1$, $k_r \geq 1$, $k_i \geq 1$, *and*

$$\rho_{r_1}\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \rho_{r_2}\alpha_p,$$

$$\rho_{i_1}\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{j-1} \leq k_i\beta_j \geq \beta_{j+1} \geq \cdots \geq \rho_{i_2}\beta_p, q \leq j \leq p.$$

Then all zeros of $P(z)$ *lie in the disk*

$$\begin{aligned} |z| \leq \; & \frac{1}{|a_n|}\Bigg(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j \\ & + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\ & + |\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Bigg) \end{aligned}$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 4.1. Let $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_j z^j + \cdots + a_p z^p + \cdots + a_n z^n$ be a polynomial of degree n with complex coefficients such that for $0 < \rho_{r_1} \leq 1$,

$0 < \rho_{r_2} \le 1$, $0 < \rho_{i_1} \le 1$, $0 < \rho_{i_2} \le 1$, $k_r \ge 1$, $k_i \ge 1$ then $\rho_{r_1}\alpha_q \le \alpha_{q+1} \le \cdots \le \alpha_{j-1} \le k_r\alpha_j \ge \alpha_{j+1} \ge \cdots \ge \rho_{r_2}\alpha_p$, and $\rho_{i_1}\beta_q \le \beta_{q+1} \le \cdots \le \beta_{j-1} \le k_i\beta_j \ge \beta_{j+1} \ge \cdots \ge \rho_{i_2}\beta_p$ for $q \le j \le p$. Define f to be the equation

$$\begin{aligned} P(z)(1-z) &= a_0 + (a_1 - a_0)z + \cdots + (a_q - a_{q-1})z^q + \cdots + (a_p - a_{p-1})z^p + \cdots \\ &\quad + (a_n - a_{n-1})z^n - a_n z^{n+1} \\ &= f - a_n z^{n+1}. \end{aligned}$$

Then for $|z| = 1$

$$\begin{aligned} |f(z)| &\le |a_0| + |a_1 - a_0| + \cdots + |a_{q+1} - a_q| + \cdots + |a_j - a_{j-1}| + \cdots + |a_p - a_{p-1}| \\ &\quad + \cdots + |a_n - a_{n-1}|. \end{aligned}$$

Let $\alpha_\ell = \mathrm{Re}(a_\ell)$ and $\beta_\ell = \mathrm{Im}(a_\ell)$ for $q \le \ell \le p$. Thus

$$\begin{aligned} |f(z)| &\le |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |\alpha_{q+1} + i\beta_{q+1} - \alpha_q - i\alpha_q| + \cdots \\ &\quad + |\alpha_j + i\beta_j - \alpha_{j-1} - i\beta_{j-1}| + |\alpha_{j+1} + i\beta_{j+1} - \alpha_j - i\beta_j| + \cdots \\ &\quad + |\alpha_p + i\beta_p - \alpha_{p-1} - i\beta_{p-1}| + \sum_{\ell=p+1}^{n} |a_j - a_{j-1}|. \end{aligned}$$

Let $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. Hence

$$\begin{aligned} |f(z)| &\le |a_0| + M_q + |\alpha_{q+1} - \alpha_q| + \cdots + |\alpha_j - \alpha_{j-1}| + |\alpha_{j+1} - \alpha_j| + \cdots \\ &\quad + |\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \beta_q| + \cdots + |\beta_j - \beta_{j-1}| + |\beta_{j+1} - \beta_j| + \cdots \\ &\quad + |\beta_p - \beta_{p-1}| + M_p \\ &= |a_0| + M_q + |\alpha_{q+1} - \rho_{r_1}\alpha_q + \rho_{r_1}\alpha_q - \alpha_q| + \cdots + |\alpha_j - k_r\alpha_j + k_r\alpha_j - \alpha_{j-1}| \\ &\quad + |\alpha_{j+1} - k_r\alpha_j + k_r\alpha_j - \alpha_j| + \cdots + |\alpha_p - \rho_{r_2}\alpha_p + \rho_{r_2}\alpha_p - \alpha_{p-1}| \end{aligned}$$

$$
\begin{aligned}
&+|\beta_{q+1}-\rho_{i_1}\beta_q+\rho_{i_1}\beta_q-\beta_q|+\cdots+|\beta_j-k_i\beta_j+k_i\beta_j-\beta_{j-1}| \\
&+|\beta_{j+1}-k_i\beta_j+k_i\beta_j-\beta_j|+\cdots+|\beta_p-\rho_{i_2}\beta_p+\rho_{i_2}\beta_p-\beta_{p-1}|+M_p \\
\leq\ & |a_0|+M_q+|\alpha_{q+1}-\rho_{r_1}\alpha_q|+|\rho_{r_1}\alpha_q-\alpha_q|+\cdots+|\alpha_j-k_r\alpha_j| \\
&+|k_r\alpha_j-\alpha_{j-1}|+|\alpha_{j+1}-k_r\alpha_j|+|k_r\alpha_j-\alpha_j|+\cdots+|\alpha_p-\rho_{r_2}\alpha_p| \\
&+|\rho_{r_2}\alpha_p-\alpha_{p-1}|+|\beta_{q+1}-\rho_{i_1}\beta_q|+|\rho_{i_1}\beta_q-\beta_q|+\cdots+|\beta_j-k_i\beta_j| \\
&+|k_i\beta_j-\beta_{j-1}|+|\beta_{j+1}-k_i\beta_j|+|k_i\beta_j-\beta_j|+\cdots+|\beta_p-\rho_{i_2}\beta_p| \\
&+|\rho_{i_2}\beta_p-\beta_{p-1}|+M_p \\
=\ & |a_0|+M_q+(\alpha_{q+1}-\rho_{r_1}\alpha_q)+|\alpha_q|(1-\rho_{r_1})+\cdots+|\alpha_j|(k_r-1) \\
&+(k_r\alpha_j-\alpha_{j-1})+(k_r\alpha_j-\alpha_{j+1})+|a_j|(k_r-1)+\cdots+|\alpha_p|(1-\rho_{r_2}) \\
&+(a_{p-1}-\rho_{r_2}\alpha_p)+(\beta_{q+1}-\rho_{i_1}\beta_q)+|\beta_q|(1-\rho_{i_1})+\cdots+|\beta_j|(k-1) \\
&+(k_i\beta_j-\beta_{j-1})+(k_i\beta_j-\beta_{j+1})+|\beta_j|(k_i-1)+\cdots+|\beta_p|(1-\rho_{i_2}) \\
&+(\beta_{p-1}-\rho_{i_2}\beta_p)+M_p \\
=\ & |a_0|+M_q-\rho_{r_1}\alpha_q+|\alpha_q|(1-\rho_{r_1})+2|\alpha_j|(k_r-1)+2k_r\alpha_j+|\alpha_p|(1-\rho_{r_2}) \\
&-\rho_{r_2}\alpha_p-\rho_{i_1}\beta_q+|\beta_q|(1-\rho_{i_1})+2|\beta_j|(k_i-1)+2k_i\beta_j+|\beta_p|(1-\rho_{i_2}) \\
&-\rho_{i_2}\beta_p+M_p.
\end{aligned}
$$

We can notice $z^n f\left(\frac{1}{z}\right)=\sum_{\ell=0}^{n}(a_\ell-a_{\ell-1})z^{n-\ell}$ where $a_{-1}=0$ has the same bound on $|z|=1$ as $f(z)$. Namely,

$$
\begin{aligned}
\left|z^n f\left(\frac{1}{z}\right)\right| \leq\ & |a_0|+M_q-\rho_{r_1}\alpha_q+|\alpha_q|(1-\rho_{r_1})+2|\alpha_j|(k_r-1)+2k_r\alpha_j \\
&+|\alpha_p|(1-\rho_{r_2})-\rho_{r_2}\alpha_p-\rho_{i_1}\beta_q+|\beta_q|(1-\rho_{i_1})+2|\beta_j|(k_i-1)+2k_i\beta_j \\
&+|\beta_p|(1-\rho_{i_2})-\rho_{i_2}\beta_p+M_p
\end{aligned}
$$

is analytic in $|z|\leq 1$ where we consider this function to have the value a_n-a_{n-1} at

$z = 0$. We have

$$\begin{aligned}\left|z^n f\left(\frac{1}{z}\right)\right| &\leq |a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j \\ &\quad +|\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\ &\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\end{aligned}$$

for $|z| \leq 1$ by the Maximum Modulus Theorem. Thus

$$\begin{aligned}\left|f\left(\frac{1}{z}\right)\right| &\leq \frac{1}{|z|^n}\Big(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j \\ &\quad +|\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\ &\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Big)\end{aligned}$$

for $|z| \leq 1$. Replacing z with $\frac{1}{z}$ we have

$$\begin{aligned}\left|f\left(\frac{1}{z}\right)\right| &\leq \Big(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j \\ &\quad +|\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\ &\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Big)|z|^n\end{aligned}$$

for $|z| \geq 1$. We have

$$\begin{aligned}|P(z)(1-z)| &= |f(z) - a_n z^{n+1}| \\ &\geq |a_n||z^{n+1}| - |f(z)| \\ &\geq |a_n||z^{n+1}| - \Big(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) \\ &\quad +2k_r\alpha_j + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) \\ &\quad +2|\beta_j|(k_i - 1) + 2k_i\beta_j + |\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Big)|z|^n \\ &= |z^n|\Big[|a_n||z| - \Big(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1)\end{aligned}$$

$$+2k_r\alpha_j + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1})$$
$$+2|\beta_j|(k_i-1) + 2k_i\beta_j + |\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Big)\Big].$$

So if

$$\begin{aligned}|z| &> \frac{1}{|a_n|}\Big(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r-1) + 2k_r\alpha_j \\ &\quad +|\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i-1) + 2k_i\beta_j \\ &\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Big)\end{aligned}$$

then

$$\begin{aligned}0 &\neq |z^n|\Big[|a_n||z| - \Big(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r-1) + 2k_r\alpha_j \\ &\quad +|\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i-1) + 2k_i\beta_j \\ &\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Big)\Big].\end{aligned}$$

Therefore all zeros of $P(z)$ lie in

$$\begin{aligned}|z| &\leq \frac{1}{|a_n|}\Big(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r-1) + 2k_r\alpha_j \\ &\quad +|\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i-1) + 2k_i\beta_j \\ &\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Big)\end{aligned}$$

□

Notice that Theorem 4.1 reduces to Theorem 2.2 when $j = p$. By applying a reversal condition, this allows us to examine polynomials which have strictly decreasing subset of coefficients when $j = q$. We also have the following for lacunary polynomials:

Corollary 4.2. *Let $P(z)$ be an nth degree lacunary polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$, $a_0 \neq 0$. Suppose $P(z)$ has complex coefficients such that $Re(a_\ell) = \alpha_\ell$, $Im(a_\ell) = \beta_\ell$, $0 < \rho_{r_1} \leq 1$, $0 < \rho_{r_2} \leq 1$, $0 < \rho_{i_1} \leq 1$, $0 < \rho_{i_2} \leq 1$, $k_r \geq 1$, and $k_i \geq 1$ satisfies*

$$\rho_{r_1}\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \rho_{r_2}\alpha_p,$$

$$\rho_{i_1}\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{j-1} \leq k_i\beta_j \geq \beta_{j+1} \geq \cdots \geq \rho_{i_2}\beta_p, 1 \leq m \leq q \leq j \leq p.$$

Then all zeros of $P(z)$ lie in the disk

$$\begin{aligned} |z| \quad > \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j \\ & +|\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\ & +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p\Big) \end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Notice that Corollary 4.2 reduces to Theorem 4.1 when $m = 1$. Notice the only change is M_q. Similar to Corollary 2.4 we can consider polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{m'} + a_n z^n$, where $a_0 \neq 0$, and $a_n \neq 0$. Will have the same M_q as in Corollary 4.2 and $M_p = \sum_{\ell=p}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$. Now that we have an outer bound for locations of zeros with a reversal, we will now find the inner bound to which no zeros will be located. Thus we get:

Theorem 4.3. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n such that $\alpha_i = Re(a_i)$ and $\beta_i = Im(a_i)$, $0 < \rho_{r_1} \leq 1$, $0 < \rho_{r_2} \leq 1$, $0 < \rho_{i_1} \leq 1$, $0 < \rho_{i_2} \leq 1$, $k_r \geq 1$, $k_i \geq 1$, and*

$$\rho_{r_1}\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \rho_{r_2}\alpha_p,$$

$$\rho_{i_1}\beta_q \le \beta_{q+1} \le \cdots \le \beta_{j-1} \le k_i\beta_j \ge \beta_{j+1} \ge \cdots \ge \rho_{i_2}\beta_p, q \le j \le p.$$

Then $P(z)$ does not vanish in

$$\begin{aligned}
|z| \quad < \quad & \min\left\{1, |a_0| \Big/ \Big(M_q + |\alpha_q|(1-\rho_{r_1}) - \rho_{r_1}\alpha_q + |\beta_q|(1-\rho_{i_1}) - \rho_{r_1}\beta_q + 2k_r\alpha_j \right. \\
& +2|\alpha_j|(k_r-1) + 2k_i\beta_j + 2|\beta_j|(k_i-1) + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p + |\beta_p|(1-\rho_{i_2}) \\
& \left. -\rho_{i_2}\beta_p + M_p + |a_n| \Big) \right\},
\end{aligned}$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 4.3. Consider the reciprocal polynomial

$$S(z) = z^n P\left(\frac{1}{z}\right) = a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + \cdots + a_p z^{n-p} + \cdots + a_{n-1} z + a_n.$$

Let

$$\begin{aligned}
H(z) &= (1-z)S(z) \\
&= -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_q - a_{q+1})z^{n-q} \\
&\quad + \cdots + (a_p - a_{p+1})z^{n-p} + \cdots + (a_{n-2} - a_{n-1})z^2 + (a_{n-1} - a_n)z + a_n.
\end{aligned}$$

This gives

$$\begin{aligned}
|H(z)| \quad \ge \quad & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_q - a_{q+1}||z|^{n-q} \\
& + \cdots + |a_{j-1} - a_j||z|^{n-j+1} + |a_j - a_{j+1}||z^{n-j}| + \cdots + |a_p - a_{p+1}||z|^{n-p} \\
& +|a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n| \Big) \\
= \quad & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots \\
& +|\alpha_q + i\beta_q - \alpha_{q+1} - i\beta_{q+1}||z|^{n-q} + \cdots \\
& +|\alpha_{j-1} + i\beta_{j-1} - \alpha_j - i\beta_j||z|^{n-j+1} + |\alpha_j + i\beta_j - \alpha_{j+1} - i\beta_{j+1}||z^{n-j}|
\end{aligned}$$

$$+\cdots+|\alpha_p+i\beta_p-\alpha_{p+1}-i\beta_{p+1}||z|^{n-p}$$
$$+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big)$$

where $\alpha_\ell = \mathrm{Re}(a_\ell)$ and $\beta_\ell = \mathrm{Im}(a_\ell)$ for $q \le \ell \le p$. This gives

$$\begin{aligned}
|H(z)| \;\ge\;& |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots+|\alpha_q-\alpha_{q+1}||z|^{n-q}\\
&+|\beta_q-\beta_{q+1}||z|^{n-q}+\cdots+|\alpha_{j-1}-\alpha_j||z|^{n-j+1}+|\beta_{j-1}-\beta_j||z|^{n-j+1}\\
&+|\alpha_j-\alpha_{j+1}||z|^{n-j}+|\beta_j-\beta_{j+1}||z|^{n-j}+\cdots+|\alpha_p-\alpha_{p+1}||z|^{n-p}\\
&+|\beta_p-\beta_{p+1}||z|^{n-p}+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big)\\
=\;& |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots\\
&+|\alpha_q-\rho_{r_1}\alpha_q+\rho_{r_1}\alpha_q-\alpha_{q+1}||z|^{n-q}+|\beta_q-\rho_{i_1}\beta_q+\rho_{i_1}\beta_q-\beta_{q+1}||z|^{n-q}\\
&+\cdots+|\alpha_{j-1}-k_r\alpha_j+k_r\alpha_j-\alpha_j||z|^{n-j+1}\\
&+|\beta_{j-1}-k_i\beta_j+k_i\beta_j-\beta_j||z|^{n-j+1}+|\alpha_j-k_r\alpha_j+k_r\alpha_j-\alpha_{j+1}||z|^{n-j}\\
&+|\beta_j-k_i\beta_j+k_i\beta_j-\beta_{j+1}||z|^{n-j}+\cdots\\
&+|\alpha_p-\rho_{r_2}\alpha_p+\rho_{r_2}\alpha_p-\alpha_{p+1}||z|^{n-p}+|\beta_p-\rho_{i_2}\beta_p+\rho_{i_2}\beta_p-\beta_{p+1}||z|^{n-p}\\
&+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big)\\
\ge\;& |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots+|\alpha_q-\rho_{r_1}\alpha_q||z|^{n-q}\\
&+|\rho_{r_1}\alpha_q-\alpha_{q+1}||z|^{n-q}+|\beta_q-\rho_{i_1}\beta_q||z|^{n-q}+|\rho_{i_1}\beta_q-\beta_{q+1}||z|^{n-q}+\cdots\\
&+|\alpha_{j-1}-k_r\alpha_j||z|^{n-j+1}+|k_r\alpha_j-\alpha_j||z|^{n-j+1}+|\beta_{j-1}-k_i\beta_j||z|^{n-j+1}\\
&+|k_i\beta_j-\beta_j||z|^{n-j+1}+|\alpha_j-k_r\alpha_j||z|^{n-j}+|k_r\alpha_j-\alpha_{j+1}||z|^{n-j}\\
&+|\beta_j-k_i\beta_j||z|^{n-j}+|k_i\beta_j-\beta_{j+1}||z|^{n-j}+\cdots+|\alpha_p-\rho_{r_2}\alpha_p||z|^{n-p}\\
&+|\rho_{r_2}\alpha_p-\alpha_{p+1}||z|^{n-p}+|\beta_p-\rho_{i_2}\beta_p||z|^{n-p}+|\rho_{i_2}\beta_p-\beta_{p+1}||z|^{n-p}
\end{aligned}$$

$$
\begin{aligned}
& +\cdots+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big) \\
= \; & |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots+|\alpha_q|(1-\rho_{r_1})|z|^{n-q} \\
& +(\alpha_{q+1}-\rho_{r_1}\alpha_q)|z|^{n-q}+|\beta_q|(1-\rho_{i_1})|z|^{n-q}+(\beta_{q+1}-\rho_{r_1}\beta_q)|z|^{n-q}+\cdots \\
& +(k_r\alpha_j-\alpha_{j-1})|z|^{n-j+1}+|\alpha_j|(k_r-1)|z|^{n-j+1}+(k_i\beta_j-\beta_{j-1})|z|^{n-j+1} \\
& +|\beta_j|(k_i-1)|z|^{n-j+1}+|\alpha_j|(k_r-1)|z|^{n-j}+(k_r\alpha_j-\alpha_{j+1})|z|^{n-j} \\
& +|\beta_j|(k_i-1)|z|^{n-j}+(k_i\beta_j-\beta_{j+1})|z|^{n-j}+\cdots+|\alpha_p|(1-\rho_{r_2})|z|^{n-p} \\
& +(\alpha_{p+1}-\rho_{r_2}\alpha_p)|z|^{n-p}+|\beta_p|(1-\rho_{i_2})|z|^{n-p}+(\beta_{p+1}-\rho_{i_2}\beta_p)|z|^{n-p} \\
& +\cdots+|a_{n-2}-a_{n-1}||z|^2+|a_{n-1}-a_n||z|+|a_n|\Big) \\
= \; & |z|^n\Bigg[|a_0||z|-\Bigg(|a_0-a_1|+\frac{|a_1-a_2|}{|z|}+\cdots+\frac{|\alpha_q|(1-\rho_{r_1})}{|z|^q} \\
& +\frac{\alpha_{q+1}-\rho_{r_1}\alpha_q}{|z|^q}+\frac{|\beta_q|(1-\rho_{i_1})}{|z|^q}+\frac{\beta_{q+1}-\rho_{r_1}\beta_q}{|z|^q}+\cdots+\frac{k_r\alpha_j-\alpha_{j-1}}{|z|^{j-1}} \\
& +\frac{|\alpha_j|(k_r-1)}{|z|^{j-1}}+\frac{k_i\beta_j-\beta_{j-1}}{|z|^{j-1}}+\frac{|\beta_j|(k_i-1)}{|z|^{j-1}}+\frac{|\alpha_j|(k_r-1)}{|z|^j} \\
& +\frac{k_r\alpha_j-\alpha_{j+1}}{|z|^j}+\frac{|\beta_j|(k_i-1)}{|z|^j}+\frac{k_i\beta_j-\beta_{j+1}}{|z|^j}+\cdots+\frac{|\alpha_p|(1-\rho_{r_2})}{|z|^p} \\
& +\frac{\alpha_{p+1}-\rho_{r_2}\alpha_p}{|z|^p}+\frac{|\beta_p|(1-\rho_{i_2})}{|z|^p}+\frac{\beta_{p+1}-\rho_{i_2}\beta_p}{|z|^p}+\cdots+\frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}} \\
& +\frac{|a_{n-1}-a_n|}{|z|^{n-1}}+\frac{|a_n|}{|z|^n}\Bigg)\Bigg].
\end{aligned}
$$

Now for $|z|>1$ so that $\dfrac{1}{|z|^{n-\ell}}<1$, for $0\le \ell<n$ we have

$$
\begin{aligned}
|H(z)| \;\geq\; & |z|^n\Big[|a_0||z|-\Big(|a_0-a_1|+|a_1-a_2|+\cdots+|\alpha_q|(1-\rho_{r_1}) \\
& +(\alpha_{q+1}-\rho_{r_1}\alpha_q)+|\beta_q|(1-\rho_{i_1})+(\beta_{q+1}-\rho_{r_1}\beta_q)+\cdots+(k_r\alpha_j-\alpha_{j-1}) \\
& +|\alpha_j|(k_r-1)+(k_i\beta_j-\beta_{j-1})+|\beta_j|(k_i-1)+|\alpha_j|(k_r-1) \\
& +(k_r\alpha_j-\alpha_{j+1})+|\beta_j|(k_i-1)+(k_i\beta_j-\beta_{j+1})+\cdots+|\alpha_p|(1-\rho_{r_2})
\end{aligned}
$$

$$
\begin{aligned}
&+(\alpha_{p+1} - \rho_{r_2}\alpha_p) + |\beta_p|(1-\rho_{i_2}) + (\beta_{p+1} - \rho_{i_2}\beta_p) + \cdots + |a_{n-2} - a_{n-1}| \\
&+|a_{n-1} - a_n| + |a_n|\Big)\Big] \\
= \; & |z|^n\Bigg[|a_0||z| - \Bigg(\sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}| + |\alpha_q|(1-\rho_{r_1}) - \rho_{r_1}\alpha_q + |\beta_q|(1-\rho_{i_1}) \\
&-\rho_{r_1}\beta_q + 2k_r\alpha_j + 2|\alpha_j|(k_r - 1) + 2k_i\beta_j + 2|\beta_j|(k_i-1) + |\alpha_p|(1-\rho_{r_2}) \\
&-\rho_{r_2}\alpha_p + |\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}| + |a_n|\Bigg)\Bigg] \\
> \; & 0
\end{aligned}
$$

if

$$
\begin{aligned}
|z| \; > \; & \frac{1}{|a_0|}\Bigg(M_q + |\alpha_q|(1-\rho_{r_1}) - \rho_{r_1}\alpha_q + |\beta_q|(1-\rho_{i_1}) - \rho_{r_1}\beta_q + 2k_r\alpha_j \\
&+2|\alpha_j|(k_r-1) + 2k_i\beta_j + 2|\beta_j|(k_i-1) + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p + |\beta_p|(1-\rho_{i_2}) \\
&-\rho_{i_2}\beta_p + M_p + |a_n|\Bigg)
\end{aligned}
$$

where $M_q = \sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}|$. Thus all zeros of $H(z)$ whose modulus is greater than 1 lie in

$$
\begin{aligned}
|z| \; \leq \; & \frac{1}{|a_0|}\Bigg(M_q + |\alpha_q|(1-\rho_{r_1}) - \rho_{r_1}\alpha_q + |\beta_q|(1-\rho_{i_1}) - \rho_{r_1}\beta_q + 2k_r\alpha_j \\
&+2|\alpha_j|(k_r-1) + 2k_i\beta_j + 2|\beta_j|(k_i-1) + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p + |\beta_p|(1-\rho_{i_2}) \\
&-\rho_{i_2}\beta_p + M_p + |a_n|\Bigg).
\end{aligned}
$$

Hence all zeros of $H(z)$ and hence of $S(z)$ lie in

$$
\begin{aligned}
|z| \; \leq \; & \max\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |\alpha_q|(1-\rho_{r_1}) - \rho_{r_1}\alpha_q + |\beta_q|(1-\rho_{i_1}) - \rho_{r_1}\beta_q + 2k_r\alpha_j \\
&+2|\alpha_j|(k_r-1) + 2k_i\beta_j + 2|\beta_j|(k_i-1) + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p + |\beta_p|(1-\rho_{i_2}) \\
&-\rho_{i_2}\beta_p + M_p + |a_n|\Bigg)\Bigg\}.
\end{aligned}
$$

Therefore all the zeros of $P(z)$ lie in

$$\begin{aligned}|z| \quad &\geq \quad \min\left\{1, |a_0|\Big/\Big(M_q + |\alpha_q|(1-\rho_{r_1}) - \rho_{r_1}\alpha_q + |\beta_q|(1-\rho_{i_1}) - \rho_{r_1}\beta_q + 2k_r\alpha_j\right. \\ &\quad +2|\alpha_j|(k_r-1) + 2k_i\beta_j + 2|\beta_j|(k_i-1) + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p + |\beta_p|(1-\rho_{i_2}) \\ &\quad \left. -\rho_{i_2}\beta_p + M_p + |a_n|\Big)\right\}.\end{aligned}$$

Thus the polynomial $P(z)$ does not vanish in

$$\begin{aligned}|z| \quad &< \quad \min\left\{1, |a_0|\Big/\Big(M_q + |\alpha_q|(1-\rho_{r_1}) - \rho_{r_1}\alpha_q + |\beta_q|(1-\rho_{i_1}) - \rho_{r_1}\beta_q + 2k_r\alpha_j\right. \\ &\quad +2|\alpha_j|(k_r-1) + 2k_i\beta_j + 2|\beta_j|(k_i-1) + |\alpha_p|(1-\rho_{r_2}) - \rho_{r_2}\alpha_p + |\beta_p|(1-\rho_{i_2}) \\ &\quad \left. -\rho_{i_2}\beta_p + M_p + |a_n|\Big)\right\}.\end{aligned}$$

□

Notice that Theorem 4.3 will reduce to Theorem 2.5 when $j = p$. We also have the ability to examine the coefficients with strictly decreasing monotone behavior when $j = q$. Similarly, for a lacunary polynomial with similar parameters we have:

Corollary 4.4. *Let* $P(z) = a_0 + \sum_{\ell=m}^{n}$, $a_0 \neq 0$ *be a polynomial of degree* n *such that* $\alpha_i = Re(a_i)$ *and* $\beta_i = Im(a_i)$, $0 < \rho_{r_1} \leq 1$, $0 < \rho_{r_2} \leq 1$, $0 < \rho_{i_1} \leq 1$, $0 < \rho_{i_2} \leq 1$, $k_r \geq 1$, $k_i \geq 1$, *and*

$$\rho_{r_1}\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \rho_{r_2}\alpha_p,$$

$$\rho_{i_1}\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{j-1} \leq k_i\beta_j \geq \beta_{j+1} \geq \cdots \geq \rho_{i_2}\beta_p, q \leq j \leq p.$$

Then $P(z)$ *does not vanish in*

$$|z| \quad < \quad \min\left\{1, |a_0|\Big/\Big(M_q + |\alpha_q|(1-\rho_{r_1}) - \rho_{r_1}\alpha_q + |\beta_q|(1-\rho_{i_1}) - \rho_{r_1}\beta_q + 2k_r\alpha_j\right.$$

$+2|\alpha_j|(k_r-1)+2k_i\beta_j+2|\beta_j|(k_i-1)+|\alpha_p|(1-\rho_{r_2})-\rho_{r_2}\alpha_p+|\beta_p|(1-\rho_{i_2})$
$-\rho_{i_2}\beta_p+M_p+|a_n|\Big)\Big\}$,

where $M_q=\sum_{\ell=m}^{q}|a_\ell-a_{\ell-1}|$ *and* $M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$.

Notice Corollary 4.4 reduces to Theorem 4.3 when $m=1$. We can also examine a dual gap polynomial, and we will have the same bound, except $M_p=\sum_{\ell=p}^{m'}|a_\ell-a_{\ell-1}|+|a_n-a_{n-1}|$. Now that the parameters have been applied to the *real* and *imaginary* parts of the polynomial, we will now apply these parameters to the moduli of the coefficients and thus we get:

Theorem 4.5. *Let* $P(z)=\sum_{\ell=0}^{n}a_\ell z^\ell$ *be a polynomial of degree* n *with complex coefficients such that* $|\arg a_\ell-\beta|\leq\alpha\leq\pi/2$, $\ell=0,1,2,\ldots,n$ *for some real* β, $k\geq 1$, $0<\rho_1\leq 1$, $0<\rho_2\leq 1$,

$$\rho_1|a_q|\leq|a_{q+1}|\leq\cdots\leq|a_{j-1}|\leq k|a_j|\geq|a_{j+1}|\geq\cdots\geq\rho_2|a_p|, q\leq j\leq p.$$

Then all zeros of $P(z)$ *lie in the disk*

$$\begin{aligned}|z| \;\leq\; & \frac{1}{|a_n|}\Bigg(|a_0|+M_q+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+|a_q|+2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha\\ & +2k|a_j|(\sin\alpha+\cos\alpha+1)-2|a_j|-2|a_{j+1}|\cos\alpha+2|a_{p-1}|\cos\alpha\\ & +2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p\Bigg),\end{aligned}$$

where $M_q=\sum_{\ell=1}^{q}|a_\ell-a_{\ell-1}|$ *and* $M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$.

Proof of Theorem 4.5. Let $P(z)=a_0+a_1z+\cdots+a_qz^q+\cdots+a_jz^j+\cdots+a_pz^p+\cdots+a_nz^n$ be a polynomial of degree n with complex coefficients such that $|\arg a_\ell-\beta|\leq\alpha\leq\pi/2$,

$\ell = 0, 1, 2, \ldots, n$ for some real β, $\rho_1|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{j-1}| \leq k|a_j| \geq |a_{j+1}| \geq \cdots \geq \rho_2|a_p|$ for $q \leq j \leq p$ where $0 < \rho_1 \leq 1$, $0 < \rho_2 \leq 1$, $k \geq 1$. Without loss of generality assume $\beta = 0$. Consider

$$\begin{aligned}
P(z)(1-z) &= a_0 + (a_1 - a_0)z + \cdots + (a_{q+1} - a_q)z^q + \cdots + (a_j - a_{j-1})z^j \\
&\quad +(a_{j+1} - a_j)z^{j+1} + \cdots + (a_p - a_{p-1})z^p + \cdots + (a_n - a_{n-1})z^n \\
&\quad -a_n z^{n+1} \\
&= f(z) - a_n z^{n+1}.
\end{aligned}$$

Then if $|z| = 1$, then

$$\begin{aligned}
|f(z)| &\leq |a_0| + |a_1 - a_0| + \cdots + |a_{q+1} - a_q| + \cdots + |a_j - a_{j-1}| + |a_{j+1} - a_j| + \cdots \\
&\quad +|a_p - a_{p-1}| + \cdots + |a_n - a_{n-1}| \\
&= |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |a_{q+1} - \rho_1 a_q + \rho_1 a_q - a_q| + \cdots \\
&\quad +|a_j - ka_j + ka_j - a_{j-1}| + |a_{j+1} - ka_j + ka_j - a_j| + \cdots \\
&\quad +|a_p - \rho_2 a_p + \rho_2 a_p - a_{p-1}| + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|.
\end{aligned}$$

For $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$ then

$$\begin{aligned}
|f(z)| &\leq |a_0| + M_q + |a_{q+1} - \rho_1 a_q| + |\rho_1 a_q - a_q| + \cdots + |a_j - ka_j| + |ka_j - a_{j-1}| \\
&\quad +|a_{j+1} - ka_j| + |ka_j - a_j| + \cdots + |a_p - \rho_2 a_p| + |\rho_2 a_p - a_{p-1}| + M_p \\
&\leq |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho_1|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho_1|a_q|\sin\alpha \\
&\quad +|a_q|(1-\rho_1) + \sum_{\ell=q+2}^{j-1} |a_\ell - a_{\ell-1}| + |a_j|(k-1) + k|a_j|\cos\alpha - |a_{j-1}|\cos\alpha \\
&\quad +k|a_j|\sin\alpha + |a_{j-1}|\sin\alpha + k|a_j|\cos\alpha - |a_{j+1}|\cos\alpha + k|a_j|\sin\alpha \\
&\quad +|a_{j+1}|\sin\alpha + |a_j|(k-1) + \sum_{\ell=j+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p|(1-\rho_2) + |a_{p-1}|\cos\alpha
\end{aligned}$$

$$
\begin{aligned}
& -\rho_2|a_p|\cos\alpha + |a_{p-1}|\sin\alpha + \rho_2|a_p|\sin\alpha + M_p \\
\leq\ & |a_0| + M_q + |a_{q+1}|(\cos\alpha + \sin\alpha) + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + |a_q| \\
& + \sum_{\ell=q+2}^{j-1} (|a_\ell| - |a_{\ell-1}|)\cos\alpha + \sum_{\ell=q+2}^{j-1} (|a_\ell| + |a_{\ell-1}|)\sin\alpha \\
& +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + |a_{j-1}|(\sin\alpha - \cos\alpha) \\
& +|a_{j+1}|(\sin\alpha - \cos\alpha) + \sum_{\ell=j+2}^{p-1} (|a_{\ell-1}| - |a_\ell|)\cos\alpha \\
& + \sum_{\ell=j+2}^{p-1} (|a_{\ell-1}| + |a_\ell|)\sin\alpha + \rho_2|a_p|(\sin\alpha - \cos\alpha - 1) \\
& +|a_{p-1}|(\sin\alpha + \cos\alpha) + |a_p| + M_p \\
=\ & |a_0| + M_q + |a_{q+1}|(\cos\alpha + \sin\alpha) + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + |a_q| \\
& + \sum_{\ell=q+2}^{j-1} |a_\ell|\cos\alpha - \sum_{\ell=q+2}^{j-1} |a_{\ell-1}|\cos\alpha + \sum_{\ell=q+2}^{j-1} |a_\ell|\sin\alpha + \sum_{\ell=q+2}^{j-1} |a_{\ell-1}|\sin\alpha \\
& +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + |a_{j-1}|(\sin\alpha - \cos\alpha) \\
& +|a_{j+1}|(\sin\alpha - \cos\alpha) + \sum_{\ell=j+2}^{p-1} |a_{\ell-1}|\cos\alpha - \sum_{\ell=j+2}^{p-1} |a_\ell|\cos\alpha \\
& + \sum_{\ell=j+2}^{p-1} |a_{\ell-1}|\sin\alpha + \sum_{\ell=j+2}^{p-1} |a_\ell|\sin\alpha + \rho_2|a_p|(\sin\alpha - \cos\alpha - 1) \\
& +|a_{p-1}|(\sin\alpha + \cos\alpha) + |a_p| + M_p \\
=\ & |a_0| + M_q + |a_{q+1}|(\cos\alpha + \sin\alpha) + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + |a_q| \\
& +|a_{j-1}|\cos\alpha + \sum_{\ell=q+2}^{j-2} |a_{q+1}|\cos\alpha - |a_{q+1}|\cos\alpha - \sum_{\ell=q+2}^{j-2} |a_\ell|\cos\alpha \\
& +|a_{j-1}|\sin\alpha + \sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha + |a_{q+1}|\sin\alpha + \sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha \\
& +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + |a_{j-1}|(\sin\alpha - \cos\alpha) \\
& +|a_{j+1}|(\sin\alpha - \cos\alpha) + |a_{j+1}|\cos\alpha + \sum_{\ell=j+2}^{p-2} |a_\ell|\cos\alpha - |a_{p-1}|\cos\alpha
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{\ell=j+2}^{p-2} |a_\ell| \cos\alpha + |a_{j+1}| \sin\alpha + \sum_{\ell=j+2}^{p-2} |a_\ell| \sin\alpha + |a_{p-1}| \sin\alpha \\
& + \sum_{\ell=j+2}^{p-2} |a_\ell| \sin\alpha + \rho_2 |a_p| (\sin\alpha - \cos\alpha - 1) + |a_{p-1}| (\sin\alpha + \cos\alpha) \\
& + |a_p| + M_p \\
= \; & |a_0| + M_q + 2|a_{q+1}| \sin\alpha + \rho_1 |a_q| (\sin\alpha - \cos\alpha - 1) + |a_q| + 2|a_{j-1}| \sin\alpha \\
& + 2 \sum_{\ell=q+2}^{j-2} |a_\ell| \sin\alpha + 2k|a_j| (\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2|a_{j+1}| \sin\alpha \\
& + 2 \sum_{\ell=j+2}^{p-2} |a_\ell| \sin\alpha + 2|a_{p-1}| \sin\alpha + \rho_2 |a_p| (\sin\alpha - \cos\alpha - 1) + |a_p| + M_p \\
= \; & |a_0| + M_q + \rho_1 |a_q| (\sin\alpha - \cos\alpha - 1) + |a_q| + 2 \sum_{\ell=q+1}^{j-1} |a_\ell| \sin\alpha \\
& + 2k|a_j| (\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2 \sum_{\ell=j+1}^{p-1} |a_\ell| \sin\alpha \\
& + \rho_2 |a_p| (\sin\alpha - \cos\alpha - 1) + |a_p| + M_p.
\end{aligned}
$$

Hence also,

$$
\begin{aligned}
\left| z^n P\left(\frac{1}{z}\right) \right| \leq \; & |a_0| + M_q + \rho_1 |a_q| (\sin\alpha - \cos\alpha - 1) + |a_q| + 2 \sum_{\ell=q+1}^{j-1} |a_\ell| \sin\alpha \\
& + 2k|a_j| (\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2 \sum_{\ell=j+1}^{p-1} |a_\ell| \sin\alpha \\
& + \rho_2 |a_p| (\sin\alpha - \cos\alpha - 1) + |a_p| + M_p
\end{aligned}
$$

for $|z| = 1$. By the Maximum Modulus Theorem

$$
\begin{aligned}
\left| z^n P\left(\frac{1}{z}\right) \right| \leq \; & |a_0| + M_q + \rho_1 |a_q| (\sin\alpha - \cos\alpha - 1) + |a_q| + 2 \sum_{\ell=q+1}^{j-1} |a_\ell| \sin\alpha \\
& + 2k|a_j| (\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2 \sum_{\ell=j+1}^{p-1} |a_\ell| \sin\alpha \\
& + \rho_2 |a_p| (\sin\alpha - \cos\alpha - 1) + |a_p| + M_p
\end{aligned}
$$

holds inside the unit circle $|z| \leq 1$. If $R > 1$, then $\frac{1}{R}e^{-i\theta}$ lies inside the unit circle for every real θ. Thus it follows

$$\begin{aligned}|P(Re^{i\theta})| \leq & \Bigg(|a_0| + M_q + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + |a_q| + 2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha \\ & +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha \\ & +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p\Bigg)R^n\end{aligned}$$

for every $R \geq 1$ and θ real. Thus for every $|z| = R > 1$,

$$\begin{aligned}|P(z)(1-z)| &= |-a_n z^{n+1} + f(z)| \\ &\geq |a_n|R^{n+1} - |f(z)| \\ &\geq |a_n|R^{n+1} - \Bigg(|a_0| + M_q + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + |a_q| \\ &\quad +2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha + 2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| \\ &\quad +2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha + \rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p\Bigg)R^n \\ &= R^n\Bigg[|a_n|R - \Bigg|a_0| + M_q + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + |a_q| \\ &\quad +2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha + 2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| \\ &\quad +2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha + \rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p\Bigg)\Bigg] \\ &> 0\end{aligned}$$

if

$$R > \frac{1}{|a_n|}\Bigg(|a_0| + M_q + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + |a_q| + 2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha$$

$$+2k|a_j|(\sin\alpha+\cos\alpha+1)-2|a_j|+2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha$$

$$+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p\Bigg).$$

Therefore all zeros of $P(z)$ lie within

$$|z| \leq \frac{1}{|a_n|}\Bigg(|a_0|+M_q+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+|a_q|+2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha$$

$$+2k|a_j|(\sin\alpha+\cos\alpha+1)-2|a_j|+2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha$$

$$+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p\Bigg).$$

□

Notice when $j = p$, then Theorem 4.5 reduces to Theorem 2.8. With additional parameters as discussed in Chapter 2, it will reduce down to Theorem 1.4. Now these parameters can be applied to a lacunary polynomial:

Corollary 4.6. *Let* $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$*,* $a_0 \neq 0$ *be a polynomial of degree* n *with complex coefficients such that* $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2$*,* $\ell = 0, 1, 2, \ldots, n$ *for some real* β*,* $k \geq 1$*,* $0 < \rho_1 \leq 1$*,* $0 < \rho_2 \leq 1$*,* $m \leq q \leq j \leq p$*,*

$$\rho_1|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{j-1}| \leq k|a_j| \geq |a_{j+1}| \geq \cdots \geq \rho_2|a_p|.$$

Then all zeros of $P(z)$ *lie in the disk*

$$|z| \leq \frac{1}{|a_n|}\Bigg(|a_0|+M_q+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+|a_q|+2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha$$

$$+2k|a_j|(\sin\alpha+\cos\alpha+1)-2|a_j|+2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha$$

$$+\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p\Big),$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

These parameters may also be applied to a dual gap polynomial as seen in Corollary 2.10, and we will have the same bound as found except $M_p = \sum_{\ell=p}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$. Now that we have the outer bound given these parameters, we will now find a bound in which no zeros exist, giving:

Theorem 4.7. *Let* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *be a polynomial of degree* n *with complex coefficients. Suppose that, for some positive numbers* k, ρ_1, ρ_2, p *and* q *with* $k \geq 1$, $0 < \rho_1 \leq 1$, $0 < \rho_2 \leq 1$, *and* $0 \leq q \leq j \leq p \leq n$, *the coefficients satisfy* $|\arg(a_\ell) - \beta| \leq \alpha \leq \pi/2$ *for some real* β *and for* $q \leq \ell \leq p$, *and*

$$\rho_1|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{j-1}| \leq k|a_j| \geq |a_{j+1}| \geq \cdots \geq \rho_2|a_p|.$$

Then $P(z)$ *does not vanish in*

$$\begin{aligned} |z| \;&<\; \min\Big\{1, |a_0| \Big/ \Big(M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1} |a_\ell|\sin\alpha \\ &\quad +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2\sum_{\ell=j+1}^{p-1} |a_\ell|\sin\alpha \\ &\quad +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Big), \end{aligned}$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 4.7. Consider the reciprocal polynomial

$$S(z) = z^n P\left(\frac{1}{z}\right) \;=\; a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + \cdots + a_j z^{n-j} + \cdots + a_p z^{n-p}$$

$$+\cdots+a_{n-1}z+a_n.$$

Let

$$\begin{aligned}
H(z) &= (1-z)S(z)\\
&= -a_0z^{n+1}+(a_0-a_1)z^n+(a_1-a_2)z^{n-1}+\cdots+(a_q-a_{q+1})z^{n-q}+\cdots\\
&\quad +(a_{j-1}-a_j)z^{n-j+1}+(a_j-a_{j+1})z^{n-j}+\cdots+(a_{p-1}-a_p)z^{n-p+1}+\cdots\\
&\quad +(a_{n-1}-a_n)z+a_n.
\end{aligned}$$

This gives

$$\begin{aligned}
|H(z)| &\geq |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots+|a_q-a_{q+1}||z|^{n-q}\\
&\quad +\cdots+|a_{j-1}-a_j||z|^{n-j+1}+|a_j-a_{j+1}||z|^{n-j}+\cdots+|a_{p-1}-a_p||z|^{n-p+1}\\
&\quad +\cdots+|a_{n-1}-a_n||z|+|a_n|\Big)\\
&= |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots\\
&\quad +|a_q-\rho_1a_q+\rho_1a_q-a_{q+1}||z|^{n-q}+\cdots+|a_{j-1}-ka_j+ka_j-a_j||z|^{n-j+1}\\
&\quad +|a_j-ka_j+ka_j-a_{j+1}||z|^{n-j}+\cdots+|a_{p-1}-\rho_2a_p+\rho_2a_p-a_p||z|^{n-p+1}\\
&\quad +\cdots+|a_{n-1}-a_n||z|+|a_n|\Big)\\
&\geq |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots+|a_q-\rho_1a_q||z|^{n-q}\\
&\quad +|\rho_1a_q-a_{q+1}||z|^{n-q}+\cdots+|a_{j-1}-ka_j||z|^{n-j+1}+|ka_j-a_j||z|^{n-j+1}\\
&\quad +|a_j-ka_j||z|^{n-j}+|ka_j-a_{j+1}||z|^{n-j}+\cdots+|a_{p-1}-\rho_2a_p||z|^{n-p+1}\\
&\quad +|\rho_2a_p-a_p||z|^{n-p+1}+\cdots+|a_{n-1}-a_n||z|+|a_n|\Big).
\end{aligned}$$

Since $0<\rho_1\leq 1$, $0<\rho_2\leq 1$, $k\geq 1$ then

$$|H(z)| \geq |a_0||z|^{n+1}-\Big(|a_0-a_1||z|^n+|a_1-a_2||z|^{n-1}+\cdots+|a_q|(1-\rho_1)|z|^{n-q}$$

$$
\begin{aligned}
& +|a_{q+1}-\rho_1 a_q||z|^{n-q}+\cdots+|ka_j-a_{j-1}||z|^{n-j+1}+|a_j|(k-1)|z|^{n-j+1} \\
& +|a_j|(k-1)|z|^{n-j}+|ka_j-a_{j+1}||z|^{n-j}+\cdots+|a_{p-1}-\rho_2 a_p||z|^{n-p+1} \\
& +|a_p|(1-\rho_2)|z|^{n-p+1}+\cdots+|a_{n-1}-a_n||z|+|a_n|\Big) \\
= & \; |z|^n\Bigg[|a_0||z|-\Bigg(|a_0-a_1|+\frac{|a_1-a_2|}{|z|}+\cdots+\frac{|a_q|(1-\rho_1)}{|z|^q}+\frac{|a_{q+1}-\rho_1 a_q|}{|z|^q} \\
& +\cdots+\frac{|ka_j-a_{j-1}|}{|z|^{j-1}}+\frac{|a_j|(k-1)}{|z|^{j-1}}+\frac{|a_j|(k-1)}{|z|^j}+\frac{|ka_j-a_{j+1}|}{|z|^j}+\cdots \\
& +\frac{|a_{p-1}-\rho_2 a_p|}{|z|^{p-1}}+\frac{|a_p|(1-\rho_2)}{|z|^{p-1}}+\cdots+\frac{|a_{n-1}-a_n|}{|z|^{n-1}}+\frac{|a_n|}{|z|^n}\Bigg)\Bigg].
\end{aligned}
$$

Now for $|z|>1$ so that $\dfrac{1}{|z|^{n-\ell}}<1$, for $0\le \ell<n$ we have and by Lemma 2.1

$$
\begin{aligned}
|H(z)| \geq & \; |z|^n\Bigg[|a_0||z|-\Bigg(|a_0-a_1|+|a_1-a_2|+\cdots+|a_{q-1}-a_q|+|a_q|(1-\rho_1) \\
& +|a_{q+1}-\rho_1 a_q|+\cdots+|ka_j-a_{j-1}|+|a_j|(k-1)+|a_j|(k-1) \\
& +|ka_j-a_{j+1}|+\cdots+|a_{p-1}-\rho_2 a_p|+|a_p|(1-\rho_2)+\cdots+|a_{n-1}-a_n| \\
& +|a_n|\Bigg)\Bigg] \\
= & \; |z|^n\Bigg[|a_0||z|-\Bigg(\sum_{\ell=1}^{q}|a_\ell-a_{\ell-1}|+|a_q|(1-\rho_1)+|a_{q+1}-\rho_1 a_q|+\cdots \\
& +|ka_j-a_{j-1}|+|a_j|(k-1)+|a_j|(k-1)+|ka_j-a_{j+1}|+\cdots \\
& +|a_{p-1}-\rho_2 a_p|+|a_p|(1-\rho_2)+\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|+|a_n|\Bigg)\Bigg] \\
\geq & \; |z|^n\Bigg[|a_0||z|-\Bigg(M_q+|a_q|(1-\rho_1)+(|a_{q+1}|-|\rho_1 a_q|)\cos\alpha \\
& +(|a_{q+1}|+|\rho_1 a_q|)\sin\alpha+\sum_{\ell=q+2}^{j-1}|a_\ell-a_{\ell-1}|+(|ka_j|-|a_{j-1}|)\cos\alpha \\
& +(|ka_j|+|a_{j-1}|)\sin\alpha+|a_j|(k-1)+|a_j|(k-1)+(|ka_j|-|a_{j+1}|)\cos\alpha \\
& +(|ka_j|+|a_{j+1}|)\sin\alpha+\sum_{\ell=j+2}^{p-1}|a_\ell-a_{\ell-1}|+(|a_{p-1}|-|\rho_2 a_p|)\cos\alpha \\
& +(|a_{p-1}|+|\rho_2 a_p|)\sin\alpha+|a_p|(1-\rho_2)+M_p+|a_n|\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\geq\ & |z|^n\Bigg[|a_0||z| - \Bigg(M_q + |a_q|(1-\rho_1) + |a_{q+1}|\cos\alpha - \rho_1|a_q|\cos\alpha \\
& + |a_{q+1}|\sin\alpha + \rho_1|a_q|\sin\alpha + \sum_{\ell=q+2}^{j-1}(|a_\ell| - |a_{\ell-1}|)\cos\alpha \\
& + \sum_{\ell=q+2}^{j-1}(|a_\ell| + |a_{\ell-1}|)\sin\alpha + 2k|a_j|\cos\alpha - |a_{j-1}|\cos\alpha + 2k|a_j|\sin\alpha \\
& + |a_{j-1}|\sin\alpha + 2|a_j|(k-1) - |a_{j+1}|\cos\alpha + |a_{j+1}|\sin\alpha \\
& + \sum_{\ell=j+2}^{p-1}(|a_{\ell-1}| - |a_\ell|)\cos\alpha + \sum_{\ell=j+2}^{p-1}(|a_{\ell-1}| + |a_\ell|)\sin\alpha + |a_{p-1}|\cos\alpha \\
& - \rho_2|a_p|\cos\alpha + |a_{p-1}|\sin\alpha + \rho_2|a_p|\sin\alpha + |a_p|(1-\rho_2) + M_p + |a_n|\Bigg)\Bigg] \\
=\ & |z|^n\Bigg[|a_0||z| - \Bigg(M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) \\
& + |a_{q+1}|(\sin\alpha + \cos\alpha) + \sum_{\ell=q+2}^{j-1}|a_\ell|\cos\alpha - \sum_{\ell=q+2}^{j-1}|a_{\ell-1}|\cos\alpha \\
& + \sum_{\ell=q+2}^{j-1}|a_\ell|\sin\alpha + \sum_{\ell=q+2}^{j-1}|a_{\ell-1}|\sin\alpha + 2k|a_j|(\sin\alpha + \cos\alpha + 1) \\
& + |a_{j-1}|(\sin\alpha - \cos\alpha) - 2|a_j| + |a_{j+1}|(\sin\alpha - \cos\alpha) + \sum_{\ell=j+2}^{p-1}|a_{\ell-1}|\cos\alpha \\
& - \sum_{\ell=j+2}^{p-1}|a_\ell|\cos\alpha + \sum_{\ell=j+2}^{p-1}|a_{\ell-1}|\sin\alpha + \sum_{\ell=j+2}^{p-1}|a_\ell|\sin\alpha + |a_{p-1}|\cos\alpha \\
& + |a_{p-1}|\sin\alpha + \rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Bigg)\Bigg] \\
=\ & |z|^n\Bigg[|a_0||z| - \Bigg(M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) \\
& + |a_{q+1}|(\sin\alpha + \cos\alpha) + |a_{j-1}|\cos\alpha + \sum_{\ell=q+2}^{j-2}|a_\ell|\cos\alpha - |a_{q+1}|\cos\alpha \\
& - \sum_{\ell=q+2}^{j-2}|a_\ell|\cos\alpha + |a_{j-1}|\sin\alpha + \sum_{\ell=q+2}^{j-2}|a_\ell|\sin\alpha + |a_{q+1}|\sin\alpha \\
& + \sum_{\ell=q+2}^{j-2}|a_\ell|\sin\alpha + 2k|a_j|(\sin\alpha + \cos\alpha + 1) + |a_{j-1}|(\sin\alpha - \cos\alpha) - 2|a_j|
\end{aligned}
$$

$$
\begin{aligned}
& +|a_{j+1}|(\sin\alpha-\cos\alpha)+|a_{j+1}|\cos\alpha+\sum_{\ell=j+2}^{p-2}|a_\ell|\cos\alpha-|a_{p-1}|\cos\alpha \\
& -\sum_{\ell=j+2}^{p-2}|a_\ell|\cos\alpha+|a_{j+1}|\sin\alpha+\sum_{\ell=j+2}^{p-2}|a_\ell|\sin\alpha+|a_{p-1}|\sin\alpha \\
& +\sum_{\ell=j+2}^{p-2}|a_\ell|\sin\alpha+|a_{p-1}|\cos\alpha+|a_{p-1}|\sin\alpha+\rho_2|a_p|(\sin\alpha-\cos\alpha-1) \\
& +|a_p|+M_p+|a_n|\Bigg)\Bigg] \\
= \;& |z|^n\Bigg[|a_0||z|-\Bigg(M_q+|a_q|+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+2|a_{q+1}|\sin\alpha \\
& +2|a_{j-1}|\sin\alpha+2\sum_{\ell=q+2}^{j-2}|a_\ell|\sin\alpha+2k|a_j|(\sin\alpha+\cos\alpha+1)-2|a_j| \\
& +2|a_{j+1}|\sin\alpha+2\sum_{\ell=j+2}^{p-2}|a_\ell|\sin\alpha+2|a_{p-1}|\sin\alpha \\
& +\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p+|a_n|\Bigg)\Bigg] \\
= \;& |z|^n\Bigg[|a_0||z|-\Bigg(M_q+|a_q|+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha \\
& +2k|a_j|(\sin\alpha+\cos\alpha+1)-2|a_j|+2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha \\
& +\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p+|a_n|\Bigg)\Bigg]
\end{aligned}
$$

if

$$
\begin{aligned}
|z| \;>\; & \frac{1}{|a_0|}\Bigg(M_q+|a_q|+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha \\
& +2k|a_j|(\sin\alpha+\cos\alpha+1)-2|a_j|+2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha \\
& +\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p+|a_n|\Bigg).
\end{aligned}
$$

Thus all zeros of $H(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}|z| \;\leq\; & \frac{1}{|a_0|}\Bigg(M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha \\ & +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha \\ & +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Bigg).\end{aligned}$$

Hence all zeros of $H(z)$ and of $S(z)$ lie in

$$\begin{aligned}|z| \;\leq\; & \max\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha \\ & +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha \\ & +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Bigg).\end{aligned}$$

Therefore all zeros of $P(z)$ lie in

$$\begin{aligned}|z| \;\geq\; & \min\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha \\ & +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha \\ & +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Bigg).\end{aligned}$$

Thus the polynomial $P(z)$ does not vanish in

$$\begin{aligned}|z| \;<\; & \min\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha \\ & +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha \\ & +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Bigg).\end{aligned}$$

□

Notice when $j = p$ then Theorem 4.7 reduces to Theorem 2.11. Now consider these parameters applied to a lacunary polynomial:

Corollary 4.8. *Let $P(z)$ be an nth degree lacunary polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$, $a_0 \neq 0$. Suppose that, for some positive numbers k, ρ_1, ρ_2, p and q with $k \geq 1$, $0 < \rho_1 \leq 1$, $0 < \rho_2 \leq 1$, and $1 \leq m \leq q \leq j \leq p \leq n$, the complex coefficients satisfy $|\arg(a_\ell) - \beta| \leq \alpha \leq \pi/2$ for some real β and for $q \leq \ell \leq p$, and*

$$\rho_1|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{j-1}| \leq k|a_j| \geq |a_{j+1}| \geq \cdots \geq \rho_2|a_p|.$$

Then $P(z)$ does not vanish in

$$\begin{aligned}|z| \quad < \quad & \min\left\{1, |a_0| \middle/ \left(M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1} |a_\ell| \sin\alpha\right.\right. \\ & +2k|a_j|(\sin\alpha + \cos\alpha + 1) - 2|a_j| + 2\sum_{\ell=j+1}^{p-1} |a_\ell| \sin\alpha \\ & \left.\left. +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\right)\right\}.\end{aligned}$$

where $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

These parameters may also be applied to dual gap polynomial as seen in Corollary 2.10 and we will have the same inner bound as seen in Corollary 4.8 except $M_p = \sum_{\ell=p}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

One major benefit of including a reversal into the hypotheses for the coefficients is it allows us to shift the point in which that reversal exists. Since j is not defined in relation to either the q or p we are able to consider a larger set of complex polynomials. Notice that when $j = p$, then our results resemble the properties found in Chapter 2, but when $j = q$, then instead of monotonically increasing coefficients, we are able to consider coefficients that are decreasing monotonically. We will further explore these types of behaviors for counting zeros within a bound, and for quaternionic polynomials.

5 NUMBER OF ZEROS, MONOTONE WITH A REVERSAL

In Chapter 3, we discussed counting the number of zeros which appear within a bound given a polynomial which has a subsection with monotonically increasing coefficients on either the *real* and *imaginary* parts, or on the moduli. In this chapter, we will discuss when the subsection of coefficients is monotone with a reversal as seen in Chapter 4. Hence we have the following:

Theorem 5.1. *Let* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *be complex polynomial of degree* n *with complex coefficients where* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$ *which satisfies for some real* $\rho_{r_1}, \rho_{r_2}, \rho_{i_1}, \rho_{i_2}, k_r,$ *and* k_i *where* $0 < \rho_{r_1} \leq 1$, $0 < \rho_{r_2} \leq 1$, $0 < \rho_{i_1} \leq 1$, $0 < \rho_{i_2} \leq 1$, $k_r \geq 1$, *and* $k_i \geq 1$

$$\rho_{r_1}\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_{p-1} \geq \rho_{r_2}\alpha_p$$

$$\rho_{i_1}\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{j-1} \leq k_i\beta_j \geq \beta_{j+1} \geq \cdots \geq \beta_{p-1} \geq \rho_{i_2}\beta_p.$$

Then the number of zeros of $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

for $0 < \delta < 1$,

$$\begin{aligned} M &= |a_0| + M_q + \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j + |\alpha_p|(1-\rho_{r_2}) \\ &\quad -\rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\ &\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p + |a_n|, \end{aligned}$$

$M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 5.1. Consider

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= a_0 + (a_1 - a_0)z + \cdots + (a_{q+1} - a_q)z^q + \cdots + (a_j - a_{j-1})z^j \\
&\quad + (a_{j+1} - a_j)z^{j+1} + \cdots + (a_p - a_{p-1})z^p + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}.
\end{aligned}$$

For $|z| = 1$ we have,

$$\begin{aligned}
|F(z)| &\le |a_0| + |a_1 - a_0||z| + \cdots + |a_{q+1} - a_q||z|^q + \cdots + |a_j - a_{j-1}||z|^j \\
&\quad + |a_{j+1} - a_j||z|^{j+1} + \cdots + |a_p - a_{p-1}||z|^p + \cdots + |a_n - a_{n-1}||z|^n \\
&\quad + |a_n||z|^{n+1} \\
&= |a_0| + |a_1 - a_0| + \cdots + |a_{q+1} - a_q| + \cdots + |a_j - a_{j-1}| + |a_{j+1} - a_j| \\
&\quad + \cdots + |a_p - a_{p-1}| + \cdots + |a_n - a_{n-1}| + |a_n| \\
&= |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |\alpha_{q+1} + i\beta_{q+1} - \alpha_q - i\beta_q| + \cdots \\
&\quad + |\alpha_j + i\beta_j - \alpha_{j-1} - i\beta_{j-1}| + |\alpha_{j+1} + i\beta_{j+1} - \alpha_j - i\beta_j| + \cdots \\
&\quad + |\alpha_p + i\beta_p - \alpha_{p-1} - i\beta_{p-1}| + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| + |a_n| \\
&\le |a_0| + M_q + |\alpha_{q+1} - \alpha_q| + \cdots + |\alpha_j - \alpha_{j-1}| + |\alpha_{j+1} - \alpha_j| + \cdots \\
&\quad + |\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \beta_q| + \cdots + |\beta_j - \beta_{j-1}| + |\beta_{j+1} - \beta_j| + \cdots \\
&\quad + |\beta_p - \beta_{p-1}| + M_p + |a_n|,
\end{aligned}$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. So for $|z| = 1$, we have

$$\begin{aligned}
|F(z)| &\le |a_0| + M_q + |\alpha_{q+1} + \rho_{r_1}\alpha_q + \rho_{r_1}\alpha_q - \alpha_q| + \cdots \\
&\quad + |\alpha_j - k_r\alpha_j + k_r\alpha_j - \alpha_{j-1}| + |\alpha_{j+1} - k_r\alpha_j + k_r\alpha_j - \alpha_j| + \cdots \\
&\quad + |\alpha_p - \rho_{r_2}\alpha_p + \rho_{r_2}\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_{i_1}\beta_q + \rho_{i_1}\beta_q - \beta_q| + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+|\beta_j - k_i\beta_j + k_i\beta_j - \beta_{j-1}| + |\beta_{j+1} - k_i\beta_j + k_i\beta_j - \beta_j| + \cdots \\
&+|\beta_p - \rho_{i_2}\beta_p + \rho_{i_2}\beta_p - \beta_{p-1}| + M_p + |a_n| \\
\leq\ & |a_0| + M_q + |\alpha_{q+1} + \rho_{r_1}\alpha_q| + |\rho_{r_1}\alpha_q - \alpha_q| + \cdots + |\alpha_j - k_r\alpha_j| \\
&+|k_r\alpha_j - \alpha_{j-1}| + |\alpha_{j+1} - k_r\alpha_j| + |k_r\alpha_j - \alpha_j| + \cdots + |\alpha_p - \rho_{r_2}\alpha_p| \\
&+|\rho_{r_2}\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_{i_1}\beta_q| + |\rho_{i_1}\beta_q - \beta_q| + \cdots + |\beta_j - k_i\beta_j| \\
&+|k_i\beta_j - \beta_{j-1}| + |\beta_{j+1} - k_i\beta_j| + |k_i\beta_j - \beta_j| + \cdots + |\beta_p - \rho_{i_2}\beta_p| \\
&+|\rho_{i_2}\beta_p - \beta_{p-1}| + M_p + |a_n| \\
=\ & |a_0| + M_q + (\alpha_{q+1} + \rho_{r_1}\alpha_q) + |\alpha_q|(1 - \rho_{r_1}) + \cdots + |\alpha_j|(k_r - 1) \\
&+(k_r\alpha_j - \alpha_{j-1}) + (k_r\alpha_j - \alpha_{j+1}) + |\alpha_j|(k_r - 1) + \cdots + |\alpha_p|(1 - \rho_{r_2}) \\
&+(\alpha_{p-1} - \rho_{r_2}\alpha_p) + (\beta_{q+1} - \rho_{i_1}\beta_q) + |\beta_q|(1 - \rho_{i_1}) + \cdots + |\beta_j|(k_i - 1) \\
&+(k_i\beta_j - \beta_{j-1}) + (k_i\beta_j - \beta_{j+1}) + |\beta_j|(k_i - 1) + \cdots + |\beta_p|(1 - \rho_{i_2}) \\
&+(\beta_{p-1} - \rho_{i_2}\beta_p) + M_p + |a_n| \\
=\ & |a_0| + M_q + \rho_{r_1}\alpha_q + |\alpha_q|(1 - \rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j + |\alpha_p|(1 - \rho_{r_2}) \\
&-\rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1 - \rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\
&+|\beta_p|(1 - \rho_{i_2}) - \rho_{i_2}\beta_p + M_p + |a_n|.
\end{aligned}
$$

Since $F(z)$ is analytic in $|z| \leq 1$, and $|F(z)| \leq |a_0| + M_q + \rho_{r_1}\alpha_q + |\alpha_q|(1 - \rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j + |\alpha_p|(1 - \rho_{r_2}) - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1 - \rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j + |\beta_p|(1 - \rho_{i_2}) - \rho_{i_2}\beta_p + M_p + |a_n|$ for $|z| = 1$. So by Theorem 1.14 and the Maximum Modulus Theorem, the number of zeros of $F(z)$ (and hence of $P(z)$) in $|z| \leq \delta$ is less than or equal to

$$\frac{1}{\log {}^1\!/\!_\delta} \log \frac{M}{|a_0|}$$

where $0 < \delta < 1$ and

$$
\begin{aligned}
M &= |a_0| + M_q + \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j + |\alpha_p|(1-\rho_{r_2}) \\
&\quad -\rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\
&\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p + |a_n|.
\end{aligned}
$$

□

Notice when $j = p$, then Theorem 5.1 reduces to Theorem 3.1. We can also consider when $j = q$, hence we would have a strictly decreasing monotone behavior on the real and imaginary parts of the coefficients. Now that we have considered a reversal on the parts of the coefficient we will consider when a reversal is applied to the moduli of the coefficients:

Theorem 5.2. *Let* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *be a polynomial of degree* n *with complex coefficients satisfying* $|\arg a_\ell - \beta| \le \alpha \le \pi/2$, $\ell = q, q+1, \ldots, p$, *such that for real* k, ρ_1, ρ_2, *where* $k \ge 1$, $0 < \rho_1 \le 1$, $0 < \rho_2 \le 1$,

$$\rho_1|a_q| \le |a_{q+1}| \le \cdots \le |a_{j-1}| \le k|a_j| \ge |a_{j+1}| \ge \cdots \ge |a_{p-1}| \ge \rho_2|a_p|.$$

Then the number of zeros of $P(z)$ *in the disk* $|z| \le \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

for $0 < \delta < 1$, *and*

$$
\begin{aligned}
M &= |a_0| + M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1} |a_\ell|\sin\alpha - 2|a_j| \\
&\quad +2k|a_j|(\cos\alpha + \sin\alpha + 1) + 2\sum_{\ell=j+1}^{p-1} |a_\ell|\sin\alpha + |a_p|
\end{aligned}
$$

$$+\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + M_p + |a_n|$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 5.2. Consider

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= a_0 + (a_1 - a_0)z + \cdots + (a_{q+1} - a_q)z^q + \cdots + (a_j - a_{j-1})z^j \\ &\quad + (a_{j+1} - a_j)z^{j+1} + \cdots + (a_p - a_{p-1})z^p + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}. \end{aligned}$$

For $|z| = 1$ we have,

$$\begin{aligned} |F(z)| &\leq |a_0| + |a_1 - a_0||z| + \cdots + |a_{q+1} - a_q||z|^q + \cdots + |a_j - a_{j-1}||z|^j \\ &\quad + |a_{j+1} - a_j||z|^{j+1} + \cdots + |a_p - a_{p-1}||z|^p + \cdots + |a_n - a_{n-1}||z|^n \\ &\quad + |a_n||z|^{n+1} \\ &= |a_0| + |a_1 - a_0| + \cdots + |a_{q+1} - a_q| + \cdots + |a_j - a_{j-1}| + |a_{j+1} - a_j| \\ &\quad + \cdots + |a_p - a_{p-1}| + \cdots + |a_n - a_{n-1}| + |a_n| \\ &= |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |a_{q+1} - a_q| + \sum_{\ell=q+2}^{j-1} |a_\ell - a_{\ell-1}| + |a_j - a_{j-1}| \\ &\quad + |a_{j+1} - a_j| + \sum_{\ell=j+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| + |a_n| \\ &= |a_0| + M_q + |a_{q+1} - \rho_1 a_q + \rho_1 a_q - a_q| + \sum_{\ell=q+2}^{j-1} |a_\ell - a_{\ell-1}| \\ &\quad + |a_j - ka_j + ka_j - a_{j-1}| + |a_{j+1} - ka_j| + |ka_j - a_j| + \sum_{\ell=j+2}^{p-1} |a_\ell - a_{\ell-1}| \\ &\quad + |a_p - \rho_2 a_p + \rho_2 a_p - a_{p-1}| + M_p + |a_n| \\ &\leq |a_0| + M_q + |a_{q+1} - \rho_1 a_q| + |\rho_1 a_q - a_q| + \sum_{\ell=q+2}^{j-1} |a_\ell - a_{\ell-1}| \end{aligned}$$

$$
\begin{aligned}
&+|a_j - ka_j| + |ka_j - a_{j-1}| + |a_{j+1} - ka_j| + |ka_j - a_j| + \sum_{\ell=j+2}^{p-1} |a_\ell - a_{\ell-1}| \\
&+|a_p - \rho_2 a_p| + |\rho_2 a_p - a_{p-1}| + M_p + |a_n| \\
\leq\ & |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho_1|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho_1|a_q|\sin\alpha \\
&+|a_q|(1-\rho_1) + \sum_{\ell=q+2}^{j-1} |a_\ell|\cos\alpha - \sum_{\ell=q+2}^{j-1} |a_{\ell-1}|\cos\alpha + \sum_{\ell=q+2}^{j-1} |a_\ell|\sin\alpha \\
&+ \sum_{\ell=q+2}^{j-1} |a_{\ell-1}|\sin\alpha + |a_j|(k-1) + k|a_j|\cos\alpha - |a_{j-1}|\cos\alpha + k|a_j|\sin\alpha \\
&+|a_{j-1}|\sin\alpha + k|a_j|\cos\alpha - |a_{j+1}|\cos\alpha + k|a_j|\sin\alpha + |a_{j+1}|\sin\alpha \\
&+|a_j|(k-1) + \sum_{\ell=j+2}^{p-1} |a_{\ell-1}|\cos\alpha - \sum_{\ell=j+2}^{p-1} |a_\ell|\cos\alpha + \sum_{\ell=j+2}^{p-1} |a_{\ell-1}|\sin\alpha \\
&+ \sum_{\ell=j+2}^{p-1} |a_\ell|\sin\alpha + |a_p|(1-\rho_2) + |a_{p-1}|\cos\alpha - \rho_2|a_p|\cos\alpha + |a_{p-1}|\sin\alpha \\
&+\rho_2|a_p|\sin\alpha + M_p + |a_n|,
\end{aligned}
$$

by Lemma 2.1. Hence

$$
\begin{aligned}
|F(z)| \ \leq\ & |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho_1|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho_1|a_q|\sin\alpha \\
&+|a_q|(1-\rho_1) + |a_{j-1}|\cos\alpha + \sum_{\ell=q+2}^{j-2} |a_\ell|\cos\alpha - |a_{q+1}|\cos\alpha \\
&- \sum_{\ell=q+2}^{j-2} |a_\ell|\cos\alpha + |a_{j-1}|\sin\alpha + \sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha + |a_{q+1}|\sin\alpha \\
&+ \sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha + |a_j|(k-1) + k|a_j|\cos\alpha - |a_{j-1}|\cos\alpha + k|a_j|\sin\alpha \\
&+|a_{j-1}|\sin\alpha + k|a_j|\cos\alpha - |a_{j+1}|\cos\alpha + k|a_j|\sin\alpha + |a_{j+1}|\sin\alpha \\
&+|a_j|(k-1) + |a_{j+1}|\cos\alpha + \sum_{\ell=j+2}^{p-2} |a_\ell|\cos\alpha - |a_{p-1}|\cos\alpha \\
&- \sum_{\ell=j+2}^{p-2} |a_\ell|\cos\alpha + |a_{j+1}|\sin\alpha + \sum_{\ell=j+2}^{p-2} |a_\ell|\sin\alpha + |a_{p-1}|\sin\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\ell=j+2}^{p-2}|a_\ell|\sin\alpha+|a_p|(1-\rho_2)+|a_{p-1}|\cos\alpha-\rho_2|a_p|\cos\alpha+|a_{p-1}|\sin\alpha \\
&+\rho_2|a_p|\sin\alpha+M_p+|a_n| \\
= \;& |a_0|+M_q+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+2|a_{q+1}|\sin\alpha+|a_q| \\
&+2\sum_{\ell=q+2}^{j-2}|a_\ell|\sin\alpha-2|a_j|+2k|a_j|(\cos\alpha+\sin\alpha+1)+2|a_{j-1}|\sin\alpha \\
&+2|a_{j+1}|\sin\alpha+2\sum_{\ell=j+2}^{p-2}|a_\ell|\sin\alpha+2|a_{p-1}|\sin\alpha+|a_p| \\
&+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+M_p+|a_n| \\
= \;& |a_0|+M_q+|a_q|+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha-2|a_j| \\
&+2k|a_j|(\cos\alpha+\sin\alpha+1)+2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha+|a_p| \\
&+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+M_p+|a_n|.
\end{aligned}
$$

Since $F(z)$ is analytic in $|z|\le 1$, and $|F(z)|\le |a_0|+M_q+|a_q|+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha-2|a_j|+2k|a_j|(\cos\alpha+\sin\alpha+1)+2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha+|a_p|+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+M_p+|a_n|$ for $|z|=1$, so by Theorem 1.14 and the Maximum Modulus Theorem, the number of zeros of $F(z)$ (and hence of $P(z)$) in $|z|\le\delta$ is less than or equal to

$$\frac{1}{\log 1/\delta}\log\frac{M}{|a_0|}$$

where $0<\delta<1$, and

$$
\begin{aligned}
M \;=\;& |a_0|+M_q+|a_q|+\rho_1|a_q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=q+1}^{j-1}|a_\ell|\sin\alpha-2|a_j| \\
&+2k|a_j|(\cos\alpha+\sin\alpha+1)+2\sum_{\ell=j+1}^{p-1}|a_\ell|\sin\alpha+|a_p| \\
&+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+M_p+|a_n|.
\end{aligned}
$$

□

We can notice Theorem 5.2 reduces to Theorem 3.4 when $j = p$. Similarly, we can consider the monotone decreasing on the moduli when $j = q$. If we apply this condition to a lacunary polynomial, we will have the same bound except $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$. If we apply this to a polynomial with two gaps, then we will have $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

The addition of the reversal allows us to shift the point in which the reversal occurs as there are no limitations for the occurrence of our j subscripted coefficients. The ability to move this point allows us to apply the theorems found within this chapter to a larger set of polynomials.

6 LOCATIONS OF ZEROS IN THE QUATERNIONS

6.1 Quaternions Background

In the field of mathematics it is rare for there to exist an exact date in which major discoveries were found. Sir William Rowan Hamilton attempted to discover a triplet number system as his sons, Archibald Henry Hamilton and William Edwin Hamilton, would attend breakfast and ask Hamilton if he was able to multiply triplets. On 16 October 1843, Hamilton created a four dimensional number system which we now refer to as the quaternions, denoted $\mathbb{H}$ in honor of Hamilton. While Hamilton never succeeded in being able to multiply triplets, the quaternions produced the four dimensional number system [6]. The standard notation $\mathbb{H} = \{\alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$, where i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1$. We can note the quaternions are the standard example of a noncommutative division ring.

A number in the quaternions, usually denoted with q where $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ contains one *real part*, α and three *imaginary parts*, β, γ and δ. The conjugate in the quaternions is defined as $\overline{q} = \alpha - \beta i - \gamma j - \delta k$ and the modulus is defined as $|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The modulus is a norm on $\mathbb{H}$. We also define the ball $B(0, r) = \{q \in \mathbb{H} \mid |q| < r\}$, for $r > 0$. Now define the angle between two quaternions q_1 and q_2 by treating them as if they were vectors in $\mathbb{R}^4$. For $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$, the angle between q_1 and q_2 is

$$\measuredangle(q_1, q_2) = \cos^{-1}\left(\frac{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 + \delta_1\delta_2}{|q_1||q_2|}\right).$$

Since commutivity fails in the quaternions, a polynomial of the form aq^n is different from the polynomial $a_0 q a_1 q \cdots q a_n$, where $a = a_0 a_1 \cdots a_n$. By convention, we take indeterminate value on the left and coefficients on the right, so we would write the quaternionic polynomial $P_1(q) = \sum_{\ell=0}^{n} q^\ell a_n$. We also define for such P_1 and $P_2(q) = \sum_{\ell=0}^{m} q^\ell b_n$, the regular product is $(P_1 * P_2)(q) = \sum_{i=0,1,\ldots,n; j=0,1,\ldots,m} q^{i+j} a_i b_j$. Unlike in the complex field, the lack of commutivity also causes the Factor Theorem to fail. For example, a second degree polynomial q^2+1 has an infinite number of zeros; namely, $q = \beta i + \gamma j + \delta k$ where $\beta^2+\gamma^2+\delta^2 = 1$. Thus we can denote the solution set of the equation q^2+1 q as $\mathbb{S}$: $\mathbb{S} = \{\beta i + \gamma j + \delta k \mid \beta^2+\gamma^2+\delta^2 = 1\}$. Also we have [21]:

Theorem 6.1. *Let f and g be given quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.*

The following appears in the same paper:

Theorem 6.2. *Let $\sum_{\ell=0}^{\infty} q^\ell a_\ell$ be a given quaternionic power series with radius of convergence R. Suppose that there exists $x_0, y_0 \in \mathbb{R}$ and $I, J \in \mathbb{S}$ with $I \neq J$ such that $\sum_{\ell=0}^{\infty} (x_0 + y_0 I)^\ell a_\ell = 0$ and $\sum_{\ell=0}^{\infty} (x_0 + y_0 J)^\ell a_\ell = 0$. Then for all $L \in \mathbb{S}$ we have $\sum_{\ell=0}^{\infty} (x_0 + y_0 L)^\ell a_\ell = 0$.*

Notice Theorem 6.2 illustrates how the 2-sphere $\mathbb{S}$ plays a fundamental role in zeros of a quaternionic series, and thus of polynomials [14]. Gentili and Struppa gave a definition for the multiplicity of zeros for which when counted by their multiplicity equals the degree of the polynomial [15]. Gentili and Struppa also introduced a Maximum Modulus Theorem for regular functions [14]:

Theorem 6.3. *Let $B = B(0, r)$ be a ball in $\mathbb{H}$ with center 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is constant on B.*

6.2 Necessary Lemma

As a corollary to Theorem 6.1, we can extract the following lemma:

Lemma 6.4. *Let $P(q)$ be a quaternionic polynomial. Then the only zeros $P(q)*(1-q)$ are $q = 1$ and the zeros of $P(q)$.*

Proof of Lemma 6.4. By Theorem 6.1, $P(q)*(1-q) = 0$ if and only if either $P(q) = 0$, or $P(q) \neq 0$ implies $P(q)^{-1}qP(q)-1 = 0$. Notice that $P(q)^{-1}qP(q)-1 = 0$ is equvalent to $P(q)^{-1}qP(q) = 1$ and, if $P(q) \neq 0$, this implies that $q = 1$. So the only zeros of $P(q) * (1 - q)$ are $q = 1$ and the zeros of $P(q)$. □

Similar to Theorem 2.1, there exists a similar property for quaternionic variables. That is [3]:

Lemma 6.5. *Let $q_1, q_2 \in \mathbb{H}$ where $q_1 = \alpha_1+\beta_1 i+\gamma_1 j+\delta_1 k$ and $q_2 = \alpha_2+\beta_2 i+\gamma_2 j+\delta_2 k$, $\measuredangle(q_1, q_1) = 2\theta' \leq 2\theta$, and $|q_1| \leq |q_2|$. Then*

$$|q_2 - q_1| \leq (|q_2| - |q_1|)\cos\theta + (|q_2| + |q_1|)\sin\theta.$$

6.3 Locations of Zeros for Quaternionic Polynomials

We will now consider applying our ρ, k, p and q parameters to a quaternionic polynomial with quaternionic coefficients. We first look at the monotonicity condition applied to the *real* and *imaginary* parts of the coefficients. Hence we have:

Theorem 6.6. *Let $P(q) = a_0 + qa_1 + \cdots + q^n a_n$ be a polynomial of degree n with quaternionic coefficients. That is $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$, for $\ell = 0, 1, 2, \ldots, n$ and*

$\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell$ are real, $q \leq p$, $0 < \rho_R \leq 1$, $0 < \rho_I \leq 1$, $0 < \rho_J \leq 1$, $0 < \rho_K \leq 1$ and $k_R \geq 1$, $k_I \geq 1$, $k_J \geq 1$, $k_K \geq 1$,

$$\rho_R \alpha_Q \leq \alpha_{Q+1} \leq \cdots \leq k_R \alpha_p,$$

$$\rho_I \beta_Q \leq \beta_{Q+1} \leq \cdots \leq k_I \beta_p,$$

$$\rho_J \gamma_Q \leq \gamma_{Q+1} \leq \cdots \leq k_J \gamma_p,$$

$$\rho_K \delta_Q \leq \delta_{Q+1} \leq \cdots \leq k_K \delta_p,$$

$$M_q = \sum_{\ell=1}^{Q} |a_\ell - a_{\ell-1}| \text{ and } M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|.$$

Then all zeros of P lie in

$$\begin{aligned} |q| \leq \ & \frac{1}{|a_n|}\bigg(|a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) \\ & + k_I\beta_p + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q \\ & + (1 - \rho_R)|\alpha_Q| - \rho_I\beta_Q + (1 - \rho_I)|\beta_Q| - \rho_J\gamma_Q + (1 - \rho_J)|\gamma_Q| \\ & - \rho_K\delta_Q + (1 - \rho_K)|\delta_Q| + M_p \bigg). \end{aligned}$$

Proof of Theorem 6.6. Define f by the equation

$$\begin{aligned} P(q) * (1 - q) &= a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\ &= f(q) - q^{n+1}a_n. \end{aligned}$$

By Lemma 6.4, the only zeros of $P(q) * (1 - q)$ are $q = 1$ and the zeros of $P(q)$. So, for $|q| = 1$, we have

$$\begin{aligned} |f(q)| &= |a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(a_n - a_{n-1})| \\ &\leq |a_0| + |q||a_1 - a_0| + |q|^2|a_2 - a_1| + \cdots + |q|^{Q+1}|a_{Q+1} - a_Q| \end{aligned}$$

$$
\begin{aligned}
&+\cdots+|q|^p|a_p-a_{p-1}|+\cdots+|q|^n|a_n-a_{n-1}| \\
= \;& |a_0|+|a_1-a_0|+|a_2-a_1|+\cdots+|a_{Q+1}-a_Q|+\cdots+|a_p-a_{p-1}| \\
&+\cdots+|a_n-a_{n-1}| \\
= \;& |a_0|+\sum_{\ell=1}^{Q}|a_\ell-a_{\ell-1}|+\sum_{\ell=Q+1}^{p}|a_\ell-a_{\ell-1}|+\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}| \\
= \;& |a_0|+M_q+\sum_{\ell=Q+1}^{p}|a_\ell-a_{\ell-1}|+M_p
\end{aligned}
$$

where

$$M_q=\sum_{\ell=1}^{Q}|a_\ell-a_{\ell-1}| \text{ and } M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|.$$

Thus

$$
\begin{aligned}
|f(z)| \;\leq\;& |a_0|+M_q+\sum_{\ell=Q}^{p}|a_\ell-a_{\ell-1}|+M_p \\
=\;& |a_0|+M_q \\
&+\sum_{\ell=Q+1}^{p}\sqrt{(\alpha_\ell-\alpha_{\ell-1})^2+(\beta_\ell-\beta_{\ell-1})^2+(\gamma_\ell-\gamma_{\ell-1})^2+(\delta_\ell-\delta_{\ell-1})^2}+M_p \\
\leq\;& |a_0|+M_q+\sum_{\ell=Q+1}^{p}\left(|\alpha_\ell-\alpha_{\ell-1}|+|\beta_\ell-\beta_{\ell-1}|+|\gamma_\ell-\gamma_{\ell-1}|+|\delta_\ell-\delta_{\ell-1}|\right) \\
&+M_p \\
=\;& |a_0|+M_q+|\alpha_p-\alpha_{p-1}|+|\beta_p-\beta_{p-1}|+|\gamma_p-\gamma_{p-1}|+|\delta_p-\delta_{p-1}| \\
&+\sum_{\ell=Q+2}^{p-1}\left(|\alpha_\ell-\alpha_{\ell-1}|+|\beta_\ell-\beta_{\ell-1}|+|\gamma_\ell-\gamma_{\ell-1}|+|\delta_\ell-\delta_{\ell-1}|\right) \\
&+|\alpha_{Q+1}-\alpha_Q|+|\beta_{Q+1}-\beta_Q|+|\gamma_{Q+1}-\gamma_Q|+|\delta_{Q+1}-\delta_Q|+M_p \\
=\;& |a_0|+M_q+|\alpha_p-k_R\alpha_p+k_R\alpha_p-\alpha_{p-1}|+|\beta_p-k_I\beta_p+k_I\beta_p-\beta_{p-1}| \\
&+|\gamma_p-k_J\gamma_p+k_J\gamma_p-\gamma_{p-1}|+|\delta_p-k_K\delta_p+k_K\delta_p-\delta_{p-1}| \\
&+\sum_{\ell=Q+2}^{p-1}\left(|\alpha_\ell-\alpha_{\ell-1}|+|\beta_\ell-\beta_{\ell-1}|+|\gamma_\ell-\gamma_{\ell-1}|+|\delta_\ell-\delta_{\ell-1}|\right) \\
&+|\alpha_{Q+1}-\rho_R\alpha_Q+\rho_R\alpha_Q-\alpha_Q|+|\beta_{Q+1}-\rho_I\beta_Q+\rho_I\beta_Q-\beta_Q|
\end{aligned}
$$

$$
\begin{aligned}
&+|\gamma_{Q+1}-\rho_J\gamma_Q+\rho_J\gamma_Q-\gamma_Q|+|\delta_{Q+1}-\rho_K\delta_Q+\rho_K\delta_Q-\delta_Q|+M_p \\
\leq\ & |a_0|+M_q+|\alpha_p-k_R\alpha_p|+|k_R\alpha_p-\alpha_{p-1}|+|\beta_p-k_I\beta_p|+|k_I\beta_p-\beta_{p-1}| \\
&+|\gamma_p-k_J\gamma_p|+|k_J\gamma_p-\gamma_{p-1}|+|\delta_p-k_K\delta_p|+|k_K\delta_p-\delta_{p-1}| \\
&+\sum_{\ell=Q+2}^{p-1}(|\alpha_\ell-\alpha_{\ell-1}|+|\beta_\ell-\beta_{\ell-1}|+|\gamma_\ell-\gamma_{\ell-1}|+|\delta_\ell-\delta_{\ell-1}|) \\
&+|\alpha_{Q+1}-\rho_R\alpha_Q|+|\rho_R\alpha_Q-\alpha_Q|+|\beta_{Q+1}-\rho_I\beta_Q|+|\rho_I\beta_Q-\beta_Q| \\
&+|\gamma_{Q+1}-\rho_J\gamma_Q|+|\rho_J\gamma_Q-\gamma_Q|+|\delta_{Q+1}-\rho_K\delta_Q|+|\rho_K\delta_Q-\delta_Q|+M_p \\
=\ & |a_0|+M_q+|\alpha_p||1-k_R|+(k_R\alpha_p-\alpha_{p-1})+|\beta_p||1-k_I|+(k_I\beta_p-\beta_{p-1}) \\
&+|\gamma_p||1-k_J|+(k_J\gamma_p-\gamma_{p-1})+|\delta_p||1-k_K|+(k_K\delta_p-\delta_{p-1}) \\
&+\sum_{\ell=Q+2}^{p-1}(\alpha_\ell-\alpha_{\ell-1})+\sum_{\ell=Q+2}^{p-1}(\beta_\ell-\beta_{\ell-1})+\sum_{\ell=Q+2}^{p-1}(\gamma_\ell-\gamma_{\ell-1}) \\
&+\sum_{\ell=Q+2}^{p-1}(\delta_\ell-\delta_{\ell-1})+(\alpha_{Q+1}-\rho_R\alpha_Q)+|\rho_R-1||\alpha_Q|+(\beta_{Q+1}-\rho_I\beta_Q) \\
&+|\rho_I-1||\beta_Q|+(\gamma_{Q+1}-\rho_J\gamma_Q)+|\rho_J-1||\gamma_Q|+(\delta_{Q+1}-\rho_K\delta_Q) \\
&+|\rho_K-1||\delta_Q|+M_p \\
=\ & |a_0|+M_q+|\alpha_p||1-k_R|+(k_R\alpha_p-\alpha_{p-1})+|\beta_p||1-k_I|+(k_I\beta_p-\beta_{p-1}) \\
&+|\gamma_p||1-k_J|+(k_J\gamma_p-\gamma_{p-1})+|\delta_p||1-k_K|+(k_K\delta_p-\delta_{p-1})+\alpha_{p-1} \\
&-\alpha_{Q+1}+\beta_{p-1}-\beta_{Q+1}+\gamma_{p-1}-\gamma_{Q+1}+\delta_{p-1}-\delta_{Q+1}+(\alpha_{Q+1}-\rho_R\alpha_Q) \\
&+|\rho_R-1||\alpha_Q|+(\beta_{Q+1}-\rho_I\beta_Q)+|\rho_I-1||\beta_Q|+(\gamma_{Q+1}-\rho_J\gamma_Q) \\
&+|\rho_J-1||\gamma_Q|+(\delta_{Q+1}-\rho_K\delta_Q)+|\rho_K-1||\delta_Q|+M_p \\
=\ & |a_0|+M_q+|\alpha_p|(k_R-1)+k_R\alpha_p+|\beta_p|(k_I-1)+k_I\beta_p+|\gamma_p|(k_J-1) \\
&+k_J\gamma_p+|\delta_p|(k_K-1)+k_K\delta_p-\rho_R\alpha_Q+(1-\rho_R)|\alpha_Q|-\rho_I\beta_Q \\
&+(1-\rho_I)|\beta_Q|-\rho_J\gamma_Q+(1-\rho_J)|\gamma_Q|-\rho_K\delta_Q+(1-\rho_K)|\delta_Q|+M_p.
\end{aligned}
$$

We can notice $q^n f\left(\frac{1}{q}\right) = \sum_{\ell=0}^{n} q^{n-\ell}(a_\ell - a_{\ell-1})$ where $a_{-1} = 0$ has the same bound on $|q|$ as f. Namely $\left|q^n f\left(\frac{1}{q}\right)\right| \leq |a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) + k_I\beta_p + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q + (1-\rho_R)|\alpha_Q| - \rho_I\beta_Q + (1-\rho_I)|\beta_Q| - \rho_J\gamma_Q + (1-\rho_J)|\gamma_Q| - \rho_K\delta_Q + (1-\rho_K)|\delta_Q| + |M_p|$ for $|q| = 1$. Since $q^n f\left(\frac{1}{q}\right)$ is analytic in $|q| \leq 1$ we have $\left|q^n f\left(\frac{1}{q}\right)\right| \leq |a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) + k_I\beta_p + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q + (1-\rho_R)|\alpha_Q| - \rho_I\beta_Q + (1-\rho_I)|\beta_Q| - \rho_J\gamma_Q + (1-\rho_J)|\gamma_Q| - \rho_K\delta_Q + (1-\rho_K)|\delta_Q| + M_p$ for $|q| \leq 1$ by Theorem 6.3. Thus

$$\begin{aligned}\left|f\left(\frac{1}{q}\right)\right| &= \frac{1}{|q|^n}\Big(|a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) + k_I\beta_p \\ &\quad + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q + (1-\rho_R)|\alpha_Q| \\ &\quad - \rho_I\beta_Q + (1-\rho_I)|\beta_Q| - \rho_J\gamma_Q + (1-\rho_J)|\gamma_Q| - \rho_K\delta_Q + (1-\rho_K)|\delta_Q| \\ &\quad + M_p\Big)\end{aligned}$$

for $|q| \leq 1$. Replacing q with $\frac{1}{q}$ we have

$$\begin{aligned}|f(q)| &= |q|^n\Big(|a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) + k_I\beta_p \\ &\quad + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q + (1-\rho_R)|\alpha_Q| \\ &\quad - \rho_I\beta_Q + (1-\rho_I)|\beta_Q| - \rho_J\gamma_Q + (1-\rho_J)|\gamma_Q| - \rho_K\delta_Q + (1-\rho_K)|\delta_Q| \\ &\quad + M_p\Big)\end{aligned}$$

for $|q| \geq 1$. Since

$$\begin{aligned}|P(q) * (1-q)| &= |f(q) - q^{n+1}a_n| \\ &\geq |q|^{n+1}|a_n| - |f(q)|\end{aligned}$$

$$\begin{aligned}
\geq\ & |q|^{n+1}|a_n| - |q|^n\Big(|a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) \\
& + k_I\beta_p + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q \\
& + (1-\rho_R)|\alpha_Q| - \rho_I\beta_Q + (1-\rho_I)|\beta_Q| - \rho_J\gamma_Q + (1-\rho_J)|\gamma_Q| \\
& - \rho_K\delta_Q + (1-\rho_K)|\delta_Q| + M_p\Big),
\end{aligned}$$

then

$$\begin{aligned}
|q| \quad > \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) \\
& + k_I\beta_p + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q \\
& + (1-\rho_R)|\alpha_Q| - \rho_I\beta_Q + (1-\rho_I)|\beta_Q| - \rho_J\gamma_Q + (1-\rho_J)|\gamma_Q| \\
& - \rho_K\delta_Q + (1-\rho_K)|\delta_Q| + M_p\Big)
\end{aligned}$$

then $|P(q) * (1-q)| > 0$ and $P(q) * (1-q) \neq 0$. Since the only zeros of $P(q) * (1-q)$ are $q = 1$ and the zeros of P, for

$$\begin{aligned}
|q| \quad > \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) \\
& + k_I\beta_p + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q \\
& + (1-\rho_R)|\alpha_Q| - \rho_I\beta_Q + (1-\rho_I)|\beta_Q| - \rho_J\gamma_Q + (1-\rho_J)|\gamma_Q| \\
& - \rho_K\delta_Q + (1-\rho_K)|\delta_Q| + M_p\Big)
\end{aligned}$$

we have $P(q) \neq 0$. That is, all the zeros of P lie in

$$\begin{aligned}
|q| \quad \leq \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q + |\alpha_p|(k_R - 1) + k_R\alpha_p + |\beta_p|(k_I - 1) \\
& + k_I\beta_p + |\gamma_p|(k_J - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + k_K\delta_p - \rho_R\alpha_Q \\
& + (1-\rho_R)|\alpha_Q| - \rho_I\beta_Q + (1-\rho_I)|\beta_Q| - \rho_J\gamma_Q + (1-\rho_J)|\gamma_Q|
\end{aligned}$$

$$-\rho_K \delta_Q + (1-\rho_K)|\delta_Q| + M_p \Big).$$

□

Notice when $\gamma_\ell = \delta_\ell = 0$ for all $0 \leq \ell \leq n$ then Theorem 6.6 reduces to Theorem 2.2. With additional parameters previously discussed in this thesis, then Theorem 6.6 reduces to Theorem 1.4. With similar parameters we will now find an inner bound containing no zeros:

Theorem 6.7. *If* $P(q) = a_0 + qa_1 + \cdots + q^Q a_Q + \cdots + q^p a_p + \cdots + q^n a_n$ *is a polynomial of degree* n *with quaternionic coefficients. That is* $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$ *for* $\ell = 0, 1, 2, \ldots, n$ *and real* $\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell$*, where* $q \leq p$, $0 < \rho_R \leq 1$, $0 < \rho_I \leq 1$, $0 < \rho_J \leq 1$, $0 < \rho_K \leq 1$, $k_R \geq 1$, $k_I \geq 1$, $k_J \geq 1$, $k_K \geq 1$, *satisfying*

$$\rho_R \alpha_Q \leq \alpha_{Q+1} \leq \cdots \leq k_R \alpha_p$$

$$\rho_I \beta_Q \leq \beta_{Q+1} \leq \cdots \leq k_I \beta_p$$

$$\rho_J \gamma_Q \leq \gamma_{Q+1} \leq \cdots \leq k_J \gamma_p$$

$$\rho_K \delta_Q \leq \delta_{Q+1} \leq \cdots \leq k_K \delta_p.$$

Then $P(q)$ *does not vanish in*

$$\begin{aligned} |q| \;<\; \min\Big\{1, |a_0| \Big/ \Big(& M_q + |\alpha_Q|(1-\rho_R) - \rho_R \alpha_Q + |\beta_Q|(1-\rho_I) - \rho_I \beta_Q \\ & + |\gamma_Q|(1-\rho_J) - \rho_J \gamma_Q + |\delta_Q|(1-\rho_K) - \rho_K \delta_Q + k_R \alpha_p + |\alpha_p|(k_R - 1) \\ & + k_I \beta_p + |\beta_p|(k_I - 1) + k_J \gamma_p + |\delta_p|(k_K - 1) + M_p + |a_n| \Big) \Big\}, \end{aligned}$$

where $M_q = \sum_{\ell=1}^{Q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 6.7. Consider the reciprocal polynomial

$$S(q) = q^n * P\left(\frac{1}{q}\right) = q^n a_0 + q^{n-1}a_1 + \cdots + q^{n-Q}a_q + \cdots + q^{n-p}a_p + \cdots + qa_{n-1} + a_n.$$

Let

$$\begin{aligned} H(q) &= S(q) * (1-q) \\ &= -q^{n+1}a_0 + q^n(a_0 - a_1) + q^{n-1}(a_1 - a_2) + \cdots + q^{n-Q}(a_Q - a_{Q+1}) + \cdots \\ &\quad + q^{n-p}(a_p - a_{p+1}) + \cdots + q^2(a_{n-2} - a_{n-1}) + q(a_{n-1} - a_n) + a_n. \end{aligned}$$

By Lemma 6.4, the only zeros of $P(q) * (1-q)$ are $q = 1$ and the zeros of $P(q)$. Thus we have

$$\begin{aligned} |H(q)| &\geq |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}|a_Q - a_{Q+1}| \\ &\quad + \cdots + |q|^{n-p+1}|a_{p-1} - a_p| + \cdots + |q|^2|a_{n-2} - a_{n-1}| + |q||a_{n-1} - a_n| \\ &\quad + |a_n|\Big) \\ &= |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}\Big((\alpha_Q - \alpha_{Q+1})^2 \\ &\quad + (\beta_Q - \beta_{Q+1})^2 + (\gamma_Q - \gamma_{Q+1})^2 + (\delta_Q - \delta_{Q+1})^2\Big)^{\frac{1}{2}} + \cdots \\ &\quad + |q|^{n-p+1}\Big((\alpha_{p-1} - \alpha_p)^2 + (\beta_{p-1} - \beta_p)^2 + (\gamma_{p-1} - \gamma_p)^2 + (\delta_{p-1} - \delta_p)^2\Big)^{\frac{1}{2}} \\ &\quad + \cdots + |q|^2|a_{n-2} - a_{n-1}| + |q||a_{n-1} - a_n| + |a_n|\Big) \\ &\geq |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}|\alpha_Q - \alpha_{Q+1}| \\ &\quad + |q|^{n-Q}|\beta_Q - \beta_{Q+1}| + |q|^{n-Q}|\gamma_Q - \gamma_{Q+1}| + |q|^{n-Q}|\delta_Q - \delta_{Q+1}| + \cdots \\ &\quad + |q|^{n-p+1}|\alpha_{p-1} - \alpha_p| + |q|^{n-p+1}|\beta_{p-1} - \beta_p| + |q|^{n-p+1}|\gamma_{p-1} - \gamma_p| \\ &\quad + |q|^{n-p+1}|\delta_{p-1} - \delta_p| + \cdots + |q|^2|a_{n-2} - a_{n-1}| + |q||a_{n-1} - a_n| + |a_n|\Big) \end{aligned}$$

$$
\begin{aligned}
&= |q|^{n+1}|a_0| - \Bigg(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots \\
&\quad +|q|^{n-Q}|\alpha_Q - \rho_R\alpha_Q + \rho_R\alpha_Q - \alpha_{Q+1}| + |q|^{n-Q}|\beta_Q - \rho_I\beta_Q + \rho_I\beta_Q - \beta_{Q+1}| \\
&\quad +|q|^{n-Q}|\gamma_Q - \rho_J\gamma_Q + \rho_J\gamma_Q - \gamma_{Q+1}| + |q|^{n-Q}|\delta_Q - \rho_K\delta_Q + \rho_K\delta_Q - \delta_{Q+1}| \\
&\quad + \cdots + |q|^{n-p+1}|\alpha_{p-1} - k_R\alpha_p + k_R\alpha_p - \alpha_p| \\
&\quad +|q|^{n-p+1}|\beta_{p-1} - k_I\beta_p + k_I\beta_p - \beta_p| + |q|^{n-p+1}|\gamma_{p-1} - k_J\gamma_p + k_J\gamma_p - \gamma_p| \\
&\quad +|q|^{n-p+1}|\delta_{p-1} - k_K\delta_p + k_K\delta_p - \delta_p| + \cdots + |q|^2|a_{n-2} - a_{n-1}| \\
&\quad +|q||a_{n-1} - a_n| + |a_n|\Bigg) \\
&\geq |q|^{n+1}|a_0| - \Bigg(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}|\alpha_Q - \rho_R\alpha_Q| \\
&\quad +|q|^{n-Q}|\rho_R\alpha_Q - \alpha_{Q+1}| + |q|^{n-Q}|\beta_Q - \rho_I\beta_Q| + |q|^{n-Q}|\rho_I\beta_Q - \beta_{Q+1}| \\
&\quad +|q|^{n-Q}|\gamma_Q - \rho_J\gamma_Q| + |q|^{n-Q}|\rho_J\gamma_Q - \gamma_{Q+1}| + |q|^{n-Q}|\delta_Q - \rho_K\delta_Q| \\
&\quad +|q|^{n-Q}|\rho_K\delta_Q - \delta_{Q+1}| + \cdots + |q|^{n-p+1}|\alpha_{p-1} - k_R\alpha_p| \\
&\quad +|q|^{n-p+1}|k_R\alpha_p - \alpha_p| + |q|^{n-p+1}|\beta_{p-1} - k_I\beta_p| + |q|^{n-p+1}|k_I\beta_p - \beta_p| \\
&\quad +|q|^{n-p+1}|\gamma_{p-1} - k_J\gamma_p| + |q|^{n-p+1}|k_J\gamma_p - \gamma_p| + |q|^{n-p+1}|\delta_{p-1} - k_K\delta_p| \\
&\quad +|q|^{n-p+1}|k_K\delta_p - \delta_p| + \cdots + |q|^2|a_{n-2} - a_{n-1}| + |q||a_{n-1} - a_n| + |a_n|\Bigg) \\
&= |q|^{n+1}|a_0| - \Bigg(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}|\alpha_Q||1 - \rho_R| \\
&\quad +|q|^{n-Q}|\alpha_{Q+1} - \rho_R\alpha_Q| + |q|^{n-Q}|\beta_Q||1 - \rho_I| + |q|^{n-Q}|\beta_{Q+1} - \rho_I\beta_Q| \\
&\quad +|q|^{n-Q}|\gamma_Q||1 - \rho_J| + |q|^{n-Q}|\gamma_{Q+1} - \rho_J\gamma_Q| + |q|^{n-Q}|\delta_Q||1 - \rho_K| \\
&\quad +|q|^{n-Q}|\delta_{Q+1} - \rho_K\delta_Q| + \cdots + |q|^{n-p+1}|k_R\alpha_p - \alpha_{p-1}| \\
&\quad +|q|^{n-p+1}|\alpha_p||k_R - 1| + |q|^{n-p+1}|k_I\beta_p - \beta_{p-1}| + |q|^{n-p+1}|\beta_p||k_I - 1| \\
&\quad +|q|^{n-p+1}|k_J\gamma_p - \gamma_{p-1}| + |q|^{n-p+1}|\gamma_p||k_J - 1| + |q|^{n-p+1}|k_K\delta_p - \delta_{p-1}|
\end{aligned}
$$

$$
\begin{aligned}
&+|q|^{n-p+1}|\delta_p||k_K-1|+\cdots+|q|^2|a_{n-2}-a_{n-1}|+|q||a_{n-1}-a_n|+|a_n|\Big) \\
= \; & |q|^{n+1}|a_0|-\Big(|q|^n|a_0-a_1|+|q|^{n-1}|a_1-a_2|+\cdots+|q|^{n-Q}|\alpha_Q|(1-\rho_R) \\
&+|q|^{n-Q}(\alpha_{Q+1}-\rho_R\alpha_Q)+|q|^{n-Q}|\beta_Q|(1-\rho_I)+|q|^{n-Q}(\beta_{Q+1}-\rho_I\beta_Q) \\
&+|q|^{n-Q}|\gamma_Q|(1-\rho_J)+|q|^{n-Q}(\gamma_{Q+1}-\rho_J\gamma_Q)+|q|^{n-Q}|\delta_Q|(1-\rho_K) \\
&+|q|^{n-Q}(\delta_{Q+1}-\rho_K\delta_Q)+\cdots+|q|^{n-p+1}(k_R\alpha_p-\alpha_{p-1}) \\
&+|q|^{n-p+1}|\alpha_p|(k_R-1)+|q|^{n-p+1}(k_I\beta_p-\beta_{p-1})+|q|^{n-p+1}|\beta_p|(k_I-1) \\
&+|q|^{n-p+1}(k_J\gamma_p-\gamma_{p-1})+|q|^{n-p+1}|\gamma_p|(k_J-1)+|q|^{n-p+1}(k_K\delta_p-\delta_{p-1}) \\
&+|q|^{n-p+1}|\delta_p|(k_K-1)+\cdots+|q|^2|a_{n-2}-a_{n-1}|+|q||a_{n-1}-a_n|+|a_n|\Big),
\end{aligned}
$$

since $0<\rho_R\le 1, 0<\rho_I\le 1, 0<\rho_J\le 1, 0<\rho_K\le 1, k_R\ge 1, k_I\ge 1, k_J\ge 1, k_K\ge 1$, $\rho_R\alpha_Q\le\alpha_{Q+1}\le\cdots\le k_R\alpha_p$, $\rho_I\beta_Q\le\beta_{Q+1}\le\cdots\le k_R\beta_p$, $\rho_J\gamma_Q\le\gamma_{Q+1}\le\cdots\le k_R\gamma_p$, and $\rho_K\delta_Q\le\delta_{Q+1}\le\cdots\le k_K\delta_p$. Thus

$$
\begin{aligned}
|H(q)| \; \ge \; & |q|^{n+1}|a_0|-\Big(|q|^n|a_0-a_1|+|q|^{n-1}|a_1-a_2|+\cdots+|q|^{n-Q}|\alpha_Q|(1-\rho_R) \\
&+|q|^{n-Q}(\alpha_{Q+1}-\rho_R\alpha_Q)+|q|^{n-Q}|\beta_Q|(1-\rho_I)+|q|^{n-Q}(\beta_{Q+1}-\rho_I\beta_Q) \\
&+|q|^{n-Q}|\gamma_Q|(1-\rho_J)+|q|^{n-Q}(\gamma_{Q+1}-\rho_J\gamma_Q)+|q|^{n-Q}|\delta_Q|(1-\rho_K) \\
&+|q|^{n-Q}(\delta_{Q+1}-\rho_K\delta_Q)+\cdots+|q|^{n-p+1}(k_R\alpha_p-\alpha_{p-1}) \\
&+|q|^{n-p+1}|\alpha_p|(k_R-1)+|q|^{n-p+1}(k_I\beta_p-\beta_{p-1})+|q|^{n-p+1}|\beta_p|(k_I-1) \\
&+|q|^{n-p+1}(k_J\gamma_p-\gamma_{p-1})+|q|^{n-p+1}|\gamma_p|(k_J-1)+|q|^{n-p+1}(k_K\delta_p-\delta_{p-1}) \\
&+|q|^{n-p+1}|\delta_p|(k_K-1)+\cdots+|q|^2|a_{n-2}-a_{n-1}|+|q||a_{n-1}-a_n|+|a_n|\Big) \\
= \; & |q|^n\Big[|q||a_0|-\Big(|a_0-a_1|+\frac{|a_1-a_2|}{|q|}+\cdots+\frac{|\alpha_Q|(1-\rho_R)}{|q|^Q} \\
&+\frac{\alpha_{Q+1}-\rho_R\alpha_Q}{|q|^Q}+\frac{|\beta_Q|(1-\rho_I)}{|q|^Q}+\frac{\beta_{Q+1}-\rho_I\beta_Q}{|q|^Q}+\frac{|\gamma_Q|(1-\rho_J)}{|q|^Q}
\end{aligned}
$$

$$+\frac{\gamma_{Q+1}-\rho_J\gamma_Q}{|q|^Q}+\frac{|\delta_Q|(1-\rho_K)}{|q|^Q}+\frac{\delta_{Q+1}-\rho_K\delta_Q}{|q|^Q}+\cdots$$
$$+\frac{k_R\alpha_p-\alpha_{p-1}}{|q|^{p-1}}+\frac{|\alpha_p|(k_R-1)}{|q|^{p-1}}+\frac{k_I\beta_p-\beta_{p-1}}{|q|^{p-1}}+\frac{|\beta_p|(k_I-1)}{|q|^{p-1}}$$
$$+\frac{k_J\gamma_p-\gamma_{p-1}}{|q|^{p-1}}+\frac{|\gamma_p|(k_J-1)}{|q|^{p-1}}+\frac{k_K\delta_p-\delta_{p-1}}{|q|^{p-1}}+\frac{|\delta_p|(k_K-1)}{|q|^{p-1}}$$
$$+\cdots+\frac{|a_{n-2}-a_{n-1}|}{|q|^{n-2}}+\frac{|a_{n-1}-a_n|}{|q|^{n-1}}+\frac{|a_n|}{|q|^n}\Bigg)\Bigg].$$

Now for $|q|>1$ so that $\frac{1}{|q|^{n-\ell}}<1$, for $0\leq\ell<n$ we have

$$\begin{aligned}
H(q) &\geq |q|^n\Bigg[|q||a_0|-\Bigg(|a_0-a_1|+|a_1-a_2|+\cdots+|\alpha_Q|(1-\rho_R)\\
&+(\alpha_{Q+1}-\rho_R\alpha_Q)+|\beta_Q|(1-\rho_I)+(\beta_{Q+1}-\rho_I\beta_Q)+|\gamma_Q|(1-\rho_J)\\
&+(\gamma_{Q+1}-\rho_J\gamma_Q)+|\delta_Q|(1-\rho_K)+(\delta_{Q+1}-\rho_K\delta_Q)+\cdots+(k_R\alpha_p-\alpha_{p-1})\\
&+|\alpha_p|(k_R-1)+(k_I\beta_p-\beta_{p-1})+|\beta_p|(k_I-1)+(k_J\gamma_p-\gamma_{p-1})\\
&+|\delta_p|(k_K-1)+\cdots+|a_{n-2}-a_{n-1}|+|a_{n-1}-a_n|+|a_n|\Bigg)\Bigg]\\
&= |q|^n\Bigg[|q||a_0|-\Bigg(\sum_{\ell=1}^{Q}|a_\ell-a_{\ell-1}|+|\alpha_Q|(1-\rho_R)-\rho_R\alpha_Q+|\beta_Q|(1-\rho_I)\\
&-\rho_I\beta_Q+|\gamma_Q|(1-\rho_J)-\rho_J\gamma_Q+|\delta_Q|(1-\rho_K)-\rho_K\delta_Q+k_R\alpha_p\\
&+|\alpha_p|(k_R-1)+k_I\beta_p+|\beta_p|(k_I-1)+k_J\gamma_p+|\delta_p|(k_K-1)\\
&+\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|+|a_n|\Bigg)\Bigg]\\
&= |q|^n\Bigg[|q||a_0|-\Bigg(M_q+|\alpha_Q|(1-\rho_R)-\rho_R\alpha_Q+|\beta_Q|(1-\rho_I)-\rho_I\beta_Q\\
&+|\gamma_Q|(1-\rho_J)-\rho_J\gamma_Q+|\delta_Q|(1-\rho_K)-\rho_K\delta_Q+k_R\alpha_p+|\alpha_p|(k_R-1)\\
&+k_I\beta_p+|\beta_p|(k_I-1)+k_J\gamma_p+|\delta_p|(k_K-1)+M_p+|a_n|\Bigg)\Bigg],
\end{aligned}$$

where $M_q = \sum_{\ell=1}^{Q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. Now

$$\begin{aligned}
|H(q)| &\geq |q|^n \Bigg[|q||a_0| - \Bigg(M_q + |\alpha_Q|(1-\rho_R) - \rho_R\alpha_Q + |\beta_Q|(1-\rho_I) - \rho_I\beta_Q \\
&\quad + |\gamma_Q|(1-\rho_J) - \rho_J\gamma_Q + |\delta_Q|(1-\rho_K) - \rho_K\delta_Q + k_R\alpha_p + |\alpha_p|(k_R - 1) \\
&\quad + k_I\beta_p + |\beta_p|(k_I - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + M_p + |a_n| \Bigg) \Bigg] \\
&> 0
\end{aligned}$$

if

$$\begin{aligned}
|q| &> \frac{1}{|a_0|} \Bigg(M_q + |\alpha_Q|(1-\rho_R) - \rho_R\alpha_Q + |\beta_Q|(1-\rho_I) - \rho_I\beta_Q \\
&\quad + |\gamma_Q|(1-\rho_J) - \rho_J\gamma_Q + |\delta_Q|(1-\rho_K) - \rho_K\delta_Q + k_R\alpha_p + |\alpha_p|(k_R - 1) \\
&\quad + k_I\beta_p + |\beta_p|(k_I - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + M_p + |a_n| \Bigg).
\end{aligned}$$

Thus all zeros of $H(q)$ whose modulus is greater than 1 lie in

$$\begin{aligned}
|q| &\leq \frac{1}{|a_0|} \Bigg(M_q + |\alpha_Q|(1-\rho_R) - \rho_R\alpha_Q + |\beta_Q|(1-\rho_I) - \rho_I\beta_Q \\
&\quad + |\gamma_Q|(1-\rho_J) - \rho_J\gamma_Q + |\delta_Q|(1-\rho_K) - \rho_K\delta_Q + k_R\alpha_p + |\alpha_p|(k_R - 1) \\
&\quad + k_I\beta_p + |\beta_p|(k_I - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + M_p + |a_n| \Bigg).
\end{aligned}$$

Hence all zeros of $H(q)$ and hence of $S(q)$ lie in

$$\begin{aligned}
|q| &\leq \max \Bigg\{ 1, |a_0| \Bigg/ \Bigg(M_q + |\alpha_Q|(1-\rho_R) - \rho_R\alpha_Q + |\beta_Q|(1-\rho_I) - \rho_I\beta_Q \\
&\quad + |\gamma_Q|(1-\rho_J) - \rho_J\gamma_Q + |\delta_Q|(1-\rho_K) - \rho_K\delta_Q + k_R\alpha_p + |\alpha_p|(k_R - 1) \\
&\quad + k_I\beta_p + |\beta_p|(k_I - 1) + k_J\gamma_p + |\delta_p|(k_K - 1) + M_p + |a_n| \Bigg) \Bigg\}.
\end{aligned}$$

Therefore all the zeros of $P(q)$ lie in

$$|q| \geq \min \Bigg\{ 1, |a_0| \Bigg/ \Bigg(M_q + |\alpha_Q|(1-\rho_R) - \rho_R\alpha_Q + |\beta_Q|(1-\rho_I) - \rho_I\beta_Q$$

$$+|\gamma_Q|(1-\rho_J) - \rho_J\gamma_Q + |\delta_Q|(1-\rho_K) - \rho_K\delta_Q + k_R\alpha_p + |\alpha_p|(k_R-1)$$
$$+k_I\beta_p + |\beta_p|(k_I-1) + k_J\gamma_p + |\delta_p|(k_K-1) + M_p + |a_n|\Big)\Big\}.$$

Thus the polynomial $P(q)$ does not vanish in

$$\begin{aligned}|q| \quad < \quad & \min\Big\{1, |a_0| \Big/ \Big(M_q + |\alpha_Q|(1-\rho_R) - \rho_R\alpha_Q + |\beta_Q|(1-\rho_I) - \rho_I\beta_Q \\ & +|\gamma_Q|(1-\rho_J) - \rho_J\gamma_Q + |\delta_Q|(1-\rho_K) - \rho_K\delta_Q + k_R\alpha_p + |\alpha_p|(k_R-1) \\ & +k_I\beta_p + |\beta_p|(k_I-1) + k_J\gamma_p + |\delta_p|(k_K-1) + M_p + |a_n|\Big)\Big\}.\end{aligned}$$

$\square$

Similarly, when $\gamma_\ell = \delta_\ell$ for all $0 \leq \ell \leq n$, then Theorem 6.7 reduces to Theorem 2.5. Which further reduces to Theorem 1.13. Since we considered monotonicity on the parts we will now consider monotonicity on the moduli:

Theorem 6.8. *If $P(q) = a_0 + qa_1 + \cdots + q^Q a_Q + \cdots + q^p a_p + \cdots + q^n a_n$ is a polynomial of degree n with quaternionic coefficients satisfying, for a nonzero quaternion b, $\measuredangle(a_\ell, b) \leq \alpha \leq \frac{\pi}{2}$ for some α and $\ell = Q, Q+1, \ldots, p$. Assume*

$$\rho|a_Q| \leq |a_{Q+1}| \leq \cdots \leq k|a_p|.$$

Then all zeros $P(q)$ lie in

$$\begin{aligned}|q| \quad \leq \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha - |a_p| \\ & +k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p\Big)\end{aligned}$$

where $M_q = \sum_{\ell=1}^{Q}|a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}|$.

Proof of Theorem 6.8. Let $P(z) = a_0 + qa_1 + \cdots + q^Q a_Q + \cdots + q^p a_p + \cdots + q^n a_n$ be a polynomial of degree n with quaternionic coefficients. Without loss of generality assume $\beta = 0$. Consider

$$\begin{aligned} P(q) * (1-q) &= a_0 + q(a_1 - a_0) + \cdots + q^{Q+1}(a_{Q+1} - a_Q) \\ &\quad + \cdots + q^p(a_p - a_{p-1}) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\ &= f(q) - q^{n+1}a_n \end{aligned}$$

By Lemma 6.4, the only zeros of $P(q) * (1-q)$ are $q = 1$ and the zeros of $P(q)$. So, for $|q| = 1$,

$$\begin{aligned} |f(q)| &\leq |a_0 + q(a_1 - a_0) + \cdots + q^{Q+1}(a_{Q+1} - a_Q) + \cdots \\ &\quad + q^p(a_p - a_{p-1}) + \cdots + q^n(a_n - a_{n-1})| \\ &\leq |a_0| + |q||a_1 - a_0| + \cdots + |q|^{Q+1}|a_{Q+1} - a_Q| + \cdots \\ &\quad + |q|^p|a_p - a_{p-1}| + \cdots + |q|^n|a_n - a_{n-1}| \\ &= |a_0| + |a_1 - a_0| + \cdots + |a_{Q+1} - a_Q| + \cdots + |a_p - a_{p-1}| \\ &\quad + \cdots + |a_n - a_{n-1}| \\ &= |a_0| + \sum_{\ell=1}^{Q} |a_\ell - a_{\ell-1}| + |a_{Q+1} - a_Q| + \sum_{\ell=Q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| \\ &\quad + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| \\ &= |a_0| + M_q + |a_{Q+1} - a_Q| + \sum_{\ell=Q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| + M_p \end{aligned}$$

where $M_q = \sum_{\ell=1}^{Q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. Thus

$$|f(q)| \leq |a_0| + M_q + |a_{Q+1} - \rho a_Q + \rho a_Q - a_Q| + \sum_{\ell=Q+2}^{p-1} |a_\ell - a_{\ell-1}|$$

$$
\begin{aligned}
&\quad +|a_p - ka_p + ka_p - a_{p-1}| + M_p \\
&\leq \quad |a_0| + M_q + |a_{Q+1} - \rho a_Q| + |\rho a_Q - a_Q| + \sum_{\ell=Q+2}^{p-1} |a_\ell - a_{\ell-1}| \\
&\quad +|a_p - ka_p| + |ka_p - a_{p-1}| + M_p.
\end{aligned}
$$

So by Lemma 6.5,

$$
\begin{aligned}
|f(q)| \quad \leq \quad & |a_0| + M_q + (|a_{Q+1}| - |\rho a_Q|)\cos\alpha + (|a_{Q+1}| + |\rho a_Q|)\sin\alpha + |a_Q|(1-\rho) \\
& + \sum_{\ell=Q+2}^{p-1} (|a_\ell| - |a_{\ell-1}|)\cos\alpha + \sum_{\ell=Q+2}^{p-1} (|a_\ell| + |a_{\ell-1}|)\sin\alpha + |a_p|(k-1) \\
& +(|ka_p| - |a_{p-1}|)\cos\alpha + (|ka_p| + |a_{p-1}|)\sin\alpha + M_p \\
= \quad & |a_0| + M_q + |a_{Q+1}|\cos\alpha - \rho|a_Q|\cos\alpha + |a_{Q+1}|\sin\alpha + \rho|a_Q|\sin\alpha \\
& +|a_Q|(1-\rho) + \sum_{\ell=Q+2}^{p-1} |a_\ell|\cos\alpha - \sum_{\ell=Q+2}^{p-1} |a_{\ell-1}|\cos\alpha + \sum_{\ell=Q+2}^{p-1} |a_\ell|\sin\alpha \\
& + \sum_{\ell=Q+2}^{p-1} |a_{\ell-1}|\sin\alpha + |a_p|(k-1) + k|a_p|\cos\alpha - |a_{p-1}|\cos\alpha + k|a_p|\sin\alpha \\
& +|a_{p-1}|\sin\alpha + M_p \\
= \quad & |a_0| + M_q + |a_{Q+1}|\cos\alpha - \rho|a_Q|\cos\alpha + |a_{Q+1}|\sin\alpha + \rho|a_Q|\sin\alpha \\
& +|a_Q|(1-\rho) + |a_{p-1}|\cos\alpha + \sum_{\ell=Q+2}^{p-2} |a_\ell|\cos\alpha - |a_{Q+1}|\cos\alpha \\
& - \sum_{\ell=Q+2}^{p-2} |a_\ell|\cos\alpha + |a_{p-1}|\sin\alpha + \sum_{\ell=Q+2}^{p-2} |a_\ell|\sin\alpha + |a_{Q+1}|\sin\alpha \\
& + \sum_{\ell=Q+2}^{p-2} |a_\ell|\sin\alpha + |a_p|(k-1) + k|a_p|\cos\alpha - |a_{p-1}|\cos\alpha + k|a_p|\sin\alpha \\
& +|a_{p-1}|\sin\alpha + M_p \\
= \quad & |a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2|a_{Q+1}|\sin\alpha + |a_Q| \\
& +2|a_{p-1}|\sin\alpha + 2\sum_{\ell=Q+2}^{p-2} |a_\ell|\sin\alpha - |a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p
\end{aligned}
$$

$$
\begin{aligned}
= \quad & |a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1} |a_\ell| \sin\alpha - |a_p| \\
& + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p.
\end{aligned}
$$

Hence also,

$$
\begin{aligned}
\left| q^n f\left(\frac{1}{q}\right)\right| \quad \leq \quad & |a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1} |a_\ell| \sin\alpha - |a_p| \\
& + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p
\end{aligned}
$$

for $|q| = 1$. By Theorem 6.3

$$
\begin{aligned}
\left| q^n f\left(\frac{1}{q}\right)\right| \quad \leq \quad & |a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1} |a_\ell| \sin\alpha - |a_p| \\
& + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p
\end{aligned}
$$

also holds inside the unit circle $|q| \leq 1$ as well. Thus

$$
\begin{aligned}
\left| f\left(\frac{1}{q}\right)\right| \quad \leq \quad & \frac{1}{|q|^n}\Bigg(|a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1} |a_\ell| \sin\alpha \\
& - |a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p \Bigg)
\end{aligned}
$$

for $|q| \leq 1$. Replacing q with $\dfrac{1}{q}$ we have

$$
\begin{aligned}
|f(q)| \quad \leq \quad & |q|^n \Bigg(|a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1} |a_\ell| \sin\alpha - |a_p| \\
& + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p \Bigg)
\end{aligned}
$$

for $|q| \geq 1$. Thus we have

$$
\begin{aligned}
|P(q) * (1-q)| \quad = \quad & |f(q) - q^{n+1} a_n| \\
\geq \quad & |q|^{n+1}|a_n| - |f(q)|
\end{aligned}
$$

$$\begin{aligned}
&\geq |q|^{n+1}|a_n| - |q|^n\Big(|a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| \\
&\quad +2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha - |a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p\Big) \\
&= |q|^n\Big[|q||a_n| - \Big(|a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| \\
&\quad +2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha - |a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p\Big)\Big].
\end{aligned}$$

So if

$$\begin{aligned}
|q| &> \frac{1}{|a_n|}\Big(|a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha - |a_p| \\
&\quad +k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p\Big),
\end{aligned}$$

then

$$\begin{aligned}
0 &\neq |q|^n\Big[|q||a_n| - \Big(|a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha \\
&\quad -|a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p\Big)\Big].
\end{aligned}$$

Therefore all zeros of $P(q)$ lie in

$$\begin{aligned}
|q| &\leq \frac{1}{|a_n|}\Big(|a_0| + M_q + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + |a_Q| + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha - |a_p| \\
&\quad +k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p\Big).
\end{aligned}$$

□

Notice when $\gamma_\ell = \delta_\ell = 0$ for all $0 \leq \ell \leq n$, then Theorem 6.8 reduces to Theorem 2.8. We can also define an inner bound of a quaternionic polynomial with quaternionic coefficients as:

Theorem 6.9. *If* $P(q) = a_0 + qa_1 + \cdots + q^Q a_Q + \cdots + a^p a_p + \cdots + a^n a_n$ *is a polynomial of degree* n *with quaternionic coefficients satisfying, for a nonzero quaternion* b, $\measuredangle(a_\ell, b) \le \alpha \le \dfrac{\pi}{2}$ *for some* α *and* $\ell = Q, Q+1, \ldots, p$. *Assume*

$$\rho|a_Q| \le |a_{Q+1}| \le \cdots \le k|a_p|.$$

Then $P(q)$ *does not vanish in*

$$\begin{aligned} |q| \;&<\; \min\left\{1, |a_0| \Big/ \left(M_q + |a_Q| + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{p-1} |a_\ell| \sin\alpha \right.\right. \\ &\qquad \left.\left. + |a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p + |a_n|\right)\right\}, \end{aligned}$$

where $M_q = \displaystyle\sum_{\ell=1}^{Q} |a_\ell - a_{\ell-1}|$ *and* $M_p = \displaystyle\sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Proof of Theorem 6.9. Consider the reciprocal polynomial

$$S(q) = q^n * P\left(\frac{1}{q}\right) = q^n a_0 + q^{n-1} a_1 + \cdots + q a_{n-1} + a_n.$$

Let

$$\begin{aligned} H(q) &= S(q) * (1-q) \\ &= -q^{n+1} a_0 + q^n (a_0 - a_1) + q^{n-1}(a_1 - a_2) + \cdots + q^{n-Q}(a_Q - a_{Q+1}) \\ &\quad + \cdots + q^{n-p}(a_p - a_{p+1}) + \cdots + q(a_{n-1} - a_n) + a_n. \end{aligned}$$

By Lemma 6.4, the only zeros of $P(q) * (1-q)$ are $q = 1$ and the zeros of $P(q)$. This gives

$$|H(q)| \;\ge\; |q|^{n+1}|a_0| - \Big(|q|^n |a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}|a_Q - a_{Q+1}|$$

$$
\begin{aligned}
&+\cdots+|q|^{n-p+1}|a_{p-1}-a_p|+\cdots+|q||a_{n-1}-a_n|+|a_n|\Big) \\
= \;& |q|^{n+1}|a_0|-\Big(|q|^n|a_0-a_1|+|q|^{n-1}|a_1-a_2|+\cdots \\
&+|q|^{n-Q}|a_Q-\rho a_Q+\rho a_Q-a_{Q+1}|+\cdots+|q|^{n-p+1}|a_{p-1}-ka_p+ka_p-a_p| \\
&+\cdots+|q||a_{n-1}-a_n|+|a_n|\Big) \\
\geq \;& |q|^{n+1}|a_0|-\Big(|q|^n|a_0-a_1|+|q|^{n-1}|a_1-a_2|+\cdots \\
&+|q|^{n-Q}|a_Q-\rho a_Q|+|q|^{n-Q}|\rho a_Q-a_{Q+1}|+\cdots+|q|^{n-p+1}|a_{p-1}-ka_p| \\
&+|q|^{n-p+1}|ka_p-a_p|+\cdots+|q||a_{n-1}-a_n|+|a_n|\Big).
\end{aligned}
$$

By our hypotheses,

$$
\begin{aligned}
|H(q)| \;\geq\;& |q|^{n+1}|a_0|-\Big(|q|^n|a_0-a_1|+|q|^{n-1}|a_1-a_2|+\cdots \\
&+|q|^{n-Q}|a_Q|(1-\rho a_Q)+|q|^{n-Q}|a_{Q+1}-\rho a_Q|+\cdots+|q|^{n-p}|a_p|(k-1) \\
&+|q|^{n-p}|ka_p-a_{p+1}|+\cdots+|q||a_{n-1}-a_n|+|a_n|\Big) \\
=\;& |q|^n\Bigg[|q||a_0|-\Bigg(|a_0-a_1|+\frac{|a_1-a_2|}{|q|}+\cdots+\frac{|a_Q|(1-\rho)}{|q|^Q}+\frac{|a_{Q+1}-\rho a_Q|}{|q|^Q} \\
&+\cdots+\frac{|ka_p-a_{p-1}|}{|q|^p}+\frac{|a_p|(k-1)}{|q|^p}+\cdots+\frac{|q||a_{n-1}-a_n|}{|q|^{n-1}}+\frac{|a_n|}{|q|^n}\Bigg)\Bigg].
\end{aligned}
$$

Now for $|q|>1$, so that $\frac{1}{|q|^{n-\ell}}<1$, for $0\leq \ell<n$ we have

$$
\begin{aligned}
|H(q)| \;\geq\;& |q|^n\Big[|q||a_0|-\Big(|a_0-a_1|+|a_1-a_2|+\cdots \\
&+|a_Q|(1-\rho)+|a_{Q+1}-\rho a_Q|+\cdots+|ka_p-a_{p-1}| \\
&+|a_p|(k-1)+\cdots+|a_{n-1}-a_n|+|a_n|\Big)\Big] \\
=\;& |q|^n\Big[|q||a_0|-\Big(\sum_{\ell=1}^{Q}|a_\ell-a_{\ell-1}|+|a_Q|(1-\rho)+|a_{Q+1}-\rho a_Q|
\end{aligned}
$$

$$+\sum_{\ell=Q+2}^{p-1}|a_\ell-a_{\ell-1}|+|ka_p-a_{p-1}|+|a_p|(k-1)+\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$$
$$+|a_n|\Big)\Big].$$

So by Lemma 6.5,

$$\begin{aligned}
|H(q)| &\geq |q|^n\Big[|q||a_0|-\Big(M_q+|a_Q|(1-\rho)+(|a_{Q+1}|-|\rho a_Q|)\cos\alpha \\
&\quad +(|a_{Q+1}|+|\rho a_Q|)\sin\alpha+\sum_{\ell=Q+2}^{p-1}(|a_\ell|-|a_{\ell-1}|)\cos\alpha \\
&\quad +\sum_{\ell=Q+2}^{p-1}(|a_\ell|+|a_{\ell-1}|)\sin\alpha+|a_p|(k-1)+(|ka_p|-|a_{p-1}|)\cos\alpha \\
&\quad +(|ka_p|+|a_{p-1}|)\sin\alpha+M_p+|a_n|\Big)\Big] \\
&= |q|^n\Big[|q||a_0|-\Big(M_q+|a_Q|(1-\rho)+|a_{Q+1}|\cos\alpha-\rho|a_Q|\cos\alpha \\
&\quad +|a_{Q+1}|\sin\alpha+\rho|a_Q|\sin\alpha+\sum_{\ell=Q+2}^{p-1}|a_\ell|\cos\alpha-\sum_{\ell=Q+2}^{p-1}|a_{\ell-1}|\cos\alpha \\
&\quad +\sum_{\ell=Q+2}^{p-1}|a_\ell|\sin\alpha+\sum_{\ell=Q+2}^{p-1}|a_{\ell-1}|\sin\alpha+|a_p|(k-1)+k|a_p|\cos\alpha \\
&\quad -|a_{p-1}|\cos\alpha+k|a_p|\sin\alpha+|a_{p-1}|\sin\alpha+M_p+|a_n|\Big)\Big] \\
&= |q|^n\Big[|q||a_0|-\Big(M_q+|a_Q|(1-\rho)+|a_{Q+1}|\cos\alpha-\rho|a_Q|\cos\alpha \\
&\quad +|a_{Q+1}|\sin\alpha+\rho|a_Q|\sin\alpha+|a_{p-1}|\cos\alpha+\sum_{\ell=Q+2}^{p-2}|a_\ell|\cos\alpha-|a_{Q+1}|\cos\alpha \\
&\quad -\sum_{\ell=Q+2}^{p-2}|a_\ell|\cos\alpha+|a_{p-1}|\sin\alpha+\sum_{\ell=Q+2}^{p-2}|a_\ell|\sin\alpha+|a_{Q+1}|\sin\alpha \\
&\quad +\sum_{\ell=Q+2}^{p-2}|a_\ell|\sin\alpha+|a_p|(k-1)+k|a_p|\cos\alpha-|a_{p-1}|\cos\alpha+k|a_p|\sin\alpha \\
&\quad +|a_{p-1}|\sin\alpha+M_p+|a_n|\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= |q|^n\Bigg[|q||a_0| - \Bigg(M_q + |a_Q| + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2|a_{Q+1}|\sin\alpha \\
&\quad +2|a_{p-1}|\sin\alpha + 2\sum_{\ell=Q+2}^{p-2}|a_\ell|\sin\alpha + |a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p \\
&\quad +|a_n|\Bigg)\Bigg] \\
&= |q|^n\Bigg[|q||a_0| - \Bigg(M_q + |a_Q| + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha \\
&\quad +|a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p + |a_n|\Bigg)\Bigg] \\
&> 0
\end{aligned}
$$

if

$$
\begin{aligned}
|q| &> \frac{1}{|a_0|}\Bigg(M_q + |a_Q| + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha \\
&\quad +|a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p + |a_n|\Bigg).
\end{aligned}
$$

Thus all zeros of $H(q)$ whose modulus is greater than 1 lie in

$$
\begin{aligned}
|q| &\le \frac{1}{|a_0|}\Bigg(M_q + |a_Q| + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha \\
&\quad +|a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p + |a_n|\Bigg).
\end{aligned}
$$

Then all zeros of $H(q)$ and hence of $S(q)$ lie in

$$
\begin{aligned}
|q| &\le \max\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_Q| + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha \\
&\quad +|a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p + |a_n|\Bigg)\Bigg\}.
\end{aligned}
$$

Therefore all the zeros of $P(q)$ lie in

$$
|q| \ge \min\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_Q| + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{p-1}|a_\ell|\sin\alpha
$$

$$+|a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p + |a_n|\Big)\Big\}.$$

Thus the polynomial $P(q)$ does not vanish in

$$|q| \quad < \quad \min\left\{1, |a_0| \Big/ \Big(M_q + |a_Q| + \rho|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{p-1} |a_\ell|\sin\alpha\right.$$
$$\left. +|a_p| + k|a_p|(\sin\alpha + \cos\alpha + 1) + M_p + |a_n|\Big)\right\}.$$

□

Note, when $\gamma_\ell = \delta_\ell = 0$ for all $0 \le \ell \le n$, then Theorem 6.9 reduces to Theorem 2.11. We can also note all theorems discussed in this chapter may be applied to quaternionic lacunary polynomials with quaternionic coefficients and we will have the same bounds except $M_q = \sum_{\ell=m}^{Q} |a_\ell - a_{\ell-1}|$. If we consider a dual gap quaternionic polynomial then we will have the new $M_q = \sum_{\ell=m}^{Q} |a_\ell - a_{\ell-1}|$ as well as $M_p = \sum_{\ell=p}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

6.4 Locations for Quaternionic Polynomials with a Reversal

In Section 6.3, we found bounds for polynomials with a strictly increasing monotonicity on the real parts and the imaginary parts of the coefficients, as well as on the moduli of the coefficients. In this section, we will include a reversal into the monotone behavior of the coefficients. When applied to the real part and imaginary parts of the coefficients we get:

Theorem 6.10. *If* $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ *is a polynomial of degree* n *with quaternionic coefficients, that is* $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$*, where for real* $\rho_{R_1}, \rho_{R_2}, \rho_{I_1}, \rho_{I_2}, \rho_{J_1}, \rho_{J_2}, \rho_{K_1},$

$\rho_{K_2}, k_R, k_I, k_J,$ *and* k_K *where* $0 < \rho_{R_1} \leq 1, 0 < \rho_{R_2} \leq 1, 0 < \rho_{I_1} \leq 1, 0 < \rho_{I_2} \leq 1, 0 < \rho_{J_1} \leq 1, 0 < \rho_{J_2} \leq 1, 0 < \rho_{K_1} \leq 1, 0 < \rho_{K_2} \leq 1, k_R \geq 1, k_I \geq 1, k_J \geq 1,$ $k_K \geq 1,$ *and* $Q \leq \eta \leq p$ *satisfying*

$$\rho_{R_1}\alpha_Q \leq \alpha_{Q+1} \leq \cdots \leq \alpha_{\eta-1} \leq k_R\alpha_\eta \geq \alpha_{\eta+1} \geq \cdots \geq \rho_{R_2}\alpha_p$$

$$\rho_{I_1}\beta_Q \leq \beta_{Q+1} \leq \cdots \leq \beta_{\eta-1} \leq k_I\beta_\eta \geq \beta_{\eta+1} \geq \cdots \geq \rho_{I_2}\beta_p$$

$$\rho_{J_1}\gamma_Q \leq \gamma_{Q+1} \leq \cdots \leq \gamma_{\eta-1} \leq k_J\gamma_\eta \geq \gamma_{\eta+1} \geq \cdots \geq \rho_{J_2}\gamma_p$$

$$\rho_{K_1}\delta_Q \leq \delta_{Q+1} \leq \cdots \leq \delta_{\eta-1} \leq k_K\delta_\eta \geq \delta_{\eta+1} \geq \cdots \geq \rho_{K_2}\delta_p.$$

Then all zeros of $P(q)$ *lie in*

$$\begin{aligned}|q| \quad \leq \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q - \rho_{R_1}\alpha_Q + |\alpha_Q|(1-\rho_{R_1}) + 2|\alpha_\eta|(k_R - 1) + 2k_R\alpha_\eta \\ & +|\alpha_p|(1-\rho_{R_2}) - \rho_{R_2}\alpha_p - \rho_{I_1}\beta_Q + |\beta_Q|(1-\rho_{I_1}) + 2|\beta_\eta|(k_I - 1) \\ & +2k_I\beta_\eta + |\beta_p|(1-\rho_{I_2}) - \rho_{I_2}\beta_p - \rho_{J_1}\gamma_Q + |\gamma_Q|(1-\rho_{J_1}) + 2|\gamma_\eta|(k_J - 1) \\ & +2k_J\gamma_\eta + |\gamma_p|(1-\rho_{J_2}) - \rho_{J_2}\gamma_p - \rho_{K_1}\delta_Q + |\delta_Q|(1-\rho_{K_1}) + 2|\delta_\eta|(k_K - 1) \\ & +2k_K\delta_\eta + |\delta_p|(1-\rho_{K_2}) - \rho_{K_2}\delta_p + M_p\Big).\end{aligned}$$

Proof of Theorem 6.10. Consider the polynomial $P(q) = \sum_{\ell=1}^{n} q^\ell a_\ell$ where $0 < \rho_{R_1} \leq 1, 0 < \rho_{R_2} \leq 1, 0 < \rho_{I_1} \leq 1, 0 < \rho_{I_2} \leq 1, 0 < \rho_{J_1} \leq 1, 0 < \rho_{J_2} \leq 1, 0 < \rho_{K_1} \leq 1, 0 < \rho_{K_2} \leq 1, k_R \geq 1, k_I \geq 1, k_J \geq 1,$ and $k_K \geq 1$ satisfying

$$\rho_{R_1}\alpha_Q \leq \alpha_{Q+1} \leq \cdots \leq \alpha_{\eta-1} \leq k_R\alpha_\eta \geq \alpha_{\eta+1} \geq \cdots \geq \rho_{R_2}\alpha_p$$

$$\rho_{I_1}\beta_Q \leq \beta_{Q+1} \leq \cdots \leq \beta_{\eta-1} \leq k_I\beta_\eta \geq \beta_{\eta+1} \geq \cdots \geq \rho_{I_2}\beta_p$$

$$\rho_{J_1}\gamma_Q \leq \gamma_{Q+1} \leq \cdots \leq \gamma_{\eta-1} \leq k_J\gamma_\eta \geq \gamma_{\eta+1} \geq \cdots \geq \rho_{J_2}\gamma_p$$

$$\rho_{K_1}\delta_Q \le \delta_{Q+1} \le \cdots \le \delta_{\eta-1} \le k_K\delta_\eta \ge \delta_{\eta+1} \ge \cdots \ge \rho_{K_2}\delta_p.$$

Now define $f(q)$ as

$$\begin{aligned} P(q) * (1-q) &= a_0 + q(a_1 - a_0) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\ &= f(q) - q^{n+1}a_n. \end{aligned}$$

By Lemma 6.4, the only zeros of $P(q) * (1-q)$ are $q = 1$ and the zeros of $P(q)$. So, for $|q| = 1$,

$$\begin{aligned} |f(q)| &= |a_0 + q(a_1 - a_0) + \cdots + q^n(a_n - a_{n-1})| \\ &\le |a_0| + |q||a_1 - a_0| + \cdots + |q|^n|a_n - a_{n-1}| \\ &= |a_0| + |a_1 - a_0| + \cdots + |a_{Q+1} - a_Q| + \cdots + |a_\eta - a_{\eta-1}| + |a_{\eta+1} - a_\eta| \\ &\quad + \cdots + |a_p - a_{p-1}| + \cdots + |a_n - a_{n-1}| \\ &= |a_0| + \sum_{\ell=1}^{Q}|a_\ell - a_{\ell-1}| + |a_{Q+1} - a_Q| + \cdots + |a_\eta - a_{\eta-1}| + |a_{\eta+1} - a_\eta| \\ &\quad + \cdots + |a_p - a_{p-1}| + \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}| \\ &= |a_0| + M_q + |a_{Q+1} - a_Q| + \cdots + |a_\eta - a_{\eta-1}| + |a_{\eta+1} - a_\eta| + \cdots \\ &\quad + |a_p - a_{p-1}| + M_p \end{aligned}$$

where $M_q = \sum_{\ell=1}^{Q}|a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}|$. Thus

$$\begin{aligned} |f(q)| \le\ & |a_0| + M_q \\ & + \Big((\alpha_{Q+1} - \alpha_Q)^2 + (\beta_{Q+1} - \beta_Q)^2 + (\gamma_{Q+1} - \gamma_Q)^2 + (\delta_{Q+1} - \delta_Q)^2\Big)^{\frac{1}{2}} + \cdots \\ & + \Big((\alpha_\eta - \alpha_{\eta-1})^2 + (\beta_\eta - \beta_{\eta-1})^2 + (\gamma_\eta - \gamma_{\eta-1})^2 + (\delta_\eta - \delta_{\eta-1})^2\Big)^{\frac{1}{2}} \\ & + \Big((\alpha_{\eta+1} - \alpha_\eta)^2 + (\beta_{\eta+1} - \beta_\eta)^2 + (\gamma_{\eta+1} - \gamma_\eta)^2 + (\delta_{\eta+1} - \delta_\eta)^2\Big)^{\frac{1}{2}} + \cdots \end{aligned}$$

$$
\begin{aligned}
& +\Big((\alpha_p-\alpha_{p-1})^2+(\beta_p-\beta_{p-1})^2+(\gamma_p-\gamma_{p-1})^2+(\delta_p-\delta_{p-1})^2\Big)^{\frac{1}{2}}+M_p \\
\leq\ & |a_0|+M_q+|\alpha_{Q+1}-\alpha_Q|+\cdots+|\alpha_\eta-\alpha_{\eta-1}|+|\alpha_{\eta+1}-\alpha_\eta|+\cdots \\
& +|\alpha_p-\alpha_{p-1}|+|\beta_{Q+1}-\beta_Q|+\cdots+|\beta_\eta-\beta_{\eta-1}|+|\beta_{\eta+1}-\beta_\eta|+\cdots \\
& +|\beta_p-\beta_{p-1}|+|\gamma_{Q+1}-\gamma_Q|+\cdots+|\gamma_\eta-\gamma_{\eta-1}|+|\gamma_{\eta+1}-\gamma_\eta|+\cdots \\
& +|\gamma_p-\gamma_{p-1}|+|\delta_{Q+1}-\delta_Q|+\cdots+|\delta_\eta-\delta_{\eta-1}|+|\delta_{\eta+1}-\delta_\eta|+\cdots \\
& +|\delta_p-\delta_{p-1}|+M_p \\
=\ & |a_0|+M_q+|\alpha_{Q+1}-\rho_{R_1}\alpha_Q+\rho_{R_1}\alpha_Q-\alpha_Q|+\cdots \\
& +|\alpha_\eta-k_R\alpha_\eta+k_R\alpha_\eta-\alpha_{\eta-1}|+|\alpha_{\eta+1}-k_R\alpha_\eta+k_R\alpha_\eta-\alpha_\eta|+\cdots \\
& +|\alpha_p-\rho_{R_2}\alpha_p+\rho_{R_2}\alpha_p-\alpha_{p-1}|+|\beta_{Q+1}-\rho_{I_1}\beta_Q+\rho_{I_1}\beta_Q-\beta_Q|+\cdots \\
& +|\beta_\eta-k_I\beta_\eta+k_I\beta_\eta-\beta_{\eta-1}|+|\beta_{\eta+1}-k_I\beta_\eta+k_I\beta_\eta-\beta_\eta|+\cdots \\
& +|\beta_p-\rho_{I_2}\beta_p+\rho_{I_2}\beta_p-\beta_{p-1}|+|\gamma_{Q+1}-\rho_{J_1}\gamma_Q+\rho_{J_1}\gamma_Q-\gamma_Q|+\cdots \\
& +|\gamma_\eta-k_J\gamma_\eta+k_J\gamma_\eta-\gamma_{\eta-1}|+|\gamma_{\eta+1}-k_J\gamma_\eta+k_J\gamma_\eta-\gamma_\eta|+\cdots \\
& +|\gamma_p-\rho_{J_2}\gamma_p+\rho_{J_2}\gamma_p-\gamma_{p-1}|+|\delta_{Q+1}-\rho_{K_1}\delta_Q+\rho_{K_1}\delta_Q-\delta_Q|+\cdots \\
& +|\delta_\eta-k_K\delta_\eta+k_K\delta_\eta-\delta_{\eta-1}|+|\delta_{\eta+1}-k_K\delta_\eta+k_K\delta_\eta-\delta_\eta|+\cdots \\
& +|\delta_p-\rho_{K_2}\delta_p+\rho_{K_2}\delta_p-\delta_{p-1}|+M_p \\
\leq\ & |a_0|+M_q+|\alpha_{Q+1}-\rho_{R_1}\alpha_Q|+|\rho_{R_1}\alpha_Q-\alpha_Q|+\cdots+|\alpha_\eta-k_R\alpha_\eta| \\
& +|k_R\alpha_\eta-\alpha_{\eta-1}|+|\alpha_{\eta+1}-k_R\alpha_\eta|+|k_R\alpha_\eta-\alpha_\eta|+\cdots+|\alpha_p-\rho_{R_2}\alpha_p| \\
& +|\rho_{R_2}\alpha_p-\alpha_{p-1}|+|\beta_{Q+1}-\rho_{I_1}\beta_Q|+|\rho_{I_1}\beta_Q-\beta_Q|+\cdots+|\beta_\eta-k_I\beta_\eta| \\
& +|k_I\beta_\eta-\beta_{\eta-1}|+|\beta_{\eta+1}-k_I\beta_\eta|+|k_I\beta_\eta-\beta_\eta|+\cdots+|\beta_p-\rho_{I_2}\beta_p| \\
& +|\rho_{I_2}\beta_p-\beta_{p-1}|+|\gamma_{Q+1}-\rho_{J_1}\gamma_Q|+|\rho_{J_1}\gamma_Q-\gamma_Q|+\cdots+|\gamma_\eta-k_J\gamma_\eta|
\end{aligned}
$$

$$
\begin{aligned}
& +|k_J\gamma_\eta - \gamma_{\eta-1}| + |\gamma_{\eta+1} - k_J\gamma_\eta| + |k_J\gamma_\eta - \gamma_\eta| + \cdots + |\gamma_p - \rho_{J_2}\gamma_p| \\
& +|\rho_{J_2}\gamma_p - \gamma_{p-1}| + |\delta_{Q+1} - \rho_{K_1}\delta_Q| + |\rho_{K_1}\delta_Q - \delta_Q| + \cdots + |\delta_\eta - k_K\delta_\eta| \\
& +|k_K\delta_\eta - \delta_{\eta-1}| + |\delta_{\eta+1} - k_K\delta_\eta| + |k_K\delta_\eta - \delta_\eta| + \cdots + |\delta_p - \rho_{K_2}\delta_p| \\
& +|\rho_{K_2}\delta_p - \delta_{p-1}| + M_p \\
= \; & |a_0| + M_q + (\alpha_{Q+1} - \rho_{R_1}\alpha_Q) + |\alpha_Q|(1-\rho_{R_1}) + \cdots + |\alpha_\eta|(k_R - 1) \\
& +(k_R\alpha_\eta - \alpha_{\eta-1}) + (k_R\alpha_\eta - \alpha_{\eta+1}) + |\alpha_\eta|(k_R-1) + \cdots + |\alpha_p|(1-\rho_{R_2}) \\
& +(\alpha_{p-1} - \rho_{R_2}\alpha_p) + (\beta_{Q+1} - \rho_{I_1}\beta_Q) + |\beta_Q|(1-\rho_{I_1}) + \cdots + |\beta_\eta|(k_I - 1) \\
& +(k_I\beta_\eta - \beta_{\eta-1}) + (\beta_{\eta+1} - k_I\beta_\eta) + |\beta_\eta|(k_I-1) + \cdots + |\beta_p|(1-\rho_{I_2}) \\
& +(\beta_{p-1} - \rho_{I_2}\beta_p) + (\gamma_{Q+1} - \rho_{J_1}\gamma_Q) + |\gamma_Q|(1-\rho_{J_1}) + \cdots + |\gamma_\eta|(k_J - 1) \\
& +(k_J\gamma_\eta - \gamma_{\eta-1}) + |\gamma_{\eta+1}|(k_J-1) + |\gamma_\eta|(k_J-1) + \cdots + |\gamma_p|(1-\rho_{J_2}) \\
& +(\gamma_{p-1} - \rho_{J_2}\gamma_p) + (\delta_{Q+1} - \rho_{K_1}\delta_Q) + |\delta_Q|(1-\rho_{K_1}) + \cdots + |\delta_\eta|(k_K - 1) \\
& +(k_K\delta_\eta - \delta_{\eta-1}) + (k_K\delta_\eta - \delta_{\eta+1}) + |\delta_\eta|(k_K-1) + \cdots + |\delta_p|(1-\rho_{K_2}) \\
& +(\delta_{p-1} - \rho_{K_2}\delta_p) + M_p
\end{aligned}
$$

by our hypotheses. Thus we have

$$
\begin{aligned}
|f(q)| \;\leq\; & |a_0| + M_q - \rho_{R_1}\alpha_Q + |\alpha_Q|(1-\rho_{R_1}) + 2|\alpha_\eta|(k_R-1) + 2k_R\alpha_\eta \\
& +|\alpha_p|(1-\rho_{R_2}) - \rho_{R_2}\alpha_p - \rho_{I_1}\beta_Q + |\beta_Q|(1-\rho_{I_1}) + 2|\beta_\eta|(k_I-1) \\
& +2k_I\beta_\eta + |\beta_p|(1-\rho_{I_2}) - \rho_{I_2}\beta_p - \rho_{J_1}\gamma_Q + |\gamma_Q|(1-\rho_{J_1}) + 2|\gamma_\eta|(k_J-1) \\
& +2k_J\gamma_\eta + |\gamma_p|(1-\rho_{J_2}) - \rho_{J_2}\gamma_p - \rho_{K_1}\delta_Q + |\delta_Q|(1-\rho_{K_1}) + 2|\delta_\eta|(k_K-1) \\
& +2k_K\delta_\eta + |\delta_p|(1-\rho_{K_2}) - \rho_{K_2}\delta_p + M_p.
\end{aligned}
$$

We can notice $q^n f\left(\frac{1}{q}\right) = \sum_{\ell=0}^{n} q^{n-\ell}(a_\ell - a_{\ell-1})$ where $a_{-1} = 0$ has the same bound on

$|q| = 1$ as $f(q)$. Namely

$$\begin{aligned}\left|q^n f\left(\frac{1}{q}\right)\right| \leq\ & |a_0| + M_q - \rho_{R_1}\alpha_Q + |\alpha_Q|(1-\rho_{R_1}) + 2|\alpha_\eta|(k_R - 1) + 2k_R\alpha_\eta \\ & +|\alpha_p|(1-\rho_{R_2}) - \rho_{R_2}\alpha_p - \rho_{I_1}\beta_Q + |\beta_Q|(1-\rho_{I_1}) + 2|\beta_\eta|(k_I - 1) \\ & +2k_I\beta_\eta + |\beta_p|(1-\rho_{I_2}) - \rho_{I_2}\beta_p - \rho_{J_1}\gamma_Q + |\gamma_Q|(1-\rho_{J_1}) \\ & +2|\gamma_\eta|(k_J - 1) + 2k_J\gamma_\eta + |\gamma_p|(1-\rho_{J_2}) - \rho_{J_2}\gamma_p - \rho_{K_1}\delta_Q \\ & +|\delta_Q|(1-\rho_{K_1}) + 2|\delta_\eta|(k_K - 1) + 2k_K\delta_\eta + |\delta_p|(1-\rho_{K_2}) - \rho_{K_2}\delta_p \\ & +M_p\end{aligned}$$

is analytic in $|q| \leq 1$ where we consider this function to have the vaule $a_n - a_{n-1}$ at $q = 0$ we have

$$\begin{aligned}\left|q^n f\left(\frac{1}{q}\right)\right| \leq\ & |a_0| + M_q - \rho_{R_1}\alpha_Q + |\alpha_Q|(1-\rho_{R_1}) + 2|\alpha_\eta|(k_R - 1) + 2k_R\alpha_\eta \\ & +|\alpha_p|(1-\rho_{R_2}) - \rho_{R_2}\alpha_p - \rho_{I_1}\beta_Q + |\beta_Q|(1-\rho_{I_1}) + 2|\beta_\eta|(k_I - 1) \\ & +2k_I\beta_\eta + |\beta_p|(1-\rho_{I_2}) - \rho_{I_2}\beta_p - \rho_{J_1}\gamma_Q + |\gamma_Q|(1-\rho_{J_1}) \\ & +2|\gamma_\eta|(k_J - 1) + 2k_J\gamma_\eta + |\gamma_p|(1-\rho_{J_2}) - \rho_{J_2}\gamma_p - \rho_{K_1}\delta_Q \\ & +|\delta_Q|(1-\rho_{K_1}) + 2|\delta_\eta|(k_K - 1) + 2k_K\delta_\eta + |\delta_p|(1-\rho_{K_2}) - \rho_{K_2}\delta_p \\ & +M_p\end{aligned}$$

for $|q| \leq 1$ by Theorem 6.3. Thus

$$\begin{aligned}\left|f\left(\frac{1}{q}\right)\right| \leq\ & \frac{1}{|q|^n}\Big(|a_0| + M_q - \rho_{R_1}\alpha_Q + |\alpha_Q|(1-\rho_{R_1}) + 2|\alpha_\eta|(k_R - 1) + 2k_R\alpha_\eta \\ & +|\alpha_p|(1-\rho_{R_2}) - \rho_{R_2}\alpha_p - \rho_{I_1}\beta_Q + |\beta_Q|(1-\rho_{I_1}) + 2|\beta_\eta|(k_I - 1) \\ & +2k_I\beta_\eta + |\beta_p|(1-\rho_{I_2}) - \rho_{I_2}\beta_p - \rho_{J_1}\gamma_Q + |\gamma_Q|(1-\rho_{J_1}) \\ & +2|\gamma_\eta|(k_J - 1) + 2k_J\gamma_\eta + |\gamma_p|(1-\rho_{J_2}) - \rho_{J_2}\gamma_p - \rho_{K_1}\delta_Q\end{aligned}$$

$$+|\delta_Q|(1-\rho_{K_1})+2|\delta_\eta|(k_K-1)+2k_K\delta_\eta+|\delta_p|(1-\rho_{K_2})-\rho_{K_2}\delta_p$$
$$+M_p\Big)$$

for $|q|\leq 1$. Replacing q with $\dfrac{1}{q}$ we have

$$\begin{aligned}
|f(q)| \leq\ & |q|^n\Big(|a_0|+M_q-\rho_{R_1}\alpha_Q+|\alpha_Q|(1-\rho_{R_1})+2|\alpha_\eta|(k_R-1)\\
&+2k_R\alpha_\eta+|\alpha_p|(1-\rho_{R_2})-\rho_{R_2}\alpha_p-\rho_{I_1}\beta_Q\\
&+|\beta_Q|(1-\rho_{I_1})+2|\beta_\eta|(k_I-1)+2k_I\beta_\eta+|\beta_p|(1-\rho_{I_2})\\
&-\rho_{I_2}\beta_p-\rho_{J_1}\gamma_Q+|\gamma_Q|(1-\rho_{J_1})+2|\gamma_\eta|(k_J-1)+2k_J\gamma_\eta\\
&+|\gamma_p|(1-\rho_{J_2})-\rho_{J_2}\gamma_p-\rho_{K_1}\delta_Q+|\delta_Q|(1-\rho_{K_1})+2|\delta_\eta|(k_K-1)\\
&+2k_K\delta_\eta+|\delta_p|(1-\rho_{K_2})-\rho_{K_2}\delta_p+M_p\Big)
\end{aligned}$$

for $|q|\geq 1$. We have

$$\begin{aligned}
|P(q)*(1-q)| =\ & |f(q)-q^{n+1}a_n|\\
\geq\ & |q^{n+1}||a_n|-|f(q)|\\
\geq\ & |q^{n+1}||a_n|-|q|^n\Big(|a_0|+M_q-\rho_{R_1}\alpha_Q+|\alpha_Q|(1-\rho_{R_1})\\
&+2|\alpha_\eta|(k_R-1)+2k_R\alpha_\eta+|\alpha_p|(1-\rho_{R_2})-\rho_{R_2}\alpha_p-\rho_{I_1}\beta_Q\\
&+|\beta_Q|(1-\rho_{I_1})+2|\beta_\eta|(k_I-1)+2k_I\beta_\eta+|\beta_p|(1-\rho_{I_2})-\rho_{I_2}\beta_p\\
&-\rho_{J_1}\gamma_Q+|\gamma_Q|(1-\rho_{J_1})+2|\gamma_\eta|(k_J-1)+2k_J\gamma_\eta+|\gamma_p|(1-\rho_{J_2})\\
&-\rho_{J_2}\gamma_p-\rho_{K_1}\delta_Q+|\delta_Q|(1-\rho_{K_1})+2|\delta_\eta|(k_K-1)+2k_K\delta_\eta\\
&+|\delta_p|(1-\rho_{K_2})-\rho_{K_2}\delta_p+M_p\Big)\\
=\ & |q^n|\Big[|q||a_n|-\Big(|a_0|+M_q-\rho_{R_1}\alpha_Q+|\alpha_Q|(1-\rho_{R_1})\\
&+2|\alpha_\eta|(k_R-1)+2k_R\alpha_\eta+|\alpha_p|(1-\rho_{R_2})-\rho_{R_2}\alpha_p-\rho_{I_1}\beta_Q
\end{aligned}$$

$$+|\beta_Q|(1-\rho_{I_1})+2|\beta_\eta|(k_I-1)+2k_I\beta_\eta+|\beta_p|(1-\rho_{I_2})-\rho_{I_2}\beta_p$$
$$-\rho_{J_1}\gamma_Q+|\gamma_Q|(1-\rho_{J_1})+2|\gamma_\eta|(k_J-1)$$
$$+2k_J\gamma_\eta+|\gamma_p|(1-\rho_{J_2})-\rho_{J_2}\gamma_p-\rho_{K_1}\delta_Q+|\delta_Q|(1-\rho_{K_1})$$
$$+2|\delta_\eta|(k_K-1)+2k_K\delta_\eta+|\delta_p|(1-\rho_{K_2})-\rho_{K_2}\delta_p+M_p\Big)\Big].$$

So if

$$\begin{aligned}|q| &> \frac{1}{|a_n|}\Big(|a_0|+M_q-\rho_{R_1}\alpha_Q+|\alpha_Q|(1-\rho_{R_1})+2|\alpha_\eta|(k_R-1)+2k_R\alpha_\eta \\ &+|\alpha_p|(1-\rho_{R_2})-\rho_{R_2}\alpha_p-\rho_{I_1}\beta_Q+|\beta_Q|(1-\rho_{I_1})+2|\beta_\eta|(k_I-1) \\ &+2k_I\beta_\eta+|\beta_p|(1-\rho_{I_2})-\rho_{I_2}\beta_p-\rho_{J_1}\gamma_Q+|\gamma_Q|(1-\rho_{J_1})+2|\gamma_\eta|(k_J-1) \\ &+2k_J\gamma_\eta+|\gamma_p|(1-\rho_{J_2})-\rho_{J_2}\gamma_p-\rho_{K_1}\delta_Q+|\delta_Q|(1-\rho_{K_1})+2|\delta_\eta|(k_K-1) \\ &+2k_K\delta_\eta+|\delta_p|(1-\rho_{K_2})-\rho_{K_2}\delta_p+M_p\Big)\end{aligned}$$

then

$$\begin{aligned}0 &\neq |q^n|\Big[|q||a_n|-\Big(|a_0|+M_q-\rho_{R_1}\alpha_Q+|\alpha_Q|(1-\rho_{R_1})+2|\alpha_\eta|(k_R-1)+2k_R\alpha_\eta \\ &+|\alpha_p|(1-\rho_{R_2})-\rho_{R_2}\alpha_p-\rho_{I_1}\beta_Q+|\beta_Q|(1-\rho_{I_1})+2|\beta_\eta|(k_I-1) \\ &+2k_I\beta_\eta+|\beta_p|(1-\rho_{I_2})-\rho_{I_2}\beta_p-\rho_{J_1}\gamma_Q+|\gamma_Q|(1-\rho_{J_1})+2|\gamma_\eta|(k_J-1) \\ &+2k_J\gamma_\eta+|\gamma_p|(1-\rho_{J_2})-\rho_{J_2}\gamma_p-\rho_{K_1}\delta_Q+|\delta_Q|(1-\rho_{K_1})+2|\delta_\eta|(k_K-1) \\ &+2k_K\delta_\eta+|\delta_p|(1-\rho_{K_2})-\rho_{K_2}\delta_p+M_p\Big)\Big].\end{aligned}$$

Therefore all zeros of $P(q)$ lie in

$$\begin{aligned}|q| &\leq \frac{1}{|a_n|}\Big(|a_0|+M_q-\rho_{R_1}\alpha_Q+|\alpha_Q|(1-\rho_{R_1})+2|\alpha_\eta|(k_R-1)+2k_R\alpha_\eta \\ &+|\alpha_p|(1-\rho_{R_2})-\rho_{R_2}\alpha_p-\rho_{I_1}\beta_Q+|\beta_Q|(1-\rho_{I_1})+2|\beta_\eta|(k_I-1) \\ &+2k_I\beta_\eta+|\beta_p|(1-\rho_{I_2})-\rho_{I_2}\beta_p-\rho_{J_1}\gamma_Q+|\gamma_Q|(1-\rho_{J_1})+2|\gamma_\eta|(k_J-1)\end{aligned}$$

$$+2k_J\gamma_\eta + |\gamma_p|(1-\rho_{J_2}) - \rho_{J_2}\gamma_p - \rho_{K_1}\delta_Q + |\delta_Q|(1-\rho_{K_1}) + 2|\delta_\eta|(k_K - 1)$$
$$+2k_K\delta_\eta + |\delta_p|(1-\rho_{K_2}) - \rho_{K_2}\delta_p + M_p\Big).$$

□

When $\gamma_\ell = \delta_\ell = 0$, for all $0 \le \ell \le n$ then Theorem 6.10 reduces to Theorem 2.2. Notice when $\rho_{R_1} = \rho_{R_2} = \rho_{I_1} = \rho_{I_2} = \rho_{J_1} = \rho_{J_2} = \rho_{K_1} = \rho_{K_2} = k_R = k_I = k_J = k_K = 1$ then we get the following corollary:

Corollary 6.11. *If* $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ *is a polynomial of degree* n *with quaternionic coefficients, that is* $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$*, where* $Q \le \eta \le p$ *satisfying*

$$\alpha_Q \le \alpha_{Q+1} \le \cdots \le \alpha_{\eta-1} \le \alpha_\eta \ge \alpha_{\eta+1} \ge \cdots \ge \alpha_p$$

$$\beta_Q \le \beta_{Q+1} \le \cdots \le \beta_{\eta-1} \le \beta_\eta \ge \beta_{\eta+1} \ge \cdots \ge \beta_p$$

$$\gamma_Q \le \gamma_{Q+1} \le \cdots \le \gamma_{\eta-1} \le \gamma_\eta \ge \gamma_{\eta+1} \ge \cdots \ge \gamma_p$$

$$\delta_Q \le \delta_{Q+1} \le \cdots \le \delta_{\eta-1} \le \delta_\eta \ge \delta_{\eta+1} \ge \cdots \ge \delta_p.$$

Then all zeros of $P(q)$ *lie in*

$$|q| \le \frac{1}{|a_n|}\Big(|a_0| + M_q - (\alpha_Q + \beta_Q + \gamma_Q + \delta_Q) + 2(\alpha_\eta + \beta_\eta + \gamma_e ta + \delta_\eta)$$
$$-(\alpha_p + \beta_p + \gamma_p + \delta_p) + M_p\Big).$$

By applying the same parameters to a quaternionic polynomial with quaternionic coefficient we get the inner bound:

Theorem 6.12. *If* $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ *is a polynomial of degree* n *with quaternionic coefficients, that is* $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$*, where for real* $\rho_{R_1}, \rho_{R_2}, \rho_{I_1}, \rho_{I_2}, \rho_{J_1}, \rho_{J_2}, \rho_{K_1}$*,*

$\rho_{K_2}, k_R, k_I, k_J,$ *and* k_K *where* $0 < \rho_{R_1} \leq 1, 0 < \rho_{R_2} \leq 1, 0 < \rho_{I_1} \leq 1, 0 < \rho_{I_2} \leq 1,$ $0 < \rho_{J_1} \leq 1, 0 < \rho_{J_2} \leq 1, 0 < \rho_{K_1} \leq 1, 0 < \rho_{K_2} \leq 1, k_R \geq 1, k_I \geq 1, k_J \geq 1, k_K \geq 1$ *and* $Q \leq \eta \leq p$ *satisfying*

$$\rho_{R_1}\alpha_Q \leq \alpha_{Q+1} \leq \cdots \leq \alpha_{\eta-1} \leq k_R\alpha_\eta \geq \alpha_{\eta+1} \geq \cdots \geq \rho_{R_2}\alpha_p$$

$$\rho_{I_1}\beta_Q \leq \beta_{Q+1} \leq \cdots \leq \beta_{\eta-1} \leq k_I\beta_\eta \geq \beta_{\eta+1} \geq \cdots \geq \rho_{I_2}\beta_p$$

$$\rho_{J_1}\gamma_Q \leq \gamma_{Q+1} \leq \cdots \leq \gamma_{\eta-1} \leq k_J\gamma_\eta \geq \gamma_{\eta+1} \geq \cdots \geq \rho_{J_2}\gamma_p$$

$$\rho_{K_1}\delta_Q \leq \delta_{Q+1} \leq \cdots \leq \delta_{\eta-1} \leq k_K\delta_\eta \geq \delta_{\eta+1} \geq \cdots \geq \rho_{K_2}\delta_p.$$

Then $P(q)$ *does not vanish in*

$$\begin{aligned} |q| \quad < \quad & \min\left\{1, |a_0| \Big/ \Big(M_q + |\alpha_Q|(1-\rho_{R_1}) - \rho_{R_1}\alpha_Q + |\beta_Q|(1-\rho_{I_1}) - \rho_{I_1}\beta_Q \right. \\ & + |\gamma_Q|(1-\rho_{J_1}) - \rho_{J_1}\gamma_Q + |\delta_Q|(1-\rho_{K_1}) - \rho_{K_1}\delta_Q + 2k_R\alpha_\eta + 2|\alpha_\eta|(k_R-1) \\ & + 2k_I\beta_\eta + 2|\beta_\eta|(k_I-1) + 2k_J\gamma_\eta + 2|\gamma_\eta|(k_J-1) + 2k_K\delta_\eta + 2|\delta_\eta|(k_K-1) \\ & - \rho_{R_2}\alpha_p + |\alpha_p|(1-\rho_{R_2}) - \rho_{I_2}\beta_p + |\beta_p|(1-\rho_{R_2}) - \rho_{J_2}\gamma_p + |\gamma_p|(1-\rho_{J_2}) \\ & \left. - \rho_{K_2}\delta_p + |\delta_p|(1-\rho_{K_2}) + M_p + |a_n| \Big) \right. . \end{aligned}$$

Proof of Theorem 6.12. Consider the reciprocal polynomial

$$\begin{aligned} S(q) &= q^n * P\left(\frac{1}{q}\right) \\ &= q^n a_0 + q^{n-1}a_1 + \cdots + q^{n-Q}a_Q + \cdots + q^{n-\eta}a_\eta + \cdots + q^{n-p}a_p + \cdots + qa_{n-1} \\ &\quad + a_n. \end{aligned}$$

Let

$$H(q) = S(q) * (1-q)$$

$$
\begin{aligned}
&= -a_0q^{n+1} + q^n(a_0 - a_1) + a^{n-1}(a_1 - a_2) + \cdots + q^{n-Q}(a_Q - a_{Q+1}) + \cdots \\
&\quad + q^{n-\eta}a_\eta + \cdots + q^{n-p}(a_p - a_{p+1}) + \cdots + q^2(a_{n-2} - a_{n-1}) + q(a_{n-1} - a_n) \\
&\quad + a_n.
\end{aligned}
$$

By Lemma 6.4, the only zeros of $P(q) * (1-q)$ are $q = 1$ and the zeros of $P(q)$. Thus we have

$$
\begin{aligned}
|H(q)| &\geq |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}|a_Q - a_{Q+1}| \\
&\quad + \cdots + |q|^{n-p+1}|a_{p-1} - a_p| + \cdots + |q|^2|a_{n-2} - a_{n-1}| + |q||a_{n-1} - a_n| \\
&\quad + |a_n| \Big) \\
&= |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}\Big((\alpha_Q - \alpha_{Q+1})^2 \\
&\quad + (\beta_Q + \beta_{Q+1})^2 + (\gamma_Q - \gamma_{Q+1})^2 + (\delta_Q - \delta_{Q+1})^2 \Big)^{\frac{1}{2}} + \cdots \\
&\quad + |q|^{n-\eta+1}\Big((\alpha_{\eta-1} - \alpha_\eta)^2 + (\beta_{\eta-1} - \beta_\eta)^2 + (\gamma_{\eta-1} - \gamma_\eta)^2 + (\delta_{\eta-1} - \delta_\eta)^2 \Big)^{\frac{1}{2}} \\
&\quad + |q|^{n-\eta}\Big((\alpha_\eta - \alpha_{\eta+1})^2 + (\beta_\eta - \beta_{\eta+1})^2 + (\gamma_\eta - \gamma_{\eta+1})^2 + (\delta_\eta - \delta_{\eta+1})^2 \Big)^{\frac{1}{2}} \\
&\quad + \cdots + |q|^{n-p+1}\Big((\alpha_{p-1} - \alpha_p)^2 + (\beta_{p-1} - \beta_p)^2 + (\gamma_{p-1} - \gamma_p)^2 \\
&\quad + (\delta_{p-1} - \delta_p)^2 \Big)^{\frac{1}{2}} + \cdots + |q|^2|a_{n-2} - a_{n-1}| + |q||a_{n-1} - a_n| + |a_n| \Big) \\
&= |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}|\alpha_Q - \alpha_{Q+1}| \\
&\quad + |q|^{n-Q}|\beta_Q + \beta_{Q+1}| + |q|^{n-Q}|\gamma_Q - \gamma_{Q+1}| + |q|^{n-Q}|\delta_Q - \delta_{Q+1}| + \cdots \\
&\quad + |q|^{n-\eta+1}|\alpha_{\eta-1} - \alpha_\eta| + |q|^{n-\eta+1}|\beta_{\eta-1} - \beta_\eta| + |q|^{n-\eta+1}|\gamma_{\eta-1} - \gamma_\eta| \\
&\quad + |q|^{n-\eta+1}|\delta_{\eta-1} - \delta_\eta| + |q|^{n-\eta}|\alpha_\eta - \alpha_{\eta+1}| + |q|^{n-\eta}|\beta_\eta - \beta_{\eta+1}| \\
&\quad + |q|^{n-\eta}|\gamma_\eta - \gamma_{\eta+1}| + |q|^{n-\eta}|\delta_\eta - \delta_{\eta+1}| + \cdots + |q|^{n-p+1}|\alpha_{p-1} - \alpha_p|
\end{aligned}
$$

$$
\begin{aligned}
& +|q|^{n-p+1}|\beta_{p-1}-\beta_p| + |q|^{n-p+1}|\gamma_{p-1}-\gamma_p| + |q|^{n-p+1}|\delta_{p-1}-\delta_p| + \cdots \\
& +|q|^2|a_{n-2}-a_{n-1}| + |q||a_{n-1}-a_n| + |a_n|\Big) \\
= \; & |q|^{n+1}|a_0| - \Big(|q|^n|a_0-a_1| + |q|^{n-1}|a_1-a_2| + \cdots \\
& +|q|^{n-Q}|\alpha_Q - \rho_{R_1}\alpha_Q + \rho_{R_1}\alpha_Q - \alpha_{Q+1}| \\
& +|q|^{n-Q}|\beta_Q - \rho_{I_1}\beta_Q + \rho_{I_1}\beta_Q + \beta_{Q+1}| \\
& +|q|^{n-Q}|\gamma_Q - \rho_{J_1}\gamma_Q + \rho_{J_1}\gamma_Q - \gamma_{Q+1}| \\
& +|q|^{n-Q}|\delta_Q - \rho_{K_1}\delta_Q + \rho_{K_1}\delta_Q - \delta_{Q+1}| + \cdots \\
& +|q|^{n-\eta+1}|\alpha_{\eta-1} - k_R\alpha_\eta + k_R\alpha_\eta - \alpha_\eta| \\
& +|q|^{n-\eta+1}|\beta_{\eta-1} - k_I\beta_\eta + k_I\beta_\eta - \beta_\eta| \\
& +|q|^{n-\eta+1}|\gamma_{\eta-1} - k_J\gamma_\eta + k_J\gamma_\eta - \gamma_\eta| \\
& +|q|^{n-\eta+1}|\delta_{\eta-1} - k_K\delta_\eta + k_K\delta_\eta - \delta_\eta| \\
& +|q|^{n-\eta}|\alpha_\eta - k_R\alpha_\eta + k_R\alpha_\eta - \alpha_{\eta+1}| \\
& +|q|^{n-\eta}|\beta_\eta - k_I\beta_\eta + k_I\beta_\eta - \beta_{\eta+1}| \\
& +|q|^{n-\eta}|\gamma_\eta - k_J\gamma_\eta + k_J\gamma_\eta - \gamma_{\eta+1}| \\
& +|q|^{n-\eta}|\delta_\eta - k_K\delta_\eta + k_K\delta_\eta - \delta_{\eta+1}| + \cdots \\
& +|q|^{n-p+1}|\alpha_{p-1} - \rho_{R_2}\alpha_p + \rho_{R_2}\alpha_p - \alpha_p| \\
& +|q|^{n-p+1}|\beta_{p-1} - \rho_{I_2}\beta_p + \rho_{R_2}\beta_p - \beta_p| \\
& +|q|^{n-p+1}|\gamma_{p-1} - \rho_{J_2}\gamma_p + \rho_{J_2}\gamma_p - \gamma_p| \\
& +|q|^{n-p+1}|\delta_{p-1} - \rho_{K_2}\delta_p + \rho_{K_2}\delta_p - \delta_p| + \cdots + |q|^2|a_{n-2}-a_{n-1}| \\
& +|q||a_{n-1}-a_n| + |a_n|\Big)
\end{aligned}
$$

$$
\begin{aligned}
\geq\ & |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots \\
& + |q|^{n-Q}|\alpha_Q - \rho_{R_1}\alpha_Q| + |q|^{n-Q}|\rho_{R_1}\alpha_Q - \alpha_{Q+1}| + |q|^{n-Q}|\beta_Q - \rho_{I_1}\beta_Q| \\
& + |q|^{n-Q}|\rho_{I_1}\beta_Q - \beta_{Q+1}| + |q|^{n-Q}|\gamma_Q - \rho_{J_1}\gamma_Q| + |q|^{n-Q}|\rho_{J_1}\gamma_Q - \gamma_{Q+1}| \\
& + |q|^{n-Q}|\delta_Q - \rho_{K_1}\delta_Q| + |q|^{n-Q}|\rho_{K_1}\delta_Q - \delta_{Q+1}| + \cdots \\
& + |q|^{n-\eta+1}|\alpha_{\eta-1} - k_R\alpha_\eta| + |q|^{n-\eta+1}|k_R\alpha_\eta - \alpha_\eta| + |q|^{n-\eta+1}|\beta_{\eta-1} - k_I\beta_\eta| \\
& + |q|^{n-\eta+1}|k_I\beta_\eta - \beta_\eta| + |q|^{n-\eta+1}|\gamma_{\eta-1} - k_J\gamma_\eta| + |q|^{n-\eta+1}|k_J\gamma_\eta - \gamma_\eta| \\
& + |q|^{n-\eta+1}|\delta_{\eta-1} - k_K\delta_\eta| + |q|^{n-\eta+1}|k_K\delta_\eta - \delta_\eta| + |q|^{n-\eta}|\alpha_\eta - k_R\alpha_\eta| \\
& + |q|^{n-\eta}|k_R\alpha_\eta - \alpha_{\eta+1}| + |q|^{n-\eta}|\beta_\eta - k_I\beta_\eta| + |q|^{n-\eta}|k_I\beta_\eta - \beta_{\eta+1}| \\
& + |q|^{n-\eta}|\gamma_\eta - k_J\gamma_\eta| + |q|^{n-\eta}|k_J\gamma_\eta - \gamma_{\eta+1}| + |q|^{n-\eta}|\delta_\eta - k_K\delta_\eta| \\
& + |q|^{n-\eta}|k_K\delta_\eta - \delta_{\eta+1}| + \cdots + |q|^{n-p+1}|\alpha_{p-1} - \rho_{R_2}\alpha_p| \\
& + |q|^{n-p+1}|\rho_{R_2}\alpha_p - \alpha_p| + |q|^{n-p+1}|\beta_{p-1} - \rho_{I_2}\beta_p| + |q|^{n-p+1}|\rho_{R_2}\beta_p - \beta_p| \\
& + |q|^{n-p+1}|\gamma_{p-1} - \rho_{J_2}\gamma_p| + |q|^{n-p+1}|\rho_{J_2}\gamma_p - \gamma_p| + |q|^{n-p+1}|\delta_{p-1} - \rho_{K_2}\delta_p| \\
& + |q|^{n-p+1}|\rho_{K_2}\delta_p - \delta_p| + \cdots + |q|^2|a_{n-2} - a_{n-1}| + |q||a_{n-1} - a_n| + |a_n| \Big) \\
=\ & |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots \\
& + |q|^{n-Q}|\alpha_Q|(1 - \rho_{R_1}) + |q|^{n-Q}(\alpha_{Q+1} - \rho_{R_1}\alpha_Q) + |q|^{n-Q}|\beta_Q|(1 - \rho_{I_1}) \\
& + |q|^{n-Q}(\beta_{Q+1} - \rho_{I_1}\beta_Q) + |q|^{n-Q}|\gamma_Q|(1 - \rho_{J_1}) + |q|^{n-Q}(\gamma_{Q+1} - \rho_{J_1}\gamma_Q) \\
& + |q|^{n-Q}|\delta_Q|(1 - \rho_{K_1}) + |q|^{n-Q}(\delta_{Q+1} - \rho_{K_1}\delta_Q) + \cdots \\
& + |q|^{n-\eta+1}(k_R\alpha_\eta - \alpha_{\eta-1}) + |q|^{n-\eta+1}|\alpha_\eta|(k_R - 1) + |q|^{n-\eta+1}(k_I\beta_\eta - \beta_{\eta-1}) \\
& + |q|^{n-\eta+1}|\beta_\eta|(k_I - 1) + |q|^{n-\eta+1}(k_J\gamma_\eta - \gamma_{\eta-1}) + |q|^{n-\eta+1}|\gamma_\eta|(k_J - 1) \\
& + |q|^{n-\eta+1}(k_K\delta_\eta - \delta_{\eta-1}) + |q|^{n-\eta+1}|\delta_\eta|(k_K - 1) + |q|^{n-\eta+1}|\alpha_\eta|(k_R - 1)
\end{aligned}
$$

$$
\begin{aligned}
&+|q|^{n-\eta}(k_R\alpha_\eta-\alpha_\eta)+|q|^{n-\eta}(k_R\alpha_\eta-\alpha_{\eta+1})+|q|^{n-\eta}|\beta_\eta|(k_I-1)\\
&+|q|^{n-\eta}(k_I\beta_\eta-\beta_{\eta+1})+|q|^{n-\eta}|\gamma_\eta|(k_J-1)+|q|^{n-\eta}(k_J\gamma_\eta-\gamma_{\eta+1})\\
&+|q|^{n-\eta}|\delta_\eta|(k_K-1)+|q|^{n-\eta}(k_K\delta_\eta-\delta_{\eta+1})+\cdots+|q|^{n-p+1}(\alpha_{p-1}-\rho_{R_2}\alpha_p)\\
&+|q|^{n-p+1}|\alpha_p|(1-\rho_{R_2})+|q|^{n-p+1}(\beta_{p-1}-\rho_{I_2}\beta_p)+|q|^{n-p+1}|\beta_p|(1-\rho_{R_2})\\
&+|q|^{n-p+1}(\gamma_{p-1}-\rho_{J_2}\gamma_p)+|q|^{n-p+1}|\gamma_p|(1-\rho_{J_2})+|q|^{n-p+1}(\delta_{p-1}-\rho_{K_2}\delta_p)\\
&+|q|^{n-p+1}|\delta_p|(1-\rho_{K_2})+\cdots+|q|^2|a_{n-2}-a_{n-1}|+|q||a_{n-1}-a_n|+|a_n|\Big),
\end{aligned}
$$

by our hypotheses. Thus

$$
\begin{aligned}
|H(q)| \;\geq\; & |q|^n\Bigg[|q||a_0|-\Bigg(|a_0-a_1|+\frac{|a_1-a_2|}{|q|}+\cdots+\frac{|\alpha_Q|(1-\rho_{R_1})}{|q|^Q}\\
&+\frac{\alpha_{Q+1}-\rho_{R_1}\alpha_Q}{|q|^Q}+\frac{|\beta_Q|(1-\rho_{I_1})}{|q|^Q}+\frac{\beta_{Q+1}-\rho_{I_1}\beta_Q}{|q|^Q}+\frac{|\gamma_Q|(1-\rho_{J_1})}{|q|^Q}\\
&+\frac{\gamma_{Q+1}-\rho_{J_1}\gamma_Q}{|q|^Q}+\frac{|\delta_Q|(1-\rho_{K_1})}{|q|^Q}+\frac{\delta_{Q+1}-\rho_{K_1}\delta_Q}{|q|^Q}+\cdots\\
&+\frac{k_R\alpha_\eta-\alpha_{\eta-1}}{|q|^{\eta-1}}+\frac{|\alpha_\eta|(k_R-1)}{|q|^{\eta-1}}+\frac{k_I\beta_\eta-\beta_{\eta-1}}{|q|^{\eta-1}}+\frac{|\beta_\eta|(k_I-1)}{|q|^{\eta-1}}\\
&+\frac{k_J\gamma_\eta-\gamma_{\eta-1}}{|q|^{\eta-1}}+\frac{|\gamma_\eta|(k_J-1)}{|q|^{\eta-1}}+\frac{k_K\delta_\eta-\delta_{\eta-1}}{|q|^{\eta-1}}+\frac{|\delta_\eta|(k_K-1)}{|q|^{\eta-1}}\\
&+\frac{|\alpha_\eta|(k_R-1)}{|q|^{\eta-1}}+\frac{k_R\alpha_\eta-\alpha_\eta}{|q|^{\eta}}+\frac{k_R\alpha_\eta-\alpha_{\eta+1}}{|q|^{\eta}}+\frac{|\beta_\eta|(k_I-1)}{|q|^{\eta}}\\
&+\frac{k_I\beta_\eta-\beta_{\eta+1}}{|q|^{\eta}}+\frac{|\gamma_\eta|(k_J-1)}{|q|^{\eta}}+\frac{k_J\gamma_\eta-\gamma_{\eta+1}}{|q|^{\eta}}+\frac{|\delta_\eta|(k_K-1)}{|q|^{\eta}}\\
&+\frac{k_K\delta_\eta-\delta_{\eta+1}}{|q|^{\eta}}+\cdots+\frac{\alpha_{p-1}-\rho_{R_2}\alpha_p}{|q|^{p-1}}+\frac{|\alpha_p|(1-\rho_{R_2})}{|q|^{p-1}}+\frac{\beta_{p-1}-\rho_{I_2}\beta_p}{|q|^{p-1}}\\
&+\frac{|\beta_p|(1-\rho_{R_2})}{|q|^{p-1}}+\frac{\gamma_{p-1}-\rho_{J_2}\gamma_p}{|q|^{p-1}}+\frac{|\gamma_p|(1-\rho_{J_2})}{|q|^{p-1}}+\frac{\delta_{p-1}-\rho_{K_2}\delta_p}{|q|^{p-1}}\\
&+\frac{|\delta_p|(1-\rho_{K_2})}{|q|^{p-1}}+\cdots+\frac{|a_{n-2}-a_{n-1}|}{|q|^{n-2}}+\frac{|a_{n-1}-a_n|}{|q|^{n-1}}+\frac{|a_n|}{|q|^n}\Bigg)\Bigg].
\end{aligned}
$$

Now for $|q|>1$, so that $\dfrac{1}{|q|^{n-\ell}}<1$, for $0\leq\ell<n$, we have

$$
|H(q)| \;\geq\; |q|^n\Bigg[|q||a_0|-\Bigg(|a_0-a_1|+|a_1-a_2|+\cdots+|\alpha_Q|(1-\rho_{R_1})
$$

$$
\begin{aligned}
& +(\alpha_{Q+1}-\rho_{R_1}\alpha_Q)+|\beta_Q|(1-\rho_{I_1})+(\beta_{Q+1}-\rho_{I_1}\beta_Q)+|\gamma_Q|(1-\rho_{J_1}) \\
& +(\gamma_{Q+1}-\rho_{J_1}\gamma_Q)+|\delta_Q|(1-\rho_{K_1})+(\delta_{Q+1}-\rho_{K_1}\delta_Q)+\cdots \\
& +(k_R\alpha_\eta-\alpha_{\eta-1})+|\alpha_\eta|(k_R-1)+(k_I\beta_\eta-\beta_{\eta-1})+|\beta_\eta|(k_I-1) \\
& +(k_J\gamma_\eta-\gamma_{\eta-1})+|\gamma_\eta|(k_J-1)+(k_K\delta_\eta-\delta_{\eta-1})+|\alpha_\eta|(k_R-1) \\
& +(k_R\alpha_\eta-\alpha_\eta)+(k_R\alpha_\eta-\alpha_{\eta+1})+|\beta_\eta|(k_I-1)+(k_I\beta_\eta-\beta_{\eta+1}) \\
& +|\gamma_\eta|(k_J-1)+(k_J\gamma_\eta-\gamma_{\eta+1})+|\delta_\eta|(k_K-1)+(k_K\delta_\eta-\delta_{\eta+1})+\cdots \\
& +(\alpha_{p-1}-\rho_{R_2}\alpha_p)+|\alpha_p|(1-\rho_{R_2})+(\beta_{p-1}-\rho_{I_2}\beta_p)+|\beta_p|(1-\rho_{R_2}) \\
& +(\gamma_{p-1}-\rho_{J_2}\gamma_p)+|\gamma_p|(1-\rho_{J_2})+(\delta_{p-1}-\rho_{K_2}\delta_p)+|\delta_p|(1-\rho_{K_2})+\cdots \\
& +|a_{n-2}-a_{n-1}|+|a_{n-1}-a_n|+|a_n|\Big)\Big] \\
= & \ |q|^n\Big[|q||a_0|-\Big(\sum_{\ell=1}^{Q}|a_\ell-a_{\ell-1}|+|\alpha_Q|(1-\rho_{R_1})-\rho_{R_1}\alpha_Q+|\beta_Q|(1-\rho_{I_1}) \\
& -\rho_{I_1}\beta_Q+|\gamma_Q|(1-\rho_{J_1})-\rho_{J_1}\gamma_Q+|\delta_Q|(1-\rho_{K_1})-\rho_{K_1}\delta_Q+k_R\alpha_\eta \\
& +|\alpha_\eta|(k_R-1)+k_I\beta_\eta+|\beta_\eta|(k_I-1)+k_J\gamma_\eta+|\gamma_\eta|(k_J-1)+k_K\delta_\eta \\
& +|\delta_\eta|(k_K-1)+|\alpha_\eta|(k_R-1)+k_R\alpha_\eta+k_R\alpha_\eta+|\beta_\eta|(k_I-1)+k_I\beta_\eta \\
& +k_J\gamma_\eta+k_J\gamma_\eta+|\gamma_\eta|(k_J-1)+|\delta_\eta|(k_K-1)+k_K\delta_\eta-\rho_{R_2}\alpha_p \\
& +|\alpha_p|(1-\rho_{R_2})-\rho_{I_2}\beta_p+|\beta_p|(1-\rho_{R_2})-\rho_{J_2}\gamma_p+|\gamma_p|(1-\rho_{J_2})-\rho_{K_2}\delta_p \\
& +|\delta_p|(1-\rho_{K_2})+\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|+|a_n|\Big)\Big] \\
= & \ |q|^n\Big[|q||a_0|-\Big(M_q+|\alpha_Q|(1-\rho_{R_1})-\rho_{R_1}\alpha_Q+|\beta_Q|(1-\rho_{I_1})-\rho_{I_1}\beta_Q \\
& +|\gamma_Q|(1-\rho_{J_1})-\rho_{J_1}\gamma_Q+|\delta_Q|(1-\rho_{K_1})-\rho_{K_1}\delta_Q+2k_R\alpha_\eta+2|\alpha_\eta|(k_R-1) \\
& +2k_I\beta_\eta+2|\beta_\eta|(k_I-1)+2k_J\gamma_\eta+2|\gamma_\eta|(k_J-1)+2k_K\delta_\eta+2|\delta_\eta|(k_K-1) \\
& -\rho_{R_2}\alpha_p+|\alpha_p|(1-\rho_{R_2})-\rho_{I_2}\beta_p+|\beta_p|(1-\rho_{R_2})-\rho_{J_2}\gamma_p+|\gamma_p|(1-\rho_{J_2})
\end{aligned}
$$

$$-\rho_{K_2}\delta_p + |\delta_p|(1-\rho_{K_2}) + M_p + |a_n|\Big)\Big],$$

where $M_q = \sum_{\ell=1}^{Q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. Notice

$$\begin{aligned}
|H(q)| &\geq |q|^n\Big[|q||a_0| - \Big(M_q + |\alpha_Q|(1-\rho_{R_1}) - \rho_{R_1}\alpha_Q + |\beta_Q|(1-\rho_{I_1}) - \rho_{I_1}\beta_Q \\
&\quad +|\gamma_Q|(1-\rho_{J_1}) - \rho_{J_1}\gamma_Q + |\delta_Q|(1-\rho_{K_1}) - \rho_{K_1}\delta_Q + 2k_R\alpha_\eta + 2|\alpha_\eta|(k_R-1) \\
&\quad +2k_I\beta_\eta + 2|\beta_\eta|(k_I-1) + 2k_J\gamma_\eta + 2|\gamma_\eta|(k_J-1) + 2k_K\delta_\eta + 2|\delta_\eta|(k_K-1) \\
&\quad -\rho_{R_2}\alpha_p + |\alpha_p|(1-\rho_{R_2}) - \rho_{I_2}\beta_p + |\beta_p|(1-\rho_{R_2}) - \rho_{J_2}\gamma_p + |\gamma_p|(1-\rho_{J_2}) \\
&\quad -\rho_{K_2}\delta_p + |\delta_p|(1-\rho_{K_2}) + M_p + |a_n|\Big)\Big] \\
&> 0
\end{aligned}$$

if

$$\begin{aligned}
|q| &> \frac{1}{|a_0|}\Big(M_q + |\alpha_Q|(1-\rho_{R_1}) - \rho_{R_1}\alpha_Q + |\beta_Q|(1-\rho_{I_1}) - \rho_{I_1}\beta_Q \\
&\quad +|\gamma_Q|(1-\rho_{J_1}) - \rho_{J_1}\gamma_Q + |\delta_Q|(1-\rho_{K_1}) - \rho_{K_1}\delta_Q + 2k_R\alpha_\eta + 2|\alpha_\eta|(k_R-1) \\
&\quad +2k_I\beta_\eta + 2|\beta_\eta|(k_I-1) + 2k_J\gamma_\eta + 2|\gamma_\eta|(k_J-1) + 2k_K\delta_\eta + 2|\delta_\eta|(k_K-1) \\
&\quad -\rho_{R_2}\alpha_p + |\alpha_p|(1-\rho_{R_2}) - \rho_{I_2}\beta_p + |\beta_p|(1-\rho_{R_2}) - \rho_{J_2}\gamma_p + |\gamma_p|(1-\rho_{J_2}) \\
&\quad -\rho_{K_2}\delta_p + |\delta_p|(1-\rho_{K_2}) + M_p + |a_n|\Big).
\end{aligned}$$

Thus all zeros of $H(q)$ whose modulus is greater than 1 lie in

$$\begin{aligned}
|q| &\leq \frac{1}{|a_0|}\Big(M_q + |\alpha_Q|(1-\rho_{R_1}) - \rho_{R_1}\alpha_Q + |\beta_Q|(1-\rho_{I_1}) - \rho_{I_1}\beta_Q \\
&\quad +|\gamma_Q|(1-\rho_{J_1}) - \rho_{J_1}\gamma_Q + |\delta_Q|(1-\rho_{K_1}) - \rho_{K_1}\delta_Q + 2k_R\alpha_\eta + 2|\alpha_\eta|(k_R-1) \\
&\quad +2k_I\beta_\eta + 2|\beta_\eta|(k_I-1) + 2k_J\gamma_\eta + 2|\gamma_\eta|(k_J-1) + 2k_K\delta_\eta + 2|\delta_\eta|(k_K-1) \\
&\quad -\rho_{R_2}\alpha_p + |\alpha_p|(1-\rho_{R_2}) - \rho_{I_2}\beta_p + |\beta_p|(1-\rho_{R_2}) - \rho_{J_2}\gamma_p + |\gamma_p|(1-\rho_{J_2})
\end{aligned}$$

$$-\rho_{K_2}\delta_p + |\delta_p|(1-\rho_{K_2}) + M_p + |a_n|\Big).$$

Hence all zeros of $H(q)$ and hence of $S(q)$ lie in

$$\begin{aligned} |q| \;\; &\leq \;\; \max\Bigg\{1, |a_0| \Big/ \Big(M_q + |\alpha_Q|(1-\rho_{R_1}) - \rho_{R_1}\alpha_Q + |\beta_Q|(1-\rho_{I_1}) - \rho_{I_1}\beta_Q \\ &+|\gamma_Q|(1-\rho_{J_1}) - \rho_{J_1}\gamma_Q + |\delta_Q|(1-\rho_{K_1}) - \rho_{K_1}\delta_Q + 2k_R\alpha_\eta + 2|\alpha_\eta|(k_R-1) \\ &+2k_I\beta_\eta + 2|\beta_\eta|(k_I-1) + 2k_J\gamma_\eta + 2|\gamma_\eta|(k_J-1) + 2k_K\delta_\eta + 2|\delta_\eta|(k_K-1) \\ &-\rho_{R_2}\alpha_p + |\alpha_p|(1-\rho_{R_2}) - \rho_{I_2}\beta_p + |\beta_p|(1-\rho_{R_2}) - \rho_{J_2}\gamma_p + |\gamma_p|(1-\rho_{J_2}) \\ &-\rho_{K_2}\delta_p + |\delta_p|(1-\rho_{K_2}) + M_p + |a_n|\Big)\Bigg\}. \end{aligned}$$

Therefore all zeros of $P(q)$ lie in

$$\begin{aligned} |q| \;\; &\geq \;\; \min\Bigg\{1, |a_0| \Big/ \Big(M_q + |\alpha_Q|(1-\rho_{R_1}) - \rho_{R_1}\alpha_Q + |\beta_Q|(1-\rho_{I_1}) - \rho_{I_1}\beta_Q \\ &+|\gamma_Q|(1-\rho_{J_1}) - \rho_{J_1}\gamma_Q + |\delta_Q|(1-\rho_{K_1}) - \rho_{K_1}\delta_Q + 2k_R\alpha_\eta + 2|\alpha_\eta|(k_R-1) \\ &+2k_I\beta_\eta + 2|\beta_\eta|(k_I-1) + 2k_J\gamma_\eta + 2|\gamma_\eta|(k_J-1) + 2k_K\delta_\eta + 2|\delta_\eta|(k_K-1) \\ &-\rho_{R_2}\alpha_p + |\alpha_p|(1-\rho_{R_2}) - \rho_{I_2}\beta_p + |\beta_p|(1-\rho_{R_2}) - \rho_{J_2}\gamma_p + |\gamma_p|(1-\rho_{J_2}) \\ &-\rho_{K_2}\delta_p + |\delta_p|(1-\rho_{K_2}) + M_p + |a_n|\Big)\Bigg\}. \end{aligned}$$

Thus the polynomial $P(q)$ does not vanish in

$$\begin{aligned} |q| \;\; &< \;\; \min\Bigg\{1, |a_0| \Big/ \Big(M_q + |\alpha_Q|(1-\rho_{R_1}) - \rho_{R_1}\alpha_Q + |\beta_Q|(1-\rho_{I_1}) - \rho_{I_1}\beta_Q \\ &+|\gamma_Q|(1-\rho_{J_1}) - \rho_{J_1}\gamma_Q + |\delta_Q|(1-\rho_{K_1}) - \rho_{K_1}\delta_Q + 2k_R\alpha_\eta + 2|\alpha_\eta|(k_R-1) \\ &+2k_I\beta_\eta + 2|\beta_\eta|(k_I-1) + 2k_J\gamma_\eta + 2|\gamma_\eta|(k_J-1) + 2k_K\delta_\eta + 2|\delta_\eta|(k_K-1) \\ &-\rho_{R_2}\alpha_p + |\alpha_p|(1-\rho_{R_2}) - \rho_{I_2}\beta_p + |\beta_p|(1-\rho_{R_2}) - \rho_{J_2}\gamma_p + |\gamma_p|(1-\rho_{J_2}) \\ &-\rho_{K_2}\delta_p + |\delta_p|(1-\rho_{K_2}) + M_p + |a_n|\Big)\Bigg\}. \end{aligned}$$

□

Notice when $\gamma_\ell = \delta_\ell = 0$ for all $0 \le \ell \le n$, then Theorem 6.12 reduces to Theorem 2.5 and with additional parameters will reduce down to the inner bound of Theorem 1.10. Notice when $\rho_{R_1} = \rho_{R_2} = \rho_{I_1} = \rho_{I_2} = \rho_{J_1} = \rho_{J_2} = \rho_{K_1} = \rho_{K_2} = k_R = k_I = k_J = k_K = 1$ then we get the following corollary:

Corollary 6.13. *If $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is a polynomial of degree n with quaternionic coefficients, that is $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$, where $Q \le \eta \le p$ satisfying*

$$\alpha_Q \le \alpha_{Q+1} \le \cdots \le \alpha_{\eta-1} \le \alpha_\eta \ge \alpha_{\eta+1} \ge \cdots \ge \alpha_p$$

$$\beta_Q \le \beta_{Q+1} \le \cdots \le \beta_{\eta-1} \le \beta_\eta \ge \beta_{\eta+1} \ge \cdots \ge \beta_p$$

$$\gamma_Q \le \gamma_{Q+1} \le \cdots \le \gamma_{\eta-1} \le \gamma_\eta \ge \gamma_{\eta+1} \ge \cdots \ge \gamma_p$$

$$\delta_Q \le \delta_{Q+1} \le \cdots \le \delta_{\eta-1} \le \delta_\eta \ge \delta_{\eta+1} \ge \cdots \ge \delta_p.$$

Then the polynomial $P(q)$ does not vanish in

$$\begin{aligned} |q| \;<\; & \min\left\{1, |a_0| \Big/ \Big(M_q - (\alpha_Q + \beta_Q + \gamma_Q + \delta_Q) + 2(\alpha_\eta + \beta_\eta + \gamma_\eta + \delta_\eta) \right. \\ & \left. -(\alpha_p + \beta_p + \gamma_p + \delta_p) + M_p + |a_n| \Big) \right\}. \end{aligned}$$

Now that we have found the bounds on the parts of the quaternionic coefficients, we will now consider the modulus of the coefficients:

Theorem 6.14. *Let $P(q) = a_0 + qa_1 + \cdots + q^Q a_Q + \cdots + q^p a_p + \cdots + q^n a_n$ is a polynomial of degree n with quaternionic coefficients satisfying, for a nonzero quaternion b, $\measuredangle(a_\ell, b) \le \alpha \le \frac{\pi}{2}$ for some α and $\ell = Q, Q+1, \ldots, p$. If*

$$\rho_1|a_Q| \le |a_{Q+1}| \le \cdots \le |a_{\eta-1}| \le k|a_\eta| \ge |a_{\eta+1}| \ge \cdots \ge \rho_2|a_p|$$

then all zeros of $P(q)$ lie in

$$\begin{aligned}|q| &\leq \frac{1}{|a_n|}\Big(|a_0| + M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha - \cos\alpha - 1)\\ &\quad +2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha - 2|a_\eta| + 2k|a_\eta|(\sin\alpha + \cos\alpha + 1)\\ &\quad +2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha + |a_p| + \rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + M_p\Big)\end{aligned}$$

where $M_q = \sum_{\ell=1}^{Q}|a_\ell - a_{\ell-1}|$ *and* $M_p = \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}|$.

Proof of Theorem 6.14. Let $P(z) = a_0 + qa_1 + \cdots + q^Q a_Q + \cdots + q^p a_p + \cdots + q^n a_n$ be a polynomial of degree n with quaternionic coefficients. Without loss of generality assume $\beta = 0$. Consider

$$\begin{aligned}P(q) * (1-q) &= a_0 + q(a_1 - a_0) + \cdots + q^{Q+1}(a_{Q+1} - a_Q)\\ &\quad + \cdots + q^p(a_p - a_{p-1}) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n\\ &= f(q) - q^{n+1}a_n.\end{aligned}$$

By Lemma 6.4, the only zeros of $P(q) * (1-q)$ are $q = 1$ and the zeros of $P(q)$. So, for $|q| = 1$,

$$\begin{aligned}|f(q)| &\leq |a_0 + q(a_1 - a_0) + \cdots + q^{Q+1}(a_{Q+1} - a_Q) + \cdots\\ &\quad +q^p(a_p - a_{p-1}) + \cdots + q^n(a_n - a_{n-1})|\\ &\leq |a_0| + |q||a_1 - a_0| + \cdots + |q|^{Q+1}|a_{Q+1} - a_Q| + \cdots\\ &\quad +|q|^\eta|a_\eta - a_{\eta-1}| + |q|^{\eta+1}|a_{\eta+1} - a_\eta| + \cdots + |q|^p|a_p - a_{p-1}| + \cdots\\ &\quad +|q|^n|a_n - a_{n-1}|\\ &= |a_0| + |a_1 - a_0| + \cdots + |a_{Q+1} - a_Q| + \cdots + |a_\eta - a_{\eta-1}| + |a_{\eta+1} - a_\eta|\end{aligned}$$

$$
\begin{aligned}
& +\cdots+|a_p-a_{p-1}|+\cdots+|a_n-a_{n-1}| \\
= \;& |a_0|+\sum_{\ell=1}^{Q}|a_\ell-a_{\ell-1}|+|a_{Q+1}-a_Q|+\sum_{\ell=Q+2}^{\eta-1}|a_\ell-a_{\ell-1}|+|a_\eta-a_{\eta-1}| \\
& +|a_{\eta+1}-a_\eta|+\sum_{\ell=\eta+2}^{p-1}|a_\ell-a_{\ell-1}|+|a_p-a_{p-1}|+\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}| \\
= \;& |a_0|+M_q+|a_{Q+1}-a_Q|+\sum_{\ell=Q+2}^{\eta-1}|a_\ell-a_{\ell-1}|+|a_\eta-a_{\eta-1}|+|a_{\eta+1}-a_\eta| \\
& +\sum_{\ell=\eta+2}^{p-1}|a_\ell-a_{\ell-1}|+|a_p-a_{p-1}|+M_p
\end{aligned}
$$

where $M_q=\sum_{\ell=1}^{Q}|a_\ell-a_{\ell-1}|$ and $M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$. Thus

$$
\begin{aligned}
|f(q)| \;\le\;& |a_0|+M_q+|a_{Q+1}-a_Q|+\sum_{\ell=Q+2}^{\eta-1}|a_\ell-a_{\ell-1}|+|a_\eta-a_{\eta-1}|+|a_{\eta+1}-a_\eta| \\
& +\sum_{\ell=\eta+2}^{p-1}|a_\ell-a_{\ell-1}|+|a_p-a_{p-1}|+M_p \\
=\;& |a_0|+M_q+|a_{Q+1}-\rho_1a_Q+\rho_1a_Q-a_Q|+\sum_{\ell=Q+2}^{\eta-1}|a_\ell-a_{\ell-1}| \\
& +|a_\eta-ka_\eta+ka_\eta-a_{\eta-1}|+|a_{\eta+1}-ka_\eta+ka_\eta-a_\eta|+\sum_{\ell=\eta+2}^{p-1}|a_\ell-a_{\ell-1}| \\
& +|a_p-\rho_2a_p+\rho_2a_p-a_{p-1}|+M_p \\
\le\;& |a_0|+M_q+|a_{Q+1}-\rho_1a_Q|+|\rho_1a_Q-a_Q|+\sum_{\ell=Q+2}^{\eta-1}|a_\ell-a_{\ell-1}| \\
& +|a_\eta-ka_\eta|+|ka_\eta-a_{\eta-1}|+|a_{\eta+1}-ka_\eta|+|ka_\eta-a_\eta|+\sum_{\ell=\eta+2}^{p-1}|a_\ell-a_{\ell-1}| \\
& +|a_p-\rho_2a_p|+|\rho_2a_p-a_{p-1}|+M_p \\
=\;& |a_0|+M_q+|a_{Q+1}-\rho_1a_Q|+|a_Q|(1-\rho_1)+\sum_{\ell=Q+2}^{\eta-1}|a_\ell-a_{\ell-1}| \\
& +|a_\eta|(k-1)+|ka_\eta-a_{\eta-1}|+|ka_\eta-a_{\eta+1}|+|a_\eta|(k-1)
\end{aligned}
$$

$$+\sum_{\ell=\eta+2}^{p-1}|a_{\ell-1}-a_\ell|+|a_p|(1-\rho_2)+|a_{p-1}-\rho_2a_p|+M_p.$$

So by Lemma 6.5 then

$$\begin{aligned}
|f(q)| &\le |a_0|+M_q+(|a_{Q+1}|-|\rho_1a_Q|)\cos\alpha+(|a_{Q+1}|+|\rho_1a_Q|)\sin\alpha\\
&\quad+|a_Q|(1-\rho_1)+\sum_{\ell=Q+2}^{\eta-1}(|a_\ell|-|a_{\ell-1}|)\cos\alpha+\sum_{\ell=Q+2}^{\eta-1}(|a_\ell|+|a_{\ell-1}|)\sin\alpha\\
&\quad+|a_\eta|(k-1)+(|ka_\eta|-|a_{\eta-1}|)\cos\alpha+(|ka_\eta|+|a_{\eta-1}|)\sin\alpha\\
&\quad+(|ka_\eta|-|a_{\eta+1}|)\cos\alpha+(|ka_\eta|+|a_{\eta+1}|)\sin\alpha+|a_\eta|(k-1)\\
&\quad+\sum_{\ell=\eta+2}^{p-1}(|a_{\ell-1}|-|a_\ell|)\cos\alpha+\sum_{\ell=\eta+2}^{p-1}(|a_{\ell-1}|+|a_\ell|)\sin\alpha+|a_p|(1-\rho_2)\\
&\quad+(|a_{p-1}|-|\rho_2a_p|)\cos\alpha+(|a_{p-1}|+|\rho_2a_p|)\sin\alpha+M_p\\
&= |a_0|+M_q+|a_{Q+1}|\cos\alpha-\rho_1|a_Q|\cos\alpha+|a_{Q+1}|\sin\alpha+\rho_1|a_Q|\sin\alpha\\
&\quad+|a_Q|(1-\rho_1)+\sum_{\ell=Q+2}^{\eta-1}|a_\ell|\cos\alpha-\sum_{\ell=Q+2}^{\eta-1}|a_{\ell-1}|\cos\alpha+\sum_{\ell=Q+2}^{\eta-1}|a_\ell|\sin\alpha\\
&\quad+\sum_{\ell=Q+2}^{\eta-1}|a_{\ell-1}|\sin\alpha+|a_\eta|(k-1)+k|a_\eta|\cos\alpha-|a_{\eta-1}|\cos\alpha+k|a_\eta|\sin\alpha\\
&\quad+|a_{\eta-1}|\sin\alpha+k|a_\eta|\cos\alpha-|a_{\eta+1}|\cos\alpha+k|a_\eta|\sin\alpha+|a_{\eta+1}|\sin\alpha\\
&\quad+|a_\eta|(k-1)+\sum_{\ell=\eta+2}^{p-1}|a_{\ell-1}|\cos\alpha-\sum_{\ell=\eta+2}^{p-1}|a_\ell|\cos\alpha+\sum_{\ell=\eta+2}^{p-1}|a_{\ell-1}|\sin\alpha\\
&\quad+\sum_{\ell=\eta+2}^{p-1}|a_\ell|\sin\alpha+|a_p|(1-\rho_2)+|a_{p-1}|\cos\alpha-\rho_2|a_p|\cos\alpha+|a_{p-1}|\sin\alpha\\
&\quad+\rho_2|a_p|\sin\alpha+M_p\\
&= |a_0|+M_q+|a_{Q+1}|\cos\alpha-\rho_1|a_Q|\cos\alpha+|a_{Q+1}|\sin\alpha+\rho_1|a_Q|\sin\alpha\\
&\quad+|a_Q|(1-\rho_1)+|a_{\eta-1}|\cos\alpha+\sum_{\ell=Q+2}^{\eta-2}|a_\ell|\cos\alpha-|a_{Q+1}|\cos\alpha\\
&\quad-\sum_{\ell=Q+2}^{\eta-2}|a_\ell|\cos\alpha+|a_{\eta-1}|\sin\alpha+\sum_{\ell=Q+2}^{\eta-2}|a_\ell|\sin\alpha
\end{aligned}$$

$$
\begin{aligned}
& +|a_{Q+1}|\sin\alpha + \sum_{\ell=Q+2}^{\eta-2} |a_\ell|\sin\alpha + |a_\eta|(k-1) + k|a_\eta|\cos\alpha - |a_{\eta-1}|\cos\alpha \\
& +k|a_\eta|\sin\alpha + |a_{\eta-1}|\sin\alpha + k|a_\eta|\cos\alpha - |a_{\eta+1}|\cos\alpha + k|a_\eta|\sin\alpha \\
& +|a_{\eta+1}|\sin\alpha + |a_\eta|(k-1) + |a_{\eta+1}|\cos\alpha + \sum_{\ell=\eta+2}^{p-2} |a_\ell|\cos\alpha - |a_{p-1}|\cos\alpha \\
& - \sum_{\ell=\eta+2}^{p-2} |a_\ell|\cos\alpha + |a_{\eta+1}|\sin\alpha + \sum_{\ell=\eta+2}^{p-2} |a_\ell|\sin\alpha + |a_{p-1}|\sin\alpha \\
& + \sum_{\ell=\eta+2}^{p-2} |a_\ell|\sin\alpha + |a_p|(1-\rho_2) + |a_{p-1}|\cos\alpha - \rho_2|a_p|\cos\alpha + |a_{p-1}|\sin\alpha \\
& +\rho_2|a_p|\sin\alpha + M_p \\
= \;& |a_0| + M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{\eta-1} |a_\ell|\sin\alpha \\
& -2|a_\eta| + 2k|a_\eta|(\sin\alpha + \cos\alpha + 1) + 2\sum_{\ell=\eta+1}^{p-1} |a_\ell|\sin\alpha + |a_p| \\
& +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + M_p.
\end{aligned}
$$

Hence also,

$$
\begin{aligned}
\left|q^n f\left(\frac{1}{q}\right)\right| \;\leq\; & |a_0| + M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{\eta-1} |a_\ell|\sin\alpha \\
& -2|a_\eta| + 2k|a_\eta|(\sin\alpha + \cos\alpha + 1) + 2\sum_{\ell=\eta+1}^{p-1} |a_\ell|\sin\alpha + |a_p| \\
& +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + M_p
\end{aligned}
$$

for $|q| = 1$. By Theorem 6.3

$$
\begin{aligned}
\left|q^n f\left(\frac{1}{q}\right)\right| \;\leq\; & |a_0| + M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{\eta-1} |a_\ell|\sin\alpha \\
& -2|a_\eta| + 2k|a_\eta|(\sin\alpha + \cos\alpha + 1) + 2\sum_{\ell=\eta+1}^{p-1} |a_\ell|\sin\alpha + |a_p| \\
& +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + M_p
\end{aligned}
$$

holds inside the unit disk $|q| \leq 1$ as well. If $|q| > 1$, then $\frac{1}{|q|^n} e^{-i\alpha}$ lies inside the unit circle for every real α. Thus if follows

$$\begin{aligned} |P\big(|q|^n e^{i\alpha}\big)| &\leq |q|^n \Bigg(|a_0| + M_q + |a_Q| + \rho_1 |a_Q|(\sin\alpha - \cos\alpha - 1) + 2 \sum_{\ell=Q+1}^{\eta-1} |a_\ell| \sin\alpha \\ & -2|a_\eta| + 2k|a_\eta|(\sin\alpha + \cos\alpha + 1) + 2 \sum_{\ell=\eta+1}^{p-1} |a_\ell| \sin\alpha + |a_p| \\ & +\rho_2 |a_p|(\sin\alpha - \cos\alpha - 1) + M_p \Bigg) \end{aligned}$$

for every $|q| \geq 1$ and α real. Thus for every $|q| > 1$

$$\begin{aligned} |P(q) * (1-q)| &= |-q^{n+1} a_n + f(q)| \\ &\geq |q|^{n+1}|a_n| - |f(q)| \\ &\geq |q|^{n+1}|a_n| - |q|^n \Bigg(|a_0| + M_q + |a_Q| + \rho_1 |a_Q|(\sin\alpha - \cos\alpha - 1) \\ & +2 \sum_{\ell=Q+1}^{\eta-1} |a_\ell| \sin\alpha - 2|a_\eta| + 2k|a_\eta|(\sin\alpha + \cos\alpha + 1) \\ & +2 \sum_{\ell=\eta+1}^{p-1} |a_\ell| \sin\alpha + |a_p| + \rho_2 |a_p|(\sin\alpha - \cos\alpha - 1) + M_p \Bigg) \\ &= |q|^n \Bigg[|q||a_n| - \Bigg(|a_0| + M_q + |a_Q| + \rho_1 |a_Q|(\sin\alpha - \cos\alpha - 1) \\ & +2 \sum_{\ell=Q+1}^{\eta-1} |a_\ell| \sin\alpha - 2|a_\eta| + 2k|a_\eta|(\sin\alpha + \cos\alpha + 1) \\ & +2 \sum_{\ell=\eta+1}^{p-1} |a_\ell| \sin\alpha + |a_p| + \rho_2 |a_p|(\sin\alpha - \cos\alpha - 1) + M_p \Bigg) \Bigg] \\ &> 0 \end{aligned}$$

if

$$|q| > \frac{1}{|a_n|} \Bigg(|a_0| + M_q + |a_Q| + \rho_1 |a_Q|(\sin\alpha - \cos\alpha - 1)$$

$$+2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha - 2|a_\eta| + 2k|a_\eta|(\sin\alpha+\cos\alpha+1)$$
$$+2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha + |a_p| + \rho_2|a_p|(\sin\alpha-\cos\alpha-1) + M_p\Big).$$

Therefore all zeros of $P(q)$ lie within

$$\begin{aligned}|q| \quad \leq \quad & \frac{1}{|a_n|}\Big(|a_0| + M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha-\cos\alpha-1) \\ & +2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha - 2|a_\eta| + 2k|a_\eta|(\sin\alpha+\cos\alpha+1) \\ & +2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha + |a_p| + \rho_2|a_p|(\sin\alpha-\cos\alpha-1) + M_p\Big).\end{aligned}$$

□

Notice when $\gamma_\ell = \delta_\ell = 0$ for $0 \leq \ell \leq n$ then Theorem 6.14 reduces to Theorem 4.7. Now apply these parameters to the moduli of quaternionic coefficients to find the inner bound:

Theorem 6.15. *Let* $P(q) = a_0 + qa_1 + \cdots + q^Q a_Q + \cdots + q^p a_p + \cdots + q^n a_n$ *is a polynomial of degree* n *with quaternionic coefficients satisfying, for a nonzero quaternion* b, $\measuredangle(a_\ell, b) \leq \alpha \leq \dfrac{\pi}{2}$ *for some* α *and* $\ell = Q, Q+1, \ldots, p$. *Suppose for real* ρ_1, ρ_2, k *where* $0 < \rho_1 \leq 1$, $0 < \rho_2 \leq 1$, $k \geq 1$ *satisfying*

$$\rho_1|a_Q| \leq |a_{Q+1}| \leq \cdots \leq |a_{\eta-1}| \leq k|a_\eta| \geq |a_{\eta+1}| \geq \cdots \geq \rho_2|a_p|.$$

Then $P(q)$ *does not vanish in*

$$|q| \quad < \quad \min\Big\{1, |a_0| \Big/ \Big(M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha-\cos\alpha-1) + 2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha$$

$$+2k|a_\eta|(\sin\alpha+\cos\alpha+1)-2|a_\eta|+2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha$$

$$+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p+|a_n|\Big)\Big\},$$

where $M_q=\sum_{\ell=1}^{Q}|a_\ell-a_{\ell-1}|$ *and* $M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$.

Proof of Theorem 6.15. Consider the reciprocal polynomial

$$S(q)=q^n*P\left(\frac{1}{q}\right)=q^na_0+q^{n-1}a_1+\cdots+qa_{n-1}+a_n.$$

Let

$$\begin{aligned}H(q) &= S(q)*(1-q)\\ &= -q^{n+1}a_0+q^n(a_0-a_1)+q^{n-1}(a_1-a_2)+\cdots+a^{n-Q}(a_Q-a_{Q+1})\\ &\quad+\cdots+q^{n-p+1}(a_{p-1}-a_p)+\cdots+q(a_{n-1}-a_n)+a_n.\end{aligned}$$

By Lemma 6.4, the only zeros of $P(q)*(1-q)$ are $q=1$ and the zeros of $P(q)$. This gives

$$\begin{aligned}|H(q)| &\geq |q|^{n+1}|a_0|-\Big(|q|^n|a_0-a_1|+|q|^{n-1}|a_1-a_2|+\cdots+|q|^{n-Q}|a_Q-a_{Q+1}|\\ &\quad+\cdots+|q|^{n-p+1}|a_{p-1}-a_p|+\cdots+|q||a_{n-1}-a_n|+|a_n|\Big)\\ &= |q|^{n+1}|a_0|-\Big(|q|^n|a_0-a_1|+|q|^{n-1}|a_1-a_2|+\cdots\\ &\quad+|q|^{n-Q}|a_Q-\rho_1a_Q+\rho_1a_Q-a_{Q+1}|+\cdots\\ &\quad+|q|^{n-\eta+1}|a_{\eta-1}-ka_\eta+ka_\eta-a_\eta|+|q|^\eta|a_\eta-ka_\eta+ka_\eta-a_{\eta+1}|+\cdots\\ &\quad+|q|^{n-p+1}|a_{p-1}-\rho_2a_p+\rho_2a_p-a_p|+\cdots+|q||a_{n-1}-a_n|+|a_n|\Big)\\ &\geq |q|^{n+1}|a_0|-\Big(|q|^n|a_0-a_1|+|q|^{n-1}|a_1-a_2|+\cdots+|q|^{n-Q}|a_Q-\rho_1a_Q|\end{aligned}$$

$$+|q|^{n-Q}|\rho_1 a_Q - a_{Q+1}| + \cdots + |q|^{n-\eta+1}|a_{\eta-1} - ka_\eta| + |q|^{n-\eta+1}|ka_\eta - a_\eta|$$
$$+|q|^{\eta}|a_\eta - ka_\eta| + |q|^{\eta}|ka_\eta - a_{\eta+1}| + \cdots + |q|^{n-p+1}|a_{p-1} - \rho_2 a_p|$$
$$+|q|^{n-p+1}|\rho_2 a_p - a_p| + \cdots + |q||a_{n-1} - a_n| + |a_n|\Big).$$

Thus by our hypotheses,

$$\begin{aligned}
|H(q)| \geq\ & |q|^{n+1}|a_0| - \Big(|q|^n|a_0 - a_1| + |q|^{n-1}|a_1 - a_2| + \cdots + |q|^{n-Q}|a_Q|(1-\rho_1) \\
& +|q|^{n-Q}|a_{Q+1} - \rho_1 a_Q| + \cdots + |q|^{n-\eta+1}|ka_\eta - a_{\eta-1}| + |q|^{n-\eta+1}|a_\eta|(k-1) \\
& +|q|^{\eta}|a_\eta|(k-1) + |q|^{\eta}|ka_\eta - a_{\eta+1}| + \cdots + |q|^{n-p+1}|a_{p-1} - \rho_2 a_p| \\
& +|q|^{n-p+1}|a_p|(1-\rho_2) + \cdots + |q||a_{n-1} - a_n| + |a_n|\Big) \\
=\ & |q|^n\Big[|q||a_0| - \Big(|a_0 - a_1| + \frac{|a_1 - a_2|}{|q|} + \cdots + \frac{|a_Q|(1-\rho_1)}{|q|^Q} \\
& + \frac{|a_{Q+1} - \rho_1 a_Q|}{|q|^Q} + \cdots + \frac{|ka_\eta - a_{\eta-1}|}{|q|^{\eta-1}} + \frac{|a_\eta|(k-1)}{|q|^{\eta-1}} + \frac{|a_\eta|(k-1)}{|q|^{\eta}} \\
& + \frac{|ka_\eta - a_{\eta+1}|}{|q|^{\eta}} + \cdots + \frac{|a_{p-1} - \rho_2 a_p|}{|q|^{p-1}} + \frac{|a_p|(1-\rho_2)}{|q|p-1} + \cdots + \frac{|a_{n-1} - a_n|}{|q|^{n-1}} \\
& + \frac{|a_n|}{|q|^n}\Big)\Big].
\end{aligned}$$

Now for $|q| > 1$, so that $\dfrac{1}{|q|^{n-\ell}} < 1$, for $0 \leq \ell < n$ we have

$$\begin{aligned}
|H(q)| \geq\ & |q|^n\Big[|q||a_0| - \Big(|a_0 - a_1| + |a_1 - a_2| + \cdots + |a_Q|(1-\rho_1) + |a_{Q+1} - \rho_1 a_Q| \\
& + \cdots + |ka_\eta - a_{\eta-1}| + |a_\eta|(k-1) + |a_\eta|(k-1) + |ka_\eta - a_{\eta+1}| + \cdots \\
& +|a_{p-1} - \rho_2 a_p| + |a_p|(1-\rho_2) + \cdots + |a_{n-1} - a_n| + |a_n|\Big)\Big] \\
=\ & |q|^n\Big[|q||a_0| - \Big(\sum_{\ell=1}^{Q}|a_\ell - a_{\ell-1}| + |a_Q|(1-\rho_1) + |a_{Q+1} - \rho_1 a_Q| \\
& + \sum_{\ell=Q+2}^{\eta-1}|a_\ell - a_{\ell-1}| + |ka_\eta - a_{\eta-1}| + 2|a_\eta|(k-1) + |ka_\eta - a_{\eta+1}|
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{\ell=\eta+2}^{p-1} |a_{\ell-1} - a_\ell| + |a_{p-1} - \rho_2 a_p| + |a_p|(1-\rho_2) + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| \\
& +|a_n|\Big)\Big] \\
\geq \ & |q|^n\Big[|q||a_0| - \Big(M_1 + |a_Q|(1-\rho_1) + (|a_{Q+1}| - |\rho_1 a_Q|)\cos\alpha \\
& +(|a_{Q+1}| + |\rho_1 a_Q|)\sin\alpha + \sum_{\ell=Q+2}^{\eta-1} (|a_\ell| - |a_{\ell-1}|)\cos\alpha \\
& + \sum_{\ell=Q+2}^{\eta-1} (|a_\ell| + |a_{\ell-1}|)\sin\alpha + (|ka_\eta| - |a_{\eta-1}|)\cos\alpha + (|ka_\eta| + |a_{\eta-1}|)\sin\alpha \\
& +2|a_\eta|(k-1) + (|ka_\eta| - |a_{\eta+1}|)\cos\alpha + (|ka_\eta| + |a_{\eta+1}|)\sin\alpha \\
& + \sum_{\ell=\eta+2}^{p-1} (|a_{\ell-1}| - |a_\ell|)\cos\alpha + \sum_{\ell=\eta+2}^{p-1} (|a_{\ell-1}| + |a_\ell|)\sin\alpha \\
& +(|a_{p-1}| - |\rho_2 a_p|)\cos\alpha + (|a_{p-1}| + |\rho_2 a_p|)\sin\alpha + |a_p|(1-\rho_2) + M_p \\
& +|a_n|\Big)\Big] \\
= \ & |q|^n\Big[|q||a_0| - \Big(M_q + |a_Q|(1-\rho_1) + |a_{Q+1}|\cos\alpha - \rho_1|a_Q|\cos\alpha \\
& +|a_{Q+1}|\sin\alpha + \rho_1|a_Q|\sin\alpha + \sum_{\ell=Q+2}^{\eta-1} |a_\ell|\cos\alpha - \sum_{\ell=Q+2}^{\eta-1} |a_{\ell-1}|\cos\alpha \\
& + \sum_{\ell=Q+2}^{\eta-1} |a_\ell|\sin\alpha + \sum_{\ell=Q+2}^{\eta-1} |a_{\ell-1}|\sin\alpha + k|a_\eta|\cos\alpha - |a_{\eta-1}|\cos\alpha \\
& +k|a_\eta|\sin\alpha + |a_{\eta-1}|\sin\alpha + 2|a_\eta|(k-1) + k|a_\eta|\cos\alpha - |a_{\eta+1}|\cos\alpha \\
& +k|a_\eta|\sin\alpha + |a_{\eta+1}|\sin\alpha + \sum_{\ell=\eta+2}^{p-1} |a_{\ell-1}|\cos\alpha - \sum_{\ell=\eta+2}^{p-1} |a_\ell|\cos\alpha \\
& + \sum_{\ell=\eta+2}^{p-1} |a_{\ell-1}|\sin\alpha + \sum_{\ell=\eta+2}^{p-1} |a_\ell|\sin\alpha + |a_{p-1}|\cos\alpha - \rho_2|a_p|\cos\alpha \\
& +|a_{p-1}|\sin\alpha + \rho_2|a_p|\sin\alpha + |a_p|(1-\rho_2) + M_p + |a_n|\Big)\Big],
\end{aligned}
$$

by our hypotheses, Lemma 6.5 and $M_q = \sum_{\ell=1}^{Q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Hence we have

$$
\begin{aligned}
|H(q)| \geq\ & |q|^n\Bigg[|q||a_0|-\Bigg(M_q+|a_Q|(1-\rho_1)+|a_{Q+1}|\cos\alpha-\rho_1|a_Q|\cos\alpha \\
& +|a_{Q+1}|\sin\alpha+\rho_1|a_Q|\sin\alpha+|a_{\eta-1}|\cos\alpha+\sum_{\ell=Q+2}^{\eta-2}|a_\ell|\cos\alpha-|a_{Q+1}|\cos\alpha \\
& -\sum_{\ell=Q+2}^{\eta-2}|a_\ell|\cos\alpha+|a_{\eta-1}|\sin\alpha+\sum_{\ell=Q+2}^{\eta-2}|a_\ell|\sin\alpha+|a_{Q+1}|\sin\alpha \\
& +\sum_{\ell=Q+2}^{\eta-2}|a_\ell|\sin\alpha+k|a_\eta|\cos\alpha-|a_{\eta-1}|\cos\alpha+k|a_\eta|\sin\alpha+|a_{\eta-1}|\sin\alpha \\
& +2|a_\eta|(k-1)+k|a_\eta|\cos\alpha-|a_{\eta+1}|\cos\alpha+k|a_\eta|\sin\alpha+|a_{\eta+1}|\sin\alpha \\
& +|a_{\eta+1}|\cos\alpha+\sum_{\ell=\eta+2}^{p-2}|a_\ell|\cos\alpha-|a_{p-1}|\cos\alpha-\sum_{\ell=\eta+2}^{p-2}|a_\ell|\cos\alpha \\
& +|a_{\eta+1}|\sin\alpha+\sum_{\ell=\eta+2}^{p-2}|a_\ell|\sin\alpha+|a_{p-1}|\sin\alpha+\sum_{\ell=\eta+2}^{p-2}|a_\ell|\sin\alpha \\
& +|a_{\eta-1}|\cos\alpha-\rho_2|a_p|\cos\alpha+|a_{p-1}|\sin\alpha+\rho_2|a_p|\sin\alpha+|a_p|(1-\rho_2) \\
& +M_p+|a_n|\Bigg)\Bigg] \\
=\ & |q|^n\Bigg[|q||a_0|-\Bigg(M_q+|a_Q|+\rho_1|a_Q|(\sin\alpha-\cos\alpha-1)+2|a_{Q+1}|\sin\alpha \\
& +2|a_{\eta-1}|\sin\alpha+2\sum_{\ell=Q+2}^{j-2}|a_\ell|\sin\alpha+2k|a_\eta|(\sin\alpha+\cos\alpha+1)-2|a_\eta| \\
& +2|a_{\eta+1}|\sin\alpha+2\sum_{\ell=\eta+2}^{p-2}|a_\ell|\sin\alpha+2|a_{p-1}|\sin\alpha \\
& +\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p+|a_n|\Bigg)\Bigg] \\
=\ & |q|^n\Bigg[|q||a_0|-\Bigg(M_q+|a_Q|+\rho_1|a_Q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha \\
& +2k|a_\eta|(\sin\alpha+\cos\alpha+1)-2|a_\eta|+2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha \\
& +\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p+|a_n|\Bigg)\Bigg]
\end{aligned}
$$

$$> 0$$

if

$$\begin{aligned}|q| &> \frac{1}{|a_0|}\Bigg(M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha \\ &+2k|a_\eta|(\sin\alpha + \cos\alpha + 1) - 2|a_\eta| + 2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha \\ &+\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Bigg).\end{aligned}$$

Thus all zeros of $H(q)$ whose modulus is greater than 1 lie in

$$\begin{aligned}|q| &\leq \frac{1}{|a_0|}\Bigg(M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha \\ &+2k|a_\eta|(\sin\alpha + \cos\alpha + 1) - 2|a_\eta| + 2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha \\ &+\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Bigg).\end{aligned}$$

Therefore all zeros of $H(q)$ and hence of $S(q)$ lie in

$$\begin{aligned}|q| &\leq \max\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha \\ &+2k|a_\eta|(\sin\alpha + \cos\alpha + 1) - 2|a_\eta| + 2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha \\ &+\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + |a_p| + M_p + |a_n|\Bigg)\Bigg\}.\end{aligned}$$

Therefore all zeros of $P(q)$ lie in

$$\begin{aligned}|q| &\geq \min\Bigg\{1, |a_0|\Bigg/\Bigg(M_q + |a_Q| + \rho_1|a_Q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha \\ &+2k|a_\eta|(\sin\alpha + \cos\alpha + 1) - 2|a_\eta| + 2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha\end{aligned}$$

$$+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p+|a_n|\Bigg)\Bigg\}.$$

Thus the polynomial $P(q)$ does not vanish in

$$\begin{aligned}|q| \;<\; \min\Bigg\{1, |a_0|\Bigg/\Bigg(&M_q+|a_Q|+\rho_1|a_Q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=Q+1}^{\eta-1}|a_\ell|\sin\alpha\\ &+2k|a_\eta|(\sin\alpha+\cos\alpha+1)-2|a_j|+2\sum_{\ell=\eta+1}^{p-1}|a_\ell|\sin\alpha\\ &+\rho_2|a_p|(\sin\alpha-\cos\alpha-1)+|a_p|+M_p+|a_n|\Bigg)\Bigg\}.\end{aligned}$$

□

Notice when under similar conditions as before Theorem 6.15 reduces to Theorem 4.7 and therefore given the right parameters this will also reduce to Theorem 1.13. Similar to what we have discussed previously in this thesis, lacunary quaternionic polynomials with similar parameters applied to the quaternionic coefficients will produce the similar bound with the exception of $M_q = \sum_{\ell=m}^{Q}|a_\ell - a_{\ell-1}|$. Similarly for quaternionic dual gap polynomials the bound will be similar with the new M_q and $M_p = \sum_{\ell=p+1}^{m'}|a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

Quaternionic polynomials with quaternionic coefficients are still in the earlier stages of being studied. This allows us to further what we already know as well as to continue discovering new ideas within this field. The ability to have a reversal applied to the monotone behavior of the coefficients allows us to manipulate this turning point. We are able to shift this point of reversal to any value between q and p. With this ability, larger sets of polynomials may be explored to set bounds on the locations of zeros. While we are unable to explore the number of zeros for quaternionic polynomials at this time, as the quaternionic field continues to be studied we may eventually be able use the hypotheses found within this chapter to count the number of zeros within a bound for these types of polynomial.

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