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Decompositions of the Complete Mixed Graph by Mixed Stars

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Decompositions of the Complete Mixed Graph by Mixed Stars

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ABSTRACT

Decomposition on Complete Mixed Graphs by Mixed Stars

by

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In the study of mixed graphs, a common question is: What are the necessary and sufficient conditions for the existence of a decomposition of the complete mixed graph into isomorphic copies of a given mixed graph? Since the complete mixed graph has twice as many arcs as edges, then an obvious necessary condition is that the isomorphic copies have twice as many arcs as edges. We will prove necessary and sufficient conditions for the existence of a decomposition of the complete mixed graphs into mixed stars with two edges and four arcs. We also consider some special cases of decompositions of the complete mixed graph into partially oriented stars with twice as many arcs as edges. We employ difference methods in most of our constructions when showing sufficiency.

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1 INTRODUCTION AND DEFINITIONS

An active area of mathematical research is the study of graph designs. This area of mathematics predominately studies graph decompositions [4, 6] . We consider a set of *points,* called *vertices*, as well as a relationship between pairs of vertices called adjacency. The relationship between two vertices is categorized by either *edges* or *arcs*. These concepts lead to graphs, digraphs, and mixed graphs. A *graph*, denoted G, is a finite nonempty set of vertices denoted $V(G)$, together with a set of unordered pairs of distinct vertices called edges denoted $E(G)$. For vertices u, $v \in G$, an edge is denoted by $[u, v]$ [5].

Note that in the definition of a graph G every two distinct vertices are joined by either a single edge or no edge of G; these graphs are referred to as *simple graphs.* For this, thesis we will be focusing on the decomposition of a simple mixed graph by a simple mixed star. Note, that a simple graph contains no loops nor multi-edges. Furthermore, given a finite nonempty set of all vertices in a *graph* G, we obtain a *directed graph* denoted D by assigning a direction or *orientation* to each edge in G. A directed graph's structure is determined by a set of ordered pairs of distinct vertices $u, v \in D$, that are called *arcs*. If $a = (u, v)$ is an arc in the graph D, then the arc a is said to be incident to v and incident from u.

A *partial orientation* of a digraph or a mixed graph is obtained from G by replacing each edge $[u, v] \in E(G)$ with either (u, v), (v, u), or $[u, v]$ or some combination of these. We restrict our attention here to partial orientations that contain twice as many arcs as edges because we consider the decompositions of the complete mixed graph on v vertices, denoted M_v, and M_v contains twice as many arcs as edges (namely, $n(n-1)$ arcs and $n(n-1)/2$ edges).

The degree denoted $d(v)$ of vertex v is the number of vertices adjacent to v. The edge $[u, v]$ is said to be incident with vertex u and v [5].

Additionally, the *complete symmetric directed graph* is denoted as D_n of order n. We have $A(D_n)$ contains both ordered pairs (u, v) and (v, u) for every two distinct vertices $u, v \in V(D)$ [5]. This leads us to the definition of a *mixed graph* M, which is an ordered triple $(V(M), E(M), A(M))$, where $V(M)$ is a set whose elements are called *vertices*, $E(M)$ is a set disjoint from $V(M)$ whose elements are called *edges*, and $A(M)$ is a set disjoint from both $V(M)$ and $E(M)$, whose elements are called *arcs* and are ordered pairs of, not necessarily different, vertices u and v.

Similarly, the *complete mixed graph* on n vertices, denoted M_n, is the digraph where $|V(M_n)| = n$, $E(M_n) = \{[u, v] \mid u, v \in V(M_n), u \neq v\}$, and $A(M_n) = \{(u, v), (v, u) \mid u, v \in V(M_n), u \neq v\}$ [3]. An example of a complete mixed graph is shown in Figure 1. The *converse* of a mixed graph M or directed graph D is obtained by reversing the direction of every arc of M or D. Note, if $a = (u, v)$ is an arc of a digraph D, or mixed graph M, then u is said to be *adjacent to* v and v is *adjacent from* u. In both directed as well as mixed graphs, a vertex has out-degree of $od(u) = |\{(u, w) \in A(G) \mid w \in V(G)\}|$. This then refers to the number of vertices of either a directed or mixed graph that are adjacent from vertex u in the graph. Each vertex in a directed or mixed graph also has an in-degree of $id(u) = |\{(w, u) \in A(G) \mid w \in V(G)\}|$. This refers to the number of vertices of either a directed or mixed graph that are adjacent to vertex u in the graph.

We will be focusing predominately on complete mixed graphs. The edge degree denoted as $d(u)$ of the vertex u in a mixed graph is the $d(u) = |\{[u, v] \in E(M) \mid v \in V(M), u \neq v\}|$. The *total degree* of vertex u is determined by the sum: $od(u) + id(u) + d(u)$ [2].

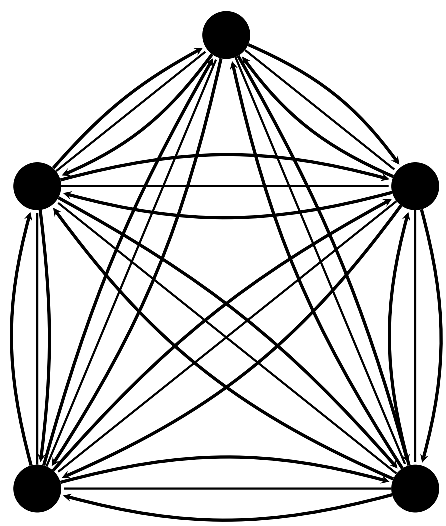


Figure 1: Complete mixed graph on 5 vertices

A *decomposition* $\mathbb{D}$ of a graph G is a collection of $\{H_1, H_2, ..., H_t\}$ of nonempty subgraphs called *blocks*, such that $V(H_i) \subset V(G)$ for all $i = 1, 2, ..., t$, and

$$\dot{\bigcup_{i=1}^{t}} E(H_i) = E(G),$$

where $\dot{\bigcup}$ represents a disjoint union. By convention we require that no subgraph H_i in a decomposition of G contains any isolated vertex. If $\mathbb{D}$ is a decomposition of G, then we say G is *decomposed* into the subgraphs $H_1, H_2, H_3, ..., H_t$. Furthermore, an *automorphism* of a graph G is an isomorphism from G to itself. Thus, an *automorphism* of G is a permutation $\pi : V(G)$ that preserves adjacency, and non-adjacency. Of course, the identity function ι on $V(G)$ is an *automorphism* of G. The inverse of an *automorphism* of G is also an *automorphism* of G, as is the composition of two *automorphisms* of G [5].

Now, if $\mathbb{D}$ is a decomposition of a graph G where each subgraph H_i is a spanning subgraph of G, then $\{H_1, H_2, ..., H_t\}$ is a *factorization* of G. However, every factorization of a nonempty graph G is also a decomposition of G [5]. Furthermore, an *isomorphic decomposition* of a graph G is a decomposition $\mathbb{D} = \{H_1, H_2, \ldots, H_i\}$ where each subgraph H_i is isomorphic to some subgraph H of G [5]. We call such an isomorphic decomposition an "H-decomposition of G." We similarly define H-decompositions of directed graphs D where H is a sub-digraph of D, and H-decompositions of mixed graph M where H is a sub-mixed graph of M.

Now, in an H-decomposition of a graph, digraph, or mixed graph, $\mathbb{D} = \{H_1, H_2, \ldots, H_i\}$, each H_i is called a *block* of the decomposition. A graph is a *star* if it is isomorphic to the complete bipartite graph $K_{1,n}$. A graph is a *complete bipartite graph* if $V(G)$ can be partitioned into two sets U and W, so that $[u, w] \in E(G)$ if and only if $u \in U$ and $w \in W$ [5]. We denote a star as $S_n = K_{1,n}$. The vertex of S_n of degree n is the *center* of the star and the remaining vertices make up the *corona* of the star. We will consider partial orientations of the star S_{3k}. We denote the partial orientation of S_{3k} in which the center has in-degree i and out-degree $2k - i$ where $0 \leq i \leq 2k$ as S_{3k}^i. We consider such partial orientations since we consider the S_{3k}^i-decompositions of complete mixed graph M_v and M_v has twice as many arcs as edges.[9]

2 HISTORY OF TRIPLE SYSTEMS

To motivate the topic of this thesis, we now state results concerning decompositions of complete graphs, digraphs and mixed graphs. In each case, we consider isomorphic decompositions based on either 3-cycles, orientations of 3-cycles, or partial orientations of 3-cycles.

2.1 3-cycles

A *cycle* is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, ..., v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, ..., n-1$ [5]. A representation of a $3 - cycle$ is given in Figure 2.

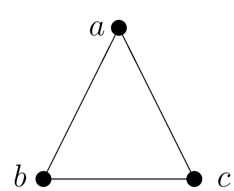


Figure 2: A representation of the $3 - cycle$ with edge set $\{[a, b], [b, c], [a, c]\}$

A graph, or digraph, decomposition into isomorphic copies of a graph, respectively directed graph, on three vertices is referred to as a triple system [5].

2.2 Steiner Triple Systems

An early result deals with the decomposition of complete graphs into three cycles, which are called *Steiner Triple Systems*. A *Steiner Triple System* of order v, denoted $STS(v)$, corresponds to a K_3-decomposition of K_v. This consists of a set S of v elements and a collection T of 3-elements subsets of S, called triples, such that every pair of elements of S belongs to exactly one triple in T [5]. Furthermore, Steiner triple systems are named after the Swiss mathematician Jacob Steiner [13]. The original conjecture was to determine the integers v such that a triple system of order v exists. However, Jacob Steiner was not the first person to propose this conjecture. The conjecture was first proposed by Reverend Wesley S. B. Woolhouse. The conjecture was published and answered by Reverend Thomas P. Kirkman in 1847. Kirkman proved the following [11].

Theorem 2.1 [11] *There exists a Steiner triple system of order v if and only if $v \equiv 1$ or $3\,(mod\ 6)$.*

2.3 Mendelsohn Triple Systems

In 1973 Nathan S. Mendelsohn considered triple systems based on digraphs. He noted that there are two orientations of a 3-cycle [12]. A representation of the two orientations of a 3-cycle is given in Figure 3. In 1973 Mendelsohn produced the following result. A decomposition of the complete digraph on v vertices into 3-circuits is called a *Mendelsohn triple system* of order v. A decomposition of the complete

Figure 3: A representation of a 3-circuit and a transitive triple with arc sets $\{(a,c),(c,b),(b,a)\}$ and $\{(a,c),(b,c),(b,a)\}$ respectively.

digraph on v vertices into transitive triples is called a directed triple system of order v.

Theorem 2.2 [12] *A Mendelsohn triple system of order v exists if and only if $v \equiv 0 \;or\; 1\,(mod\; 3), v \neq 6$.*

Theorem 2.3 [10] *A directed triple system of order v exists if and only if $v \equiv 0 \;or\; 1\,(mod\; 3),\; v \neq 6$.*

Note, that a $\lambda - fold$ *complete directed graph* on v vertices, denoted λD_v, is a directed multi-graph where, for every pair of distinct vertices $u, v \in V(D_v)$ contains λ copies of $\{(u,v),(v,u)\} \subset C$ [12]. In 1986 Hartman and Mendelsohn considered the decomposition of the $\lambda-$fold complete directed graph λD_v into every possible simple digraph on three vertices in a paper called "The Last of the Triples Systems" [7]. However, in 1999 the idea of triple systems was extended to include decompositions of the complete mixed graph, denoted M_v [8].

2.4 Mixed Triples Systems

Mixed triple systems were defined and necessary and sufficient conditions for their existence was given in [8]. These triangle systems are based on the three partial orientations of a $3-cycle$ which contain twice as many arcs as edges as shown in Figure 4.

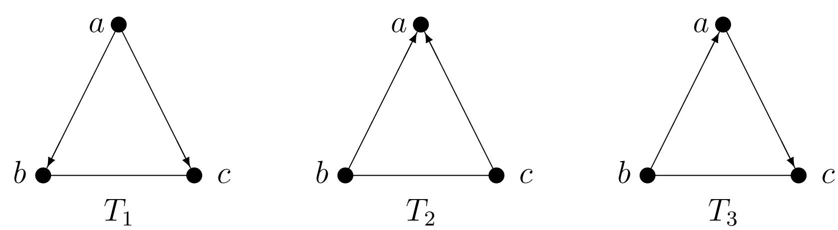


Figure 4: A representation of the mixed triples with $E(T_1) = E(T_2) = E(T_3) = \{[b,c]\}$, $A(T_1) = \{(a,c),(a,b)\}$, $A(T_2) = \{(c,a),(b,a)\}$, and $A(T_3) = \{(b,a),(a,c)\}$.

A T_i- triple system of order v is a T_i- decomposition of M_v. We then have the following results [8].

Theorem 2.4 [8] *A T_1-triple system of order v exists if and only if $v \equiv 1\,(mod\;2)$*

Theorem 2.5 [8] *A T_2-triple system of order v exists if and only if $v \equiv 1\,(mod\;2)$.*

And finally,

Theorem 2.6 [8] *A T_3- triple systems of order v exists if and only if $v \equiv 1\,(mod\;2)$, $v \notin \{3,5\}$.*

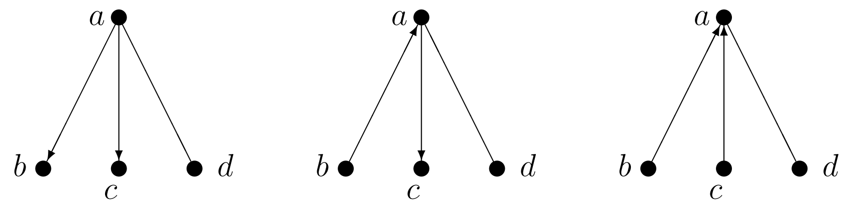
Now we will move on to the main topic of this thesis, dealing with the decomposition of mixed graphs by partial oriented mixed stars.

3 DECOMPOSITION INVOLVING PARTIAL ORIENTATIONS OF S^i_{3k}

In this chapter, we give the main results of this thesis. We give the necessary and sufficient conditions for the existence of S^i_6- decompositions of the complete mixed graph M_v for each partial orientation of the star S_6, where the center has in-degree i and out-degree $4-i$.

3.1 Introduction

An *automorphism* of a graph, digraph, or mixed graph decomposition $\mathbb{D} = \{H_1, H_2, \ldots, H_t\}$ of graph, digraph, or mixed graph G is a permutation π of $V(G)$ such that $\pi(H_i) \in \mathbb{D}$ for all $i = 1, 2, \ldots, t$. A *cyclic* decomposition is one admitting an automorphism consisting of a single cycle of length $|V(G)|$. A *rotational* decomposition is one admitting an automorphism consisting of a cycle of length $|V(G)| - 1$ and a single fixed point. When dealing with cyclic decompositions of G, we take $V(G) = \{0, 1, 2, \ldots, v-1\}$ where $v = |V(G)|$, and where $\pi(i) = i + 1 \pmod{v}$ for each $i \in V(G)$. When dealing with a rotational decomposition of G, we take $V(G) = \{\infty, 0, 1, 2, \ldots, v-2\}$ where $v = |V(G)|$, and where $\pi(\infty) = \infty$ and $\pi(i) = i + 1 \pmod{v-1}$ for each $i \in V(G) \setminus \{\infty\}$. Since there are $3k+1$ vertices in a S^i_{3k} star and the star is simple, then there are $2k+1$ possible partial orientations of the $A(S^i_{3k})$ with twice as many arcs as edges (up to isomorphism). Furthermore, an example of the partial orientations of a S^i_{3k} where $k = 1$ is shown in Figure 5.



$S_3^0 = [a, d; b, c]_3^0$ $S_3^1 = [a, d; b, c]_3^1$ $S_3^2 = [a, d; b, c]_3^2$

Figure 5: Mixed $S_3^i = [a, b; c, d]_3^i$.

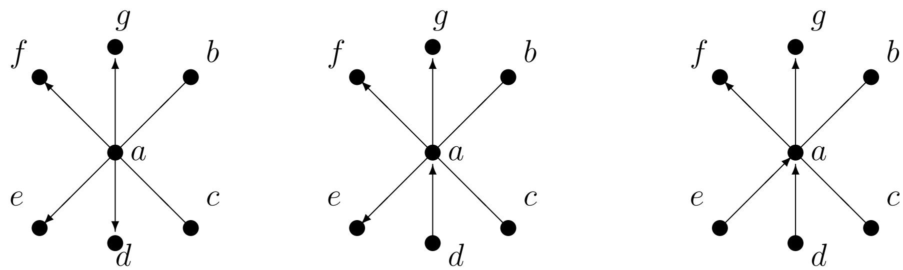
Note that S_3^0 and S_3^2 are converses of each other, and note that S_3^1 is self converse. Also, S_{3k}^0 and S_{3k}^{2k} are converses of each other and S_{3k}^k is self-converse. Hence, there are $k+1$ partial-orientations up to isomorphism of S_{3k}^i with twice as many arcs as edges. The first encounter with a decomposition of M_v by partially oriented stars was published in 2009, and gave the existence of a decomposition of M_v and the decomposition of λM_v by a partially oriented 3-star [1]. Recall that the $\lambda - fold$ *complete mixed graph* on v vertices, denoted λM_v, is a mixed multi-graph where, for every pair of distinct vertices $u, v \in V(M_v)$, λM_v, contains λ copies of $\{(u, v), (v, u), [u, v]\}$ [1]. We have the following [1],

Theorem 3.1 [1] *There exists a S_3^1 or S_3^2 decomposition of λM_v if and only if $\lambda(v - 1) \equiv 0\,(mod\ 2)$ and $v \geq 4$.*

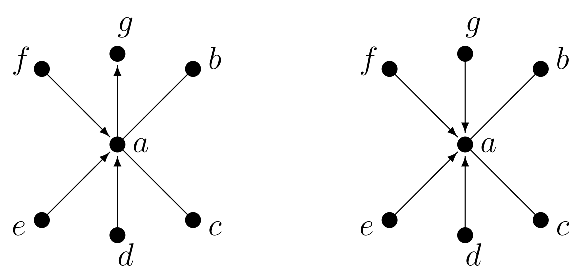
Theorem 3.2 [1] *There exists a S_3^3-decomposition of λM_v if and only if $v \geq 4$ except when $(\lambda, v) = (2k+1, 4)$.*

Now, we will give the necessary and sufficient conditions for the existence of a decomposition of a M_v into partially oriented S_6^i where $i = 0, 1, 2, 3, 4$.

There are 5 partial orientations including converses of S_6^i as shown in Figure 6. We let $[a, b, c; d, e, f, g]_6^i$ denote S_6^i with $V(G) = \{a, b, c, d, e, f, g\}$, and the edge and arc sets as illustrated in Figure 6.



$S_6^0 = [a, b, c; d, e, f, g]_6^0$ $S_6^1 = [a, b, c; d, e, f, g]_6^1$ $S_6^2 = [a, b, c; d, e, f, g]_6^2$



$S_6^3 = [a, b, c; d, e, f, g]_6^3$ $S_6^4 = [a, b, c; d, e, f, g]_6^4$

Figure 6: The representations of the five partial orientations of S_6^i with twice as many arcs as edges.

Notice that S_6^0 and S_6^4 are converses of each other, S_6^1 and S_6^3 are converses of each other, and S_6^2 is self-converse.

3.2 S_6^i decomposition results

Our technique of construction is based on the idea of difference methods. In a cyclic decomposition of G, we take $V(G) = \{0, 1, 2, \ldots, v-1\}$ where $v = |V(G)|$ (as described above). We associate the *arc difference* $b - a \pmod{v}$ with arc (a, b) and we associate the *edge difference* $\min\{(b - a(\text{mod } v), (a-b)(\text{mod } v))\}$ with edge $[a, b]$. Notice that if we have an arc or edge in a particular copy of some H_i in the decomposition with some associated difference, then the *orbit* of H_i (that is, the image of H_i under the powers of the cyclic permutation) will include all edges or arcs of G that have that same associated difference. This allows us to address the existence question (sometimes, at least) by partitioning up the sets of distinct edge differences and arc differences. In this cyclic case, notice that the set of arc differences is $\{0, 1, 2, \ldots, v-1\}$ and the set of edge differences is $\{0, 1, 2, ..., \lfloor \frac{v-1}{2} \rfloor\}$.

In a rotational decomposition of G, we take $V(G) = \{\infty, 0, 1, 2, \ldots, v-2\}$ where $v = |V(G)|$ (as described above), we associate the *arc differences* $(b - a) \pmod{v-1}$ with arc (a, b), and we associate the *edge difference* $\min\{(b-a) \pmod{v-1}, (a - b) \pmod{v}\}$ with edge $[a, b]$. As with cyclic decomposition, we devise to partition the sets of edge differences and arc differences in such a way that all arcs or edges associated with a particular difference are generated by some smaller set of special blocks called *base blocks* with respect to the permutation. In the case of a rotational decomposition, we also might address edges of the forms $[a, \infty]$ where $a \in \{0, 1, 2, \ldots, v-2\}$ and arcs of the form (a, ∞) and (∞, a) where $a, b \in \{0, 1, \ldots, v-2\}$. We will now proceed with preliminary lemmas before the presentation of the constructions. Let us begin with the necessary conditions for S_6^0- decomposition, S_6^4- decomposition, S_6^1- decomposition, S_6^3- decomposition, and S_6^2-decomposition of M_v.

Lemma 3.3 *A necessary condition for the existence of a S_6^0- decomposition of M_v is that $v \equiv 0\,(mod\ 4)$. This condition is also necessary for the existence of the converse S_6^4- decomposition of M_v.*

Proof. Note that every vertex of S_6^0 is of out-degree four. Note that every vertex in M_v is of out-degree $v-1$. Therefore, if a S_6^0-decomposition of M_v exists, then it is necessary that $v-1 \equiv 0\,(\text{mod } 4)$, or $v \equiv 1\,(\text{mod } 4)$. A similar argument shows that this is also a necessary condition for the existence of a S_6^4-decomposition of M_v since S_6^4 is the converse of S_6^0. $\square$

Lemma 3.4 *If $v \equiv 1(mod\ 4)$ and $v \geq 9$, then there exists a S_6^0-decomposition, and there exists a S_6^4-decomposition of M_v.*

Proof. Let $v = 4k+1$ where $k \geq 2$, and let the vertex set of M_v be $\{0, 1, 2, \ldots, 4k\}$. Consider the set of copies of S_6^0:

$$B = \{[0, 4k-1, 4k; 1, 2, 3, 4]_6^0\} \cup$$

$$\{[0, 3+2j, 4+2j; 5+4j, 6+4j, 7+4j, 8+4j]_6^0 \mid j = 0, 1, ..., k-4\}.$$

The copies of S_6^0, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 4k)$, form a S_6^0-decomposition of M_v where $v = 4k+1$, as claimed. Furthermore, since S_6^4 is the converse of S_6^0, we have that there is also a S_6^4-decomposition of M_v where $v = 4k+1$. $\square$

To confirm that the set of base blocks given in Lemma 3.4 do in fact form a decomposition of M_v, we need to make sure that set B has arcs and edges that correspond to each edge difference and each arc difference modulo $v = 4k+1$. Notice that the blocks include the edge and arc differences shown in Table 1.

Table 1: The edge and arc difference of Lemma 3.4.

Base Block	**Edge Differences**	**Arc Differences**
$[0, 4k-1, 4k; 1, 2, 3, 4]_6^0$	$1, 2$	$1, 2, 3, 4$
$[0, 3+2j, 4+2j; 5+4j, 6+4j, 7+4j, 8+4j]_6^0$	$3+2j : 3, 5, 7, \ldots, 2k-1$ $4+2j : 4, 6, 8, \ldots, 2k$	$5+4j : 5, 9, 13, \ldots, 4k-3$ $6+4j : 6, 10, 14, \ldots, 4k-2$ $7+4j : 7, 11, 15, \ldots, 4k-1$ $8+4j : 8, 12, 16, \ldots, 4k$

So all edge differences modulo $4k+1$, namely edge differences $1, 2, \ldots, 2k$, and all arc differences modulo $4k+1$, namely $1, 2, \ldots, 4k$, are present exactly once, justifying the construction.

Now, by Lemma 3.3, and Lemma 3.4, we obtain the following theorem.

Theorem 3.5 *A S_6^0-decomposition and a S_6^4- decomposition of M_v exists if and only if $v \equiv 1\,(mod\ 4)$ and $v \geq 9$.*

We now address, the existence of a S_6^1- decomposition, S_6^2-decomposition, and S_6^3- decompositions of M_v. We start with a necessary condition.

Lemma 3.6 *A S_6^i-decomposition where $i \in \{1,2,3\}$ of M_v does not exist when $v \equiv 2$ or $3\,(mod\ 4)$.*

Proof. Let $v = 4k+3$ where $k \geq 2$, and let the vertex set of M_v be $\{0, 1, 2, \ldots, 4k+2\}$. Consider the set of copies of S_6^i where $i = \{1, 2, 3\}$. However, graph M_v contains $(4k+3)(4k+2)$ arcs. Therefore, the number of arcs in M_v satisfies

$$(4k+3)(4k+2) = 16k^2 + 20k + 6 \equiv 2\,(\text{mod } 4).$$

However, S_6^i has 4 arcs. Thus, in an S_6^i- decomposition of M_v we need $|A(M_v)| \equiv 0\,(\text{mod } 4)$. However, we see that $(4k+3)(4k+2) \not\equiv 0\,(\text{mod } 4)$. Hence, there does not exist a set of blocks of S_6^i that decomposes M_{4k+3}. Similarly, we see that $(4k+2)(4k+1) \not\equiv 0\,(\text{mod } 4)$. Thus, there does not exist a set of blocks of S_6^i that decomposes M_{4k+2}.$\square$

Lemma 3.7 *Neither a S_6^1-decomposition, S_6^2-decomposition, S_6^3-decomposition of M_v exists when $v = 8$.*

Proof. Let $v = 8$, and let the vertex set of M_v be $\{0, 1, 2, \ldots, 7\}$. Now, note that M_8 has $\binom{8}{2} = 28$ edges; and a S_6^i where $i = 1, 2, 3$ has two edges in an S_6^i-decomposition of M_8. Then there are $28/2 = 14$ copies of S_6^i in such a decomposition. Note, that by definition, two vertices are incident with an edge if and only if either a or b is the center of S_6^i. Now, for any pair of disjoint vertices a and b of M_8 to get edge $[a, b]$, arc (a, b), and arc (b, a), we need the sum of the number of times vertex a and the number of times vertex b is the center of a copy of S_6^i to be at least 3. However, for M_8 we have that there are 8 vertices and 14 copies of S_6^i in a decomposition. This implies that each vertex cannot be the center twice (since this would require a minimum of 16 blocks). Therefore, at least two vertices cannot be the center of S_6^i more than once. Therefore, no such S_6^i- decomposition of M_8 exists.$\square$

Lemma 3.8 *A S_6^1-decomposition and a S_6^3-decomposition of M_v exist when $v \equiv 0\,(mod\ 4)$, $v \geq 12$.*

Proof. Let $v = 4k$ where $k \geq 3$, and let the vertex set of M_v be $\{0, 1, 2, \ldots, 4k-2, \infty\}$. Consider the set of copies of S_6^1:

$$B = \{[0, \infty, 1; 4k-2, 2, 3, 4]_6^1, [0, 2, 3; \infty, 5, 6, 7]_6^1, [0, 4, 5; 1, 8, 9, \infty]_6^1\}$$

$$\cup\{[0, 6+j, 2k-1-j; 2+4j, 10+4j, 11+4j, 12+4j]_6^1 \mid j = 0, 1, ..., k-4\}.$$

The copies of S_6^1, along with their images under the powers of the permutation $(\infty)(0, 1, 2, \ldots, 4k-2)$, form a S_6^1-decomposition of M_v where $v = 4k$, as claimed. $\square$

To confirm that the set of base blocks given in Lemma 3.8 do in fact form a decomposition of M_v, we need to make sure that set B has arcs and edges that correspond to each edge difference and each arc difference modulo $v = 4k-1$, along with an edge of the form $[a, \infty]$, and arcs of the form (∞, b) and (c, ∞) for some $a, b, c \in \{0, 1, 2, \ldots, 4k-1\}$. Notice that:

- $[0, \infty, 1; 4k-2, 2, 3, 4]^1_6$ contains an edge of the form $[a, \infty]$ where $a \in \{0, 1, \ldots, 4k-2\}$,
- $[0, 2, 3; \infty, 5, 6, 7]^1_6$ contains an arc of the form (∞, b) where $b \in \{0, 1, \ldots, 4k-2\}$, and
- $[0, 4, 5; 1, 8, 9, \infty]^1_6$ contains an arc of the form (c, ∞) where $c \in \{0, 1, \ldots, 4k-2\}$.

Notice that the blocks include the edge and arc differences shown in Table 2.

Table 2: The edge and arc differences of Lemma 3.8.

Base Block	**Edge Differences**	**Arc Differences**
$[0, \infty, 1; 4k-2, 2, 3, 4]^1_6$	1	$1, 2, 3, 4$
$[0, 2, 3; \infty, 5, 6, 7]^1_6$	$2, 3$	$5, 6, 7$
$[0, 4, 5; 1, 8, 9, \infty]^1_6$	$4, 5$	$4k-2, 8, 9$
$[0, 6+j, 2k-1-j; 10+4j, 11+4j, 12+4j, 13+4j]^1_6$	$6+j : 6, 7, \ldots, k+2$ $2k-1-j : k+3, \ldots, 2k-2, 2k-1$	$4k-3-4j : 13, \ldots, 4k-11, 4k-7, 4k-3$ $10+4j : 10, 14, 18, \ldots, 4k-6$ $11+4j : 11, 15, 19, \ldots, 4k-5$ $12+4j : 12, 16, 20, \ldots, 4k-4$

So all edge differences modulo $4k-1$, namely $1, 2, \ldots, 2k-1$, and all arc differences module $4k-1$, namely $1, 2, \ldots, 4k-2$, are present exactly once, justifying the construction.

Lemma 3.9 *A S_6^1-decomposition, S_6^3-decomposition of M_v exists when $v \equiv 1\,(mod\ 4$), $v \geq 9$.*

Proof. Let $v = 4k+1$ where $k \geq 2$, and let the vertex set of M_v be $\{0, 1, 2, ..., 4k\}$. Consider the set of copies of S_6^1

$$B = \{[0, 1, 4k-1; 4k, 2, 3, 4]_6^1\}$$

$$\cup\{[0, 3+2j, 4+2j; 1+j, 5+3j, 6+3j, 7+3j]_6^1 \mid j = 0, 1, ..., k-2\}.$$

The copies of S_6^1 along with their images under the powers of the permutation $(0,\ 1,\ 2,\ ...,\ 4k-1)$. Form a S_6^1 decomposition of M_v where $v = 4k$, as claimed. □

To confirm that the set of base blocks given in Lemma 3.9 do in fact form a decomposition of M_v, we need to make sure that set B has arcs and edges that correspond to each edge difference and each arc difference modulo $v = 4k+1$. Notice that the blocks include the edge and arc differences shown in Table 3.

Table 3: The edge and arc differences of Lemma 3.9.

Base Block	**Edge Differences**	**Arc Differences**
$[0, 1, 4k-1; 4k, 2, 3, 4]_6^1$	$1, 2$	$1, 2, 3, 4$
$[0, 3+2j, 4+2j; 1+j, 5+3j, 6+3j, 7+3j]_6^1$	$3+2j : 3, 5, 7, \ldots, 2k-1$ $4+2j : 4, 6, 8, \ldots, 2k$	$4k-j : 3k+2, 3k+3, \ldots, 4k$ $5+3j : 5, 8, 11, \ldots, 3k-1$ $6+3j : 6, 9, 12, \ldots, 3k$ $7+3j : 7, 10, 13, \ldots, 3k+1$

So all edge differences modulo $4k+1$, namely $1, 2, \ldots, 2k$, and all arc differences modulo $4k+1$, namely $1, 2, \ldots, 4k$, are present exactly once, justifying the construction.

By Lemma 3.6, Lemma 3.7, Lemma 3.8, and Lemma 3.9 we have the following theorem.

Theorem 3.10 *A S_6^1-decomposition, S_6^3-decomposition of M_v exists if and only if $v \equiv 0,$ or $1\,(mod\ 4)$ and $v \geq 9$.*

Finally, we will address S_6^2-decompositions of M_v. We start with a special case.

Lemma 3.11 *A S_6^2-decomposition of M_v exists when $v = 12$.*

Proof. Let $v = 12$, and let the vertex set of M_{12} be $\{0, 1, 2, \ldots, 10, \infty\}$. Consider the set of copies of S_6^2:

$$B = \{[0, \infty, 1; 10, 2, 3, 4]_6^2, [0, 2, 3; \infty, 1, 5, 6]_6^2, [0, 4, 5; 3, 9, 7, \infty]_6^2\}.$$

The copies of S_6^2, along with their images under the powers of the permutation $(\infty)(0, 1, 2, \ldots, 10)$, form a S_6^2-decomposition of M_v where $v = 12$, as claimed. $\square$

Lemma 3.12 *A S_6^2-decomposition of M_v exists when $v \equiv 0\,(mod\ 4)$, $v \geq 16$.*

Proof. Let $v = 4k$ where $k \geq 4$, and let the vertex set of M_v be $\{0, 1, 2, \ldots, 4k-2, \infty\}$. Consider the set of copies of S_6^2:

$$B = \{[0, \infty, 1; 4k-2, 2, 3, 4]_6^2, [0, 2, 3; \infty, 1, 5, 6]_6^2, [0, 4, 5; 7, 8, 2, \infty]_6^2\}$$

$$\cup\{[0, 6+2j, 7+2j; 4k-6-4j, 4k-5-4j, 9+4j, 12+4j]_6^2 \mid j = 0, 1, ..., k-4\}.$$

The copies of S_6^2, along with their images under the powers of the permutation $(\infty)(0,1,2,\ldots,4k-2)$, form a S_6^2-decomposition of M_v where $v=4k$, as claimed. $\square$

To confirm that the set of base blocks given in Lemma 3.12 do in fact form a decomposition of M_v, we need to make sure that set B has arcs and edges that correspond to each edge difference and each arc difference modulo $v=4k-1$, along with an edge of the form $[a,\infty]$, and arcs of the forms (∞,b) and (c,∞) for some $a,b,c\in\{0,1,2,\ldots,4k-1\}$. Notice that:

- $[0,\infty,1;4k-2,2,3,4]_6^2$ contains an edge of the form $[a,\infty]$ where $a\in\{0,1,\ldots,4k-2\}$,
- $[0,2,3;\infty,1,5,6]_6^2$ contains an arc of the form (∞,b) where $b\in\{0,1,\ldots,4k-2\}$, and
- $[0,4,5;7,8,2,\infty]_6^2$ contains an arc of the form (c,∞) where $c\in\{0,1,\ldots,4k-2\}$.

Notice that the blocks include the edge and arc differences shown in Table 4.

Table 4: The edge and arc differences of Lemma 3.12.

Base Block	**Edge Differences**	**Arc Differences**
$[0,\infty,1;4k-2,2,3,4]_6^2$	1	$1,4k-2;3,4$
$[0,2,3;\infty,1,5,6]_6^2$	$2,3$	$4k-1;5,6$
$[0,4,5;7,8,2,\infty]_6^2$	$4,5$	$7,8;2$
$[0,6+2j,7+2j;4k-6-4j,4k-5-4j,9+4j,12+4j]_6^2$	$6+2j:6,8,\ldots,2k-2$ $7+2j:7,9,11,\ldots,2k-1$	$4k-6-4j:10,\ldots,4k-10,4k-6$ $4k-5-4j:11,\ldots 4k-9,4k-5$ $9+4j:9,13,17,\ldots,4k-7$ $12+4j:12,16,\ldots,4k-4$

So all edge differences modulo $4k-1$, namely $1, 2, \ldots, 2k-1$, and all arc differences modulo $4k-1$, namely $1, 2, \ldots, 4k-2$, are present exactly once, justifying the construction.

Lemma 3.13 *A S_6^2-decomposition of M_v exists when $v \equiv 1\,(mod\ 4)$, $v \geq 9$.*

Proof. Let $v = 4k+1$, where $k \geq 2$, and let the vertex set M_v be $\{0, 1, ..., 4k\}$. Let us consider the set of copies of S_6^2

$$B = \{[0, 4k-1, 4k; 2, 3, 1, 4]_6^2\}$$

$$\cup\{[0, 3+2j, 4+2j; 6+4j, 7+4j, 5+4j, 8+4j]_6^2 \mid j = 0, 1, ..., k-2\}.$$

The copies of S_6^2 along with their images under the powers of the permutation $(0, 1, 2, 3, ..., 4k)$, form a S_6^2 decomposition of M_v where $v = 4k+1$, as claimed. $\square$

To confirm that the set of base blocks given in Lemma 3.13 do in fact form a decomposition of M_v, we need to make sure that set B has arcs and edges that correspond to each edge difference and each arc difference modulo $v = 4k+1$. Notice that the blocks include the edge and arc differences shown in Table 5.

Table 5: The edge and arc differences of Lemma 3.13.

Base Block	**Edge Differences**	**Arc Differences**
$[0, 4k-1, 4k; 2, 3, 1, 4]_6^2$	$1, 2$	$2, 3; 1, 4$
$[0, 3+2j, 4+2j; 6+4j, 7+4j, 5+4j, 8+4j]_6^2$	$3+2j : 3, 5, \ldots, 2k-1$ $4+2j : 4, 6, \ldots, 2k$	$6+4j : 6, 10, 14, \ldots, 4k-2$ $7+4j : 7, 11, 15, \ldots, 4k-1$ $5+4j : 5, 9, 13, \ldots, 4k-3$ $8+4j : 8, 12, 16, \ldots, 4k$

So all edge differences modulo $4k+1$, namely $1, 2, \ldots, 2k$, and all arc differences modulo $4k+1$, namely $1, 2, \ldots, 4k$, are present exactly once, justifying the construction. Then by Lemma 3.11, Lemma 3.12, and Lemma 3.13 we obtain the following theorem.

Theorem 3.14 *A S_6^2 decomposition of M_v exists if and only if*

$v \equiv 0$ *or* $1 \,(mod\ 4)$, *and* $v \geq 9$.

In conclusion, by Theorem 3.5, Theorem 3.10, and Theorem 3.14, we have the main result of this thesis.

Theorem 3.15 *A S_6^i decomposition of M_v exists*

- *for* $i = 0$ *or* 4 *if and only if* $v \equiv 1 \,(mod\ 4)$, *and* $v \geq 9$.
- *for* $i = 1$ *or* 3 *if and only if* $v \equiv 0,\ 1,\ (mod\ 4)$, *and* $v \geq 9$.
- *for* $i = 2$ *if and only if* $v \equiv 0,\ 1 \,(mod\ 4)$, *and* $v \geq 9$.

4 S_{3k}^i DECOMPOSITIONS

We now give a few results concerning S_{3k}^i- decompositions of M_v for $k > 2$.

4.1 Introduction

We now explore a few additional results concerning S_{3k}^i-decomposition of M_v where S_{3k}^i is a partial orientation of S_{3k} where the center has edge degree k, in-degree i, and out-degree $2k - i$ (as defined in Chapter 1). We use the notation

$$S_{3k}^i = [c, e_1, e_2, \ldots, e_k; a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{2k}]_{3k}^i$$

for the partial orientation of S_{3k}^i with edge set $E(S_{3k}^i) = \{[c, e_1], [c, e_2], \ldots, [c, e_k]\}$ and arc set $A(S_{3k}^i) = \{(c, a_1), (c, a_2), \ldots, (c, a_i), (a_{i+1}, c), (a_{i+2}, c), \ldots, (a_{3k}, c)\}$, where $i \in \{0, 1, 2, \ldots, 2k\}$, shown in Figure 7.

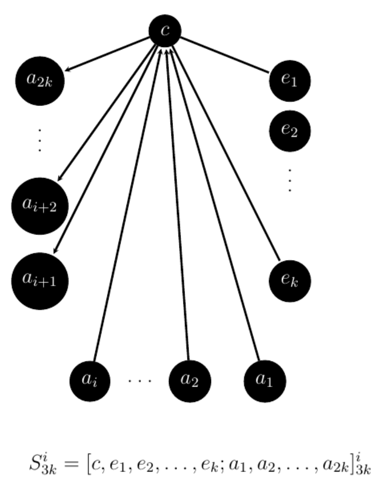


Figure 7: The representation we use for the partial orientation of S_{3k}^i

Similarly, as in Chapter 3 there are $k+1$ partial orientations of S_{3k} with twice as many arcs as edges; namely S_{3k}^i for $i = [0, 1, \ldots, 2k]$. Now, we will give the sufficient conditions for the existence of a decomposition of a M_v into some partial oriented S_{3k}^i where $k \geq 2$.

4.2 S_{3k}^i- decomposition results

Our first result is as follows.

Theorem 4.1 *There exists a S_{3k}^0- decomposition of M_v when $v = 4k+1$.*

Proof. Lets denote a S_{3k}^i as $[v_0, v_1, v_2, \ldots, v_k;\ v_{k+1}, v_{k+2}, \ldots, v_{3k}]$ where the edges set is $E(M_v) = [v_0, v_1], [v_0, v_2], \ldots, [v_0, v_k]$ and the arcs set is $A(M_v) = (v_0, v_{k+1}), (v_0, v_{k+2})$ $, \ldots, (v_0, v_{3k})$. Therefore, let the vertex set $V(M_{4k+1}) = \{0, 1, 2, \ldots, 4k\}$. Consider the set of copies of S_{3k}^0:

$$B = \{[0, 4k, 4k-1, \ldots, 3k+1; 1, 2, 3, \ldots, 2k]_{3k}^0\} \cup$$

$$\{[0, 2k, 2k-1, \ldots, k+1; 2k+1, 2k+2, \ldots, 4k]_{3k}^0\}.$$

The copies of S_{3k}^0, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 4k)$, form a S_{3k}^0-decomposition of M_v where $v = 4k+1$, as claimed. Furthermore, since S_{3k}^{2k} is the converse of S_{3k}^0 we have that there is also a S_{3k}^{2k}-decomposition of M_v where $v = 4k+1$. $\square$

To confirm that the set of base blocks given in the Theorem 4.1 do in fact form a decomposition of M_v, we need to make sure that set B has arcs and edges that correspond to each edge difference and each arc difference modulo $v = 4k + 1$, since we have fixed a point. Notice that: the blocks include the edge and arc differences shown in Table 6.

Table 6: The edge and arc difference of Theorem 4.1

Base Block	**Edge Differences**	**Arc Differences**
$[0, 4k, 4k-1, \ldots, 3k+1; 1, 2, 3, \ldots, 2k]^0_{3k}$	$1, 2, \ldots, k$	$1, 2, \ldots, 2k$
$[0, 2k, 2k-1, \ldots, k+1; 2k+1, 2k+2, \ldots, 4k]^0_{3k}$	$k+1, k+2, \ldots, 2k$	$2k+1, 2k+2, \ldots, 4k.$

So all edge differences modulo $4k+1$, namely edge differences $1, 2, \ldots, 2k$, and all arc differences modulo $4k$, namely $1, 2, \ldots, 4k$, are present exactly once, justifying the construction.

Corollary 4.2 *There exists a $S^i_{3k}-$ decomposition of M_v when $v = 4k + 1$ for each $i = 0, 1, 2, \ldots, 2k$.*

Proof Note, given the set B from Theorem 4.1, we have

$$B = \{[0, 4k, 4k-1, \ldots, 3k+1; 1, 2, \ldots, 2k]^0_{3k}\}$$

$$\cup\{[0, 2k, 2k-1, \ldots, k+1; 2k+1, 2k+2, ..., 4k]^0_{3k}\}$$

as shown in the construction above the edge differences generated by base block one $\{[0, 4k, 4k-1, \ldots, 3k+1; 1, 2, \ldots, 2k]^0_{3k}\}$ are $-1, -2, \ldots, -k \equiv 1, 2, \ldots, k$. The edge differences generated by base block two $\{[0, 2k, 2k-1, \ldots, k+1; 2k+1, 2k+2, ..., 4k]^0_{3k}\}$

are $k+1, k+2, \ldots, 2k$. Therefore, as we seen in Theorem 4.1 no edges are generated twice. Now, we focus on the arc difference. The arc differences generated by base block 1, are $1, 2, \ldots, 2k$, and the arc differences generated by base block 2 are $-(2k+1), -(2k+2), \ldots, -(4k) \equiv 2k+1, 2k+2, \ldots, 4k$. Since the arc differences in block two are the negatives of block one. We can the reverse each pair $(1, 2k+1), (2, 2k+2), \ldots, (2k, 4k)$ to generate the remaining $S^i_{3k}-$ decompositions of the M_{4k+1} for all $i = \{0, 1, 2, \ldots, 2k\}$. Namely, an $S^i_{3k}-$decomposition of M_v where $v = 4k+1$ is given is given by considering the set of copies of S^i_{3k} whose base set:

$$B = \{[0, 4k, 4k-1, \ldots, 3k+1; 1, 2, 3, \ldots, 2k]^i_{3k}\} \cup$$

$$\{[0, 2k, 2k-1, \ldots, k+1; 2k+1, 2k+2, \ldots, 4k]^i_{3k}\}.$$

The copies of S^i_{3k}, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 4k)$, form a S^i_{3k}-decomposition of M_v where $v = 4k+1$, as claimed. $\square$

Theorem 4.3 *There exists a $S^i_{6k+3}-$ decomposition of M_v when $v = 12k+7$ and for each $i = 0, 2, \ldots, 4k+2$*

Proof Let us denote a S^i_{6k+3} as $[v_0, v_1, v_2, \ldots, v_{2k+1}; v_{2k+2}, v_{2k+3}, \ldots, v_{4k+2}]$ where the edges set is $E(M_v) = \{[v_0, v_1], [v_0, v_2], \ldots, [v_0, v_{2k+1}]\}$ and the arcs set is $A(M_v) = \{(v_0, v_{2k+2}), (v_0, v_{2k+3}), \ldots, (v_0, v_{4k+2})\}$, and vertex set $V(M_{12k+7}) = \{0, 1, 2, \ldots, 12k+6\}$. Consider the set of copies of S^i_{6k+3}:

$B = \{[0, 12k+6, 12k+5, \ldots, 10k+6; 1, 2, 3, \ldots, 4k+2]^i_{6k+3}$, $[0, 8k+4, 8k+3, \ldots, 6k+4;$

$8k{+}5, 8k{+}6, \ldots, 12k{+}6]^i_{6k+3}\}$, $[0, 10k{+}5, 10k{+}4, \ldots 8k{+}5; 4k{+}3, 4k{+}4, \ldots, 8k{+}4]^i_{6k+3}$.

The copies of S^i_{6k+3}, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 12k{+}6)$, form a S^i_{6k+3}-decomposition of M_v where $v = 12k{+}7$, as claimed. Furthermore, since S^{4k+2}_{6k+3} is the converse of S^0_{6k+3}, we have that there is also a S^i_{6k+3}-decomposition of M_v where $v = 12k + 7$. □

To confirm that the set of base blocks given in Theorem 4.3 do in fact form a decomposition of M_v, we need to make sure that set B has arcs and edges that correspond to each edge difference and each arc difference modulo $v = 12k + 7$. Notice that the blocks include the edge and arc differences shown in Table 7.

Table 7: The edge and arc differences of Lemma 4.3.

Base Block	**Edge Differences**	**Arc Differences**
$[0, 12k+6, 12k+5, \ldots, 10k+6; 1, 2, 3, \ldots, 4k+2]^i_{6k+3}$	$1, 2, \ldots, 2k+1$	$1, 2, \ldots, 4k+2$
$[0, 8k+4, 8k+3, \ldots, 6k+4; 8k+5, 8k+6, \ldots, 12k+6]^i_{6k+3}$	$4k+3, 4k+4, \ldots, 6k+3$	$8k+5, 8k+6, \ldots, 12k+6$
$[0, 10k+5, 10k+4, \ldots 8k+5; 4k+3, 4k+4, \ldots, 8k+4]^i_{6k+3}$	$2k+2, 2k+3, \ldots, 4k+2$	$4k+3, 4k+4, \ldots, 8k+4$

So all edge differences modulo $12k + 7$, namely edge differences $1, 2, \ldots, 6k + 3$, and all arc differences modulo $12k + 6$, namely $1, 2, \ldots, 12k + 6$, are present exactly once, justifying the construction.

Corollary 4.4 *There exists a $S^i_{6k+3}-$ decomposition of M_v when $v = 12k + 7$ for each $i = 0, 2, 4, \ldots, 4k + 2$.*

Proof Note, given the set B from Theorem 4.3, we have

$$B = \{[0, 12k + 6, 12k + 5, \ldots, 10k + 6; 1, 2, 3, \ldots, 4k + 2]^0_{6k+3},$$

$$[0, 10k + 5, 10k + 4, \ldots 8k + 5; 4k + 3, 4k + 4, \ldots, 8k + 4]^0_{6k+3},$$

$$[0, 8k + 4, 8k + 3, \ldots, 6k + 4; 8k + 5, 8k + 6, \ldots, 12k + 6]^0_{6k+3}\}.$$

As we see from Theorem 4.3, the edge differences do not interfere with each other. Now we want to see if by changing i will the arc differences interfere with the construction. Base block one $\{[0, 12k + 6, 12k + 5, \ldots, 10k + 6; 1, 2, 3, \ldots, 4k + 2]^i_{6k+3}\}$ generates arc differences $-1, -2, \ldots, -(4k + 2) \equiv 1, 2, \ldots, 4k + 2$, base block two $\{[0, 10k + 5, 10k + 4, \ldots 8k + 5; 4k + 3, 4k + 4, \ldots, 8k + 4]^i_{6k+3}\}$ generates arc differences, $4k+3, 4k+4, \ldots, 8k+4$, and base block three $\{[0, 8k+4, 8k+3, \ldots, 6k+4; 8k+5, 8k+6, \ldots, 12k+6]^i_{6k+3}\}$ generates arc differences $8k+5, 8k+6, \ldots, 12k+6$. However, since base block 1 is the negative differences of base block 3, we then can reverse grouped pairs $[(1, 8k+5), (2, 8k+6)], \ldots, [(4k+1, 12k+5), (4k+2, 12k+6)]$ to generate the remaining $S^i_{6k+3}-$decompositions of the M_{12k+7} for all $i = \{0, 2, 4, \ldots, 4k + 2\}$. Namely, an $S^i_{6k+3}-$decomposition of M_v where $v = 12k + 7$ is given by considering the set of copies of S^i_{6k+3} whose base set:

$$B = \{[0, 12k + 6, 12k + 5, \ldots, 10k + 6; 1, 2, 3, \ldots, 4k + 2]^i_{6k+3},$$

$$[0, 10k + 5, 10k + 4, \ldots 8k + 5; 4k + 3, 4k + 4, \ldots, 8k + 4]^i_{6k+3},$$

$$[0, 8k + 4, 8k + 3, \ldots, 6k + 4; 8k + 5, 8k + 6, \ldots, 12k + 6]^i_{6k+3}\}.$$

Where the copies of S^i_{6k+3}, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 12k+6)$, form a S^i_{6k+3}-decomposition of M_v where $v = 12k+7$, as claimed.$\square$

5 FUTURE RESEARCH

The world of mathematics is ever expanding and new avenues of graph design are being discovered each day. Some of the new questions that arise in the the field of mixed graph decomposition by S^i_{3k} of M_v are: Does there exist a decomposition of $\lambda(M_v)$ where $\lambda > 1$? When an $S^i_{3k}-$decomposition of M_v does not exist, we can ask "Can we efficiently remove isomorphic copies of a given S^i_{3k} from M_v such that the leave, denoted as L is minimized, where

$$L = E(G) \setminus \dot{\bigcup_{i=1}^{t}} E(H_i)?\text{''}$$

This is called a S^i_{3k}-*packing* of the complete mixed graph on v vertices. We can also ask: "Can we efficiently place isomorphic copies of S^i_{3k} into M_v such that the padding P is minimized, where

$$|P| = \dot{\bigcup_{i=1}^{t}} E(H_i) \setminus E(G)/\text{''}$$

This is called a $S^i_{3k}-$*covering* of the complete mixed graph on v vertices [2].

We defined the automorphism of a decomposition in Section 3.1. Since we have used cyclic and rotational decompositions in several of our constructions in Chapter 3, we have sufficient conditions for the existence of some cyclic and rotational S^i_6-decompositions of M_v. One could address the more general question of necessary and sufficient conditions for the existence of cyclic and rotational $S^i_{3k}-$decompositions of M_v. In addition, the permutation of the vertex set could be explored in connection with the automorphism question.

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