5-2020

Trees with Unique Italian Dominating Functions of Minimum Weight

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Trees with Unique Italian Dominating Functions of Minimum Weight

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

by

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May 2020

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Keywords: graph theory, Italian domination, unique Italian domination

ABSTRACT

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An Italian dominating function, abbreviated IDF, of G is a function $f\colon V(G) \to \{0,1,2\}$ satisfying the condition that for every vertex $v \in V(G)$ with $f(v) = 0$, we have $\sum_{u\in N(v)} f(u) \geq 2$. That is, either v is adjacent to at least one vertex u with $f(u) = 2$, or to at least two vertices x and y with $f(x) = f(y) = 1$. The Italian domination number, denoted γ_I(G), is the minimum weight of an IDF in G. In this thesis, we use operations that join two trees with a single edge in order to build trees with unique γ_I-functions.

ACKNOWLEDGMENTS

I would like to thank my committee co-chairs, Dr. Teresa Haynes and Dr. Rodney Keaton, for their patience, encouragement, and feedback throughout this process. Their support and encouragement throughout my time as a graduate student kept me hopeful and motivated to keep going. I would also like to acknowledge Dr. Robert Gardner, for his support and guidance through the graduate program over the past two years. I am also very grateful to my fellow classmates for keeping me sane throughout this seemingly endless quest of graduate school. I would also like to pay special regards to Darrell, who has continuously supported and encouraged me throughout this journey. This has been an invaluable experience, and I am very grateful to everyone who has assisted me along the way.

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1 INTRODUCTION

Let us begin by establishing the definitions and standard notations that will be presented in this paper. Let $G = (V, E)$ be a graph with vertex set $V(G) = V$ of *order* $n = |V(G)|$ and edge set $E(G) = E$ of *size* $m = |E(G)|$. The open neighborhood of $v \in V$ is the set $N_G(v) = \{u \in V \,|\, uv \in E\}$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V(G)$ is the set of all neighbors of vertices in S, denoted $N_G(S)$, whereas the *closed neighborhood* of S is $N_G[S] = N_G(S) \cup S$. For a set $S \subseteq V(G)$, the subgraph induced by S in G is denoted $G[S]$. Further, the graph obtained from G by deleting the vertices in S and all edges incident with S is denoted by $G - S$.

The *degree* of v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. If v is a support vertex of a tree T, then L_v will denote the set of the leaves attached at v.

A *path*, denoted P_n, is a graph of order n and size $n - 1$ with vertices labelled $v_1, v_2, ..., v_n$ and edges $v_i v_{i+1}$ for $i = 1, 2, ..., n - 1$. A *star*, denoted $K_{1,t}$, is a tree in which one vertex v has $N[v] = V(G)$, and every other vertex u has $N(u) = \{v\}$. For a positive integer $t \geq 2$, a *wounded spider* is a star $K_{1,t}$ with at most $t-1$ of its edges subdivided, and a *healthy spider* is a star $K_{1,t}$ with all of its edges subdivided.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function*, abbreviated RDF, of G if every vertex $u \in V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G, and an RDF with weight $\gamma_R(G)$ is called a γ_R-function of G.

Figure 1 depicts examples of γ_R-functions of graphs C_4 and H. We can see from this figure that $\gamma_R(C_4) = 3$ and that $\gamma_R(H) = 3$.

(a) C_4 (b) Graph H

Figure 1: Examples of γ_R-functions.

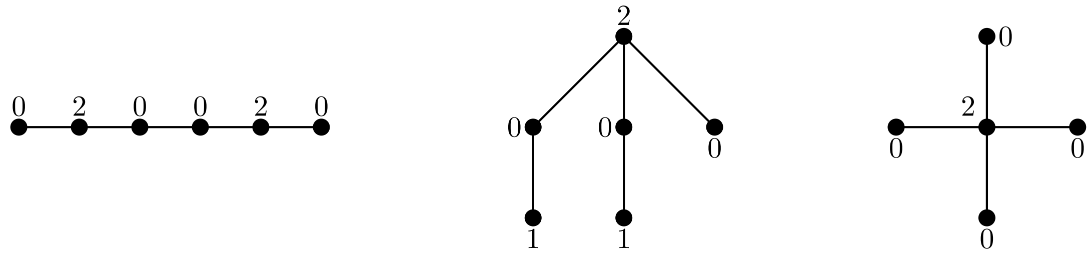
A tree T is called a *unique Roman domination tree*, or a *URD-tree*, if it has a unique γ_R-function of T. Consider the graph P_5. We can see from Figure 2 that $\gamma_R(P_5) = 4$. However, P_5 has two distinct γ_R-functions f and h of weight 4. Thus, we determine that P_5 is not a URD-tree.

(a) γ_R-function f (b) γ_R-function h

Figure 2: γ_R-functions of P_5.

Some examples that are URD-trees include paths P_{3k}, healthy spiders, wounded spiders, and stars $K_{1,t}$ where $t \geq 2$. Some of these examples and their unique γ_R-functions are depicted in Figure 3.

An *Italian dominating function*, abbreviated IDF, of G is a function $f \colon V(G) \to \{0, 1, 2\}$ satisfying the condition that for every vertex $v \in V(G)$ with $f(v) = 0$, we have $\sum_{u \in N(v)} f(u) \geq 2$. That is, either v is adjacent to at least one vertex u with $f(u) = 2$, or to at least two vertices x and y with $f(x) = f(y) = 1$. Viewed as a



(a) P_6 (b) Wounded Spider (c) $K_{1,4}$

Figure 3: URD-trees and their unique γ_R-functions.

graph labeling problem, each vertex labeled 0 must have the labels of the vertices in its closed neighborhood sum to at least 2. The *weight* of an IDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Italian domination number*, denoted γ_I(G), is the minimum weight of an IDF in G, and an IDF of G with weight $\gamma_I(G)$ is called a γ_I-function of G. For both Italian and Roman domination, let $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$. In other words, V_i is the set of vertices assigned weight i under f.

(a) C_4 (b) Graph H

Figure 4: Examples of γ_I-functions.

Figure 4 depicts examples of γ_I-functions of graphs C_4 and H. We can see from this figure that $\gamma_I(C_4) = 2$ and $\gamma_I(H) = 3$. Notice that $\gamma_I(H) = \gamma_R(H)$ even though Figure 4 (b) depicts a γ_I-function of H that is not an RDF of H. Also, we can see that $\gamma_I(C_4) < \gamma_R(C_4)$. In general, for any graph G, we have that $\gamma_I(G) \leq \gamma_R(G)$.

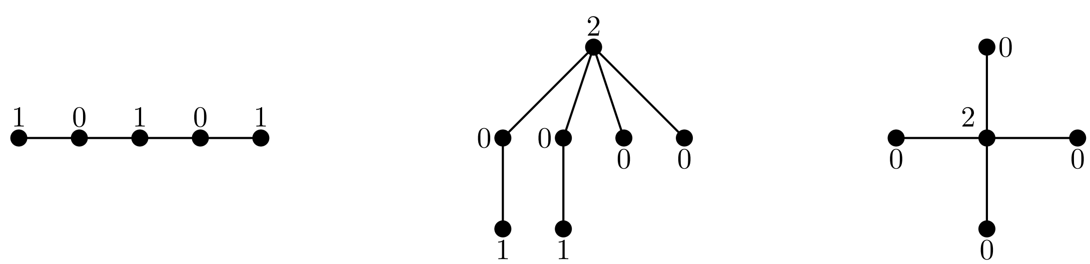
In this paper, we will be exploring trees with unique Italian dominating functions of minimum weight. A tree T will be called a *unique Italian domination tree*, abbreviated *UID-tree*, if it has a unique γ_I-function.

Consider the wounded spider T depicted in Figure 5. We can see from this figure that $\gamma_I(T) = 4$. However, T has two distinct γ_I-functions f and h of weight 4. Therefore, we can see that this wounded spider T is not a UID-tree.

(a) γ_I-function f (b) γ_I-function h

Figure 5: γ_I-functions of wounded spider T.

Some examples that are UID-trees include stars $K_{1,t}$ where $t \geq 3$, odd paths P_{2k+1} for $k \geq 2$, healthy spiders, and wounded spiders with at most $t-2$ subdivided edges. Some of these graphs and their unique γ_I-function are depicted in Figure 6.



(a) P_5 (b) Wounded Spider (c) $K_{1,4}$

Figure 6: UID-trees and their unique γ_I-functions.

2 LITERATURE SURVEY

2.1 Roman Domination

Roman domination was first introduced by Cockayne et al. [5] as a graph invariant in 2004 following a series of papers (see [19, 20, 21, 22]) on defense strategies of the ancient Roman Empire. The idea is that vertices represent cities or locations, and a vertex v of weight $f(v) = 1, 2$ represents a location with either 1 or 2 Roman legions stationed there. An adjacent vertex u, thought of as a nearby location, may be unprotected if it has no stationed legions. That is, a vertex with $f(u) = 0$ may be at risk for attack. In order to secure an unprotected location u, a neighboring location v can send one of their legions to u. However, sending a legion from v to a neighboring location should not leave v unsecured. That is, two legions must be stationed at v before a legion can be sent to an adjacent location. Hence, every vertex u with $f(u) = 0$ must be adjacent to at least one vertex v with $f(v) = 2$.

Since its introduction, over 100 papers have been published on various aspects of Roman domination in graphs. Some examples can be found in [8, 2, 1] regarding topics such as double Roman domination, perfect Roman domination, and independent Roman domination. The growing popularity of Roman domination also provided researchers with motivation to define variants of Roman domination, one of which is Italian domination.

2.2 Italian Domination

In this thesis, we will be focusing on Italian dominating functions of trees. Italian domination was first introduced as Roman $\{2\}$-domination by Chellali et al. in [4]. It was further researched and renamed Italian domination by Henning and Klostermeyer in [12]. Some researchers continue to use the notation associated with the Roman $\{2\}$-domination title; however, it is more commonly referred to as Italian domination.

Italian domination can be thought of as relaxing the Roman domination restriction placed upon a vertex u with $f(u) = 0$. As a result, Italian domination can also be thought of in reference to defending the Roman empire. This defense strategy requires that every location u with no legion must either have a neighboring location with two legions, or at least two neighboring locations with one legion each. That is, each vertex u with $f(u) = 0$ must have $\sum_{x \in N(u)} f(x) \geq 2$.

Since Italian domination is a variant of Roman domination, many of the topics that were researched and defined for Roman domination have also been extended to Italian domination. It is observed in [4] that every Roman dominating function is an Italian dominating function, thus the bound $\gamma_I(G) \leq \gamma_R(G)$ follows immediately. As a result, Martinez and Yero explored this bound in [17] and characterized trees that have $\gamma_I(T) = \gamma_R(T)$. Other Italian domination topics that have been researched include perfect Italian domination, independent Italian domination, and global Italian domination, which can be found in [11, 18, 10].

2.3 Unique Minimum Roman Dominating Functions

The topic of this thesis was inspired by [3] in which Chellali and Rad characterize trees with unique Roman dominating functions of minimum weight. In their paper, they use operations to build a family of graphs that produce URD-trees.

Let T_1 and T_2 be two vertex-disjoint URD-trees. Let f_1 be the unique γ_R-function of T_1 and f_2 the unique γ_R-function of T_2. They define the following operation that is used to link T_1 and T_2 and produces a new URD-tree.

Operation $\mathcal{O}_1$: Let T be the tree obtained from T_1 and T_2 by adding an edge joining a vertex x in T_1 with a vertex y in T_2 such that $f_1(x) = 0$ and $f_2(y) = 0$. This leads to the following lemma.

Lemma 2.1. *[3] The tree T obtained from T_1 and T_2 by performing Operation $\mathcal{O}_1$ is a URD-tree. Furthermore, f defined on $V(T)$ by $f(a) = f_1(a)$ for every $a \in V(T_1)$ and $f(b) = f_2(b)$ for every $b \in V(T_2)$ is the unique γ_R-function of T.*

They next present a constructive characterization for URD-trees. Define the family of trees as follows: Let $\mathcal{T}$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, ..., T_k$ of trees, where T_1 is a star $K_{1,t}$ with $t \geq 2$, $T = T_k$, and, if $k \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations. Let $S(T)$ denote the set of support vertices of T, $V_S(T) = \{v \in S(T) \mid \gamma_R(T - v) > \gamma_R(T)\}$, and let f_i be an RDF of T_i.

Operation $\mathcal{O}_2$: Add a new vertex x attached to a leaf y of T_i with $f_i(y) = 0$ whose support vertex belongs to $V_S(T_i)$. Let $f_{i+1}(a) = f_i(a)$ for every $a \in V(T_i)$ and $f_{i+1}(x) = 1$.

Operation $\mathcal{O}_3$: Add a star $K_{1,t}$ $(t \geq 3)$ of center vertex x attached by an edge xy at any strong support vertex y of $V_S(T_i)$. Let $f_{i+1}(a) = f_i(a)$ for every $a \in V(T_i)$, $f_{i+1}(x) = 2$, and $f_{i+1}(b) = 0$ if b is a leaf in L_x.

Operation $\mathcal{O}_4$: Add a star $K_{1,t}$ $(k \geq 2)$ of center vertex x attached by an edge xy at any strong support vertex y of T_i such that $f_i(y) = 0$ and y is adjacent to a strong support vertex z with the condition that $|L_z| \geq 3$ if a vertex in $N_{T_i}(z)$ is assigned 2. Let $f_{i+1}(a) = f_i(a)$ for every $a \in V(T_i)$, $f_{i+1}(x) = 2$, and $f_{i+1}(b) = 0$ if b is a leaf in L_x.

Operation $\mathcal{O}_5$: Add a new vertex w and k $(k \geq 1)$ stars of centers $x_1, x_2, ..., x_k$ each of order at least three attached by edges wx_j and wu at any vertex u of T_i with $f_i(u) \neq 0$. Let $f_{i+1}(x) = f_i(x)$ for every $x \in V(T_i)$, $f_{i+1}(x_j) = 2$ for every j and $f_{i+1}(a) = 0$ if $a = w$ or a is a leaf in L_{x_j}.

Lemma 2.2. *[3] If $T \in \mathcal{T}$, then T is a URD-tree.*

Theorem 2.3. *[3] A tree T is a URD-tree if and only if $T = K_1$ or $T \in \mathcal{T}$ or can be constructed from disjoint trees of $\mathcal{T}$ by a finite sequence of Operation $\mathcal{O}_1$.*

These results provided the motivation for exploring UID-trees. In this thesis, we will be using operations resembling Operation $\mathcal{O}_1$ to join two trees with a single edge and build UID-trees.

3 RESULTS

In this section, we will be defining operations that add a single edge between two vertices in order to join two UID-trees. In order to determine when a UID-tree is constructed, we will consider the various weights of the two vertices that are joined.

Let T_1 and T_2 be two vertex-disjoint UID-trees. Let f_1 be the unique γ_I-function of T_1 and f_2 the unique γ_I-function of T_2. Note that if f is an IDF on a graph G and H is a subgraph of G, then we denote the restriction of f on H by $f|_{V(H)}$. We define the following operation that can be used to join T_1 and T_2 and results in a new UID-tree.

Operation $\mathcal{O}_1$: Let T be the tree obtained from T_1 and T_2 by adding an edge between a vertex x in T_1 and a vertex y in T_2 such that $f_1(x) = 0$ and $f_2(y) = 0$.

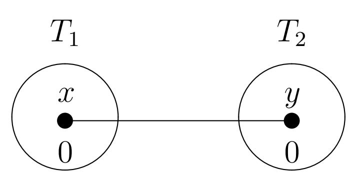


Figure 7: Operation $\mathcal{O}_1$.

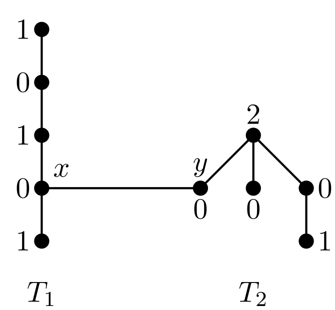


Figure 8: Example of a tree constructed from $\mathcal{O}_1$.

Figure 7 depicts the general construction of trees that are obtained by performing $\mathcal{O}_1$, and Figure 8 shows a specific example of a tree produced from this operation. In Figure 8, we can see the unique γ_I-function f_1 of $T_1 = P_5$, as well as the unique γ_I-function f_2 of the wounded spider T_2. The edge xy was added between vertex x in T_1 and vertex y in T_2 such that $f_1(x) = f_2(y) = 0$.

Proposition 3.1. *The tree T obtained from T_1 and T_2 by performing operation $\mathcal{O}_1$ is a UID-tree. Furthermore, f defined on V(T) by $f(a) = f_1(a)$ for every $a \in V(T_1)$ and $f(b) = f_2(b)$ for every $b \in V(T_2)$ is the unique γ_I-function of T.*

Proof. Clearly, the function f defined on $V(T)$ by $f(a) = f_1(a)$ for every $a \in V(T_1)$ and $f(b) = f_2(b)$ for every $b \in V(T_2)$ is an IDF of T. This implies that $\gamma_I(T) \leq \gamma_I(T_1) + \gamma_I(T_2)$.

Now let f be a γ_I-function of T. If $f(x) = f(y)$ or if $\{f(x), f(y)\} = \{1, 2\}$, then $f|_{V(T_i)}$ is an IDF for T_i and so $\gamma_I(T_i) \leq f(V(T_i))$. Thus $\gamma_I(T_1) \leq f(V(T_1))$ and $\gamma_I(T_2) \leq f(V(T_2))$, and adding these two inequalities implies that $\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_1)) + f(V(T_2)) = \gamma_I(T)$ by the assumption. Thus the equality $\gamma_I(T) = \gamma_I(T_1) + \gamma_I(T_2)$ follows.

Now consider the only remaining cases where $\{f(x), f(y)\} \in \{\{0, 2\}, \{0, 1\}\}$. Assume, without loss of generality, that $f(x) = 0$ and $f(y) \in \{1, 2\}$. Then $f|_{V(T_2)}$ is an IDF of T_2, but since $f(y) \neq f_2(y) = 0$ and T_2 is a UID-tree with unique minimum IDF f_2, we have that $\gamma_I(T_2) < f(V(T_2))$. This implies that $\gamma_I(T_2) \leq f(V(T_2)) - 1$. On the other hand, the function g defined on $V(T_1)$ by $g(u) = f(u)$ if $u \neq x$ and $g(x) = 1$ is an IDF of T_1. Thus $\gamma_I(T_1) \leq g(V(T_1)) = f(V(T_1)) + 1$. Adding the two previous inequalities gives

$$\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_2)) - 1 + f(V(T_1)) + 1 = f(V(T_2)) + f(V(T_1)) = \gamma_I(T).$$

Thus we again have the equality $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$.

Now we need to show that f is the unique γ_I-function of T. Suppose, for the purpose of contradiction, that T is not a UID-tree and let $h \neq f$ be a γ_I-function of T. Clearly, if $h(x) = h(y)$ or if $\{h(x), h(y)\} = \{1, 2\}$, then $h|_{V(T_i)}$ is a γ_I-function of T_i. This implies that either T_1 or T_2 is not a UID-tree. Thus we can assume that $\{h(x), h(y)\} \in \{\{0, 2\}, \{0, 1\}\}$, say $h(x) = 0$ and $h(y) \in \{1, 2\}$. As seen before, $h|_{V(T_2)}$ is an IDF of T_2 with weight $h(V(T_2)) \geq \gamma_I(T_2) + 1$. This, along with the fact that $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$, implies that

$$h(V(T_1)) = h(V(T)) - h(V(T_2)) \leq \gamma_I(T) - (\gamma_I(T_2) + 1) = \gamma_I(T) - \gamma_I(T_2) - 1 = \gamma_I(T_1) - 1$$

and $h|_{V(T_1)}$ is an IDF for $T_1 - x$.

Now consider the function g on $V(T)$ as follows: $g|_{V(T_2)} = f|_{V(T_2)}, g|_{V(T_1 - x)} = h|_{V(T_1 - x)}$, and $g(x) = 1$. Then we have that $g(V(T_1 - x)) = h(V(T_1 - x)) \leq \gamma_I(T_1) - 1$, implying that $g(V(T_1)) \leq \gamma_I(T_1) - 1 + 1 = \gamma_I(T_1)$. Then $g|_{V(T_1)}$ is an IDF for T_1 with weight $\gamma_I(T_1)$, that is, $g|_{V(T_1)}$ is a γ_I-function of T_1 with $g(x) = 1$. Since $g(x) = 1 \neq 0 = f_1(x)$, this contradicts the fact that T_1 is a UID-tree. Therefore, f as defined in the statement is a unique γ_I-function of T.

$\square$

In order to determine the importance of some vertices in unique γ_I-functions, we state the following definition.

Definition 3.2. Let $v \in V_i$. The *Italian external private neighbors of* v is given by $epn_i(v, V_1 \cup V_2) = \{u \in N(v) \cap V_0 \mid \sum_{x \in N(u)} f(x) = 2\}$.

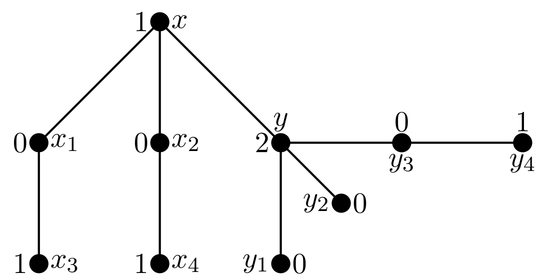


Figure 9: Example to illustrate $epn_i(v, V_1 \cup V_2)$.

We will be using Figure 9 that is labeled with γ_I-function f as an example to illustrate this definition. Let $v \in V_1$. A vertex u is in the set $epn_1(v, V_1 \cup V_2)$ if it adjacent to v and has weight 0, and has $\sum_{z \in N(u)} f(z) = 2$. In other words, u is being dominated only by its two neighbors of weight 1.

In Figure 9, we can find $epn_1(v, V_1 \cup V_2)$ for each vertex such that $f(v) = 1$. Considering the vertices of weight 1, we have that $epn_1(x, V_1 \cup V_2) = \{x_1, x_2\}$, $epn_1(x_3, V_1 \cup V_2) = \{x_1\}$, and $epn_1(x_4, V_1 \cup V_2) = \{x_2\}$. Since y_3 is also being dominated by y, we have that $epn_1(y_4, V_1 \cup V_2) = \emptyset$. In other words, there are no vertices of weight 0 that depend on y_4.

Let $v \in V_2$. A vertex u is in the set $epn_2(v, V_1 \cup V_2)$ if its adjacent to v and has weight 0, and it is not adjacent to any other vertices of weight 1 or 2. That is, u is being dominated only by v and $\sum_{z \in N(u)} f(z) = 2$. In Figure 9, y is the only vertex of weight 2. We have that $epn_2(y, V_1 \cup V_2) = \{y_1, y_2\}$. Since y_3 is also adjacent to a vertex of weight 1, we have that $y \notin epn_2(y, V_1 \cup V_2)$. In this definition, we are trying to determine which vertices of weight 0 are being dominated by only one vertex of weight 2, or are being dominated by exactly two vertices of weight 1. This leads us to the following result.

Lemma 3.3. *Let T be a UID-tree with unique γ_I-function f. If $v \in V(T)$ such that $f(v) = 2$, then $|epn_2(v, V_1 \cup V_2)| \geq 2$.*

Proof. Let T be a tree with a unique γ_I-function f. For purpose of contradiction, suppose that v is a vertex such that $f(v) = 2$ but $|epn_2(v, V_1 \cup V_2)| < 2$. This leads to the following two cases.

Case 1: $|epn_2(v, V_1 \cup V_2)| = 1$.

Let $u \in epn_2(v, V_1 \cup V_2)$, or equivalently, $\{u\} = epn_2(v, V_1 \cup V_2)$. By definition, u is the only neighboring vertex of v with weight 0 that has $N(u) \cap (V_1 \cup V_2) = \{v\}$. This implies that each vertex $x \in N(v) \setminus \{u\}$ such that $f(x) = 0$ must have $|N(x) \cap (V_1 \cup V_2)| \geq 2$.

Thus we can define a new function g as follows: $g(y) = f(y)$ if $y \in V \setminus \{u, v\}$, $g(v) = 1$, and $g(u) = 1$. This is an IDF of T that is of the same weight as f, implying that g is also a γ_I-function of T. Hence this contradicts that T has a unique γ_I-function.

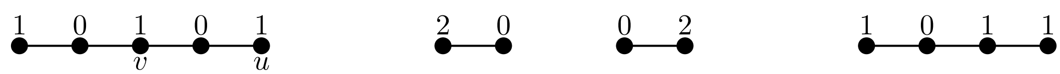
Case 2: $|epn_2(v, V_1 \cup V_2)| = 0$.

Then each vertex $x \in N(v)$ such that $f(x) = 0$ has $|N(x) \cap (V_1 \cup V_2)| \geq 2$. Therefore, a new function h can be defined as $h(x) = f(x)$ if $x \in V \setminus \{v\}$ and $h(v) = 1$. This function h is an IDF of T with smaller weight than f, contradicting that f is a γ_I-function of T. Therefore, we have that any vertex v of weight 2 in a unique γ_I-function has at least two Italian external private neighbors. $\square$

We now state a definition that will be used to determine the importance of vertices in a γ_I-function.

Definition 3.4. A vertex v is *essential* in T if $\gamma_I(T-v) > \gamma_I(T)$.

Consider the graph P_5 as depicted in Figure 10. We can see that $\gamma_I(P_5) = 3$ and $\gamma_I(P_5 - v) = 4$. In this case, removing v causes the Italian domination number to increase. Therefore, we determine that v is an essential vertex in P_5. Also, note that neither of the leaf vertices are essential. In Figure 10 (c), we can see that $\gamma_I(P_5 - u) = 3 = \gamma_I(P_5)$.



(a) P_5 (b) $P_5 - v$ (c) $P_5 - u$

Figure 10: Example of essential vertex.

We next state a proposition that will be supplemental in another proof presented in this paper.

Proposition 3.5. *Let T be a UID-tree with unique γ_I-function f. If $v \in V(T)$ such that $f(v) = 0$, then $\gamma_I(T) = \gamma_I(T-v)$.*

Proof. Since $f(v) = 0$, we have that $f|_{V(T-v)}$ is an IDF of $T-v$. This implies that $\gamma_I(T-v) \leq f(V(T-v)) = f(V(T)) = \gamma_I(T)$. Now we must show that $\gamma_I(T-v) \geq \gamma_I(T)$.

Suppose, for contradiction, that $\gamma_I(T-v) < \gamma_I(T)$. This is equivalent to $\gamma_I(T - v) \leq \gamma_I(T) - 1$. Let g be a γ_I-function of $T-v$, and so $g(V(T-v)) = \gamma_I(T-v)$. Now define a new function h on T as $h(u) = g(u)$ for $u \in V(T - v)$ and $h(v) = 1$. Now we have that h is an IDF of T, implying that $\gamma_I(T) \leq h(V(T)) = g(V(T - v)) + 1 = \gamma_I(T - v) + 1$. In particular, we have that $\gamma_I(T - v) \geq \gamma_I(T) - 1$. However, the assumption was that $\gamma_I(T - v) \leq \gamma_I(T) - 1$, which implies that $\gamma_I(T - v) + 1 = \gamma_I(T)$. Since $h(V(T)) = g(V(T - v)) + 1 = \gamma_I(T - v) + 1$, we now have that h is a γ_I-function of T. However, $h(v) = 1 \neq f(v) = 0$, contradicting that T is a UID-tree. Therefore, we have that $\gamma_I(T - v) \geq \gamma_I(T)$, resulting in the equality $\gamma_I(T - v) = \gamma_I(T)$. $\square$

From the previous result, we can conclude the following.

Lemma 3.6. *If x is an essential vertex in a UID-tree T with unique γ_I-function f, then $f(x) = 1, 2$.*

We next define another operation used to build a UID-tree.

Operation $\mathcal{O}_2$: Let T be the tree obtained from T_1 and T_2 by adding an edge between a vertex x in T_1 and a vertex y in T_2 such that x and y are essential.

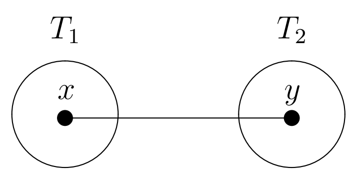


Figure 11: Operation $\mathcal{O}_2$.

Figure 11 depicts the general construction of trees that are obtained by performing $\mathcal{O}_2$, and Figure 12 shows a specific example of a tree produced from this operation. In Figure 8, we can see the unique γ_I-function f_1 of T_1 given by a healthy spider, as well as the unique γ_I-function f_2 of the wounded spider T_2. The edge xy was added between vertex x in T_1 and vertex y in T_2 such that x and y are both essential vertices.

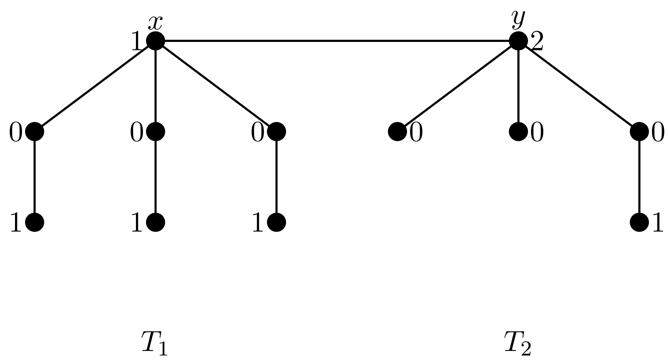


Figure 12: Example of a tree constructed from $\mathcal{O}_2$.

Proposition 3.7. *The tree T obtained from T_1 and T_2 by performing operation $\mathcal{O}_2$ is a UID-tree. Furthermore, f defined on $V(T)$ by $f(a) = f_1(a)$ for every $a \in V(T_1)$ and $f(b) = f_2(b)$ for every $b \in V(T_2)$ is the unique γ_I-function of T.*

Proof. Clearly, the function f defined on $V(T)$ by $f(a) = f_1(a)$ for every $a \in V(T_1)$ and $f(b) = f_2(b)$ for every $b \in V(T_2)$ is an IDF of T. This implies that $\gamma_I(T) \leq \gamma_I(T_1) + \gamma_I(T_2)$.

Now let f be a γ_I-function of T. If $f(x) = f(y)$ or if $\{f(x), f(y)\} = \{1, 2\}$, then $f|_{V(T_i)}$ is an IDF for T_i and so $\gamma_I(T_i) \leq f(V(T_i))$. Thus $\gamma_I(T_1) \leq f(V(T_1))$ and $\gamma_I(T_2) \leq f(V(T_2))$, and adding these two inequalities implies that $\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_1)) + f(V(T_2)) = \gamma_I(T)$ by the assumption. Hence the equality $\gamma_I(T) = \gamma_I(T_1) + \gamma_I(T_2)$ follows.

Now consider the other possibilities where $\{f(x), f(y)\} \in \{\{0, 2\}, \{0, 1\}\}$. Assume, without loss of generality, that $f(x) = 0$ and $f(y) \in \{1, 2\}$.

Case 1: $f(y) \neq f_2(y)$.

Since x and y are both essential vertices, $f_1(x) \in \{1,2\}$ and $f_2(y) \in \{1,2\}$. Since T_2 has a unique γ_I-function and $f(y) \neq f_2(y)$, then $f|_{V(T_2)}$ is an IDF of T_2 such that $\gamma_I(T_2) < f(V(T_2))$. This implies that $\gamma_I(T_2) \leq f(V(T_2)) - 1$. On the other hand, the function g defined on $V(T_1)$ by $g(u) = f(u)$ if $u \neq x$ and $g(x) = 1$ is an IDF of T_1. Thus $\gamma_I(T_1) \leq g(V(T_1)) = f(V(T_1)) + 1$. Adding the two previous inequalities gives

$$\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_2)) - 1 + f(V(T_1)) + 1 = f(V(T_2)) + f(V(T_1)) = \gamma_I(T).$$

Thus again resulting in the equality $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$.

Case 2: $f(y) = f_2(y)$.

Since $f(y) = f_2(y)$ and T_2 is a UID-tree, this implies that $f(V(T_2)) = \gamma_I(T_2)$. Assume, for the purpose of contradiction, that $\gamma_I(T) < \gamma_I(T_1) + \gamma_I(T_2)$. Equivalently, this can be expressed as $\gamma_I(T) \leq \gamma_I(T_1) + \gamma_I(T_2) - 1$. Using the fact that $f(V(T_2)) = \gamma_I(T_2)$, this implies that $\gamma_I(T) \leq \gamma_I(T_1) + f(V(T_2)) - 1$. So we have that $f(V(T_1)) + f(V(T_2)) \leq \gamma_I(T_1) + f(V(T_2)) - 1$. We then have that $f(V(T_1)) \leq \gamma_I(T_1) - 1$ which implies that $f|_{V(T_1)}$ is an IDF for $T_1 - x$. This contradicts the fact that x is an essential vertex in the unique γ_I-function of T_1. Thus we have $\gamma_I(T) \geq \gamma_I(T_1) + \gamma_I(T_2)$, again giving the equality $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$.

Now we need to show that f is the unique γ_I-function of T. Suppose, for the purpose of contradiction, that T is not a UID-tree and let $h \neq f$ be a γ_I-function of T. Clearly, if $h(x) = h(y)$ or if $\{h(x), h(y)\} = \{1,2\}$, then $h|_{V(T_i)}$ is a γ_I-function of T_i. This implies that either T_1 or T_2 is not a UID-tree. Thus we can assume that $\{h(x), h(y)\} \in \{\{0,2\},\{0,1\}\}$, say $h(x) = 0$ and $h(y) \in \{1,2\}$.

Case 1: $h(y) \neq f_2(y)$.

As seen before $h|_{V(T_2)}$ is an IDF of T_2 with weight $h(V(T_2)) \geq \gamma_I(T_2) + 1$. This along with the fact that $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$ imply that

$$h(V(T_1)) = h(V(T)) - h(V(T_2)) \leq \gamma_I(T) - (\gamma_I(T_2)+1) = \gamma_I(T) - \gamma_I(T_2) - 1 = \gamma_I(T_1) - 1$$

and $h|_{V(T_1)}$ is an IDF for $T_1 - x$. Again, this contradicts the fact that x is an essential vertex in the unique γ_I-function of T_1.

Case 2: $h(y) = f_2(y)$.

Since T_2 is a UID-tree, this implies that $h(V(T_2)) = \gamma_I(T_2)$ and that $h|_{V(T_2)} = f|_{V(T_2)}$. Thus we can assume that $h|_{V(T_1)} \neq f|_{V(T_1)}$. Clearly, $h(V(T)) = h(V(T_1)) + h(V(T_2))$, so these equations imply $\gamma_I(T) = h(V(T_1)) + \gamma_I(T_2)$. But we also know that $\gamma_I(T) = \gamma_I(T_1) + \gamma_I(T_2)$. This implies that $\gamma_I(T_2) + \gamma_I(T_1) = h(V(T_1)) + \gamma_I(T_2)$, suggesting that $\gamma_I(T_1) = h(V(T_1))$. Since $h|_{V(T_1)}$ is an IDF of $T_1 - x$, we have that $\gamma_I(T_1 - x) \leq h(V(T_1)) = \gamma_I(T_1)$. However, this implies that x is not an essential vertex in T_1, which is a contradiction. Therefore, f as defined in the statement is the unique γ_I-function of T.

□

We next define another operation that can be used to build a UID-tree.

Operation $\mathcal{O}_3$: Let T be the tree obtained from T_1 and T_2 by adding an edge between a vertex x in T_1 and a vertex y in T_2 such that $f_1(x) = 2$ and $f_2(y) = 0$.

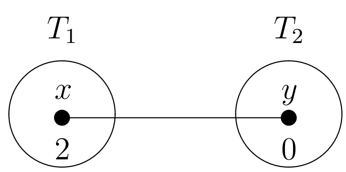
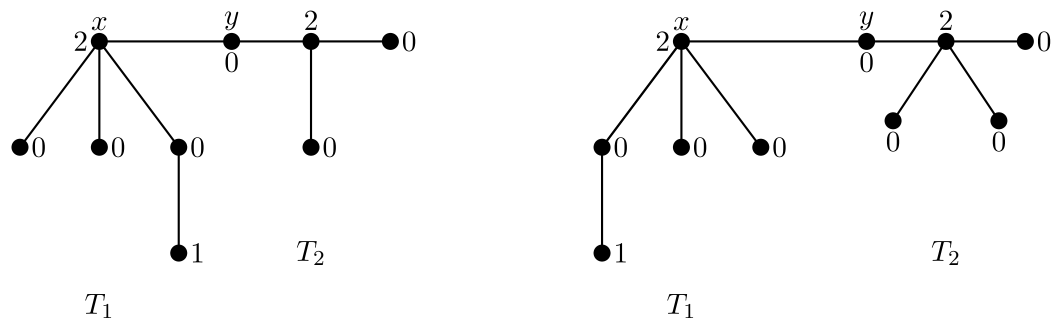


Figure 13: Operation $\mathcal{O}_3$.



(a) Example that is not a UID-tree. (b) Example that is a UID-tree.

Figure 14: Examples of trees constructed from $\mathcal{O}_3$.

Figure 13 depicts the general construction of trees that are obtained by performing $\mathcal{O}_3$, and Figure 14 shows specific examples of trees produced from this operation. In Figure 14 (a) and (b), we can see the unique γ_I-function f_1 of T_1 given by a wounded spider, as well as the unique γ_I-function f_2 of the star T_2. The edge xy was added between vertex x in T_1 and vertex y in T_2 such that $f_1(x) = 2$ and $f_2(y) = 0$.

Notice that Figure 14 (a) depicts an example of a tree produced from $\mathcal{O}_3$ that is not a UID-tree. Since the vertex y is now being dominated by x, this allows for relabelling of vertices in $T_2 - y$. This relabelling of the vertices in $T_2 - y$ is depicted in Figure 15. Therefore, there are two distinct γ_I-functions of the constructed tree T, and we can see that T is not a UID-tree.

However, the tree in Figure 14 (b) is a UID-tree, for x dominating y does not allow for any relabelling of vertices. This property can be thought of as removing y from T_2 and determining if $T_2 - y$ is a UID-tree. This leads to the following proposition.

Proposition 3.8. *If $T_2 - y$ is a UID-tree, then the tree T obtained from T_1 and T_2 performing Operation $\mathcal{O}_3$ is a UID-tree. Furthermore, f defined on $V(T)$ by $f(a) =$*

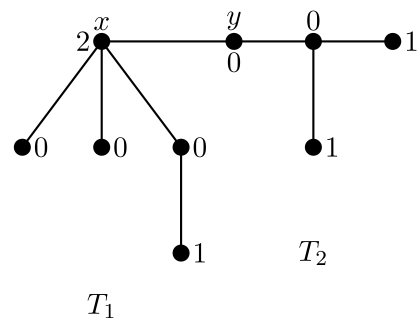


Figure 15: Another γ_I-function of the constructed tree in Figure 14 (a).

$f_1(a)$ for every $a \in V(T_1)$ and $f(b) = f_2(b)$ for every $b \in V(T_2)$ is the unique γ_I-function of T.

Proof. We must first show that $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$. We know that the function f defined on $V(T)$ by $f(a) = f_1(a)$ for every $a \in V(T_1)$ and $f(b) = f_2(b)$ for every $b \in V(T_2)$ is an IDF of T. This implies that $\gamma_I(T) \leq \gamma_I(T_1) + \gamma_I(T_2)$.

Now let f be a γ_I-function of T. If $f(x) = f(y)$ or if $\{f(x), f(y)\} = \{1, 2\}$, then $f|_{V(T_i)}$ is an IDF for T_i and so $\gamma_I(T_i) \leq f(V(T_i))$. Thus $\gamma_I(T_1) \leq f(V(T_1))$ and $\gamma_I(T_2) \leq f(V(T_2))$, and adding these two inequalities implies that $\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_1)) + f(V(T_2)) = \gamma_I(T)$ by the assumption. Thus the equality $\gamma_I(T) = \gamma_I(T_1) + \gamma_I(T_2)$ follows.

Now consider the other possibilities where $\{f(x), f(y)\} \in \{\{0, 2\}, \{0, 1\}\}$.

Case 1: $f(x) \in \{1, 2\}$ and $f(y) = 0$.

If $f(x) = 2$, then we have that $f_1(x) = f(x)$ which implies that $f|_{V(T_1)} = f_1$ and $f(V(T_1)) = f_1(V(T_1)) = \gamma_I(T_1)$. Since $f(y) = 0$, we also have that $f|_{V(T_2-y)}$ is an IDF of $T_2 - y$. This implies that $\gamma_I(T_2 - y) \leq f(V(T_2 - y) = f(V(T_2))$. From

Proposition 3.5, we also know that since T_2 is a UID-tree and $f_2(y) = 0$, it follows that $\gamma_I(T_2) = \gamma_I(T_2 - y)$. Thus we have that $\gamma_I(T_2) = \gamma_I(T_2 - y) \leq f(V(T_2))$. So adding this inequality with the previous equation gives $\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_1)) + f(V(T_2)) = \gamma_I(T)$. Thus the equality $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$ follows.

If $f(x) = 1$, then we have that $f|_{V(T_1)}$ is an IDF of T_1. This implies that $\gamma_I(T_1) \leq f(V(T_1)) - 1$ since $f(x) = 1 \neq f_1(x) = 2$. Now the function g defined on $V(T_2)$ as $g(u) = f(u)$ if $u \neq y$ and $g(y) = 1$ is an IDF of T_2. This implies that $\gamma_I(T_2) \leq g(V(T_2)) = f(V(T_2)) + 1$. Adding these two inequalities, we get that $\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_1)) + f(V(T_2)) = \gamma_I(T)$. Thus the equality $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$ follows.

Case 2: $f(x) = 0, f(y) \in \{1, 2\}$.

Assuming $f(y) \in \{1, 2\}$ implies that $f|_{V(T_2)}$ is an IDF of T_2. Since $f(y) \neq f_2(y) = 0$, we have that $\gamma_I(T_2) \leq f(V(T_2)) - 1$. Now the function g defined on $V(T_1)$ as $g(u) = f(u)$ if $u \neq x$ and $g(x) = 1$ is an IDF of T_1. This implies that $\gamma_I(T_1) \leq g(V(T_1)) = f(V(T_1)) + 1$. Adding these two inequalities, we get that

$$\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_1)) - 1 + f(V(T_2)) + 1 = f(V(T_1)) + f(V(T_2)) = \gamma_I(T).$$

Thus again resulting in the equality $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$.

Now we need to show that f is the unique γ_I-function of T. Since $f_2(y) = 0$, this implies that $f_2|_{V(T_2 - y)}$ is an IDF of $T_2 - y$ of weight $f_2(V(T_2 - y)) = f_2(V(T_2)) = \gamma_I(T_2)$. Since $f_2(y) = 0$ and T_2 is a UID-tree, we have that $\gamma_I(T_2) = \gamma_I(T_2 - y)$ from Proposition 3.5. This implies that $f_2|_{V(T_2 - y)}$ is a γ_I-function of $T_2 - y$. Moreover, since $T_2 - y$ is a UID-tree, we have that $f_2|_{V(T_2 - y)}$ is the unique γ_I-function of $T_2 - y$.

Now let $h \neq f$ be another γ_I-function of T, and consider two cases.

Case 1: $h(y) = 0$.

With $h(y) = 0$, this implies that $h|_{V(T_1)}$ is a γ_I-function of T_1. Since T_1 is a UID-tree, we have that $h|_{V(T_1)} = f_1$ and $h(V(T_1)) = \gamma_I(T_1)$. We also know that $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T) = h(V(T_1)) + h(V(T_2))$, but since we know that $h(V(T_1)) = \gamma_I(T_1)$, this equation reduces to $\gamma_I(T_2) = h(V(T_2))$. From Proposition 3.5, since $f_2(y) = 0$ and T_2 is a UID-tree, we have that $\gamma_I(T_2 - y) = \gamma_I(T_2) = h(V(T_2))$. Also, $h|_{V(T_2-y)}$ is an IDF of T_2-y of weight $h(V(T_2-y)) = h(V(T_2))$ implying that $h|_{V(T_2-y)}$ is a γ_I-function of T_2. Since T_2 is a UID-tree, it must be that $h|_{V(T_2-y)} = f_2|_{V(T_2-y)}$. Furthermore, since $h(y) = 0 = f_2(y)$, we have that $h|_{V(T_2)} = f_2$. Hence we obtain that $h = f$.

Case 2: $h(y) \in \{1, 2\}$.

This implies that $h|_{V(T_2)}$ is an IDF of T_2 such that $h(y) \neq f_2(y) = 0$, implying that $h(V(T_2)) > \gamma_I(T_2)$. We also know that $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T) = h(V(T_1)) + h(V(T_2))$, so $h(V(T_2)) > \gamma_I(T_2)$ implies that $h(V(T_1)) < \gamma_I(T_1)$ in order to satisfy the equation. Therefore, we have that $h(V(T_1)) \leq \gamma_I(T_1) - 1$.

Note that if $h(x) \in \{1, 2\}$, then $h|_{V(T_1)}$ is an IDF of T_1 implying that $\gamma_I(T_1) \leq h(V(T_1)$. This contradicts that $\gamma_I(T_1) > h(V(T_1))$, so we may assume that $h(x) = 0$. Now the function g defined on T_1 as $g(u) = h(u)$ if $u \neq x$ and $g(x) = 1$ is an IDF of T_1. This implies that $g(V(T_1)) \geq \gamma_I(T_1)$. We also know that $g(V(T_1)) = h(V(T_1)) + 1 \leq \gamma_I(T_1)$, thus implying that $g(V(T_1))$ is a γ_I-function of T_1. Since $g(x) = 1 \neq 2 = f_1(x)$, this contradicts that f_1 is the unique γ_I-function of T_1. Therefore, f as described in the statement is the unique γ_I-function of T.

$\square$

We will next state some supplemental results.

Proposition 3.9. *Let T be a UID-tree with unique γ_I-function f. If $f(v) = 1$ and $epn_1(v, V_1 \cup V_2) = \emptyset$, then $T - v$ is a UID-tree with $\gamma_I(T - v) = \gamma_I(T) - 1$ where $f|_{V(T-v)}$ is the unique γ_I-function of $T - v$.*

Proof. Since $epn(v, V_1 \cup V_2) = \emptyset$, we have that $f|_{V(T-v)}$ is an IDF of $T - v$. This implies that $\gamma_I(T - v) \leq f(V(T)) - 1 = \gamma_I(T) - 1$.

Now we must show that $\gamma_I(T-v) \geq \gamma_I(T)-1$. Let g be a γ_I-function of $T-v$, so we have that $g(V(T-v)) = \gamma_I(T-v)$. Now we can extend this function to an IDF of T by defining the function h as $h|_{V(T-v)} = g|_{V(T-v)}$ and $h(v) = 1$. We have that h is an IDF of T, implying that $\gamma_I(T) \leq h(V(T)) = g(V(T-v))+1 = \gamma_I(T-v)+1$. In particular, we have that $\gamma_I(T - v) \geq \gamma_I(T) - 1$ resulting in the equality $\gamma_I(T - v) = \gamma_I(T) - 1$.

Now we need to show that $f|_{V(T-v)}$ is the unique γ_I-function of $T - v$. Let $h \neq f|_{V(T-v)}$ be a γ_I-function of $T - v$. We can again extend this function to T by defining g as $g|_{V(T-v)} = h|_{V(T-v)}$ and $g(v) = 1$. We now have that g is an IDF of T of weight $g(V(T)) = h(V(T - v)) + 1 = \gamma_I(T - v) + 1$. We previously showed that $\gamma_I(T) = \gamma_I(T - v) + 1$, so this implies that g is a γ_I-function of T. Since $g|_{V(T-v)} \neq f|_{V(T-v)}$, we have that $g \neq f$, which contradicts that T is a UID-tree. Therefore, $f|_{V(T-v)}$ is the unique γ_I-function of $T - v$. □

Proposition 3.10. *Let T be a UID-tree with unique γ_I-function f. If $f(v) = 1$ and $|epn_1(v, V_1 \cup V_2)| = 1$, then $\gamma_I(T - v) = \gamma_I(T)$.*

Proof. Let $u \in epn_1(v, V_1 \cup V_2)$ and recall that this implies $f(u) = 0$. Now define g on $T - v$ as $g(z) = f(z)$ if $z \in V(T - \{v, u\})$ and $g(u) = 1$. We have that g is an IDF of $T - v$, implying that

$$\gamma_I(T - v) \leq f(V(T - \{v, u\}) + 1) = f(V(T)) - 1 + 1 = f(V(T)) = \gamma_I(T).$$

Now we need to show that $\gamma_I(T - v) \geq \gamma_I(T)$. Suppose for contradiction that $\gamma_I(T-v) \leq \gamma_I(T)-1$. Let g be a γ_I-function of $T-v$ so that $g(V(T-v)) = \gamma_I(T-v)$. We can extend this function to T by defining h as $h(z) = g(z)$ for $z \in V(T - v)$ and $h(v) = 1$. We now have that h is an IDF of T implying that $\gamma_I(T) \leq g(V(T-v))+1 = \gamma_I(T - v) + 1$. From the assumption, we had that $\gamma_I(T - v) \leq \gamma_I(T) - 1$. Hence the equality $\gamma_I(T - v) + 1 = \gamma_I(T)$ follows.

Since we have that $h(V(T)) = g(V(T - v)) + 1 = \gamma_I(T - v) + 1$, we have that h is the unique γ_I-function of T. We defined g as a γ_I-function of $T - v$ and let $h(z) = g(z)$ if $z \in V(T - v)$, implying that $h|_{V(T-v)}$ is also a γ_I-function of $T - v$. However, since we have $|epn_1(v, V_1 \cup V_2)| = 1$, removing v from T would leave u with $\sum_{x \in N(u)} f(x) = 1$. Thus $h|_{V(T-v)}$ being an IDF of $T-v$ is a contradiction. Therefore, $\gamma_I(T - v) = \gamma_I(T)$.

□

Combining the two previous results, we can conclude the following.

Lemma 3.11. *If T is a UID-tree with unique γ_I-function f and $v \in V(T)$ such that v is an essential vertex where $f(v) = 1$, then $|epn_1(v, V_1 \cup V_2)| \geq 2$.*

From Lemma 3.3, we know that any vertex of weight 2 in a UID-tree has at least two Italian external private neighbors. We also know that every essential vertex x in a UID-tree either has weight 1 or weight 2. Therefore, we can conclude the following.

Lemma 3.12. *If T is a UID-tree with unique γ_I-function f and $v \in V_i$ such that v is an essential vertex in T, then $|epn_i(v, V_1 \cup V_2)| \geq 2$.*

We will now define another operation that can be used to join two UID-trees.

Operation $\mathcal{O}_4$: Let T be the tree obtained from T_1 and T_2 by adding an edge between a vertex x in T_1 and a vertex y in T_2 such that $f_1(x) = 2$ and $f_2(y) = 1$, and y has $|epn_1(y, V_1 \cup V_2)| \leq 1$.

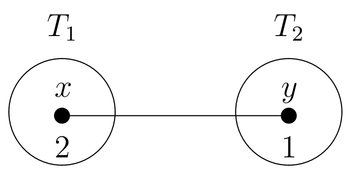
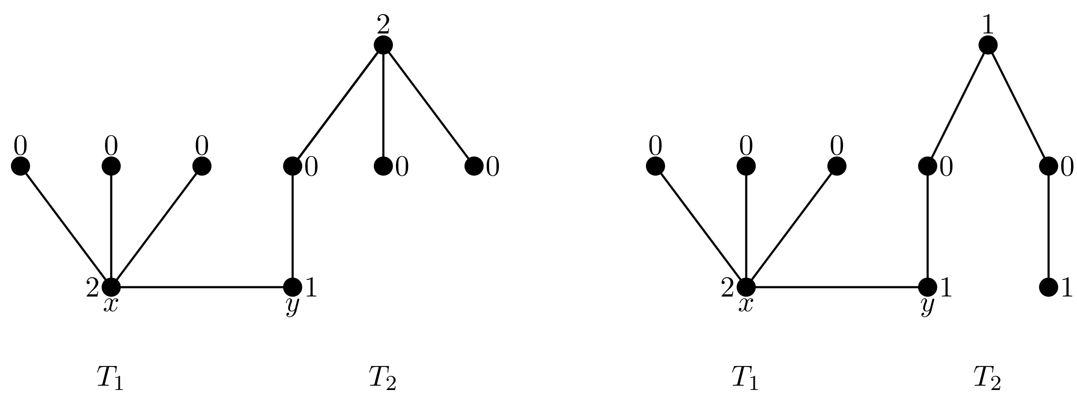


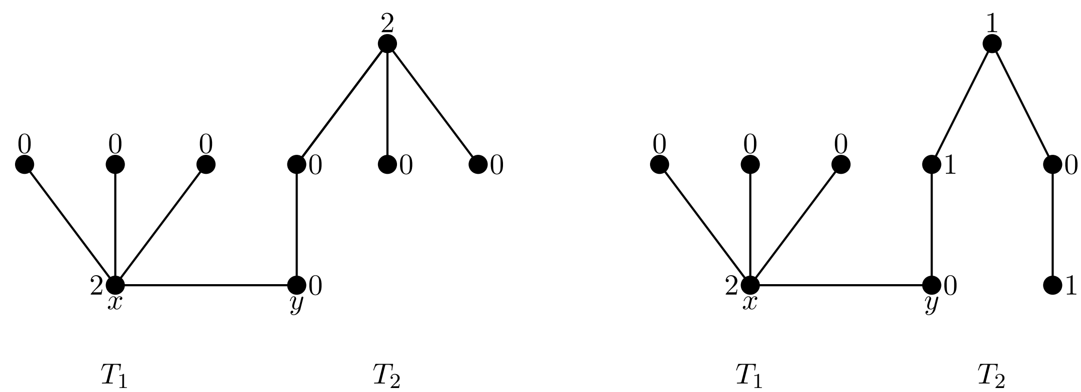
Figure 16: Operation $\mathcal{O}_4$.



(a) Example that is a UID-tree. (b) Example that is not a UID-tree.

Figure 17: Examples of trees constructed from $\mathcal{O}_4$.

Figure 16 depicts the general construction of trees that are obtained by performing $\mathcal{O}_4$, and Figure 17 shows specific examples of trees produced from this operation. Note that Operation $\mathcal{O}_4$ implies that $|epn(y, V_1 \cup V_2)| = 0, 1$ and that y is not an essential vertex in T_2. In Figure 17 (a), we can see the unique γ_I-function f_1 of T_1 given by a star, as well as the unique γ_I-function f_2 of the wounded spider T_2. This is an example where y has $|epn_1(y, V_1 \cup V_2)| = 0$, meaning that y is not required to dominate any other vertices. Thus, the currently labeled function is not a γ_I-function of T. It appears that the function f defined on $V(T)$ by $f(a) = f_1(a)$ for every $a \in V(T_1)$, $f(b) = f_2(b)$ for every b $\in V(T_2 - y)$, and $f(y) = 0$ is the unique γ_I-function of T. This γ_I-function of T is depicted in Figure 18 (a).



(a) T from Figure 17 (a). (b) T from Figure 17 (b).

Figure 18: γ_I-functions of trees from Figure 17.

In Figure 17 (b), we can see the unique γ_I-function f_1 of T_1 given by a star, as well as the unique γ_I-function f_2 of $P_5 = T_2$. This is an example where y has $|epn_1(y, V_1 \cup V_2)| = 1$. Notice that the tree T obtained is not a UID-tree.

The currently labelled function is a γ_I-function of T, but a function h where all weights remain the same except $h(y) = 0$ and $h(u) = 1$ where $u \in epn_1(V_1 \cup V_2)$ is also a γ_I-function of T. This γ_I-function is depicted in Figure 18 (b). This leads to the following result.

Proposition 3.13. *If $|epn_1(y, V_1 \cup V_2)| = 1$, then the tree T obtained from T_1 and T_2 by performing operation $\mathcal{O}_4$ is not a UID-tree.*

Proof. First we must show that $\gamma_I(T_1) + \gamma_I(T_2) = \gamma_I(T)$. We have that f defined on V(T) by $f(a) = f_1(a)$ for every $a \in V(T_1)$ and $f(b) = f_2(b)$ for every b $\in V(T_2)$ is an IDF of T. This implies that $\gamma_I(T) \leq \gamma_I(T_1) + \gamma_I(T_2)$.

Now we need to show $\gamma_I(T) \geq \gamma_I(T_1) + \gamma_I(T_2)$. Let f be a γ_I-function of T. If $f(x) = f(y)$ or if $\{f(x), f(y)\} = \{1, 2\}$, then $f|_{V(T_i)}$ is an IDF for T_i and so $\gamma_I(T_i) \leq f(V(T_i))$. Thus $\gamma_I(T_1) \leq f(V(T_1))$ and $\gamma_I(T_2) \leq f(V(T_2))$, and adding these two inequalities implies that $\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_1)) + f(V(T_2)) = \gamma_I(T)$ by the assumption. Thus the equality $\gamma_I(T) = \gamma_I(T_1) + \gamma_I(T_2)$ follows.

Now consider the other possibilities where $\{f(x), f(y)\} \in \{\{0, 2\}, \{0, 1\}\}$.

Case 1: $f(x) = 0$ and $f(y) \in \{1, 2\}$.

If $f(y) = 2$, we have that $f|_{V(T_2)}$ is an IDF of T_2. Since $f(y) = 2 \neq 1 = f_2(y)$, this implies that $\gamma_I(T_2) \leq f(V(T_2) - 1$. Now the function g defined on $V(T_1)$ as $g(u) = f(u)$ if $u \neq x$ and $g(x) = 1$ is an IDF of T_1. So we have that $\gamma_I(T_1) \leq f(V(T_1)) + 1$ and adding these two inequalities gives $\gamma_I(T_1) + \gamma_I(T_2) \leq f(V(T_1)) + 1 + f(V(T_2)) - 1 = \gamma_I(T)$.

If $f(y) = 1$, then $f|_{V(T_2)}$ is the unique γ_I-function of T_2 implying that $f(V(T_2)) = \gamma_I(T_2)$. Assume for contradiction that $\gamma_I(T) < \gamma_I(T_1) + \gamma_I(T_2)$. Substituting the previous equation, this implies that $\gamma_I(T) = f(V(T_1)) + f(V(T_2)) < \gamma_I(T_1) + f(V(T_2))$. After cancellation we are left with $f(V(T_1)) < \gamma_I(T_1)$ implying that $f(V(T_1)) \leq \gamma_I(T_1) - 1$. Notice that g defined above on $V(T_1)$ has weight $g(V(T_1)) = f(V(T_1)) + 1 =$

$\gamma_I(T_1)+1$. We just established that $f(V(T_1))+1 \leq \gamma_I(T_1)$, implying that g is also a γ_I-function of T_1. Since $g(x)=1 \neq f_1(x)=2$, this contradicts the uniqueness of f_1. Hence we have that $\gamma_I(T) \geq \gamma_I(T_1)+\gamma_I(T_2)$ and the equality $\gamma_I(T)=\gamma_I(T_1)+\gamma_I(T_2)$ follows.

Case 2: $f(x) \in \{1,2\}$ and $f(y)=0$.

If $f(x)=1$, then $f|_{V(T_1)}$ is an IDF of T_1 with $f_1(x)=2 \neq 1 = f(x)$. This implies that $\gamma_I(T_1) \leq f(V(T_1))-1$. Now the function g defined $V(T_2)$ as $g(u)=f(u)$ if $u \neq y$ and $g(y)=1$ is an IDF of T_2. Thus we have that $\gamma_I(T_2) \leq g(V(T_2)) = f(V(T_2))+1$. Adding the two previous inequalities we have $\gamma_I(T_1)+\gamma_I(T_2) \leq f(V(T_1))-1+f(V(T_2))+1=\gamma_I(T)$. Thus the equality $\gamma_I(T)=\gamma_I(T_1)+\gamma_I(T_2)$ follows.

If $f(x)=2$, then $f|_{V(T_1)}=f_1$ and $f(V(T_1))=\gamma_I(T_1)$. Since $f(y)=0$, we have that $f|_{V(T_2-y)}$ is an IDF of T_2. This implies that $\gamma_I(T_2-y) \leq f(V(T_2-y)) = f(V(T_2))$. Since $|epn(y, V_1 \cup V_2)|=1$ and $f(y)=1$, we have that $\gamma_I(T_2-y)=\gamma_I(T_1)$ from Proposition 3.10. Thus we have that $\gamma_I(T_2-y)=\gamma_I(T_2) \leq f(V(T_2))$. Adding this inequality and the fact that $f(V(T_1))=\gamma_I(T_1)$, we have that $\gamma_I(T_1)+\gamma_I(T_2) \leq f(V(T_1))+f(V(T_2))=\gamma_I(T)$. Again, the equality $\gamma_I(T)=\gamma_I(T_1)+\gamma_I(T_2)$ follows.

Now we need to show f is not a unique γ_I-function of T. Let $u \in epn(y, V_1 \cup V_2)$. Consider the function h defined as $h|_{V(T_1)}=f|_{V(T_1)}$, $h(y)=0$, $h(u)=1$, and $h|_{V(T_2-\{u,y\})}=f|_{V(T_2-\{u,y\})}$. Now h is an IDF of T of weight $h(V(T))=f(V(T_1))+f(V(T_2))-1+1=\gamma_I(T)$. Thus h is a γ_I-function of T where $h(y)=0 \neq 1 = f(y)$. Therefore, T is not a UID-tree. $\square$

We will now define the following two operations that still need to be researched.

Operation $\mathcal{O}_5$: Let T be the tree obtained from T_1 and T_2 by adding an edge between a vertex x in T_1 and a vertex y in T_2 such that $f_1(x) = 1$ and $f_2(y) = 0$.

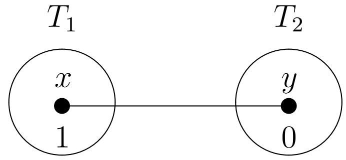
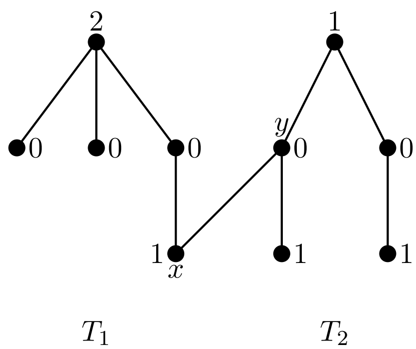
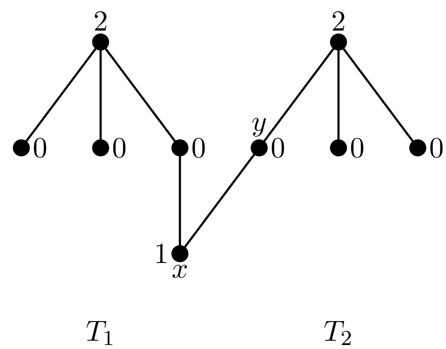


Figure 19: Operation $\mathcal{O}_5$.



(a) Example that is not a UID-tree.



(b) Example that is a UID-tree.

Figure 20: Examples of trees constructed from $\mathcal{O}_5$.

Figure 19 depicts the general construction of trees that are obtained by performing $\mathcal{O}_5$, and Figure 20 shows specific examples of trees produced from this operation. Note that this puts no restriction on x, which means that x could be essential or nonessential. The trees produced in Figure 20 are examples where x is a nonessential vertex. As we can see, the tree produced in Figure 20 (a) is not a UID-tree, but the tree constructed in (b) is a UID-tree. Similarly to how we dealt with Operation $\mathcal{O}_3$, one might consider adding the restriction that $T_2 - y$ is a UID-tree in order to guarantee the constructed tree is a UID-tree. However, $T_2 - y$ is not a UID-tree in both (a) and (b), even though one of the constructed trees is a UID-tree and the other is not. Therefore, $T_2 - y$ being a UID-tree is not a sufficient condition for Operation $\mathcal{O}_5$.

We will now define the other operation that still needs to be addressed.

Operation $\mathcal{O}_6$: Let T be the tree obtained from T_1 and T_2 by adding an edge between a vertex x in T_1 and a vertex y in T_2 such that $f_1(x) = 1$ and $f_2(y) = 1$ where at least one of x, y is not an essential vertex.

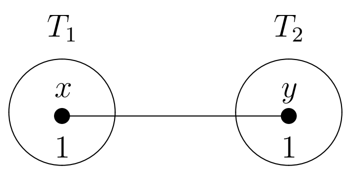
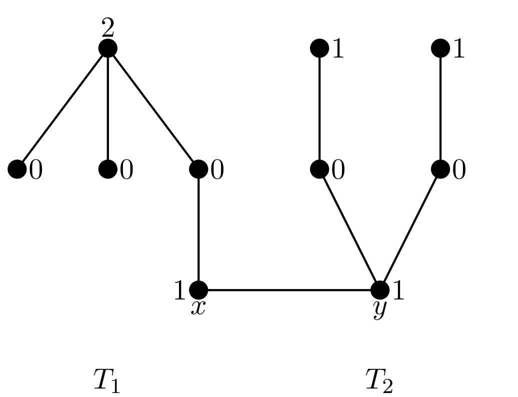
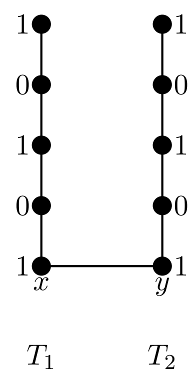


Figure 21: Operation $\mathcal{O}_6$.



(a) Example with one essential vertex.



(b) Example with nonessential vertices.

Figure 22: Examples of trees constructed from $\mathcal{O}_6$.

Figure 21 depicts the general construction of trees that are obtained by performing $\mathcal{O}_6$, and Figure 22 shows specific examples of trees produced from this operation. Notice that this operation addresses two cases for the vertices x, y. One case being that x, y are both nonessential vertices, and the other being that only one of x, y is essential. Figure 22 (a) depicts a tree constructed from one essential vertex, y in this example, and (b) shows an example where both x, y are nonessential vertices. Notice that both of the trees produced in Figure 22 are not UID-trees. Let f be the function depicted in (a). Then the function h defined as $h(u) = f(u)$ for $u \in V(T_1 - x)$, $h(x) = 0$, $h(y) = 2$, and $h(z) = f(z)$ for $z \in V(T_2 - y)$ is also a γ_I-function of T. Thus, x being a nonessential vertex allows for relabelling of weights of vertices.

Similarly, consider the function g depicted in (b). Define a new function k as $k(u) = g(u)$ for $u \in V(T_1)$, $k(y) = 0$, $k(v) = 1$ for $v \in epn_1(y, V_1 \cup V_2)$, and $k(z) = g(z)$ for $z \in V(T_2 - \{y, v\})$. Then k is also a γ_I-function of the tree depicted in (b). In both cases, it appears that having at least one nonessential vertex allows for relabelling of vertices in its closed neighborhood. Therefore, it appears that trees produced from Operation $\mathcal{O}_6$ are not UID-trees, but this remains to be proven.

4 CONCLUDING REMARKS

We considered various weights of two vertices that were used to join two UID-trees T_1 and T_2 with a single edge. We considered adding the edge between two vertices of weight 0, and between two essential vertices. The case considering two essential vertices includes: two vertices of weight 2, some cases where both vertices have weight 1, and some cases where one vertex has weight 2 and the other has weight 1. We also considered the case when the edge is added between a vertex of weight 2 and a vertex of weight 0. The last case addressed was adding this edge between a vertex of weight 2 and a nonessential vertex of weight 1 that was not self-dominating.

The following cases still remain: adding this edge between two vertices of weight 1 where at least one is nonessential, adding the edge between a vertex of weight 1 and a vertex of weight 0, and adding the edge between a vertex of weight 2 and a self-dominating vertex of weight 1. It also remains to determine if these are sufficient conditions on a UID-tree. That is, if we have a UID-tree that was obtained by adding an edge between two trees T_1 and T_2, can we determine under what conditions are T_1 and T_2 UID-trees. We conclude with problems and topics that could be used to further research of UID-trees.

4.1 Future Work

1. Find properties that characterize UID-trees.

2. Extend unique Italian domination to other topics, such as unique perfect Italian domination or unique independent Italian domination.

3. Characterize the family of trees $\mathcal{T}$ where $T \in \mathcal{T}$ if T is both a UID-tree and a URD-tree with $\gamma_I(T) = \gamma_R(T)$. That is, f is the unique γ_I-function of T and f is the unique γ_R-function of T.

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