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Partially Oriented 6-star Decomposition of Some Complete Mixed Graphs

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by

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ABSTRACT

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Let M_v denotes a complete mixed graph on v vertices, and let S_6^i denotes the partial orientation of the 6-star with twice as many arcs as edges. In this work, we state and prove the necessary and sufficient conditions for the existence of λ-fold decomposition of a complete mixed graph into S_6^i for $i \in \{1, 2, 3, 4\}$. We used the difference method for our proof in some cases. We also give some general sufficient conditions for the existence of S_6^i-decomposition of the complete bipartite mixed graph for $i \in \{1, 2, 3, 4\}$. Finally, this work introduces the decomposition of a complete mixed graph with a hole into mixed stars.

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1 INTRODUCTION AND BASIC DEFINITIONS

Graph decompositions rank among the most prominent areas of graph theory and combinatorics. Results on graph decompositions can be applied in coding theory, design of experiments, X-ray crystallography, radioastronomy, radiolocation, computer and communication networks, serology, and other fields [4].

We give a fairly comprehensive list of definitions for a better understanding of this thesis and we follow the definitions and notations of [3].

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an incidence function that associates with each edge of G an unordered pair of vertices of G. The graph G is *directed (undirected)* if all its edges are directed (undirected). A *digraph* D is a graph with directed edges called *arc*.

A *mixed graph* M on v vertices is an ordered triple $(V(M), E(M), A(M))$ where $|V(M)| = v$, $E(M)$ is a set of unordered $[x, y]$ pairs of elements of $V(M)$, and $A(M)$ is a set of ordered pairs (x, y) of elements of $V(M)$. The ordered pair (x, y) is called an *arc* and the unordered pair $[x, y]$ is called an *edge*.

The *complete mixed graph* on v vertices denoted by M_v is the mixed graph in which for every two distinct vertices x and y, the arc set contains $(x, y), (y, x)$ and the edge sets contains $[x, y]$.

The *degree* of a vertex u of G, denoted by $\deg(u)$ is the number of edges incident to u in G. In a mixed graph, the *outdegree* of a vertex u of G, denoted by $\mathrm{od}(u)$, is the number of arcs emanating from u. The *indegree*, denoted by $\mathrm{id}(u)$, is the number of arcs terminating at u. The *total degree* of a vertex u in a mixed graph M_v is the summation: $od(u) + id(u) + deg(u)$.

A graph G is *bipartite* if its vertex set can be partitioned into subsets X and Y such that every edge in G has one end in X and the other end in Y. That is if $V(G) = X \cup Y$ and $[x, y] \in E(G)$ then $x \in X$ and $y \in Y$, $X \neq \emptyset$, $Y \neq \emptyset$ and $X \cap Y = \emptyset$. A bipartite graph G with bipartition (X, Y) denoted $G[X, Y]$ is called a *complete bipartite graph* if $G[X, Y]$ is simple and every vertex in X is adjacent to every vertex in Y.

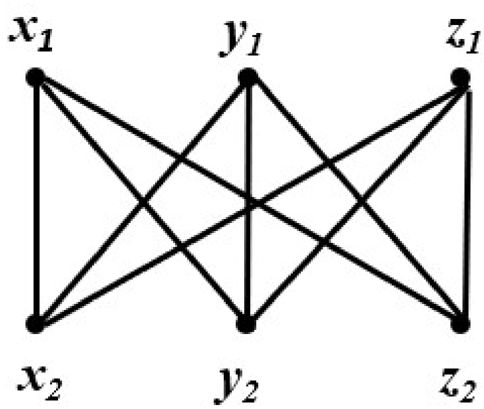

Figure 1: A complete bipartite graph

The mixed graph with vertex set V such that for every pair of distinct vertices $x \in X$ and $y \in Y$, where $V = X \cup Y$, the set of arcs contains (x, y) and (y, x), and the set of edges contains $[x, y]$ is called a *complete bipartite mixed graph*, Figure 2.

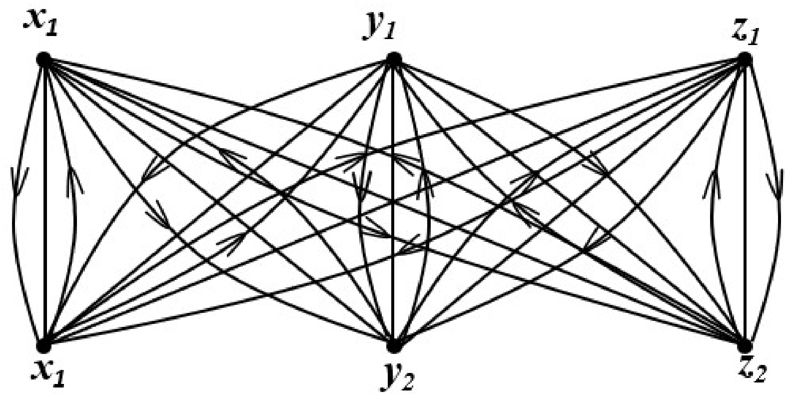

Figure 2: A complete bipartite mixed graph

For each natural number λ, the λ-*fold complete mixed graph* on V-vertices, denoted by λM_v, is the mixed multigraph where, for each pair of distinct vertices v_1 and v_2 in G, we have λ copies of (v_1, v_2), (v_2, v_1) and $[v_1, v_2]$.

The *complete mixed graph on v vertices with a hole of size w* is the graph with the vertex set $V(M_{v,w}) = V_{v-w} \cup V_w$ where $|V_{v-w}| = v - w$ and $|V_w| = w$, edge set $E(M_{v,w}) = \{[a, b]$ such that $a, b \in V(M_{v,w}), \{a, b\} \nsubseteq V_w\}$ and arc set $A(M_{v,w}) = \{(a, b), (b, a)$ such that $a, b \in V(M_{v,w})$,{a,b}$\notin V_w\}$. For a graph G, replacing each edge $[v_1v_2] \in E(G)$ with either (v_1, v_2), (v_2, v_1) or $[v_1, v_2]$ is known as *partial orientation of G*.

A decomposition of a graph G is a family $\mathcal{F}$ of edge-disjoint subgraphs of G such that $\cup_{F\in\mathcal{F}} E(\mathcal{F}) = E(G)$. Given that $E(G)$ is the edge set of G and $V(G)$ is the vertex set of G, then a decomposition of a simple graph G into isomorphic copies of a graph H is a set $\{H_1, H_2, ..., H_n\}$ where $H_i \cong H$ and $V(H_i) \subset V(G)$ for all i, $E(H_i) \cap E(H_i) = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} E(H_i) = E(G)$, where $H_i's$ are the blocks consisting of the subgraphs. Similarly, a decomposition of a digraph D is a family $\mathcal{F}$ of arc-disjoint subgraph of D such that $\cup_{F \in \mathcal{F}} A(\mathcal{F}) = A(G)$. Given that $A(G)$ is the arc set of D and $V(D)$ is the vertex set of D, then a decomposition of a simple digraph into isomorphic copies of a graph H is a set $\{H_1, H_2, \ldots, H_n\}$ where $H_i \cong g$ and $V(H_i) \subset V(D)$ for all i, $A(H_i) \cap A(H_i) = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} A(H_i) = A(D)$, where $H_i's$ are the blocks consisting of the subgraphs. That is, a subgraph in a decomposition is called *block*.

A Steiner triple system of order n is an isomorphic decomposition of a complete graph $G = K_n$ into family $\mathcal{F}$ of subgraphs of G such that each $F \in \mathcal{F}$ is isomorphic to a 3-cycle [12].

For example, an isomorphic decomposition of K_7 into 3-cycles is given in Figure 3.

An *m-star* is a complete bipartite graph $K_{1,m}$. So a decomposition of a graph into stars is a way of expressing the graph as the union of edge-disjoint stars.

Pauline Cain [7] showed that the complete graphs on rm and $rm + 1$ vertices, $r > 1$, can be decomposed into stars with m edges if and only if r is even or m is odd.

An (s,t)-*directed star* (Figure 4) is a directed graph with $s + t + 1$ vertices and $(s + t)$ arcs; s vertices have indegree 0 and outdegree 1, t vertices have indegree 1 and outdegree 0, and one has indegree s and outdegree t. So an (s,t)-directed star decomposition is a partition of the arcs of a complete directed graph of order n into (s,t) directed stars.

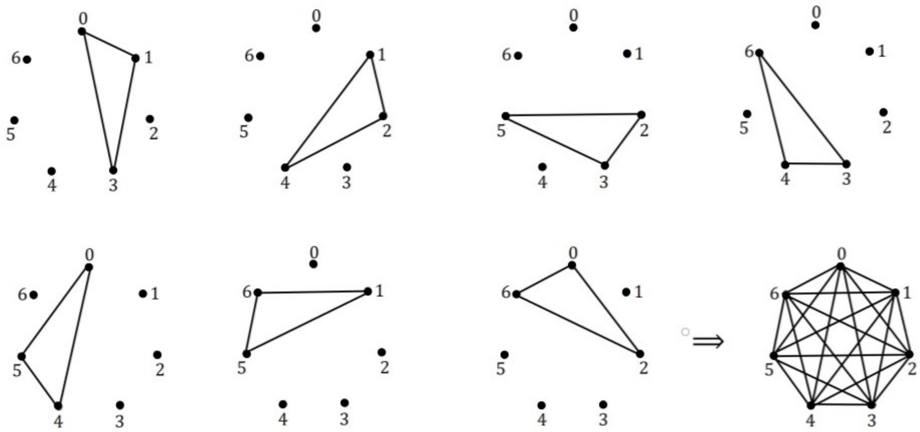

Figure 3: A decomposition of K_7 into 3-cycles

C.J. Colbourn [9] established the necessary and sufficient conditions on s, t, and n for an (s, t)-directed decomposition of the complete graph of order n to exist.

Robert Gardner [12] first addressed the decomposition of complete mixed graphs in the mixed triple systems setting where the necessary and sufficient conditions for the existence of some new triple systems were given. That is, a T_i-triple system (Figure 5) of order v exists for $i = 1, 2, 3$ if and only if $v \equiv 1 \pmod 2$, except for $i = 3$ and $v = 3, 5$.

Robert Beeler and Adam Meadows [2] gave necessary and sufficient conditions for a decomposition of the λ-fold complete mixed graph into partial orientations of P_4 and S_3, where P_4 is the path on four vertices and S_3 is the star on four vertices.

The necessary and sufficient conditions for isomorphic decompositions of the complete mixed graph into mixed stars on 7 vertices was given by Chancé Culver and Robert Gardner [10]; that is, the decomposition of M_v into copies of partial orienta-

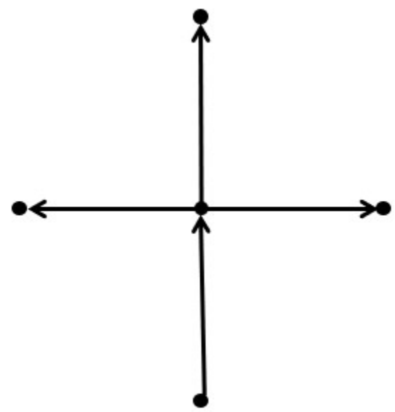

Figure 4: A (1,3)-directed star

tions of 6 stars which have two edges and four arcs. See Figure 6.

The following result was given in [10]:

Theorem 1.1 *An S_6^i-decomposition of M_v exists if and only if $v \geq 9$ and*

1. *if $i \in \{0, 4\}$, then $v \equiv 1$ (mod 4), $v \geq 9$ and*

2. *if $i \in \{1, 2, 3\}$, then $v \equiv 0$ or 1 (mod 4), $v \geq 9$.*

We are inspired by the above results to study the λ-fold decomposition of the complete mixed graphs into mixed stars. Combined with Theorem 1.1, we give the necessary and sufficient conditions for such decompositions. We also explore the decomposition of complete bipartite mixed graphs into mixed stars and also the de-

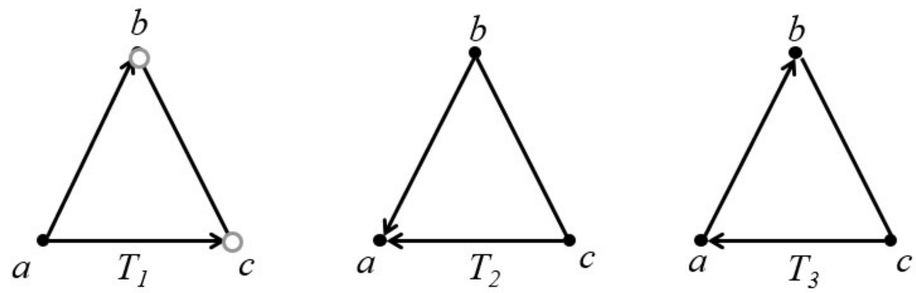

Figure 5: Mixed triples.

composition of $M_{v,h}$, complete mixed graph with v vertices and a hole of size h into mixed stars.

A complete mixed graph (and a λ-fold complete mixed graph) has twice as many arcs as edges. So any isomorphic decomposition of a complete mixed graph (or a complete λ-fold mixed graph) must involve a graph with twice as many arcs as edges.

Figure 6 shows the partial orientations of the 6-star which has two edges and four arcs where the two edges are $[a, b]$ and $[a, c]$ and the arcs are $(ad), (ag), (af)$ and (ae). Vertex a is the *center* vertex. We use the same notation S_6^i as used in [10] for each orientation where i is the indegree and $4 - i$ is the outdegree. So a S_6^2 mixed graph is a star with the center having edge degree 2, indegree 2, and outdegree 2.

We shall explore the difference method in some cases to show that all the arc and edge differences are present and we also show that all vertex labels are distinct. We also give some general sufficient conditions for the existence of S_6^i-decomposition of the complete bipartite mixed graph for $i = \{1, 2, 3, 4\}$ and then conclude this work

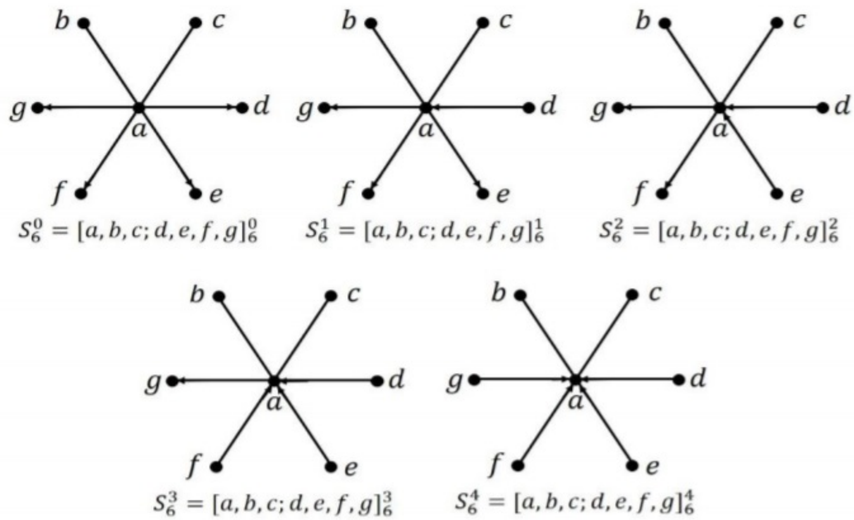

Figure 6: Partial orientations of 6-stars with two edges and four arcs

with an introduction of S_6^i-decompositions of a complete mixed graph with a hole.

2 λ-FOLD DECOMPOSITION OF COMPLETE MIXED GRAPH INTO MIXED STARS

2.1 Introduction

There are five partial orientations of 6-stars as shown in the Figure 6, which have two edges and four arcs. These are considered because there are twice as many arcs as edges in a complete mixed graph. The S_6^i block with vertex set $\{a, b, c, d, e, f, g\}$ will be denoted by $[a, b, c; d, e, f, g]_6^i$, as illustrated in Figure 6. The center of the star has indegree i and outdegree $4 - i$. So an S_6^1 is a star with the center having edge degree 2, indegree 1, and outdegree 3.

There are many results concerning the necessary and sufficient conditions for the existence of the decomposition of a graph into isomorphic subgraphs [4, 11]. In the case of this thesis, the necessary and sufficient conditions for the decomposition of a complete mixed graph into partial orientations of 6-stars were given for $\lambda = 1$ in [10]. We shall explore the use of the difference method in the construction of the decomposition of mixed graph into mixed stars in certain cases.

Consider a simple complete graph. Suppose we want to decompose K_n into $K_n = C_3$, that is, we want a collection of copies of K_3 which are edge disjoint and union to give K_n. For example, a K_3-decomposition of K_7 with the vertex set $\{0, 1, 2, 3, 4, 5, 6\}$ is given by: $[0, 1, 3], [1, 2, 4], [2, 3, 5], [3, 4, 6], [4, 5, 0], [5, 6, 1], [6, 0, 2]$ (see Figure 3). With the edge $(0, 1)$, we associate the difference $1 - 0 = 1$, with edge $(1, 3)$, we associate the difference $3 - 1 = 2$, and with the edge $(0, 3)$, we associate the difference $3 - 0 = 3$. In general, the difference associated with the edge (x, y) in K_n

with vertex set $\{0, 1, 2, ..., n-1\}$ is $|x-y|_n = min\{(x-y) \pmod{n}, (y-x) \pmod{n}\}$. The set of differences for edges of K_n is $\{1, 2, ..., \lfloor n/2 \rfloor\}$. Note that in a complete directed graph on n vertices labeled $(0, 1, 2, ..., n-1)$, we associate with arc (a, b) the difference $(b-a) \pmod{n}$. The set of arc differences is $\{1, 2, ..., n-1\}$. Under the permutation $\pi : V \to V$ defined as $\pi(i) = i + 1 \pmod{n}$, all the edges (arcs) with a given difference are in the same orbit.

We now give the necessary and sufficient conditions for the existence of a λ-fold decomposition of the complete mixed graph into mixed stars. In each case, we give a direct construction to establish sufficiency.

2.2 Necessary Conditions Lemmas

An S_6^i-decomposition of M_v does not exist when $v \equiv 3 \pmod{4}$, for $i \in \{1, 2, 3\}$ and also, a S_6^i-decomposition of M_8 for $i \in \{1, 2, 3\}$ does not exist [10]

In this subsection, we give some necessary conditions for the existence of the decomposition of a λ-fold complete graph into various mixed stars.

Lemma 2.1 *For λ odd, if an S_6^i-decomposition of λM_v exists then $v \equiv 0$ or 1 (mod 4).*

Proof. We show that if $v \equiv 2$ or $3 \pmod{4}$ and λ is odd, then an S_6^i-decomposition of λM_v does not exist.

Let $v \equiv 2 \pmod{4}$, say $v = 4k+2$. Then λM_v has

$$\lambda\binom{v}{2} = \lambda\frac{v(v-1)}{2} = \lambda\frac{(4k+2)(4k+1)}{2} = \lambda(2k+1)(4k+1)$$

edges. But then λM_v has an odd number of edges and S_6^i has 2 edges, so no S_6^i-decomposition of λM_v exists.

Let $v \equiv 3 \pmod 4$, say $v = 4k+3$. Then λM_v has

$$\lambda\binom{v}{2} = \lambda\frac{v(v-1)}{2} = \lambda\frac{(4k+3)(4k+2)}{2} = \lambda(4k+3)(2k+1)$$

edges. But then λM_v has an odd number of edges and S_6^i has 2 edges, so no S_6^i-decomposition of λM_v exists. $\square$

Lemma 2.2 *If an S_6^0-decomposition of λM_v exists, then $\lambda(v-1) \equiv 0 \pmod 4$.*

Proof. Each vertex of λM_v has out-degree $\lambda(v-1)$ and each vertex of S_6^0 has out-degree 0 (mod 4). So if an S_6^0-decomposition of λM_v exists, then we must have $\lambda(v-1) \equiv 0 \pmod 4$, as claimed. $\square$

2.3 An S_6^0-Decomposition of λM_v

In this subsection, we give the necessary and sufficient conditions for an S_6^0-decomposition of λM_v. We verify the result using the difference method and then conclude the subsection with an example.

Lemma 2.3 *An S_6^0-decomposition of λM_v, where $\lambda = 4$, exists for $v \geq 8$.*

Proof. Let $v = 4k$ where $k \geq 2$. Let $(\lambda M_v) = \{0, 1, 2, \ldots v-1\}$. Consider the blocks:

$\{2 \times [0, 2k-2, 2k-1; 2k, 4k-3, 4k-2, 4k-1]_6^0, 2 \times [0, 2k+1, 4k-1; 1, 2, 3, 2k]_6^0\}$

$\cup\{[0, 4k-1, 2k; 1, 2, 4k-3, 4k-2]_6^0, [0, 2k+2, 2k; 1, 2, 3, 4k-1]_6^0, [0, 1, 2k-2; 3, 4k-3, 4k-2, 4k-1]_6^0\} \cup\{4 \times [0, 2+2i, 3+2i; 4+2i, 5+2i, 2k+1+2i, 2k+2+2i]_6^0$ for $i =$

$0, 1, \ldots, k-3\}$. These stars along with their images under the permutation $\pi : V \rightarrow V$ defined as $\pi(i) = i + 1 \pmod{v}$, form an S_6^0-decomposition of λM_v where $\lambda = 4$ and $v \geq 6$, as claimed. $\square$

Lemma 2.4 *An S_6^0-decomposition of λM_v, where $\lambda = 4$, exists for $v \equiv 2$ (mod 4) and $v \geq 8$.*

Proof. Let $v = 4k + 2$ where $k \geq 1$. Let $(\lambda M_v) = \{0, 1, 2, \ldots v - 1\}$, consider the blocks:

$$\{2 \times [0, 1, 2k-2; 4k-2, 4k-1, 4k, 4k+1]_6^0, 2 \times [0, 2k-1, 2k+2; 1, 2, 4, 2k+1]_6^0\}$$

$$\cup\{[0, 1, 2k+3; 2, 3, 4k, 4k+1]_6^0, [0, 2k-2, 2k; 3, 2k+1, 4k-2, 4k]_6^0,$$

$$[0, 2k+1, 2k+3; 1, 2, 3, 4k-2]_6^0, [0, 1, 2k-2; 3, 4, 2k+1, 4k-1]_6^0,$$

$$[0, 2k+1, 2k+2; 1, 4, 4k-1, 4k+1]_6^0\}$$

$\cup\{4 \times [0, 2+2i, 3+2i; 5+2i, 6+2i, 2k+2+2i, 2k+3+2i]_6^0$ for $i = 0, 1, \ldots, k-3\}$.

These stars along with their images under the permutation $\pi : V \rightarrow V$ defined as $\pi(i) = i + 1 \pmod{v}$, form an S_6^0-decomposition of λM_v where $\lambda = 2$ and $v \geq 6$, as claimed. $\square$

Lemma 2.5 *A S_6^0-decomposition of λM_v exists for $v \equiv 3$ (mod 4) for $\lambda = 2$.*

Proof. Let $v = 4k + 3$ where $k \geq 1$ and $\lambda = 2$. Let $(\lambda M_v) = \{0, 1, 2, \ldots v - 1\}$, consider the blocks:

$B = [0, 4k+2, 4k+1; 1, 2, 3, 4]_6^0$,

$[0, 2k+1, 2k+2; 1, 2, 4k+1, 4k+2]_6^0$,

$[0, 1+2j, 2+2j; 3+4j, 4+4j, 5+4j, 6+4j]_6^0$, $j = 0, 1, 2, ..., k-1$,

$[0, 3+2j, 4+2j; 5+4j, 6+4j, 7+4j, 8+4j]_6^0$, $j = 0, 1, 2, ..., k-2$.

These stars along with their images under the permutation $\pi : V \to V$ defined as $\pi(i) = i + 1 \pmod{v}$, form an S_6^0-decomposition of λM_v where $\lambda = 2$ and $v \geq 6$, as claimed. □

Theorem 2.6 *An S_6^0-decomposition of λM_v exists if and only if $v \geq 7$ and*

1. $v \equiv 0$ *(mod 2) and* $\lambda \equiv 0$ *(mod 4), or*

2. $v \equiv 1$ *(mod 4) and* $\lambda \geq 1$*, or*

3. $v \equiv 3$ *(mod 4) and* $\lambda \equiv 0$ *(mod 2).*

Proof. For $v \equiv 0 \pmod 2$, we have $v - 1$ is odd and by Lemma 2.2 a necessary condition for an S_6^0-decomposition of λM_v is $\lambda(v-1) \equiv 0 \pmod 4$. So for $v \equiv 0$ (mod 2), $\lambda = 0 \pmod 4$ is necessary. For $v \equiv 3 \pmod 4$, we have $v - 1 \equiv 2 \pmod 4$ and by Lemma 2.2 a necessary condition for an S_6^0-decomposition of λM_v is $\lambda(v-1) \equiv 0$ (mod 4). So for $v \equiv 3 \pmod 4$, $\lambda = 0 \pmod 2$ is necessary.

For sufficiency, when $v \equiv 0 \pmod 4$ and $\lambda = 4$, an S_6^0-decomposition of $4M_v$ exists by Lemma 2.3. So when $\lambda \equiv 0 \pmod 4$, by taking $\lambda/4$ copies of the blocks of such a decomposition gives a decomposition of λM_v. When $v \equiv 1 \pmod 4$, an S_6^0-decomposition of M_v exists by [10]. So when $\lambda \geq 1$, taking λ copies of the blocks of such a decomposition gives a decomposition of λM_v. When $v \equiv 2 \pmod 4$ and $\lambda = 4$, an S_6^0-decomposition of $4M_v$ exists by Lemma 2.4. So when $\lambda \equiv 0$ (mod

4), taking $\lambda/4$ copies of the blocks of such a decomposition gives a decomposition of λM_v. When $v \equiv 3 \pmod 4$ and $\lambda = 2$, an S_6^0-decomposition of $2M_v$ exists by Lemma 2.5. So when $\lambda \equiv 0 \pmod 2$, taking $\lambda/2$ copies of such a decomposition gives a decomposition of λM_v. $\square$

Notice that the converse of S_6^0, obtained by reversing the orientation of all the arcs, is S_6^4. Since M_v is self converse, Theorem 2.6 also gives the necessary and sufficient conditions for an S_6^4-decomposition of λM_v where $\lambda = 2$.

Theorem 2.7 *An S_6^4-decomposition of λM_v exists if and only if $v \geq 7$ and*

1. $v \equiv 0$ *(mod 2) and* $\lambda \equiv 0$ *(mod 4), or*

2. $v \equiv 1$ *(mod 4) and* $\lambda \geq 1$, *or*

3. $v \equiv 3$ *(mod 4) and* $\lambda \equiv 0$ *(mod 2).*

2.3.1 Verification and Example

We use the difference method in the previous section in details to show how we came about Lemma 2.5 and then check that all the edges and arcs are repeated twice. For $v = 4k+3$ in terms of k, we have $2k+1$ blocks. Recall that, the set of differences for edges of K_n is $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$ and the set of arc differences is $\{1, 2, ..., n-1\}$. So for $v = 4k+3$ and $\lambda = 2$ we have the following multisets of edge and arc differences:

Edge differences: $\{1, 2, \ldots, 2k+1\}$ $(\times 2)$

Arc differences: $\{1, 2, \ldots, 4k+2\}(\times 2)$.

Now we check if all the edges and arcs are repeated twice (Table 1).

Table 1: The Edge and Arc Differences of Lemma 2.5

Blocks	Edge differences	Arc Differences
$[0, 4k+2, 4k+1; 1, 2, 3, 4]^0_6$	$1, 2$	$1, 2, 3, 4$
$[0, 2k+1, 2k+2; 1, 2, 4k+1, 4k+2]^0_6$	$2k+1, 2k+1$	$1, 2, 4k+1, 4k+2$
$[0, 1+2j, 2+2j; 3+4j, 4+4j,$ $5+4j\, 6+4j]^0_6$,$j = 0, 1, 2, ..., k-1$	$1+2j : 1, 3, 5, 7, \ldots, 2k-1$ $2+2j : 2, 4, 6, 8, \ldots, 2k$	$3+4j : 3, 7, 11, 15, \ldots, 4k-1$ $4+4j : 4, 8, 12, 16, \ldots, 4k$ $5+4j : 5, 9, 13, 17, \ldots, 4k+1$ $6+4j : 6, 10, 14, 18, \ldots, 4k+2$
$[0, 3+2j, 4+2j; 5+4j, 6+4j,$ $7+4j, 8+4j]^0_6$, $j = 0, 1, 2, ..., k-2$	$3+2j : 3, 5, 7, 9, \ldots, 2k-1$ $4+2j : 4, 6, 8, 10, \ldots, 2k$	$5+4j : 5, 9, 13, 17, \ldots, 4k-3$ $6+4j : 6, 10, 14, 18, \ldots, 4k-2$ $7+4j : 7, 11, 15, 19, \ldots, 4k-1$ $8+4j : 8, 12, 16, 20, \ldots, 4k$

Since the arc and edge differences are repeated twice, then the permutation $\pi(i) = i+1 \pmod{v}$ produces all stars in the decomposition.

For verification purposes, consider for example, $v = 11$ and $\lambda = 2$. We have:

Edge differences:$\{1, 2, 3, 4, 5\ \}\times 2$

Arc differences: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\ \times 2$.

We obtain the following blocks: $[0, 10, 9; 1, 2, 3, 4]^0_6$, $[0, 5, 6; 1, 2, 9, 10]^0_6$, $[0, 1, 2; 3, 4, 5, 6]^0_6$, $[0, 3, 4; 7, 8, 9, 10]^0_6$, $[0, 3, 4; 5, 6, 7, 8]^0_6$. The arc and edge differences generated by these blocks are shown in Table 2.

Table 2: The Edge and Arc Differences for an S^0_6-decomposition of $2M_{11}$

Blocks	Edge differences	Arc Differences
$[0, 10, 9; 1, 2, 3, 4]^0_6$	1, 2	1, 2, 3, 4
$[0, 5, 6; 1, 2, 9, 10]^0_6$	5, 5	1, 2, 9, 10
$[0, 1, 2; 3, 4, 5, 6]^0_6$	1, 2	3, 4, 5, 6
$[0, 3, 4; 7, 8, 9, 10]^0_6$	3, 4	7, 8, 9, 10
$[0, 3, 4; 5, 6, 7, 8]^0_6$	3, 4	5, 6, 7, 8

When put together, it can be seen that all the edge and arc differences $\{1,2,3,4,5\}$ $(\times 2)$ and $\{1,2,3,4,5,6,7,8,9,10\}$ $(\times 2)$ are repeated twice. Hence, the result.

2.4 An S_6^1-Decomposition of λM_v

A S_6^1-decomposition of M_v exists if and only if $v \equiv 0$ or $1 \pmod 4$ and $v \geq 9$ [10]. In this subsection, we give the necessary and sufficient conditions for the existence of a S_6^2-decomposition of λM_v, where $\lambda = 2$.

Lemma 2.8 *An S_6^1-decomposition of λM_v exists for $v \equiv 3$ (mod 8) and $\lambda = 2$.*

Proof. Let $v = 8k+3$ and $k \geq 1$. Let $(\lambda M_v) = \{0,1,2,\ldots v-1\}$, consider the following set of blocks:

$$\{2\times[0,8k+2-2j,8k+1-2j;8k-1-4j,1+4j,2+4j,3+4j]_6^1 \mid j=0,1,\ldots,k-1\}$$

$$\cup\{[0,6k+2-2j,2k+2+2j;4k-1-4j,4k+1+4j,4k+2+4j,4k+3+4j]_6^1$$

$$\mid j=0,1,\ldots,k-2,\ \text{and } j\neq k/3 \text{ if } k\equiv 0 \pmod 3\}$$

$$\cup\{[0,8k/3+1,8k/3+2;8k/3-1,16k/3+1,16k/3+2,16k/3+3]_6^1$$

$$\text{if } k\equiv 0 \pmod 3 \text{ and } j=k/3\}.$$

$$\cup\{[0,2k+1+2j,2k+2+2j;4k-3-4j,4k+3+4j,4k+4+4j,4k+5+4j]_6^1$$

$$\mid j=0,1,\ldots,k-1,\ \text{and } j\neq (k-2)/3 \text{ if } k\equiv 2 \pmod 3\}$$

$$\cup\{[0,(16k+10)/3,(8k+2)/3;(8k-1)/3,(16k+1)/3,(16k+4)/3,(16k+7)/3]_6^1$$

$$\text{if } k\equiv 2 \pmod 3 \text{ and } j=(k-2)/3\}.$$

$\cup\{[0,4k+1,4k+2;3,8k-3,8k-2,8k-1]_6^1,[0,4k-1,4k;1,4k+1,4k+2,8k+1]_6^1\}$.

These stars along with their images under the permutation $\pi : V \to V$ defined as $\pi(i) = i + 1 \pmod{v}$, form an S_6^1-decomposition of λM_v where $\lambda = 2$ and $v \geq 6$, as claimed. □

Lemma 2.9 *A S_6^1-decomposition of λM_v exists for $v \equiv 6 \pmod 8$, $v \geq 14$ and $\lambda = 2$.*

Proof. let $v = 8k+6$ and $k \geq 1$. Let $(\lambda M_v) = \{0,1,2,\ldots v-2,\infty\}$. The required decomposition is given by the set of blocks:

$[0,\infty,v-2;v-5,1,2,3]$ $\times 2$, $[0,2,3;\infty,5,6,7]$ $\times 2$, $[0,4,5;v-11,8,9,\infty]$, $[0,5,6;v-11,8,12,\infty]$, $[0,4,6;v-12,9,11,12]$,

and

$\{[0,7+4j,v-8-4j;8+8j,16+8j,13+8j,17+8j],[0,8+4j,v-9-4j;7+8j,19+8j,14+8j,18+8j],[0,9+4j,v-10-4j;6+8j,15+8j,18+8j,19+8j],[0,10+4j,v-11-4j;1+8j,16+8j,17+8j,20+8j]|j=0,1,2,\ldots,k-2\}$.

These stars along with their images under the permutation $\pi : V \to V$ defined as $\pi(i) = i + 1 \pmod{v-1}$ if $i \in \{0,1,\ldots v-2\}$, and $\pi(i) = \infty$ if $i = \infty$, form an S_6^0-decomposition of λM_v where $\lambda = 2$ and $v \geq 14$, as claimed. □

Lemma 2.10 *A S_6^1-decomposition of $2M_v$ exists for all $v \equiv 3 \pmod 4$, with $v \geq 7$*

Proof. let $v = 4k+3$ and $k \geq 1$. Let $(\lambda M_v) = \{0,1,2,\ldots v-1\}$, consider the blocks:

$\{[0,4k+2,4k+1;4,1,2,3]$,

$[0,2k+1,2k+2;4k-1,2,4k+1,4k+2]$,

$[0, 1, 2; 4k, 4k + 2, 5, 6]\}$

$$\cup\{[0, 3 + 2j, 4 + 2j; 4k - 4 - 4j, 8 + 4j, 9 + 4j, 10 + 4j] \mid j = 0, 1, \ldots, k - 2,$$

$$j \neq (2k - 4)/3 \text{ when } j \equiv 2 \pmod 3\}, \text{and} j \neq (k - 3)/2 \text{ when } j \equiv 1 \pmod 2\}$$

$$\cup\{[0, \frac{4k+1}{3}, \frac{8k+5}{3}; \frac{4k+4}{3}, \frac{8k+8}{3}, \frac{8k+11}{3}, \frac{8k+14}{3}]$$

$$\text{if } j \equiv 2 \pmod 3, \text{ and } j = (2k - 4)/3\}$$

$$\cup\{[0, k, k + 1; 2k, 2k + 2, 2k + 1, 2k + 4] \text{ if } j \equiv 1 \pmod 2, \text{ and } j = (k - 3)/2\}$$

$$\cup\{[0, 3 + 2j, 4 + 2j; 4k - 2 - 4j, 6 + 4j, 7 + 4j, 8 + 4j] \mid j = 0, 1, \ldots, k - 2,$$

$$\text{and } j \neq (2k - 3)/3 \text{ when } j \equiv 0 \pmod 3\}, \text{and} j \neq (k - 2)/2 \text{ when } j \equiv 0 \pmod 2\}$$

$$\cup\{[0, \frac{4k+3}{3}, \frac{8k+3}{3}; \frac{4k+6}{3}, \frac{8k+6}{3}, \frac{8k+9}{3}, \frac{8k+12}{3}]$$

$$\text{if } j \equiv 0 \pmod 3, \text{ and } j = (2k - 3)/3\}$$

$$\cup\{[0, k + 1, k + 2; 2k, 2k + 2, 2k + 1, 2k + 4] \text{ if } j \equiv 0 \pmod 2, \text{ and } j = (k - 2)/2\}$$

This collection of stars along with their images under the permutation $\pi(i) = i + 1 \pmod v$, form a S_6^1-decomposition of $2M_v$ where $v = 4k + 3$. □

Lemma 2.11 *An S_6^1-decomposition of $2M_v$ exists for all $v \equiv 2 \pmod 8$, with $v \geq 26$.*

Proof. let $v = 8k + 2$ and $k \geq 3$. Let $(\lambda M_v) = \{0, 1, 2, \ldots v - 1\}$. The required decomposition is given by the set of blocks:

$\{[0, \infty, v - 2; v - 5, 1, 2, 3] \times 2$, $[0, 2, 3; \infty, 5, 6, 7] \times 2$, $[0, 4, 5; v - 11, 8, 9, \infty]$, $[0, 5, 6; v - 11, 8, 12, \infty]$, $[0, 4, 6; v - 12, 9, 11, 12]$, $[0, 7, v - 8; v - 14, 13, 17, 21]$, $[0, 8, v - 9; v -$

$15, 14, 15, 22]$, $[0, 9, v-10; v-19, 18, 19, 22]$, $[0, 10, v-11; v-21, 17, 20, 24]$, $[0, 11, v-12, v-17, 16, 21, 24]$, $[0, 12, v-13; v-24, 15, 19, 23]\}$

and

$\{[0, 13+4j, v-14-4j; 7+8j, 31+8j, 26+8j, 30+8j], [0, 14+4j, v-15-4j; 8+8j, 28+8j, 25+8j, 29+8j], [0, 15+4j, v-16-4j; 1+8j, 28+8j, 29+8j, 32+8j], [0, 16+4j, v-17-4j; 6+8j, 27+8j, 30+8j, 31+8j] | j = 0, 1, 2, \ldots, k-4\}$.

These stars along with their images under the permutation $\pi : V \to V$ defined as $\pi(i) = i+1 \pmod{v}$, form an S_6^1-decomposition of λM_v where $\lambda = 4$ and $v \geq 26$, as claimed. $\square$

Theorem 2.12 *An S_6^1-decomposition of λM_v exists if and only if*

1. $v \equiv 0$ *or* 1 *(mod* $4)$ *and* $\lambda \geq 1$*, or*

2. $v \equiv 2$ *(mod* $4)$ *and* $\lambda \equiv 0$ *(mod* $2)$*, or*

3. $v \equiv 3$ *(mod* $4)$ *and* $\lambda \equiv 0$ *(mod* $2)$*.*

Proof. For $v = 2$ or $3 \pmod 4$, we have $v - 1 = 1$ or $2 \pmod 4$. Since λM_v has an odd number of edges and S_6^1 has 2 edges. Then by Lemma 2.1, $\lambda = 0 \pmod 2$ is necessary.

For sufficiency, when $v \equiv 0$ or $1 \pmod 4$, an S_6^1-decomposition of M_v exists by [10]. So when $\lambda \geq 1$, taking λ copies of the blocks of such a decomposition gives a decomposition of λM_v. When $v \equiv 2 \pmod 4$ and $\lambda = 2$, an S_6^1-decomposition of $2M_v$ exists by Lemma 2.9 and Lemma 2.11. So when $\lambda \equiv 0 \pmod 2$, taking $\lambda/2$ copies of the blocks of such a decomposition gives a decomposition of λM_v. When $v \equiv 3$ (mod

4) and $\lambda = 2$, an S_6^1-decomposition of $2M_v$ exists by Lemma 2.10. So when $\lambda \equiv 0$ (mod 2), taking $\lambda/2$ copies of such a decomposition gives a decomposition of λM_v. $\square$

The converse of S_6^1, obtained by reversing the orientation of all the arcs, is S_6^3. Since M_v is self converse, Theorem 2.12 also gives the necessary and sufficient conditions for S_6^3-decomposition of λM_v where $\lambda = 2$.

Theorem 2.13 *An S_6^3-decomposition of λM_v exists if and only if*

1. $v \equiv 0$ *or* 1 $(mod\ 4)$ *and* $\lambda \geq 1$, *or*

2. $v \equiv 2$ $(mod\ 4)$ *and* $\lambda \equiv 0$ $(mod\ 2)$, *or*

3. $v \equiv 3$ $(mod\ 4)$ *and* $\lambda \equiv 0$ $(mod\ 2)$.

2.4.1 Verification and Example

We first show that all vertex labels are distinct in the Lemma 2.8. The blocks

$$\{2 \times [0, 8k+2-2j, 8k+1-2j; 8k-1-4j, 1+4j, 2+4j, 3+4j]_6^1 \mid j = 0, 1, \dots, k-1\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rccccccc}
8k+2-2j: & 8k+2, & 8k, & \dots, & 6k+6, & 6k+4 & \text{even} \\
8k+1-2j: & 8k+1, & 8k-1, & \dots, & 6k+5, & 6k+3 & \text{odd} \\
8k-1-4j: & 8k-1, & 8k-5, & \dots, & 4k+7, & 4k+3 & 3 \text{ (mod 4)} \\
1+4j: & 1, & 5, & \dots, & 4k-7, & 4k-3 & 1 \text{ (mod 4)} \\
2+4j: & 2, & 6, & \dots, & 4k-6, & 4k-2 & 2 \text{ (mod 4)} \\
3+4j: & 3, & 7, & \dots, & 4k-5, & 4k-1 & 3 \text{ (mod 4)}
\end{array}$$

The blocks

$$\{[0, 6k+2-2j, 2k+2+2j; 4k-1-4j, 4k+1+4j, 4k+2+4j, 4k+3+4j]_6^1$$

$$| \ j = 0, 1, \ldots, k-2, \text{ and } j \neq k/3 \text{ if } k \equiv 0 \pmod 3\}$$

$$\cup\{[0, 8k/3+1, 8k/3+2; 8k/3-1, 16k/3+1, 16k/3+2, 16k/3+3]_6^1$$

$$\text{if } k \equiv 0 \pmod 3 \text{ and } j = k/3\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rccccccc}
\color{blue}{6k+2-2j:} & \color{blue}{6k+2,} & \color{blue}{6k,} & \color{blue}{\ldots,} & \color{blue}{4k+8,} & \color{blue}{4k+6} & \color{blue}{\text{even}} \\
2k+2+2j: & 2k+2, & 2k+4, & \ldots, & 4k-4, & 4k-2 & \text{even} \\
4k-1-4j: & 4k-1, & 4k-5, & \ldots, & 11, & 7 & 3 \pmod 4 \\
4k+1+4j: & 4k+1, & 4k+5, & \ldots, & 8k-11, & 8k-7 & 1 \pmod 4 \\
\color{blue}{4k+2+4j:} & \color{blue}{4k+2,} & \color{blue}{4k+6,} & \color{blue}{\ldots,} & \color{blue}{8k-10,} & \color{blue}{8k-6} & \color{blue}{2 \pmod 4} \\
4k+3+4j: & 4k+3, & 4k+7, & \ldots, & 8k-9, & 8k-5 & 3 \pmod 4
\end{array}$$

Notice that we have a potential repetition of vertex labels in the two rows in blue. If $k \equiv 0 \pmod 3$ and $j = k/3$, then we have eliminated the block $[0, \color{blue}{16k/3+2}, 8k/3+2; 8k/3-1, 16k/3+1, \color{blue}{16k/3+2}, 16k/3+3]_6^1$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, 8k/3+1, 8k/3+2; 8k/3-1, 16k/3+1, 16k/3+2, 16k/3+3]_6^1$ (which covers the same differences as the omitted block).

The blocks

$$\{[0, 2k+1+2j, 2k+2+2j; 4k-3-4j, 4k+3+4j, 4k+4+4j, 4k+5+4j]_6^1$$

$$| \ j = 0, 1, \ldots, k-1, \text{ and } j \neq (k-2)/3 \text{ if } k \equiv 2 \pmod 3\}$$

$$\cup\{[0, (16k+10)/3, (8k+2)/3; (8k-1)/3, (16k+1)/3, (16k+4)/3, (16k+7)/3]_6^1$$

$$\text{if } k \equiv 2 \pmod 3 \text{ and } j = (k-2)/3\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{lcccccc}
2k+1+2j: & 2k+1, & 2k+3, & \ldots, & 4k-3, & 4k-1 & \text{odd} \\
2k+2+2j: & 2k+2, & 2k+4, & \ldots, & 4k-2, & 4k & \text{even} \\
4k-3-4j: & 4k-3, & 4k-7, & \ldots, & 5, & 1 & 1 \pmod 4 \\
4k+3+4j: & 4k+3, & 4k+7, & \ldots, & 8k-5, & 8k-1 & 3 \pmod 4 \\
4k+4+4j: & 4k+4, & 4k+8, & \ldots, & 8k-4, & 8k & 0 \pmod 4 \\
4k+5+4j: & 4k+5, & 4k+9, & \ldots, & 8k-3, & 8k+1 & 1 \pmod 4
\end{array}$$

Notice that we have a potential repetition of vertex labels in the two rows in red. If $k \equiv 2 \pmod 3$ and $j = (k-2)/3$, then we have eliminated the block $[0, (8k-1)/3, (8k+2)/3; (8k-1)/3, (16k+1)/3, (16k+4)/3, (16k+7)/3]_6^1$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, (16k+10)/3, (8k+2)/3; (8k-1)/3, (16k+1)/3, (16k+4)/3, (16k+7)/3]_6^1$ (which covers the same differences as the omitted block).

Of course, the individual blocks $[0, 4k+1, 4k+2; 3, 8k-3, 8k-2, 8k-1]_6^1$ and $[0, 4k-1, 4k; 1, 4k+1, 4k+2, 8k+1]_6^1$ have distinct vertices. Therefore, all vertex labels are distinct.

We next show that all differences are present. The blocks

$$\{2 \times [0, 8k+2-2j, 8k+1-2j; 8k-1-4j, 1+4j, 2+4j, 3+4j]_6^1$$

$$\mid j = 0, 1, \ldots, k-1\}$$

generate the following edge differences and arc differences:

$2 \times 1 + 2j : 1, 3, \ldots, 2k-1$ odd

$2 \times 2 + 2j : 2, 4, \ldots, 2k$ even

$2 \times 4 + 4j : 4, 8, \ldots, 4k \quad 0 \pmod 4$

$2 \times 1 + 4j : 1, 5, \ldots, 4k-3 \quad 1 \pmod 4$

$2 \times 2 + 4j : 2, 6, \ldots, 4k-2 \quad 2 \pmod 4$

$2 \times 3 + 4j : 3, 7, \ldots, 4k-1 \quad 3 \pmod 4$.

The blocks

$$\{[0, 6k+2-2j, 2k+2+2j; 4k-1-4j, 4k+1+4j, 4k+2+4j, 4k+3+4j]_6^1$$

$$\mid j = 0, 1, \ldots, k-2, \text{ and } j \neq k/3 \text{ if } k \equiv 0 \pmod 3\}$$

$$\cup\{[0, 8k/3+1, 8k/3+2; 8k/3-1, 16k/3+1, 16k/3+2, 16k/3+3]_6^1$$

$$\text{if } k \equiv 0 \pmod 3 \text{ and } j = k/3\}$$

generate the following edge differences and arc differences:

$2k+1+2j : 2k+1, 2k+3, \ldots, 4k-3$ odd

$2k+2+2j : 2k+2, 2k+4, \ldots, 4k-2$ even

$4k+4+4j : 4k+4, 4k+8, \ldots, 8k-4 \quad 0 \pmod 4$

$4k+1+4j : 4k+1, 4k+5, \ldots, 8k-7 \quad 1 \pmod 4$

$4k+2+4j : 4k+2, 4k+6, \ldots, 8k-6 \quad 2 \pmod 4$

$4k+3+4j : 4k+3, 4k+7, \ldots, 8k-5 \quad 3 \pmod 4$.

The blocks

$$\{[0, 2k+1+2j, 2k+2+2j; 4k-3-4j, 4k+3+4j, 4k+4+4j, 4k+5+4j]_6^1$$

$$\mid j = 0, 1, \ldots, k-1, \text{ and } j \neq (k-2)/3 \text{ if } k \equiv 2 \pmod 3\}$$

$$\cup\{[0, (16k+10)/3, (8k+2)/3; (8k-1)/3, (16k+1)/3, (16k+4)/3, (16k+7)/3]_6^1$$

$$\text{if } k \equiv 2 \pmod 3 \text{ and } j = (k-2)/3\}$$

generate the following edge differences and arc differences:

$2k+1+2j : 2k+1, 2k+3, \ldots, 4k-1$ odd

$2k+2+2j : 2k+2, 2k+4, \ldots, 4k$ even

$4k+6+4j : 4k+6, 4k+10, \ldots, 8k+2 \quad 2 \pmod 4$

$4k+3+4j : 4k+3, 4k+7, \ldots, 8k-1 \quad 3 \pmod 4$

$4k+4+4j : 4k+4, 4k+8, \ldots, 8k \quad 0 \pmod 4$

$4k+5+4j : 4k+5, 4k+9, \ldots, 8k+1 \quad 1 \pmod 4$.

The blocks

$$\{[0, 4k+1, 4k+2; 3, 8k-3, 8k-2, 8k-1]_6^1, [0, 4k-1, 4k; 1, 4k+1, 4k+2, 8k+1]_6^1\}.$$

generate the following edge differences and arc differences:

$4k+1, 4k+1$

$8k, 8k-3k, 8k-2k, 8k-1$ and

$4k-1, 4k$

$8k+2, 4k+1, 4k+2, 8k+1$

Therefore all differences are present.

Next, we verify Lemma 2.9 using the difference method with example. Let $v = 8k+6$ and $k \geq 1$,and $\lambda = 2$. We have ∞ and cycle of length $v = 8k+5$. So, we have the following multisets of edge and arc differences:

Edge differences: $\{1, 2, 3, 4, \ldots, 4k, 4k+1, 4k+2\}$ $\times 2$

Arc differences: $\{1, 2, 3, 4, \ldots, 8k+2, 8k+3, 8k+4\}$ $\times 2$.

All the edge and arc differences are repeated twice in Table 3.

For verification purposes, consider for example, $k = 2$, $v = 22$, and $\lambda = 2$ (Table

Table 3: The Edge and Arc Differences of Lemma 2.9

Blocks	Edge differences	Arc Differences
$[0, \infty, v-2; v-5, 1, 2, 3]$ $\times 2$	$\infty, 1$ $\times 2$	$4, 1, 2, 3$ $\times 2$
$[0, 2, 3; \infty, 5, 6, 7]$ $\times 2$	$2, 3$ $\times 2$	$\infty, 5, 6, 7$ $\times 2$
$[0, 4, 5; v-11, 8, 9, \infty]$	$4, 5$	$10, 8, 9, \infty$
$[0, 5, 6; v-11, 8, 12, \infty]$	$5, 6$	$10, 8, 12, \infty$
$[0, 4, 6; v-12, 9, 11, 12]$	$4, 6$	$11, 9, 11, 12$
$[0, 7+4j, v-8-4j; 8+8j, 16+8j,$ $13+8j, 17+8j], j = 0, 1, 2, ..., k-2$	$7+4j : 7, 11, 15, \ldots, 4k-1$ $8k-2-4j : 8k-2, 8k-6, \ldots, 4k+6$ $\Longrightarrow 7, 11, 15, \ldots, 4k-1$	$8k-3-8j : 8k-3, 8k-11,$ $8k-19, \ldots, 29, 21, 13$ $16+8j : 16, 24, 32, \ldots, 8k$ $13+8j : 13, 21, 29, \ldots, 8k-3$ $17+8j : 17, 25, 33, \ldots, 8k+1$
$[0, 8+4j, v-9-4j; 7+8j, 19+8j,$ $14+8j, 18+8j], j = 0, 1, 2, ..., k-2$	$8+4j : 8, 12, 16, \ldots, 4k$ $8k-3-4j : 8k-3, 8k-7, \ldots, 4k+5$ $\Longrightarrow 8, 12, 16, \ldots, 4k$	$8k-2-8j : 8k-2, 8k-10,$ $8k-18, \ldots, 30, 22, 14$ $19+8j : 19, 27, \ldots, 8k-5, 8k+3$ $14+8j : 14, 22, \ldots, 8k-10, 8k-2$ $18+8j : 18, 26, \ldots, 8k-6, 8k+2$
$[0, 9+4j, v-10-4j; 6+8j, 15+8j,$ $18+8j, 19+8j], j = 0, 1, 2, ..., k-2$	$9+4j : 9, 13, 17, \ldots, 4k+1$ $8k-4-4j : 8k-4, 8k-8, \ldots, 4k+4$ $\Longrightarrow 9, 13, 17, \ldots, 4k+1$	$8k-1-8j : 8k-1, 8k-9,$ $8k-17, \ldots, 31, 23, 15$ $15+8j : 15, 23, \ldots, 8k-9, 8k-1$ $18+8j : 18, 26, \ldots, 8k-6, 8k+2$ $19+8j : 19, 27, \ldots, 8k-5, 8k+3$
$[0, 10+4j, v-11-4j; 1+8j, 16+8j,$ $17+8j, 20+8j], j = 0, 1, 2, ..., k-2$	$10+4j : 10, 14, 18, 22, \ldots, 4k+2$ $8k-5-4j : 8k-5, 8k-9, \ldots, 4k+3$ $\Longrightarrow 10, 14, 18, \ldots, 4k+2$	$8k+4-8j : 8k+4, 8k-4,$ $8k-12, \ldots, 28, 20$ $16+8j : 16, 24, \ldots, 8k$ $17+8j : 17, 25, \ldots, 8k+1$ $20+8j : 20, 28, \ldots, 8k-4, 8k+4$

4). In this case, we have ∞ and cycle of length 21 with the multisets of edge and arc differences $\{1, 2, 3, \ldots, 10\ (\times 2)\}$ and $\{1, 2, 3, \ldots, 20\ (\times 2)\}$ respectively.

Table 4: The Edge and Arc Differences for an S_6^1-decomposition of $2M_{22}$

Blocks	Edge differences	Arc Differences
$[0, \infty, v-2; v-5, 1, 2, 3]$ $\times 2$	$\infty, 1$ $(\times 2)$	$4, 1, 2, 3$ $(\times 2)$
$[0, 2, 3; \infty, 5, 6, 7]$ $\times 2$	$2, 3$ $(\times 2)$	$\infty, 5, 6, 7$ $(\times 2)$
$[0, 4, 5; v-11, 8, 9, \infty]$	$4, 5$	$10, 8, 9, \infty$
$[0, 5, 6; v-11, 8, 12, \infty]$	$5, 6$	$10, 8, 12, \infty$
$[0, 4, 6; v-12, 9, 11, 12]$	$4, 6$	$11, 9, 11, 12$
$[0, 7+4j, v-8-4j; 8+8j, 16+8j,$ $13+8j, 17+8j]$ for $j = 0$	$7, 7$	$13, 16, 13, 17$
$[0, 8+4j, v-9-4j; 7+8j, 19+8j,$ $14+8j, 18+8j]$ for $j = 0$	$8, 8$	$14, 19, 14, 18$
$[0, 9+4j, v-10-4j; 6+8j, 15+8j,$ $18+8j, 19+8j]$ for $j = 0$	$9, 9$	$15, 15, 18, 19$
$[0, 10+4j, v-11-4j; 1+8j, 16+8j,$ $17+8j, 20+8j]$ for $j = 0$	$10, 10$	$20, 16, 17, 20$

Combining all the edge and arc differences give the required result in Table 4.

Now, we verify Lemma 2.10 by showing that all the vertex labels are distinct. The individual blocks $[0, 4k+2, 4k+1; 4, 1, 2, 3]$, $[0, 2k+1, 2k+2; 4k-1, 2, 4k+1, 4k+2]$ and $[0, 1, 2; 4k, 4k+2, 5, 6]$ have distinct vertices.

The blocks

$$\cup\{[0, 3+2j, 4+2j; 4k-4-4j, 8+4j, 9+4j, 10+4j] \mid j = 0, 1, \ldots, k-2,$$

$$j \neq (2k-4)/3 \text{ when } j \equiv 2 \pmod 3\}, \text{ and } j \neq (k-3)/2 \text{ when } j \equiv 1 \pmod 2\}$$

$$\cup\{[0, (4k+1)/3, (8k+5)/3; (4k+4)/3, (8k+8)/3, (8k+11)/3, (8k+14)/3]$$

$$\text{if } j \equiv 2 \pmod 3, \text{ and } j = (2k-4)/3\}$$

$$\cup\{[0, k, k+1; 2k, 2k+2, 2k+1, 2k+4] \text{ if } j \equiv 1 \pmod 2, \text{ and } j = (k-3)/2\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{ccccccc}
3+2j: & 3, & 5, & 7, & \ldots, & 2k-1 & \text{odd} \\
4+2j: & 4, & 6, & 8, & \ldots, & 2k & \text{even} \\
4k-4-4j: & 4k-4, & 4k-8, & 4k-12 & \ldots, & 4 & 0 \pmod 4 \\
8+4j: & 8, & 12, & 16 & \ldots, & 4k & 0 \pmod 4 \\
9+4j: & 9, & 13, & 17, & \ldots, & 4k+1 & 1 \pmod 4 \\
10+4j: & 10, & 14, & 18, & \ldots, & 4k+2 & 2 \pmod 4).
\end{array}$$

Notice that we have a potential repetition of vertex labels in the rows $4+2j$ and $4k-4-4j$, and also in the row $4k-4-4j$ and $8+4j$. For the rows $4+2j$ and $4k-4-4j$, if $j = \frac{2k-4}{3}$, then we have eliminated the block $[0, \frac{4k+1}{3}, \frac{4k+4}{3}; \frac{4k+4}{3}, \frac{8k+8}{3}, \frac{8k+11}{3}, \frac{8k+14}{3}]$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, \frac{4k+1}{3}, \frac{4k+4}{3}; \frac{8k+5}{3}, \frac{8k+8}{3}, \frac{8k+11}{3}, \frac{8k+14}{3}]$ (which covers the same differences as the omitted block).

Similarly for $4k-4-4j$ and $8+4j$. If $j = \frac{k-3}{2}$, then we replaced the block $[0,k,k+1; 2k+2, 2k+2, 2k+3, 2k+4]$ with the block $[0,k,k+1;2k,2k+2,2k+3,2k+4]$. The blocks

$$\cup\{[0, 3+2j, 4+2j; 4k-2-4j, 6+4j, 7+4j, 8+4j] \mid j = 0, 1, \ldots, k-2,$$

$$\text{and } j \neq (2k-3)/3 \text{ when } j \equiv 0 \pmod 3\}, \text{and} j \neq (k-2)/2 \text{ when } j \equiv 0 \pmod 2\}$$

$$\cup[0, (4k+3)/3, (8k+3)/3; (4k+6)/3, (8k+6)/3, (8k+9)/3, (8k+12)/3]$$

$$\text{if } j \equiv 0 \pmod 3, \text{ and } j = (2k-3)/3\}$$

$$\cup\{[0, k+1, k+2; 2k, 2k+2, 2k+1, 2k+4] \text{ if } j \equiv 0 \pmod 2, \text{ and } j = (k-2)/2\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{ccccccc}
3+2j: & 3, & 5, & 7, & \ldots, & 2k-1 & \text{odd} \\
4+2j: & 4, & 6, & 8, & \ldots, & 2k & \text{even} \\
4k-2-4j: & 4k-2, & 4k-6, & 4k-10 & \ldots, & 2 & 2 \pmod 4 \\
6+4j: & 6, & 10, & 14 & \ldots, & 4k-2 & 2 \pmod 4 \\
7+4j: & 7, & 11, & 15, & \ldots, & 4k-1 & 3 \pmod 4 \\
8+4j: & 8, & 12, & 16, & \ldots, & 4k & 0 \pmod 4.
\end{array}$$

Notice that we have a potential repetition of vertex labels in the rows $4+2j$ and $4k-2-4j$, and also in the row $4k-2-4j$ and $6+4j$. For the rows $4+2j$ and $4k-2-4j$, if $j = \frac{2k-3}{3}$, then we have eliminated the block $[0, \frac{4k+3}{3}, \frac{4k+6}{3}; \frac{4k+6}{3}, \frac{8k+6}{3}, \frac{8k+9}{3}, \frac{8k+12}{3}]$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, \frac{4k+3}{3}, \frac{8k+3}{3}; \frac{4k+6}{3}, \frac{8k+6}{3}, \frac{8k+9}{3}, \frac{8k+12}{3}]$ (which covers the same differences as the omitted block). Similarly for $4k-2-4j$ and $6+4j$. If $j = \frac{k-2}{2}$, then we replaced the block $[0,k,k+1; 2k+2, 2k+2, 2k+3, 2k+4]$ with the block $[0,k,k+1;2k,2k+$

$2, 2k+1, 2k+4]$.

Therefore, all vertex labels are distinct.

Next, we verify Lemma 2.11 using the difference method. Let $v = 8k+2$ and $k \geq 3$, and $\lambda = 2$. We have ∞ and cycle of length $v = 8k+1$. So, we have the following multisets of edge and arc differences:

Edge differences: $\{1, 2, 3, 4, \ldots, 4k\}$ $\times 2$

Arc differences: $\{1, 2, 3, 4, \ldots, 8k-1, 8k\}$ $\times 2$.

Now, we check if all the edges and arcs are repeated twice.

Table 5: The Edge and Arc Differences of Lemma 2.11

Blocks	Edge differences	Arc Differences
$[0, \infty, v-2; v-5, 1, 2, 3]$ $\times 2$	$\infty, 1$ $\times 2$	$4, 1, 2, 3$ $\times 2$
$[0, 2, 3; \infty, 5, 6, 7]$ $\times 2$	$2, 3$ $\times 2$	$\infty, 5, 6, 7$ $\times 2$
$[0, 4, 5; v-11, 8, 9, \infty]$	$4, 5$	$10, 8, 9, \infty$
$[0, 5, 6; v-11, 8, 12, \infty]$	$5, 6$	$10, 8, 12, \infty$
$[0, 4, 6; v-12, 9, 11, 12]$	$4, 6$	$11, 9, 11, 12$
$[0, 7, v-8; v-14, 13, 17, 21]$	$7, 7$	$13, 13, 17, 21$
$[0, 8, v-9; v-15, 14, 15, 22]$	$8, 8$	$14, 14, 15, 22$
$[0, 9, v-10; v-19, 18, 19, 22]$	$9, 9$	$18, 18, 19, 22$
$[0, 10, v-11; v-21, 17, 20, 24]$	$10, 10$	$20, 17, 20, 24$
$[0, 11, v-12, v-17, 16, 21, 24]$	$11, 11$	$16, 16, 21, 24$
$[0, 12, v-13; v-24, 15, 19, 23]$	$12, 12$	$23, 15, 19, 23$
$[0, 13+4j, v-14-4j; 7+8j, 31+8j,$ $26+8j, 30+8j], j = 0, 1, 2, ..., k-4$	$13+4j : 13, 17, 21, 25, \ldots, 4k-3$ $v-14-4j = 8k-12-4j : 8k-12,$ $8k-16, \ldots, 4k+4 \implies 13, 17, \ldots, 4k-3$	$7+8j \implies 8k-8j-6 : 8k-6, 8k-14,$ $8k-22, \ldots, 26$ $31+8j : 31, 39, 47, \ldots, 8k-1$ $26+8j : 26, 34, 42, \ldots, 8k-6$ $30+8j : 30, 38, 46, \ldots, 8k-2$
$[0, 14+4j, v-15-4j; 8+8j, 28+8j,,$ $25+8j, 29+8j], j = 0, 1, 2, ..., k-4$	$14+4j : 14, 18, 22, 26, \ldots, 4k-2$ $v-15-4j = 8k-13-4j : 8k-13,$ $8k-17, \ldots, 4k+3 \implies 14, 18, \ldots, 4k-2$	$8+8j \implies 8k-8j-7 : 8k-7, 8k-15,$ $8k-23, \ldots, 25$ $28+8j : 28, 36, 44, \ldots, 8k-4$ $25+8j : 25, 33, 41, \ldots, 8k-7$ $29+8j : 29, 37, 45, \ldots, 8k-3$
$[0, 15+4j, v-16-4j; 1+8j, 28+8j,$ $29+8j, 32+8j], j = 0, 1, 2, ..., k-4$	$15+4j : 15, 19, 23, 27, \ldots, 4k-1$ $v-16-4j = 8k-14-4j : 8k-14,$ $8k-18, \ldots, 4k+2 \implies 15, 19, \ldots, 4k-1$	$1+8j \implies 8k-8j : 8k, 8k-8,$ $8k-16, \ldots, 32$ $28+8j : 28, 36, 44, \ldots, 8k-4$ $29+8j : 29, 37, 45, \ldots, 8k-3$ $32+8j : 32, 40, 48, \ldots, 8k$
$[0, 16+4j, v-17-4j; 6+8j, 27+8j,$ $30+8j, 31+8j], j = 0, 1, 2, ..., k-4$	$16+4j : 16, 20, 24, 28, \ldots, 4k$ $v-17-4j = 8k-15-4j : 8k-15,$ $8k-19, \ldots, 4k+1 \implies 16, 20, \ldots, 4k$	$6+8j \implies 8k-8j-5 : 8k-5, 8k-13,$ $8k-21, \ldots, 27$ $27+8j : 27, 35, 43, \ldots, 8k-5$ $30+8j : 30, 38, 46, \ldots, 8k-2$ $31+8j : 31, 39, 47, \ldots, 8k-1$

Combining all the arc and edge differences in Table 5, we have the required result.

2.5 An S_6^2-Decomposition of λM_v

An S_6^2-decomposition of M_v exists if and only if $v \equiv 0$ or 1 (mod 4) ansd $v \geq 9$ [10]. In this subsection, we give the necessary and sufficient conditions for the existence of a S_6^2-decomposition of λM_v, where $\lambda = 2$. As usual, we give a direct construction to establish sufficiency.

Lemma 2.14 *An S_6^2-decomposition of λM_v exists for $v \equiv 3$ (mod 4), $\lambda = 2$.*

Proof. Let $v = 4k+3$ and $k \geq 1$. Let $(\lambda M_v) = \{0, 1, 2, \ldots v-1\}$, consider the blocks:

$B = \{[0, 4k+2, 4k+1; 4, 4k, 1, 2], [0, 2k+1, 2k+2; 4k-1, 1, 2, 4k+1], [0, 1, 2; 4k, 4k-3, 4k+2, 5]\}$

$$\cup\{[0, 3+2j, 4+2j; 4k-4-4j, 4k-5-4j, 9+4j, 10+4j] \mid j = 0, 1, \ldots, k-2,$$

$$j \neq (2k-4)/3, \text{ and } j \neq (k-3)/2$$

$$\cup\{[0, (4k+1)/3, (8k+5)/3; (4k+4)/3, (4k-5)/3, (8k+8)/3, (8k+11)/3]$$

$$\text{if } j = (2k-4)/3\}$$

$$\cup\{[0, k, k+1; 2k, 2k-1, 2k+2, 2k+1] \text{ if } j = (k-3)/2\}$$

$$\cup\{[0, 3+2j, 4+2j; 4k-2-4j, 4k-3-4j, 7+4j, 8+4j] \mid j = 0, 1, \ldots, k-2,$$

$$j \neq (2k-3)/3, \text{ and } j \neq (k-2)/2$$

$$\cup\{[0, (4k+3)/3, (8k+3)/3; (4k+6)/3, (4k-9)/3, (8k+6)/3, (8k+9)/3], j = (2k-3)/3$$

$$\cup\{[0, k+1, k+2; 2k, 2k-1, 2k+2, 2k+1], j = (k-2)/2\}$$

The elements of B, along with their images under the permutation $\pi(i) = i+1 \pmod v$, form a S_6^2-decomposition of $2M_v$ where $v = 4k+3$. $\square$

Lemma 2.15 *A S_6^2-decomposition of λM_v exists for $v \equiv 6 \pmod 8$, $v \geq 14$ and $\lambda = 2$.*

Proof. Let $v = 8k+6$ and $k \geq 1$. Let $(\lambda M_v) = \{0, 1, 2, \ldots v-2, \infty\}$, consider the following blocks:

$B = \{[0, \infty, 8k+4; 8k+1, 8k+2, 1, 2] \times 2,$

$[0, 2, 3; \infty, 8k, 6, 7] \times 2,$

$[0, 4, 5; 8k-5, 8k-3, 9, \infty],$

$[0, 4, 6; 8k-6, 8k-4, 11, 12]\ \}$

$\cup\{[0, 7+4j, 8k-2-4j; 8+8j, 8k-11-8j, 13+8j, 17+8j] \mid j = 0, 1, \ldots, k-2$ and

$j \neq (k-3)/2\} \cup \{[0, 2k+1, 6k+4; 4k-4, 4k, 4k+1, 4k+4]$ if $j = (k-3)/2\}$

$\cup\{[0, 8+4j, 8k-3-4j; 7+8j, 8k-14-8j, 14+8j, 18+8j] \mid j = 0, 1, \ldots, k-2$ and

$j \neq (k-4)/2\} \cup \{[0, 2k, 6k+5; 4k-9, 4k+7, 4k+3, 4k+2]$ if $j = (k-4)/2\}$

$\cup\{[0, 9+4j, 8k-4-4j; 6+8j, 8k-10-8j, 18+8j, 19+8j] \mid j = 0, 1, \ldots, k-2$ and

$j \neq (k-2)/2\} \cup \{[0, 2k+5, 6k; 4k-5, 4k-2, 4k+7, 4k+11]$ if $j = (k-2)/2\}$

$\cup\{[0, 10+4j, 8k-5-4j; 1+8j, 8k-11-8j, 17+8j, 20+8j] \mid j = 0, 1, \ldots, k-2\}$

These stars along with their images under the permutation $\pi : V \to V$ defined as $\pi(i) = i+1 \pmod v$, form an S_6^2-decomposition of λM_v where $\lambda = 2$ and $v \geq 14$, as claimed. $\square$

Lemma 2.16 *An S_6^2-decomposition of $2M_v$ exists for all $v \equiv 2 \pmod 8$, with $v \geq 26$.*

Proof. let $v = 8k+2$ and $k \geq 3$. Let $(2M_v) = \{0, 1, 2, \dots v-2, \infty\}$ The required decomposition is given by the blocks:

$\{[0, \infty, 8k; 8k-3, 8k-2, 1, 2]\ \times 2,\ [0, 2, 3; \infty, 8k-4, 6, 7]\ \times 2,\ [0, 4, 5; 8k-9, 8k-7, 9, \infty],\ [0, 5, 6; 8k-9, 8k-7, 12, \infty],\ [0, 4, 6; 8k-10, 8k-8, 11, 12],\ [0, 7, 8k-6; 8k-12, 8k-16, 13, 21],\ [0, 8, 8k-7; 8k-13, 8k-14, 14, 22],\ [0, 9, 8k-8; 8k-17, 8k-18, 18, 22],\ [0, 10, 8k-9; 8k-19, 8k-16, 20, 24],\ [0, 11, 8k-10, 8k-15, 8k-20, 16, 24],\ [0, 12, 8k-11; 8k-22, 8k-14, 19, 23]\}$

and

$\cup\{[0, 13+4j, 8k-12-4j; 7+8j, 8k-29-8j, 31+8j, 26+8j \mid j = 0, 1, \dots, k-4]\}$

$\cup\{[0, 14+4j, 8k-13-4j; 8+8j, 8k-27-8j, 25+8j, 29+8j]_6^2| \ j = 0, 1, \dots, k-4,$ and $j \neq \frac{k-7}{2}$ if $k \equiv 7 \pmod 2)\}$

$\cup\{[0, 2k, 6k+1; 4k-20, 4k+4, 4k, 4k+1]_6^2$ if $k \equiv 7 \pmod 2$ and $j = \frac{k-7}{2}\}$

$\cup\{[0, 15+4j, 8k-14-4j; 1+8j, 8k-27-8j, 29+8j, 32+8j]_6^2 \mid j = 0, 1, \dots, k-4,$ and $j \neq \frac{k-7}{2}$ if $k \equiv 7 \pmod 2)\}$

$\cup\{[0, 2k+1, 6k; 4k-27, 4k-3, 4k+1, 4k]_6^2$ if $k \equiv 7 \pmod 2$ and $j = \frac{k-7}{2}\}$

$\cup\{[0, 16+4j, 8k-15-4j; 6+8j, 8k-29-8j, 27+8j, 31+8j]_6^2 \mid j = 0, 1, \dots, k-4,$ and $j \neq \frac{k-7}{2}$ if $k \equiv 7 \pmod 2)\}$

$\cup\{[0, 2k+2, 6k-1; 4k-22, 4k-2, 4k-1, 4k+2]_6^2$ if $k \equiv 7 \pmod 2$ and $j = \frac{k-7}{2}\}$

These stars along with their images under the permutation $\pi : V \to V$ defined as $\pi(i) = i+1 \pmod v$, form an S_6^2-decomposition of λM_v where $\lambda = 2$ and $v \geq 26$, as claimed. □

We now combine the results of this subsection to give necessary and sufficient conditions for an S_6^2-decomposition of λM_v.

Theorem 2.17 *An S_6^2-decomposition of λM_v exists if and only if*

1. $v \equiv 0$ *or* 1 $(mod\ 4)$ *and* $\lambda \geq 1$, *or*

2. $v \equiv 2$ $(mod\ 4)$ *and* $\lambda \equiv 0$ $(mod\ 2)$, *or*

3. $v \equiv 3$ $(mod\ 4)$ *and* $\lambda \equiv 0$ $(mod\ 2)$.

Proof. By Lemma 2.1, $\lambda = 0$ (mod 2) is necessary.

For sufficiency, when $v \equiv 0$ or 1 (mod 4), an S_6^1-decomposition of M_v exists by [10]. So when $\lambda \geq 1$, taking λ copies of the blocks of such a decomposition gives a decomposition of λM_v. When $v \equiv 2$ (mod 4) and $\lambda = 2$, an S_6^2-decomposition of $2M_v$ exists by Lemma 2.15 and Lemma 2.16. So when $\lambda \equiv 0$ (mod 2), taking $\lambda/2$ copies of the blocks of such a decomposition gives a decomposition of λM_v. When $v \equiv 3$ (mod 4) and $\lambda = 2$, an S_6^2-decomposition of $2M_v$ exists by Lemma 2.14. So when $\lambda \equiv 0$ (mod 2), taking $\lambda/2$ copies of such a decomposition gives a decomposition of λM_v. $\square$

2.5.1 Verification and Example

Now, we verify Lemma 2.12 by showing that all the vertex labels are distinct. The individual blocks $[0, 4k+2, 4k+1; 4, 4k, 1, 2]$, $[0, 2k+1, 2k+2; 4k-1, 1, 2, 4k+1]$ and $[0, 1, 2; 4k, 4k-3, 4k+2, 5]$ have distinct vertices.

The blocks

$$\cup\{[0, 3+2j, 4+2j; 4k-4-4j, 4k-5-4j, 9+4j, 10+4j] \mid j = 0, 1, \ldots, k-2,$$

$$j \neq (2k-4)/3, \text{and} j \neq (k-3)/2$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{ccccccc}
3+2j: & 3, & 5, & 7, & \ldots, & 2k-1 & \text{odd} \\
4+2j: & 4, & 6, & 8, & \ldots, & 2k & \text{even} \\
4k-4-4j: & 4k-4, & 4k-8, & 4k-12, & \ldots, & 4 & 0 \pmod 4 \\
4k-5-4j: & 4k-5, & 4k-9, & 4k-13, & \ldots, & 4k & 3 \pmod 4 \\
9+4j: & 9, & 13, & 17, & \ldots, & 4k+1 & 1 \pmod 4 \\
10+4j: & 10, & 14, & 18, & \ldots, & 4k+2 & 2 \pmod 4
\end{array}$$

Notice that we have a potential repetition of vertex labels in the rows $4+2j$ and $4k-4-4j$, and also in the row $3+2j$ and $4k-5-4j$. Since we reverse the orientation of one of the outdegree of S_6^1 to obtain S_6^2, we have that $j \neq \frac{2k-4}{3}$.

The blocks

$$\cup\{[0,(4k{+}1)/3,(8k{+}5)/3;(4k{+}4)/3,(4k{-}5)/3,(8k{+}8)/3,(8k{+}11)/3],\ \text{if } j=(2k{-}4)/3\}$$

and

$$\cup\{[0,k,k+1;2k,2k-1,2k+2,2k+1] \text{ if } j=(k-3)/2\}$$

have distinct vertex labels.

Similarly, the blocks

$$\cup\{[0,3+2j,4+2j;4k-2-4j,6+4j,7+4j,8+4j] \mid j=0,1,\ldots,k-2,$$

$$\text{and } j \neq (2k-3)/3 \text{ when } j \equiv 0 \pmod 3\},\ \text{and } j \neq (k-2)/2 \text{ when } j \equiv 0 \pmod 2\}$$

$$\cup\{[0,(4k+3)/3,(8k+3)/3;(4k+6)/3,(8k+6)/3,(8k+9)/3,(8k+12)/3]$$

$$\text{if } j \equiv 0 \pmod 3),\ \text{and } j=(2k-3)/3\}$$

$$\cup\{[0,k+1,k+2;2k,2k+2,2k+1,2k+4] \text{ if } j \equiv 0 \pmod 2),\ \text{and } j=(k-2)/2\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rccccccc}
3+2j: & 3, & 5, & 7, & \ldots, & 2k-1 & \text{odd} \\
4+2j: & 4, & 6, & 8, & \ldots, & 2k & \text{even} \\
4k-2-4j: & 4k-2, & 4k-6, & 4k-10 & \ldots, & 2 & 2 \pmod 4 \\
4k-3-4j: & 4k-3 & 4k-7 & 4k-11 & \ldots, & 5 & 1 \pmod 4 \\
7+4j: & 7, & 11, & 15, & \ldots, & 4k-1 & 3 \pmod 4 \\
8+4j: & 8, & 12, & 16, & \ldots, & 4k & 0 \pmod 4
\end{array}$$

Which are all distinct for the first block, since $j \neq \frac{2k-3}{3}$ and $j \neq \frac{k-2}{2}$ Therefore, all vertex labels are distinct.

Next, we show that Theorem 2.13 has distinct vertex labels. The blocks

$$\cup\{[0, 7+4j, 8k-2-4j; 8+8j, 8k-11-8j, 13+8j, 17+8j] \mid j = 0, 1, \ldots, k-2 \text{ and}$$

$$j \neq (k-3)/2\} \cup \{[0, 2k+1, 6k+4; 4k-4, 4k, 4k+1, 4k+4] \text{ if } j = (k-3)/2\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rcccccc}
7+4j: & 7, & 11, & 15, & \ldots, & 4k-5, & 4k-1 \\
8k-2-4j: & 8k-2, & 8k-6, & 8k-10 & \ldots, & 4k+10, & 4k+6 \\
8+8j: & 8, & 16, & 24, & \ldots, & 8k-16, & 8k-8 \\
8k-11-8j: & 8k-11, & 8k-19, & 8k-27, & \ldots, & 13, & 5 \\
13+8j: & 13, & 21, & 29, & \ldots, & 8k-1, & 8k-3 \\
17+8j: & 7, & 25, & 33, & \ldots, & 8k-7, & 8k+1.
\end{array}$$

Notice that we have a potential repetition of vertex labels in the two rows in blue. If $j = \frac{k-3}{2}$, then we have eliminated the block $[0, 2k+1, 6k+4; 4k-4, 4k+1, 4k+1, 4k+5]$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, 2k+1, 6k+4; 4k-4, 4k, 4k+1, 4k+4]$ (which covers the same differences as the omitted block).

The blocks

$$\cup\{[0, 8+4j, 8k-3-4j; 7+8j, 8k-14-8j, 14+8j, 18+8j] \mid j = 0, 1, \ldots, k-2 \text{ and}$$

$$j \neq (k-4)/(2)\} \cup \{[0, 2k, 6k+5; 4k-9, 4k+7, 4k+3, 4k+2] \text{ if } j = (k-4)/(2)\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rcccccc}
8+4j: & 8, & 12, & \ldots, & 4k-4, & 4k & 0 \pmod 4 \\
8k-3-4j: & 8k-3, & 8k-7, & \ldots, & 4k+9, & 4k+5 & 1 \pmod 4 \\
7+8j: & 7, & 15, & \ldots, & 8k-17, & 8k-9 & 7 \pmod 8 \\
8k-14-8j: & 8k-14, & 8k-22, & \ldots, & 10, & 2 & 2 \pmod 8 \\
14+8j: & 14, & 22, & \ldots, & 8k-10, & 8k-2 & 6 \pmod 8 \\
18+8j: & 18, & 26, & \ldots, & 8k-6, & 8k+2 & 2 \pmod 8.
\end{array}$$

Notice that we have a potential repetition of vertex labels in the two rows in blue. If $j = \frac{k-3}{2}$, then we have eliminated the block $[0, 2k, 6k+5; 4k-9, 4k+2, 4k-2, 4k+2]$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, 2k, 6k+5; 4k-9, 4k+7, 4k+3, 4k+2]$ (which covers the same differences as the omitted block).

The blocks

$$\cup\{[0, 9+4j, 8k-4-4j; 6+8j, 8k-10-8j, 18+8j, 19+8j] \mid j = 0, 1, \ldots, k-2 \text{ and}$$

$$j \neq (k-2)/2\} \cup \{[0, 2k+5, 6k; 4k-5, 4k-2, 4k+7, 4k+11] \text{ if } j = (k-2)/2\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rcccccc}
9+4j: & 9, & 13, & \ldots, & 4k-3, & 4k+1 & 1 \pmod 4 \\
8k-4-4j: & 8k-4, & 8k-8, & \ldots, & 4k+8, & 4k+4 & 0 \pmod 4 \\
6+8j: & 6, & 14, & \ldots, & 8k-18, & 8k-10 & 6 \pmod 8 \\
8k-10-8j: & 8k-10, & 8k-18, & \ldots, & 14, & 6 & 6 \pmod 8 \\
18+8j: & 18, & 26, & \ldots, & 8k-6, & 8k+2 & 2 \pmod 8 \\
19+8j: & 19, & 27, & \ldots, & 8k-5, & 8k+3, & 3 \pmod 8.
\end{array}$$

Notice that we have a potential repetition of vertex labels in the two rows in red. If $j = \frac{k-2}{2}$, then we have eliminated the block $[0, 2k+5, 6k; 4k-2, 4k-2, 4k+10, 4k+11]$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, 2k+5, 6k; 4k-5, 4k-2, 4k+7, 4k+11]$ (which covers the same differences as the omitted block).

The block

$$\cup\{[0, 10+4j, 8k-5-4j; 1+8j, 8k-11-8j, 17+8j, 20+8j] \mid j = 0, 1, \dots, k-2\}$$

generates the following distinct vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{ccccccc}
10+4j: & 10, & 14, & \dots, & 4k-2, & 4k+2 & 2 \pmod 4 \\
8k-5-4j: & 8k-5, & 8k-9, & \dots, & 4k+7, & 4k+3 & 3 \pmod 4 \\
1+8j: & 1, & 9, & \dots, & 8k-23, & 8k-15 & 1 \pmod 8 \\
8k-11-8j: & 8k-11, & 8k-19, & \dots, & 13, & 5 & 5 \pmod 8 \\
17+8j: & 17, & 25, & \dots, & 8k-7, & 8k+1 & 1 \pmod 8 \\
20+8j: & 20, & 28, & \dots, & 8k-4, & 8k+4 & 4 \pmod 8.
\end{array}$$

Next, we show that all vertex labels are distinct in the Lemma 2.16. Of course, the individual blocks: $[0, \infty, 8k; 8k-3, 8k-2, 1, 2]$ $\times 2$, $[0, 2, 3; \infty, 8k-4, 6, 7]$ $\times 2$, $[0, 4, 5; 8k-9, 8k-7, 9, \infty]$, $[0, 5, 6; 8k-9, 8k-7, 12, \infty]$, $[0, 4, 6; 8k-10, 8k-8, 11, 12]$, $[0, 7, 8k-6; 8k-12, 8k-16, 13, 21]$, $[0, 8, 8k-7; 8k-13, 8k-14, 14, 22]$, $[0, 9, 8k-8; 8k-17, 8k-18, 18, 22]$, $[0, 10, 8k-9; 8k-19, 8k-16, 20, 24]$, $[0, 11, 8k-10, 8k-15, 8k-20, 16, 24]$, and $[0, 12, 8k-11; 8k-22, 8k-14, 19, 23]$ have distinct vertices.

The blocks

$$\{2 \times [0, 13+4j, 8k-12-4j; 7+8j, 8k-29-8j, 31+8j, 26+8j]_6^2 \mid j = 0, 1, \dots, k-4\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rccccccl}
13+4j: & 13, & 17, & \dots, & 4k-7, & 4k-3 & 5 \pmod 4 \\
8k-12-4j: & 8k-12, & 8k-16, & \dots, & 4k+8, & 4k+4 & 0 \pmod 4 \\
7+8j: & 7, & 15, & \dots, & 8k-33, & 8k-25 & 7 \pmod 8 \\
8k-29-8j: & 8k-29, & 8k-37, & \dots, & 11, & 3 & 3 \pmod 8 \\
31+8j: & 31, & 39, & \dots, & 8k-9, & 8k-1 & 7 \pmod 8 \\
26+8j: & 26, & 34, & \dots, & 8k-14, & 8k-2 & 6 \pmod 8.
\end{array}$$

The blocks

$$\{[0, 14+4j, 8k-13-4j; 8+8j, 8k-27-8j, 25+8j, 29+8j]_6^2 \mid j = 0, 1, \dots, k-4,$$

$$\text{and } j \neq (k-7)/2 \text{ if } k \equiv 7 \pmod 2\}$$

$$\cup \{[0, 2k, 6k+1; 4k-20, 4k+4, 4k, 4k+1]_6^2 \text{ if } k \equiv 7 \pmod 2 \text{ and } j = (k-7)/2\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rccccccl}
14+4j: & 14, & 18, & \dots, & 4k-6, & 4k-2 & 2 \pmod 4 \\
8k-13-4j: & 8k-13, & 8k-17, & \dots, & 4k+7, & 4k+3 & 3 \pmod 4 \\
8+8j: & 8, & 16, & \dots, & 8k-32, & 8k-24 & 0 \pmod 8 \\
8k-27-8j: & 8k-27, & 8k-35, & \dots, & 13, & 5 & 5 \pmod 8 \\
25+8j: & 25, & 33, & \dots, & 8k-15, & 8k-7 & 1 \pmod 8 \\
29+8j: & 29, & 37, & \dots, & 8k-11, & 8k-3 & 5 \pmod 8.
\end{array}$$

Notice that we have a potential repetition of vertex labels in the two rows in blue. If $k \equiv 7 \pmod 2$ and $j = (k-7)/2$, then we have eliminated the block $[0, 2k, 6k+1; 4k-20, 4k+1, 4k-3, 4k+1]_6^2$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, 2k, 6k+1; 4k-20, 4k+4, 4k, 4k+1]_6^2$ (which covers the same differences as the omitted block).

The blocks

$$\{[0, 15+4j, 8k-14-4j; 1+8j, 8k-27-8j, 29+8j, 32+8j]_6^2 \mid j = 0, 1, \dots, k-4,$$

$$\text{and } j \neq (k-7)/2 \text{ if } k \equiv 7 \pmod 2\}$$

$$\cup\{[0, 2k+1, 6k; 4k-27, 4k-3, 4k+1, 4k]_6^2 \text{ if } k \equiv 7 \pmod 2 \text{ and } j = (k-7)/2\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rcccccc}
15+4j: & 15, & 19, & \ldots, & 4k-5, & 4k-1 & 3 \pmod 4\\
8k-14-4j: & 8k-14, & 8k-18, & \ldots, & 4k+6, & 4k+2 & 2 \pmod 4\\
1+8j: & 1, & 9, & \ldots, & 8k-39, & 8k-31 & 1 \pmod 8\\
{\color{red}8k-27-8j:} & {\color{red}8k-27,} & {\color{red}8k-35,} & {\color{red}\ldots,} & {\color{red}13,} & {\color{red}5} & {\color{red}5 \pmod 8}\\
{\color{red}29+8j:} & {\color{red}29,} & {\color{red}37,} & {\color{red}\ldots,} & {\color{red}8k-11,} & {\color{red}8k-3} & {\color{red}5 \pmod 8}\\
32+8j: & 32, & 40, & \ldots, & 8k-8, & 8k & 0 \pmod 8.
\end{array}$$

Notice that we have a potential repetition of vertex labels in the two rows in red. If $k \equiv 7 \pmod 2$ and $j = (k-7)/2$, then we have eliminated the block $[0, 2k+1, 6k; 4k-27, {\color{red}4k+1}, {\color{red}4k+1}, 4k+4]_6^2$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, 2k+1, 6k; 4k-27, 4k-3, 4k+1, 4k]_6^2$ (which covers the same differences as the omitted block).

The blocks

$$\{[0, 16+4j, 8k-15-4j; 6+8j, 8k-29-8j, 27+8j, 31+8j]_6^2 \mid j = 0, 1, \ldots, k-4,$$

$$\text{and } j \neq (k-7)/2 \text{ if } k \equiv 7 \pmod 2\}$$

$$\cup\{[0, 2k+2, 6k-1; 4k-22, 4k-2, 4k-1, 4k+2]_6^2 \text{ if } k \equiv 7 \pmod 2 \text{ and } j = (k-7)/2\}$$

generate the following vertex labels (vertex labels for a given value of index j are all in the same column):

$$\begin{array}{rcccccc}
16+4j: & 16, & 20, & \ldots, & 4k-4, & 4k & 0 \pmod 4\\
8k-15-4j: & 8k-15, & 8k-19, & \ldots, & 4k+5, & 4k+1 & 1 \pmod 4\\
6+8j: & 6, & 14, & \ldots, & 8k-34, & 8k-26 & 6 \pmod 8\\
{\color{blue}8k-29-8j:} & {\color{blue}8k-29,} & {\color{blue}8k-37,} & {\color{blue}\ldots,} & {\color{blue}11,} & {\color{blue}3} & {\color{blue}3 \pmod 8}\\
{\color{blue}27+8j:} & {\color{blue}27,} & {\color{blue}35,} & {\color{blue}\ldots,} & {\color{blue}8k-13,} & {\color{blue}8k-5} & {\color{blue}3 \pmod 8}\\
31+8j: & 31, & 39, & \ldots, & 8k-9, & 8k-1 & 7 \pmod 8.
\end{array}$$

Notice that we have a potential repetition of vertex labels in the two rows in blue. If $k \equiv 7 \pmod 2$ and $j = (k-7)/2$, then we have eliminated the block $[0, 2k+2, 6k-1; 4k-22, 4k-1, 4k-1, 4k+3]_6^2$ that would arise in the first set (and which repeats vertex labels), and replaced it with the block $[0, 2k+2, 6k-1; 4k-22, 4k-2, 4k-1, 4k+2]_6^2$ (which covers the same differences as the omitted block). Therefore, all vertex labels are distinct.

In this chapter, we have given the necessary and sufficient conditions for the existence of of an S_6^i-decomposition of λM_v for $i \in \{0, 1, 2, 3, 4\}$. These results are given in Theorems 2.7, 2.7, 2.12, 2.13 and 2.17.

3 DECOMPOSITION OF COMPLETE BIPARTITE MIXED GRAPHS INTO MIXED STARS

3.1 Introduction

A graph G is *bipartite* if its vertex set can be partitioned into subsets X and Y, and such that every edge in G has one end in X and the other end in Y. That is if $V(G) = X \cup Y$ and $[x, y] \in E(G)$ then $x \in X$ and $y \in Y$ such that $X \neq \emptyset$, and $Y \neq \emptyset$, and $X \cap Y = \emptyset$. If every vertex of X is adjacent to every vertex of Y, the the graph is called a complete bipartite graph, denoted by $K_{n,m}$ where $|X| = n$ and $|Y| = n$.

The mixed graph with vertex set V such that for every pair of distinct vertices $x \in X$ and $y \in Y$, where $V = X \cup Y$, the set of edges and arcs contains (x, y), (y, x) and $[x, y]$ is called a *complete bipartite mixed graph*. For positive integers n_1, n_2, M_{n_1,n_2} denotes the complete bipartite mixed graph with partite sets of sizes n_1 and n_2.

The following are the few results on the decomposition of a complete bipartite graphs into stars. The complete bipartite graph $K_{n,m}$ has S_k-decomposition if and only if $(k-1)|n^2$ and $n \geq k-1$ [15]. The necessary and sufficient condition for the decomposition of the λ-fold complete bipartite graph into stars and cycles was given in [13]. See [8] and [16] for some decomposition of complete bipartite graphs. However, nothing has been done on the decomposition of a complete bipartite mixed graphs into mixed stars.

In this chapter, we consider the existence of the decomposition of complete bipartite mixed graphs into mixed stars by giving necessary and sufficient conditions.

3.2 An S_6^0-decomposition of M_{n_1,n_2}

We give the necessary and sufficient conditions of an S_6^0-decomposition of a complete bipartite mixed graph. Let M_{n_1,n_2} denote the complete bipartite mixed graph with partite sets of sizes n_1 and n_2. For an illustration of S_6^2, see Figure 6. Recall also that a decomposition of a mixed graph G is a family $\mathcal{F}$ of edge and arc-disjoint subgraph of G such that $\cup_{F\in\mathcal{F}}E(\mathcal{F}) = E(G)$ and $\cup_{F\in\mathcal{F}}A(\mathcal{F}) = A(G)$. The necessary condition for an S_6^0-decomposition of M_{n_1,n_2} is given in Lemma 3.1 below.

Lemma 3.1 *If an S_6^0-decomposition of M_{n_1,n_2} exists, then $n_1 = n_2 = 0$ $(mod\ 4)$.*

Proof. Each vertex of S_6^0 is of out-degree 4, so in M_{n_1,n_2} each vertex must be out-degree 0 (mod 4) and hence $n_1 \equiv n_2 \equiv 0$ (mod 4) is necessary.

Lemma 3.2 *An S_6^0-decomposition of $M_{8,8}$ exists.*

Proof. Consider the partite sets $X = [1_1, 2_1, 3_1, \ldots, 8_1]$ and $Y = [1_2, 2_2, 3_2, \ldots, 8_2]$. The required decomposition is given by the set of blocks: $\{\ [i_1, 1_2, 2_2; 3_2, 4_2, 5_2, 6_2]$, $[i_1, 5_2, 6_2; 1_2, 2_2, 7_2, 8_2]$ for $i = 1, 2, 5, 6\ \}$, $\{[i_1, 3_2, 4_2; 5_2, 6_2, 7_2, 8_2]$, $[i_1, 7_2, 8_2; 1_2, 2_2, 3_2, 4_2]$ for $i = 3, 4, 7, 8\ \}$, $\{[j_2, 3_1, 4_1; 5_1, 6_1, 7_1, 8_1]$, $[j_2, 7_1, 8_1; 1_1, 2_1, 3_1, 4_1]$ for $j = 1, 2, 5, 6\}$, and $\{\ [j_2, 1_1, 2_1; 3_1, 4_1, 5_1, 6_1]$, $[j_2, 5_1, 6_1; 1_1, 2_1, 7_1, 8_1]$ for $j = 3, 4, 7, 8\}$. $\square$

Lemma 3.3 *An S_6^0-decomposition of $M_{8,12}$ exists.*

Proof. Consider the partite sets $X = [1_1, 2_1, 3_1, \ldots, 8_1]$ and $Y = [1_2, 2_2, 3_2, \ldots, 12_2]$. We need 48 stars in this decomposition and the required decomposition is given by the set of blocks: $\{[i_1, 1_2, 2_2; 3_2, 4_2, 5_2, 6_2]$, $[i_1, 5_2, 6_2; 7_2, 8_2, 9_2, 10_2]$, $[i_1, 9_2, 10_2; 1_2, 2_2, 11_2, 12_2]$ for $i = 1, 2, 5, 6\ \} \cup \{[i_1, 3_2, 4_2; 5_2, 6_2, 7_2, 8_2]$, $[i_1, 7_2, 8_2; 9_2, 10_2, 11_2, 12_2]$, $[i_1, 11_2, 12_2; 1_2, 2_2, 3_2, 4_2]$ for $i = 3, 4, 7, 8\} \cup \{[j_2, 3_1, 4_1; 5_1, 5_1, 7_1, 8_1]$, $[j_2, 7_1, 8_1; 1_1, 2_1, 3_1, 4_1]$ for $j = 1, 2, 5, 6, 9, 10\} \cup \{[j_2, 1_1, 2_1; 3_1, 4_1, 5_1, 6_1]$, $[j_2, 5_1, 6_1; 1_1, 2_1, 7_1, 8_1]$ for $j = 3, 4, 7, 8, 11, 12\}$. □

Lemma 3.4 *An S_6^0-decomposition of $M_{12,12}$ exists.*

Proof. Consider the partite sets $X = [1_1, 2_1, 3_1, \ldots, 12_1]$ and $Y = [1_2, 2_2, 3_2, \ldots, 12_2]$. The required decomposition is given by the set of blocks: $\{[i_1, 1_2, 2_2; 3_2, 4_2, 5_2, 6_2]$, $[i_1, 5_2, 6_2; 7_2, 8_2, 9_2, 10_2]$, $[i_1, 9_2, 10_2; 1_2, 2_2, 11_2, 12_2]$ for $i = 1, 2, 5, 6, 9, 10\ \} \cup \{[i_1, 3_2, 4_2; 5_2, 6_2, 7_2, 8_2]$, $[i_1, 7_2, 8_2; 9_2, 10_2, 11_2, 12_2]$, $[i_1, 11_2, 12_2; 1_2, 2_2, 3_2, 4_2]$ for $i = 3, 4, 7, 8, 11, 12\} \cup \{[j_2, 3_1, 4_1; 5_1, 6_1, 7_1, 8_1]$, $[j_2, 7_1, 8_1; 9_1, 10_1, 11_1, 12_1]$, $[j_2, 11_1, 12_1; 1_1, 2_1, 3_1, 4_1]$ for $j = 1, 2, 5, 6, 9, 10\ \} \cup \{[j_2, 1_1, 2_1; 3_1, 4_1, 5_1, 6_1]$, $[j_2, 5_1, 6_1; 7_1, 8_1, 9_1, 10_1]$, $[j_2, 9_1, 10_1; 1_1, 2_1, 11_1, 12_1]$ for $j = 3, 4, 7, 8, 11, 12\ \}$. □

Theorem 3.5 *An S_6^0-decomposition of M_{n_1,n_2} exists if and only if $n_1 \equiv n_2 \equiv 0$ (mod 4), $n_1 \geq 8$ and $n_2 \geq 8$.*

Proof. Each vertex in M_{n_1,n_2} must be of out-degree 0 (mod 4) and hence $n_1 \equiv n_2 \equiv 0$ (mod 4) by Lemma 3.1. For sufficiency, suppose $n_1 \equiv n_2 \equiv 0$ (mod 4) where $n_1 \geq 8$ and $n_2 \geq 8$. This is established in 3 cases as follows:

Case 1. Suppose $n_1 \equiv n_2 \equiv 0$ (mod 8). Let $M = n_1/8$ and $N = n_2/8$. With

$$X_i = \{1_{1,i}, 2_{1,i}, \ldots, 8_{1,i}\} \text{ for } i = 1, 2, \ldots, M \text{ and}$$

$$Y_j = \{1_{2,j}, 2_{2,j}, \ldots, 8_{2,j}\} \text{ for } j = 1, 2, \ldots, N,$$

for all $i \in \{1, 2, \ldots, M\}$ and $j \in \{1, 2, \ldots, N\}$ there is an S_6^0-decomposition of the complete bipartite mixed graph with partite sets X_i and Y_j by Lemma 3.2 (since $|X_i| = |Y_j| = 8$). This collection of MN decompositions forms an S_6^1-decomposition of M_{n_1,n_2}. See Figure 7.

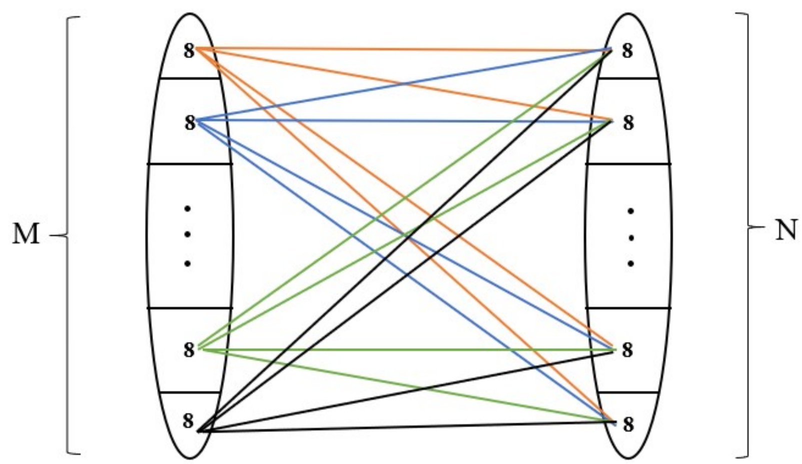

Figure 7: A schematic diagram of the copies of $M_{8,8}$ as used in case 1 of Theorem 3.5

<u>Case 2.</u> Suppose $n_1 \equiv n_2 \equiv 4 \pmod 8$, say $M = (n_1 - 12)/8$ and $N = (n_2 - 12)/8$. Let

$$X_i = \{1_{1,i}, 2_{1,i}, \ldots, 8_{1,i}\} \text{ for } i = 1, 2, \ldots, M,$$

$$Y_j = \{1_{2,j}, 2_{2,j}, \ldots, 8_{2,j}\} \text{ for } j = 1, 2, \ldots, N,$$

$$X_{M+1} = \{1_{1,M+1}, 2_{1,M+1}, \ldots, 12_{1,M+1}\} \text{ and } Y_j = \{1_{2,N+1}, 2_{2,N+1}, \ldots, 12_{2,N+1}\}.$$

An S_6^0-decomposition of the complete bipartite mixed graph with partite sets $\cup_{i=1}^{M} X_i$

and $\cup_{j=1}^{N} Y_j$ exists by Case 1 (since $\left|\cup_{i=1}^{M} X_i\right| \equiv \left|\cup_{j=1}^{N} Y_j\right| \equiv 0 \pmod{8}$). An S_6^0-decomposition of the complete bipartite mixed graph with partite sets X_i and Y_{N+1} exists for each $i = 1, 2, \ldots, M$ by Lemma 3.3 (since $|X_i| = 8$ and $|Y_{N+1}| = 12$). An S_6^0-decomposition of the complete bipartite mixed graph with partite sets Y_j and X_{M+1} exists for each $j = 1, 2, \ldots, N$ by Lemma 3.3 (since $|Y_j| = 8$ and $|X_{M+1}| = 12$). An S_6^0-decomposition of the complete bipartite mixed graph with partite sets X_{M+1} and Y_{N+1} exists by Lemma 3.4 (since $|X_{M+1}| = 12$ and $|Y_{N+1}| = 12$). This collection of decompositions form an S_6^0-decomposition of M_{n_1,n_2}. See Figure 9.

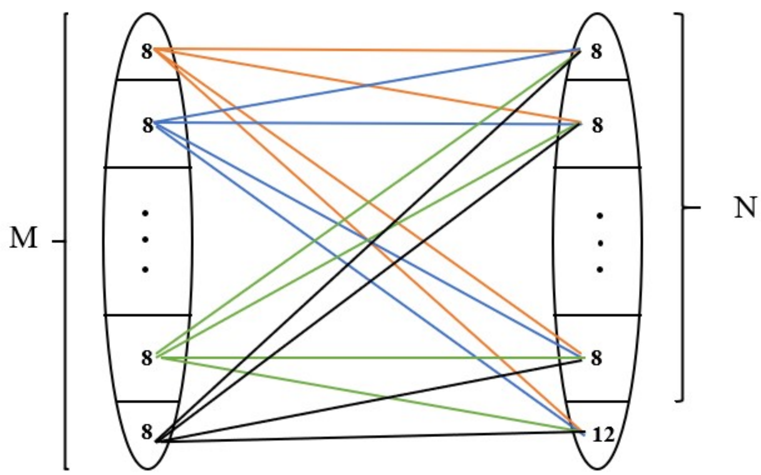

Figure 8: A schematic diagram of the copies of $M_{8,8}$, and $M_{12,8}$, $M_{8,12}$ as used in case 2 of Theorem 3.5

<u>Case 3.</u> Suppose $n_1 \equiv 0 \pmod{8}$ and $n_2 \equiv 4 \pmod{8}$, say $M = n_1/8$ and $N = (n_2 - 12)/8$. Let X_i for $i = 1, 2, \ldots, M$ be as defined in Case 1, and let Y_j for $j = 1, 2, \ldots, N+1$ be as defined in Case 2. An S_6^0-decomposition of the

complete bipartite mixed graph with partite sets $\cup_{i=1}^{M} X_i$ and $\cup_{j=1}^{N} Y_j$ exists by Case 1 (since $\left|\cup_{i=1}^{M} X_i\right| \equiv 0 \pmod 8$ and $|\cup_{j=1}^{N} Y_j| \equiv 0 \pmod 8$)). An S_6^0-decomposition of the complete bipartite mixed graph with partite sets X_i and Y_{N+1} exists for each $i = 1, 2, \ldots, M$ by Lemma 3.3 (since $|X_i| = 8$ and $|Y_{N+1}| = 12$). This collection of decompositions form an S_6^0-decomposition of M_{n_1,n_2}. See Figure 8. □

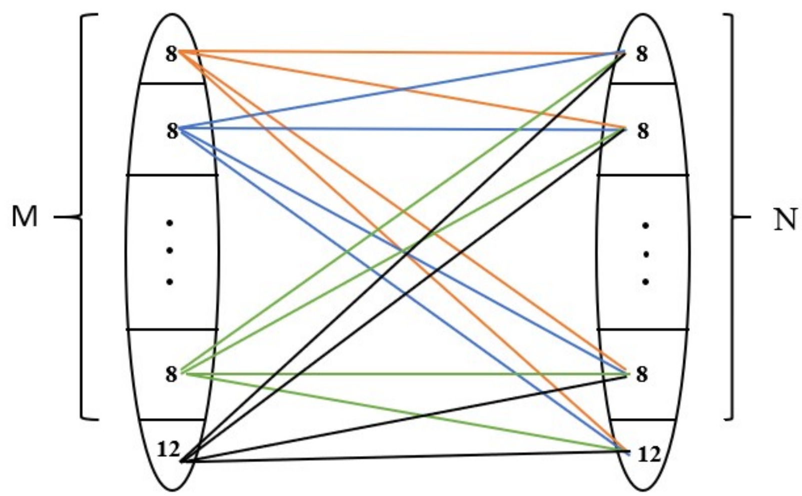

Figure 9: A schematic diagram of the copies of $M_{8,8}$, and $M_{12,8}$, $M_{8,12}$ and $M_{12,12}$ as used in case 3 of Theorem 3.5

Notice that the converse of S_6^0 is obtained by reversing the orientation of all the arcs which gives S_6^4. Since M_{n_1,n_2} is self converse, Theorem 3.5 also gives the necessary and sufficient conditions for S_6^4-decomposition of M_{n_1,n_2}.

3.3 An S_6^1-decomposition of M_{n_1,n_2}

Here we give some conditions for the existence of an S_6^1-decomposition of M_{n_1,n_2}.

Lemma 3.6 *If an S_6^1-decomposition of M_{n_1,n_2} exists, then $n_1 n_2 \equiv 0$ (mod 4).*

Proof. Let the partite sets of M_{n_1,n_2} be $X = \{1_1, 2_1, \ldots (n_1)_1\}$ and $Y = \{1_2, 2_2, \ldots (n_2)_2\}$. Define L-type and R-type stars as follows (Figure 10): The L-star is a star with partite sets $\{c_1\}$ and $\{u_2, v_2, w_2, x_2, y_2, z_2\}$ with the center c_1 "on the left" and the other partite sets with subscript 2 "on the right". Denoted $[c_1, u_2, v_2; w_2, x_2, y_2, z_2]$ with our usual notation. Similarly, the R-star is a star with partite sets $\{c_2\}$ and $\{u_1, v_1, w_1, x_1, y_1, z_1\}$ with the center c_2 "on the right" and other partite sets with subscript 2 "on the left". Denoted $[c_2, u_1, v_1; w_1, x_1, y_1, z_1]$.

Suppose an S_6^1-decomposition of M_{n_1,n_2} exists. Let L and R as the number of L-type and R-type stars in a decomposition, respectively. Now M_{n_1,n_2} has $n_1 n_2$ edges, $n_1 n_2$ left-to-right arcs (that is of the form $[v_1, v_2]$), and $n_1 n_2$ right-to-left arcs (that is $[v_2, v_1]$). Since each star contains two edges then $2L + 2R = n_1 n_2$. An L-type star has one right-to-left arc ($\leftarrow$) and an R-type star has three left-to-right arcs ($\leftarrow$). So $L + 3R = n_1 n_2$. An L-type star has three right-to-left arcs ($\leftarrow$) and an R-type star has one right-to-left arc ($\leftarrow$). So $3L + R = n_1 n_2$. Hence $n_1 n_2 = L + 3R = 3L + R$ or $2R = 2L$ or $R = L$. Also $n_1 n_2 = 2L + 2R = 4L$ and $n_1 n_2 \equiv 0$ (mod 4), as claimed. $\square$

Lemma 3.7 *An S_6^1-decomposition of $M_{8,7}$ does not exist.*

Proof. $M_{8,7}$ has 56 edges and S_6^1 has 2 edges, so an S_6^1-decomposition of $M_{8,7}$ requires 28 stars. This implies that $L = R = 14$ by Lemma 3.5. Then there is a vertex v on

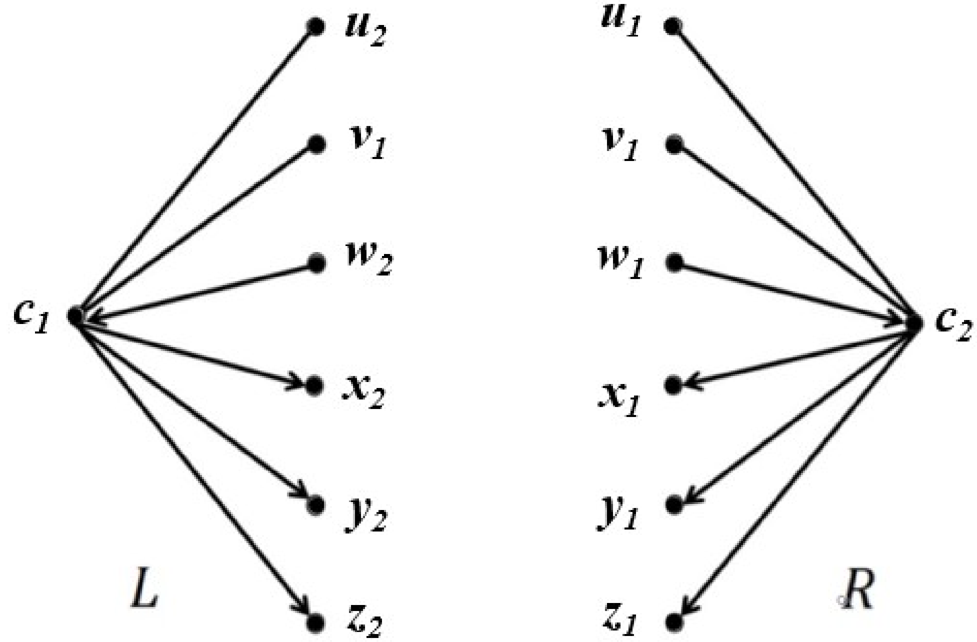

Figure 10: L-type and R-type star

the left which is the center of at most one star. This vertex v has a total degree of 21 and so must be in the corona of $21 - 5 = 15$ R-stars. But there is only 14 R-stars. Therefore an S_6^1-decomposition of $M_{8,7}$ does not exist. $\square$

Lemma 3.8 *If an S_6^1-decomposition of $M_{n,6}$ exists, then $n \equiv 0$ (mod 4).*

Proof. We know that the number of L-stars equals the number of R-stars in the decomposition by Lemma 3.6. The total number of edges in $M_{n,6}$ is $6n$. Since each star contains 2 edges then there must be a total of $3n$ stars in the decomposition;

$3n/2$ of them are L-stars and $3n/2$ of them are R-stars.

Define the *total degree* of a vertex in a mixed graph as the edge degree plus in-degree plus out-degree of the vertex. In the subgraph of $M_{n,6}$ induced by the R-stars, each of the R-vertices must be of total degree a multiple of 6. Now each L-star contributes exactly one edge or one arc to each of the R-vertices. So the number of L-stars in a decomposition must be a multiple of 6. Hence it is necessary that $3n/2 \equiv 0 \pmod 6$. That is, $n \equiv 0 \pmod 4$ is necessary, as claimed. $\square$

Lemma 3.9 *An S_6^1-decomposition of $M_{8,8}$ exists.*

Proof. Let the complete bipartite mixed graph have partite sets $\{0_1, 1_1, \ldots, 7_1\}$ and $\{0_2, 1_2, \ldots, 7_2\}$. Consider the blocks:

$\{[i_1, i_2, (i+1)_2; (i+2)_2, (i+3)_2, (i+4)_2, (i+5)_2]_6^1, [i_1, (i+2)_2, (i+3)_2; (i+1)_2, i_2, (i+6)_2, (i+7)_2]_6^1, [i_2, (i+3)_1, (i+4)_1; (i+7)_1, i_1, (i+1)_1, (i+2)_1]_6^1, [i_2, (i+1)_1, (i+2)_1; (i+6)_1, (i+3)_1, (i+4)_1, (i+5)_1]_6^1 \ |i = 0, 1, 2, \ldots, 7\}$

where vertex labels are reduced modulo 8. These form an S_6^1-decomposition of $M_{8,8}$. $\square$

Lemma 3.10 *An S_6^1-decomposition of $M_{12,12}$.*

Proof. Let the complete bipartite mixed graph have partite sets $\{0_1, 1_1, \ldots, 11_1\}$ and $\{0_2, 1_2, \ldots, 11_2\}$. Consider the blocks:

$\{[i_1, i_2, (i+1)_2; (i+2)_2, (i+3)_2, (i+4)_2, (i+5)_2]_6^1, [i_1, (i+2)_2, (i+3)_2; (i+1)_2, (i+6)_2, (i+7)_2, (i+8)_2]_6^1], [i_1, (i+4)_2, (i+5)_2; i_2, (i+9)_2, (i+10)_2, (i+11)_2]_6^1], [i_2, (i+5)_1, (i+6)_1; i_1, (i+1)_1, (i+2)_1, (i+3)_1]_6^1, [i_2, (i+1)_1, (i+2)_1; (i+11)_1, (i+4)_1, (i+$

$5)_1, (i{+}6)_1]^1_6, [i_2, (i{+}3)_1, (i{+}4)_1; (i{+}10)_1, (i{+}7)_1, (i{+}8)_1, (i{+}9)_1]^1_6 \; | i = 0, 1, 2, \ldots, 11\}$

where vertex labels are reduced modulo 12. These form an S_6^1-decomposition of $M_{12,12}$.

$\square$

Lemma 3.11 *An S_6^1-decomposition of $M_{8,6}$ exists.*

Proof. Let the partite sets of $M_{8,6}$ be $\{0_1, 1_1, \ldots, 7_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2, 5_2\}$. Consider the collection of mixed stars S_6^1:

$[0_1, 0_2, 1_2; 4_2, 2_2, 3_2, 5_2]^1_6$, $[0_1, 2_2, 3_2; 5_2, 0_2, 1_2, 4_2]^1_6$, $[1_1, 0_2, 1_2; 4_2, 2_2, 3_2, 5_2]^1_6$,

$[1_1, 2_2, 3_2; 5_2, 0_2, 1_2, 4_2]^1_6$, $[2_1, 0_2, 1_2; 2_2, 3_2, 4_2, 5_2]^1_6$, $[2_1, 4_2, 5_2; 3_2, 0_2, 1_2, 2_2]^1_6$,

$[3_1, 0_2, 1_2; 2_2, 3_2, 4_2, 5_2]^1_6$, $[3_1, 4_2, 5_2; 3_2, 0_2, 1_2, 2_2]^1_6$, $[4_1, 2_2, 3_2; 0_2, 1_2, 4_2, 5_2]^1_6$,

$[5_1, 2_2, 3_2; 0_2, 1_2, 4_2, 5_2]^1_6$, $[6_1, 4_2, 5_2; 1_2, 0_2, 2_2, 3_2]^1_6$, $[7_1, 4_2, 5_2; 1_2, 0_2, 2_2, 3_2]^1_6$,

$[0_2, 5_1, 6_1; 4_1, 2_1, 3_1, 7_1]^1_6$, $[0_2, 4_1, 7_1; 5_1, 0_1, 1_1, 6_1]^1_6$, $[1_2, 5_1, 7_1; 6_1, 2_1, 3_1, 4_1]^1_6$,

$[1_2, 4_1, 6_1; 7_1, 0_1, 1_1, 5_1]^1_6$, $[2_2, 2_1, 3_1; 4_1, 5_1, 6_1, 7_1]^1_6$, $[2_2, 6_1, 7_1; 5_1, 0_1, 1_1, 4_1]^1_6$,

$[3_2, 2_1, 3_1; 4_1, 5_1, 6_1, 7_1]^1_6$, $[3_2, 6_1, 7_1; 5_1, 0_1, 1_1, 4_1]^1_6$, $[4_2, 0_1, 1_1; 6_1, 4_1, 5_1, 7_1]^1_6$,

$[4_2, 4_1, 5_1; 7_1, 2_1, 3_1, 6_1]^1_6$, $[5_2, 0_1, 1_1; 6_1, 4_1, 5_1, 7_1]^1_6$, $[5_2, 4_1, 5_1; 7_1, 2_1, 3_1, 6_1]^1_6$.

These stars form the desired decomposition. $\square$

Lemma 3.12 *An S_6^1-decomposition of M_{n_1,n_2} exists for all $n_1 \equiv 0$ (mod 8) and $n_2 \equiv 0$ (mod 6).*

Proof. Suppose $n_1 \equiv 0 \pmod 8$ and $n_2 \equiv 0 \pmod 6$. Let $M = n_1/8$ and $N = n_2/6$. With

$$X_i = \{1_{1,i}, 2_{1,i}, \ldots, 8_{1,i}\} \text{ for } i = 1, 2, \ldots, M \text{ and}$$

$$Y_j = \{1_{2,j}, 2_{2,j}, \ldots, 6_{2,j}\} \text{ for } j = 1, 2, \ldots, N,$$

for all $i \in \{1, 2, \ldots, M\}$ and $j \in \{1, 2, \ldots, N\}$ there is an S_6^1-decomposition of the complete bipartite mixed graph with partite sets X_i and Y_j by Lemma 3.11 (since $|X_i| = 8$ and $|Y_j| = 6$). This collection of MN decompositions forms an S_6^1-decomposition of M_{n_1,n_2} for all $n_1 \equiv 0 \pmod 8$ and $n_2 \equiv 0 \pmod 6$. $\square$

Lemma 3.13 *An S_6^1-decomposition of $M_{8,12}$ exists.*

Proof. Let the partite sets of $M_{8,12}$ be $\{0_1, 1_1, \ldots, 7_1\}$ and $\{0_2, 1_2, \ldots, 11_2\}$. Consider the collection of mixed stars S_6^1:

$[0_1, 0_2, 1_2; 4_2, 2_2, 3_2, 5_2]_6^1, [0_1, 2_2, 3_2; 5_2, 0_2, 1_2, 4_2]_6^1, [0_1, 6_2, 7_2; 10_2, 8_2, 9_2, 11_2]_6^1,$

$[0_1, 8_2, 9_2; 11_2, 6_2, 7_2, 10_2]_6^1, [1_1, 0_2, 1_2; 4_2, 2_2, 3_2, 5_2]_6^1, [1_1, 2_2, 3_2; 5_2, 0_2, 1_2, 4_2]_6^1,$

$[1_1, 6_2, 7_2; 10_2, 8_2, 9_2, 11_2]_6^1, [1_1, 8_2, 9_2; 11_2, 6_2, 7_2, 10_2]_6^1, [2_1, 0_2, 1_2; 2_2, 3_2, 4_2, 5_2]_6^1,$

$[2_1, 4_2, 5_2; 3_2, 0_2, 1_2, 2_2]_6^1, [2_1, 6_2, 7_2; 8_2, 9_2, 10_2, 11_2]_6^1, [2_1, 10_2, 11_2; 9_2, 6_2, 7_2, 8_2]_6^1,$

$[3_1, 0_2, 1_2; 2_2, 3_2, 4_2, 5_2]_6^1, [3_1, 4_2, 5_2; 3_2, 0_2, 1_2, 2_2]_6^1, [3_1, 6_2, 7_2; 8_2, 9_2, 10_2, 11_2]_6^1,$

$[3_1, 10_2, 11_2; 9_2, 6_2, 7_2, 8_2]_6^1, [4_1, 2_2, 3_2; 0_2, 1_2, 4_2, 5_2]_6^1, [4_1, 8_2, 9_2; 6_2, 7_2, 10_2, 11_2]_6^1,$

$[5_1, 2_2, 3_2; 0_2, 1_2, 4_2, 5_2]_6^1, [5_1, 8_2, 9_2; 6_2, 7_2, 10_2, 11_2]_6^1, [6_1, 4_2, 5_2; 1_2, 0_2, 2_2, 3_2]_6^1,$

$[6_1, 10_2, 11_2; 7_2, 6_2, 8_2, 9_2]_6^1, [7_1, 4_2, 5_2; 1_2, 0_2, 2_2, 3_2]_6^1, [7_1, 10_2, 11_2; 7_2, 6_2, 8_2, 9_2]_6^1,$

$[0_2, 5_1, 6_1; 4_1, 2_1, 3_1, 7_1]_6^1, [0_2, 4_1, 7_1; 5_1, 0_1, 1_1, 6_1]_6^1, [6_2, 5_1, 6_1; 4_1, 2_1, 3_1, 7_1]_6^1,$

$[6_2, 4_1, 7_1; 5_1, 0_1, 1_1, 6_1]_6^1, [1_2, 5_1, 7_1; 6_1, 2_1, 3_1, 4_1]_6^1, [1_2, 4_1, 6_1; 7_1, 0_1, 1_1, 5_1]_6^1,$

$[7_2, 5_1, 7_1; 6_1, 2_1, 3_1, 4_1]_6^1, [7_2, 4_1, 6_1; 7_1, 0_1, 1_1, 5_1]_6^1, [2_2, 2_1, 3_1; 4_1, 5_1, 6_1, 7_1]_6^1,$

$[2_2, 6_1, 7_1; 5_1, 0_1, 1_1, 4_1]_6^1, [8_2, 2_1, 3_1; 4_1, 5_1, 6_1, 7_1]_6^1, [8_2, 6_1, 7_1; 5_1, 0_1, 1_1, 4_1]_6^1,$

$[3_2, 2_1, 3_1; 4_1, 5_1, 6_1, 7_1]_6^1, [3_2, 6_1, 7_1; 5_1, 0_1, 1_1, 4_1]_6^1, [9_2, 2_1, 3_1; 4_1, 5_1, 6_1, 7_1]_6^1,$

$[9_2, 6_1, 7_1; 5_1, 0_1, 1_1, 4_1]_6^1, [4_2, 0_1, 1_1; 6_1, 4_1, 5_1, 7_1]_6^1, [4_2, 4_1, 5_1; 7_1, 2_1, 3_1, 6_1]_6^1,$

$[10_2, 0_1, 1_1; 6_1, 4_1, 5_1, 7_1]_6^1, [10_2, 4_1, 5_1; 7_1, 2_1, 3_1, 6_1]_6^1, [5_2, 0_1, 1_1; 6_1, 4_1, 5_1, 7_1]_6^1,$

$[5_2, 4_1, 5_1; 7_1, 2_1, 3_1, 6_1]^1_6, [11_2, 0_1, 1_1; 6_1, 4_1, 5_1, 7_1]^1_6, [11_2, 4_1, 5_1; 7_1, 2_1, 3_1, 6_1]^1_6,$

These stars form the desired decomposition. □

We now give some general conditions for the existence of an S^1_6-decomposition of M_{n_1,n_2}.

Theorem 3.14 *An S^1_6-Decomposition of M_{n_1,n_2} exists for $n_1 \equiv n_2 \equiv 0$ (mod 4) where $n_1 \geq 8$, and $n_2 \geq 8$.*

Proof. We consider three cases.

<u>Case 1.</u> Suppose $n_1 \equiv n_2 \equiv 0$ (mod 8). Let $M = n_1/8$ and $N = n_2/8$. With

$$X_i = \{1_{1,i}, 2_{1,i}, \ldots, 8_{1,i}\} \text{ for } i = 1, 2, \ldots, M \text{ and}$$

$$Y_j = \{1_{2,j}, 2_{2,j}, \ldots, 8_{2,j}\} \text{ for } j = 1, 2, \ldots, N,$$

for all $i \in \{1, 2, \ldots, M\}$ and $j \in \{1, 2, \ldots, N\}$ there is an S^1_6-decomposition of the complete bipartite mixed graph with partite sets X_i and Y_j by Lemma 3.9 (since $|X_i| = |Y_j| = 8$). This collection of MN decompositions forms an S^1_6-decomposition of M_{n_1,n_2}. See Figure 7 again.

<u>Case 2.</u> Suppose $n_1 \equiv n_2 \equiv 4$ (mod 8), say $M = (n_1 - 12)/8$ and $N = (n_2 - 12)/8$. Let

$$X_i = \{1_{1,i}, 2_{1,i}, \ldots, 8_{1,i}\} \text{ for } i = 1, 2, \ldots, M,$$

$$Y_j = \{1_{2,j}, 2_{2,j}, \ldots, 8_{2,j}\} \text{ for } j = 1, 2, \ldots, N,$$

$$X_{M+1} = \{1_{1,M+1}, 2_{1,M+1}, \ldots, 12_{1,M+1}\} \text{ and } Y_j = \{1_{2,N+1}, 2_{2,N+1}, \ldots, 12_{2,N+1}\}.$$

An S_6^1-decomposition of the complete bipartite mixed graph with partite sets $\cup_{i=1}^{M} X_i$ and $\cup_{j=1}^{N} Y_j$ exists by Case 1 (since $\left|\cup_{i=1}^{M} X_i\right| \equiv \left|\cup_{j=1}^{N} Y_j\right| \equiv 0 \pmod 8$). An S_6^1-decomposition of the complete bipartite mixed graph with partite sets X_i and Y_{N+1} exists for each $i = 1, 2, \dots, M$ by Lemma 3.13 (since $|X_i| = 8$ and $|Y_{N+1}| = 12$). An S_6^1-decomposition of the complete bipartite mixed graph with partite sets Y_j and X_{M+1} exists for each $j = 1, 2, \dots, N$ by Lemma 3.13 (since $|Y_j| = 8$ and $|X_{M+1}| = 12$). An S_6^1-decomposition of the complete bipartite mixed graph with partite sets X_{M+1} and Y_{N+1} exists by Lemma 3.10 (since $|X_{M+1}| = 12$ and $|Y_{N+1}| = 12$). This collection of decompositions form an S_6^1-decomposition of M_{n_1,n_2}. See Figure 9 again.

<u>Case 3.</u> Suppose $n_1 \equiv 0 \pmod 8$ and $n_2 \equiv 4 \pmod 8$, say $M = n_1/8$ and $N = (n_2 - 12)/8$. Let X_i for $i = 1, 2, \dots, M$ be as defined in Case 1, and let Y_j for $j = 1, 2, \dots, N+1$ be as defined in Case 2. An S_6^1-decomposition of the complete bipartite mixed graph with partite sets $\cup_{i=1}^{M} X_i$ and $\cup_{j=1}^{N} Y_j$ exists by Case 1 (since $\left|\cup_{i=1}^{M} X_i\right| \equiv 0 \pmod 8$ and $|\cup_{j=1}^{N} Y_j| \equiv 0 \pmod 8$). An S_6^1-decomposition of the complete bipartite mixed graph with partite sets X_i and Y_{N+1} exists for each $i = 1, 2, \dots, M$ by Lemma 3.13 (since $|X_i| = 8$ and $|Y_{N+1}| = 12$). This collection of decompositions form an S_6^1-decomposition of M_{n_1,n_2}. See Figure 8 again. □

This result gives only the existence of S_6^1-decomposition of M_{n_1,n_2} where $n_1 \equiv n_2 \equiv 0 \pmod 4$, and where $n_1 \equiv 0 \pmod 8$ and $n_2 \equiv 0 \pmod 6$. The case for $n_1 \equiv n_2 \equiv 2 \pmod 4$ should also be considered for future research. We leave the case where either n_1 or n_2 is odd unaddressed, with the exception of an S_6^1-decomposition of $M_{8,7}$ which does not exist by Lemma 3.7.

3.4 An S_6^2-decomposition of M_{n_1,n_2}

Recall that a S_6^2-block with vertex set $\{b, c, g, d, f, e\}$ will be denoted by $[a, b, d; d, e, f, g]$ as illustrated in Figure 6. Let M_{n_1,n_2} be defined as in the previous section. A necessary condition for the existence of an S_6^2-decomposition of the complete mixed graph M_{n_1,n_2} is that one of n_1 and n_2 must be even. This is because in M_{n_1,n_2}, there are n_1n_2 edges and S_6^2 has 2 edges. So if a decomposition exists then we need $2|n_1n_2$. That is, at least one of n_1, n_2 must be even. As an example, Figure 11 shows an S_6^2-decomposition of $M_{1,6}$.

Lemma 3.15 *An S_6^2-decomposition of $M_{1,8}$ exists.*

Proof. Consider the partite sets $X = [1_1]$ and $Y = [1_2, 2_2, 3_2, \ldots, 8_2]$. Consider the blocks:

$[1_1, 1_2, 2_2; 3_2, 4_2, 5_2, 6_2], [1_1, 3_2, 4_2; 5_2, 6_2, 7_2, 8_2], [1_1, 5_2, 6_2; 7_2, 8_2, 1_2, 2_2]$

$[1_1, 7_2, 8_2; 1_2, 2_2, 3_2, 4_2]$. $\square$

Therefore we can decompose $M_{k,6}$ by taking k copies of the $M_{1,6}$ case. Similarly, we can decompose $M_{k,2l}$ by taking copies of the $M_{k,2l}$ case for all $k \in \mathbb{N}$, $l \in \mathbb{N}$ and $l \geq 3$.

Lemma 3.16 *An S_6^2-decomposition of $M_{2,7}$ does not exist.*

Proof. First $M_{2,7}$ has 14 edges. This implies that an S_6^2-decomposition of $M_{2,7}$ requires 7 copies of S_6^2. Now, consider the partite sets $X = [1_1, 2_1]$ and $Y = [1_2, 2_2, \ldots, 7_2]$. Each vertex in X has degree 7. Since we only have two vertices

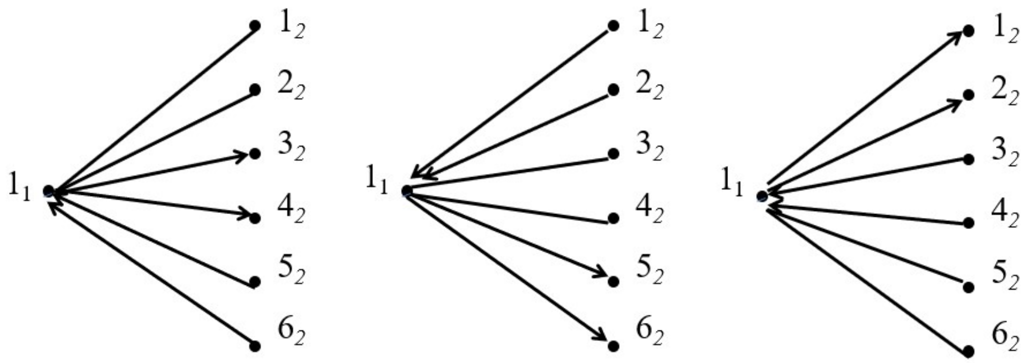

Figure 11: S_6^2-decomposition of $M_{1,6}$

in X, then each must be the centre of a star in the decomposition, and so is even degree in each star. So no decomposition exists. $\square$

The same result holds for $M_{4,7}$. In fact, the same argument holds for any M_{2,n_2} and M_{4,n_2}, where n_2 is odd.

Theorem 3.17 *An S_6^2-decomposition of M_{n_1,n_2} exists if and only if $n_1 \in \mathbb{N}$, $n_2 \equiv 0$ (mod 2), and $n_2 \geq 6$ (where n_1 and n_2 can be interchanged).*

Proof. M_{n_1,n_2} has $n_1 n_2$ edges and S_6^2 has two edges. So if a decomposition exists, then one of n_1 or n_2 must be even (say n_2). Since S_6^2 is a bipartite mixed graph with one partite set of size 6, then either n_1 or $n_2 \geq 6$. Suppose $n_2 \in \{2, 4\}$ and $n_1 \geq 6$. Then in an S_6^2-decomposition of M_{n_1,n_2}, we must have the centre of each S_6^2 as an element of Y (since $|Y| \leq 4$). Now each vertex of Y has edge degree n_1; the edge degree of the centre of S_6^2 is 2. So in an S_6^2-decomposition, n_1 must be even when $n_2 \in \{2, 4\}$ (Lemma 3.1).

For sufficiency, we consider the following cases where

1. $n_2 = \{2, 4\}$ and $n_1 \geq 6$ is even

2. $n_2 \geq 6$ is even and $n_1 \in \mathbb{N}$.

Since the decomposition exist for $M_{k,2l}$ for any $k \in \mathbb{N}$ and $l \geq 3$, then the result holds for $n_2 = 2$ and $n_1 \geq 6$ is even, if we take $k = n_2 = 2$ and $2l = n_1$. Similarly for $k = n_2 = 4$ and $2l = n_1$. Also since we can decompose $M_{k,2l}$ for any $k \in \mathbb{N}$ and $l \geq 3$, then S_6^2-decomposition of M_{n_1,n_2} exist if we take $k = n_1 \in \mathbb{N}$ and $2l = n_2$ for $l \geq 3$. $\square$

In conclusion, we have given necessary and sufficient conditions for an S_6^i-decomposition of M_{n_1,n_2} for $i \in \{0, 2, 4\}$ in Theorems 3.5 and 3.17. We have also given some S_6^1-decomposition of M_{n_1,n_2} (and hence some S_6^3-decomposition of M_{n_1,n_2}) in Theorem 3.14.

4 SOME MIXED STAR DECOMPOSITIONS OF COMPLETE MIXED GRAPHS WITH A HOLE AND CONCLUSIONS

4.1 Introduction

We recall that a mixed graph on v vertices is a graph consisting of a set of ordered and unordered pairs, denoted by (x, y) and $[x, y]$ respectively. The ordered pair (x, y) is called an *arc* and the unordered pair $[x, y]$ is called an *edge*. We also recall that the *complete mixed graph* on v vertices, denoted by M_v, is the mixed graph in which for every two distinct vertices x and y, we have the following (x, y), (y, x) and $[x, y]$. The *complete mixed graph on v vertices with a hole of size w* is the mixed graph with with vertex $V = V_{v-w} \cup V_w$, where $|V_{v-w}| = v - w$, $|V_w| = w$, and $V_{v-w} \cap V_w = \varnothing$, with edge set

$$E = \{ab \mid a \neq b, \{a, b\} \subset V \text{ and } \{a, b\} \not\subset V_w\}$$

and arc set

$$A = \{(a, b), (b, a) \mid a \neq b, \{a, b\} \subset V \text{ and } \{a, b\} \not\subset V_w\}.$$

This mixed graph is denoted $M(v, w)$. This is obtained by taking a complete mixed graph on v vertices and removing the edges and arc of a complete mixed graph on w vertices.

An example of a recently presented decomposition of complete graph with a hole can be found in [1]. A C_4 decomposition of $K(v, w)$ exists if and only if $v - w \equiv 0 \pmod 8$ and $w \equiv 1 \pmod 2$ [6]. See [5, 6, 14] for the decomposition of $K(v, w)$ into m-cycles for $m \in \{3, 4, 5, 6, 7, 8, 10, 12, 14\}$. Notice that when $m = 3$, this is equivalent to a Steiner triple system with a hole.

4.2 Some S_6^i-decompositions of $M(v, w)$.

Notice that we can decompose $M(v, w)$ into a complete mixed graph M_{v-w} and a complete bipartite mixed graph $M_{v-w,w}$. This observation allows us to use the results of Chapters 2 and 3 to get some S_6^i-decompositions of $M(v, w)$. Thus, the following results show some of the S_6^i-decompositions of complete mixed graphs with a hole.

Lemma 4.1 *An S_6^1-decomposition of $M(v, w)$ exists for $v-w \equiv 0$ (mod 8) and $w \equiv 0$ (mod 6).*

Proof. We can decompose $M(v, w)$ into M_{v-w} and $M_{v-w,w}$. Since $v - w \equiv 0$ (mod 8) then by [10] there exists an S_6^1-decomposition of M_{v-w}. Since $v - w \equiv 0$ (mod 8) and $w \equiv 0$ (mod 6) then by Lemma 3.12 there exists an S_6^1-decomposition of $M_{v-w,w}$. These two decompositions together give an S_6^1-decomposition of $M_{v,w}$, as claimed. □

Lemma 4.2 *An S_6^1-Decomposition of $M(v, w)$ exists for $v - w \equiv 0$ (mod 4), $w \equiv 0$ (mod 4), $v - w \geq 12$, and $w \geq 8$.*

Proof. We can decompose $M(v, w)$ into M_{v-w} and $M_{v-w,w}$. Since $v - w \equiv 0$ (mod 4) and $v - w \geq 12$ then by [10] there exists an S_6^1-decomposition of M_{v-w}. Since $v - w \equiv 0$ (mod 4), $w \equiv 0$ (mod 4), and $v - w \geq 8$, then by Theorem 3.14 there exists and S_6^1-decomposition of $M_{v-w,w}$. These two decompositions together give an S_6^1-decomposition of $M(v, w)$, as claimed. □

Recall that the converse of S_6^1 is obtained by reversing the orientation of all the arcs which gives S_6^3. Since $M_{v,w}$ is self converse, therefore Lemma 4.1 and Lemma 4.2 imply S_6^3 decomposition of $M(v, w)$.

Lemma 4.3 *An S_6^2-Decomposition of $M(v,w)$ exists for $v - w \equiv 0$ or 1 (mod 4), $w \equiv 0$ (mod 2), $v - w \geq 8$, and $w \geq 6$.*

Proof. We can decompose $M_{v,w}$ into M_{v-w} and $M_{v-w,w}$. Since $v - w \equiv 0$ or 1 (mod 4) and $v - w \geq 8$ then by [10] there exists an S_6^1-decomposition of M_{v-w}. Since $v - w \equiv 0$ or 1 (mod 4), $w \equiv 0$ (mod 2), and $w \geq 6$, then by Theorem 3.17 there exists an S_6^2-decomposition of $M_{v-w,w}$. These two decompositions together give an S_6^2-decomposition of $M_{v,w}$, as claimed. □

Since $M_{v,w}$ is self converse, therefore Lemma 4.3 also implies S_6^4 decomposition of $M_{v,w}$.

4.3 Future Research

In future research, one could explore necessary and sufficient conditions for the existence of an S_6^1-decomposition of the complete bipartite mixed graphs. This would complete the results for S_6^1-decomposition we give in Chapter 3. Necessary and sufficient conditions for S_6^i-decomposition of $\lambda M_{n_1,n_2}$ and $\lambda M(v,w)$ are a largely unexplored topic.

4.4 Conclusion

We have given decompositions of various complete mixed graphs into isomorphic copies of partial orientations of 6-stars which have twice as many arcs as edges.

In Chapter 1 we gave necessary and sufficient conditions for an S_6^i-decompositions of λM_v for all λ and each $i \in \{1, 2, 3, 4\}$, in Chapter 2, we gave necessary and sufficient conditions for an S_6^i-decompositions of M_{n_1,n_2} for $i \in \{0,2,4\}$ and gave some results concerning such decompositions for $i \in \{1,3\}$. In Chapter 4 we used the results from Chapter 2 and Chapter 3 to get some easy S_6^i-decompositions of $M(v,w)$ for $i \in \{1,2,3\}$.

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