12-2018

Italian Domination on Ladders and Related Products

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Italian Domination on Ladders and Related Products

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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December 2018

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Keywords: graph theory, graph products, Italian domination

ABSTRACT

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An Italian dominating function on a graph $G = (V, E)$ is a function such that $f : V \to \{0, 1, 2\}$, and for each vertex $v \in V$ for which $f(v) = 0$, we have $\sum_{u \in N(v)} f(u) \geq 2$. The weight of an Italian dominating function is $f(V) = \sum_{v \in V(G)} f(v)$. The minimum weight of all such functions on a graph G is called the Italian domination number of G. In this thesis, we will consider Italian domination in various types of products of a graph G with the complete graph K_2. We will find the value of the Italian domination number for ladders, specific families of prisms, mobius ladders and related products including categorical products $G \times K_2$ and lexicographic products $G \cdot K_2$. Finally, we will conclude with open problems.

DEDICATION

For April, my constant companion, my best friend, my biggest fan, and my forever love, without whose constant support and encouragement this would certainly not have been possible.

ACKNOWLEDGMENTS

I would like to thank my committee chair, Dr. Teresa Haynes, who not only introduced me to graph theory, but also taught me the beauty of proofs; Dr. Robert Beeler for always providing a fresh challenge and a sarcastic remark; Dr. Frederick Norwood for his inspiration, truly embodying the concept of a "Renaissance man;" and all of the ETSU Mathematics and Statistics faculty and staff for their instruction, guidance, and support over these past six years.

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1 INTRODUCTION

In this thesis, we will consider Italian domination in ladders and related "prism" type graphs. Before we proceed into our discussion, it is necessary to enumerate and clarify basic definitions and notation used. Let $G = (V, E)$ be a simple graph without directed edges having vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is the number of vertices of $V(G)$, and the *size* of G is the number of edges in $E(G)$. For vertices $x, y \in V(G)$, we say that x and y are *adjacent* if the edge $xy \in E(G)$. The *open neighborhood* of a vertex $v \in V(G)$, denoted $N(v)$, includes all vertices $u \in V(G)$ such that v and u are adjacent. The *closed neighborhood* of a vertex $v \in V(G)$ is denoted $N[v]$, is $N(v) \cup \{v\}$. The *degree* of a vertex v is the cardinality of the open neighborhood of v. That is, $deg_G(v) = |N(v)|$. The *maximum degree* of a graph G, denoted $\Delta(G)$, is $max\{deg_G(v) \mid v \in V(G)\}$. Similarly, the *minimum degree* of a graph G, denoted $\delta(G)$, is $min\{deg_G(v) \mid v \in V(G)\}$. A set of vertices $S \subseteq V(G)$ is said to be *independent* if for all $u, v \in S$, the edge $uv \notin E(G)$.

A *path graph*, denoted P_n, is a graph of order n and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are v_iv_{i+1} for $i = 1, 2, \ldots, n-1$. A *cycle graph*, denoted C_n, is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are v_1v_n and v_iv_{i+1} for $i = 1, 2, \ldots, n-1$. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. For a nonempty subset S of $V(G)$, the subgraph $G[S]$ of G *induced* by S has S as its vertex set, and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. A subgraph $H \subseteq G$ is called an *induced subgraph* of G if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. The *complete graph*, denoted

K_n, is a graph of order n in which every pair of distinct vertices are adjacent.

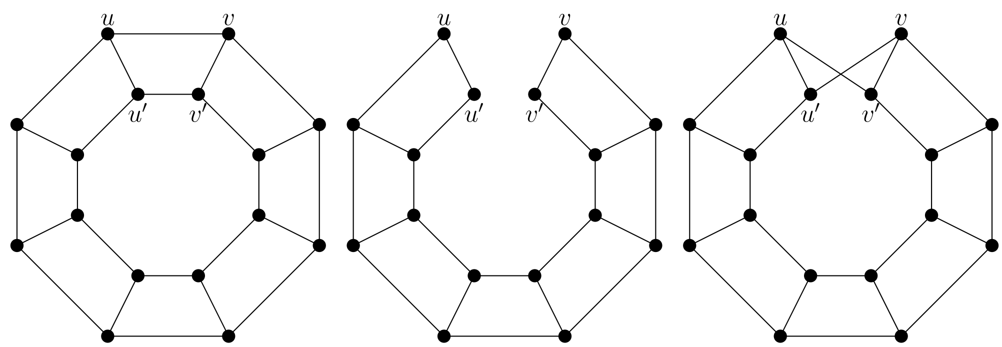
A graph is said to be *connected* if for any two vertices $u, v \in V(G)$, G contains a path connecting u and v as a subgraph. A *trivial graph* is said to be a graph with only one vertex and no edges; a graph which does not satisfy this definition is called *nontrivial.* Two graphs G and H are said to be *isomorphic*, denoted $G \cong H$, if there is a one-to-one and onto function $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.

A *star graph*, denoted $K_{1,n}$, is a graph in which one vertex v has $N[v] = V(G)$, and every other vertex u has $N(u) = \{v\}$.

The *Cartesian product* of graphs G and H, denoted $G\Box H$, with disjoint vertex sets $V(G)$ and $V(H)$ is the graph with vertex set $V(G) \times V(H)$ and (u_1, u_2) adjacent with (v_1, v_2) whenever ($u_1 = v_1$ and u_2 is adjacent to v_2) or ($u_2 = v_2$ and u_1 is adjacent to v_1). Cartesian products are examined in detail in [14].

The Cartesian product $G\Box K_2$ is called a *prism* over G, constructed by creating two copies of G labeled G and G' with vertices labeled $v \in V(G)$ and $v' \in V(G')$, and adding edges vv' between each pair of corresponding vertices of G and G'. The most common examples of prism graphs are graphs of the form $C_n\Box K_2$, denoted Π_n. The graph Π_n is an example of a *cubic graph*, a graph with every vertex having degree 3.

Note that the Cartesian product $P_n\Box K_2$ is a graph with $2n$ vertices and $3n-2$ edges. Such a graph is called a *ladder*, denoted L_n. A *Möbius ladder*, denoted M_n, is a cubic graph with an even number n of vertices, formed from a C_n by adding edges (called "rungs") v_iv_j where $i = 1, 2, \ldots, \frac{n}{2}$ and $j = i + \frac{n}{2}$. It is so-named because (with the exception of $M_6 = K_{3,3}$) M_n has exactly $\frac{n}{2}$ 4-cycles which link together by their shared edges to form a topological Möbius strip. For our purposes, a Möbius ladder may be constructed from a ladder L_n by adding edges uv' and $u'v$ as shown in Figure 1.



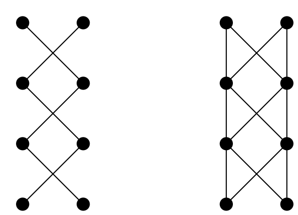
(a) Π_8 (b) L_8 (c) M_{16}

Figure 1: An octagonal prism, a ladder on 8 rungs, and a Möbius ladder with 8 rungs

The *complement* of a graph G, denoted $\overline{G}$, is a graph such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy \mid xy \notin E(G)\}$.

Complementary products were first introduced in [17] as a generalization of Cartesian products of graphs. We consider a subset of these products called complementary prisms. The *complementary prism* of a graph G, denoted $G\overline{G}$, is the disjoint union of G and $\overline{G}$ formed by adding a perfect matching between corresponding vertices of G and $\overline{G}$

The *categorical product* of graphs, also known as the tensor product or direct product, is the graph denoted $G \times H$ such that $V(G \times H) = V(G) \times V(H)$. For vertices $v_1, v_2 \in V(G)$ and $u_1, u_2 \in V(H)$, vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G \times H$ if and only if $v_1v_2 \in E(G)$ and $u_1u_2 \in E(H)$. In particular, the categorical product $G \times K_2$ is equivalent to the bipartite double graph of G, also known as a Kronecker cover or bipartite double cover, constructed as follows: Begin by by making two copies of the vertex set of a graph G, labeled G and G' and adding edges uv' and $u'v$ for every edge $uv \in E(G)$. The bipartite double cover is examined in greater detail in [13]. See Figure 2a for an example.



(a) $P_4 \times K_2$ (b) $P_4 \cdot K_2$

Figure 2: Categorical and lexicographic graph products of P_4 with K_2

The *lexicographic product* of graphs G and H, denoted $G \cdot H$, is a graph with $V(G \cdot H) = V(G) \times V(H)$, and edges as follows. For vertices $v_1, v_2 \in V(G)$ and $u_1, u_2 \in V(H)$, vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G \cdot H$ if and only if one of the following conditions is met:

i. v_1 is adjacent to v_2 in G.

ii. $v_1 = v_2$ and u_1 is adjacent to u_2 in H.

In particular, the lexicographic product $G \cdot K_2$ is equivalent to the double graph of G, constructed by making two copies of G, including its edge set, and adding edges vu' and uv' for every edge $uv \in E(G)$. Lexicographic products are examined in further detail in [29] and [30]. See Figure 2b for an example.

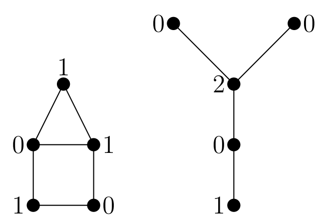
These and other kinds of graph products are explored in detail in [19].

A *dominating set* of a graph G is set $D \subseteq V(G)$ such that for all $v \in V(G)$, either $v \in D$, or $u \in N(V) \cap D$. Equivalently, a subset $D \subseteq V$ is a dominating set if and only if $|N[v] \cap D| \geq 1$ for all $v \in V(G)$. Thus, $N[D] = V(G)$. The minimum cardinality among all dominating sets of G is called the *domination number* of G and is denoted $\gamma(G)$.

Related to domination, a *2-dominating set* is a subset $D \subseteq V(G)$ such that for every vertex $v \in V(G)$, either $v \in D$ or $|N(v) \cap D| \geq 2$. The minimum cardinality among all 2-dominating sets is called the *2-domination number* of G, denoted $\gamma_2(G)$. The concept of 2-domination is first introduced in [11] and may be generalized as n-domination. See also [3, 25]. A *double dominating set* of a graph G is a subset S of $V(G)$ such that $|N[v] \cap S| \geq 2$ for every $v \in V(G)$. The minimum cardinality of such a set is called the *double domination number* of G, denoted $\gamma_{\times 2}(G)$. Double domination was introduced in [16] and is generalized as k-tuple domination in [8, 9, 15].

A *Roman dominating function*, or RDF, on a graph G is a function $f : V(G) \to \{0, 1, 2\}$, such that for every $v \in V(G)$, if $f(v) = 0$, then there is at least one $u \in N(v)$ where $f(u) = 2$. For any Roman dominating function f on a graph G, and a set $I = \{0, 1, 2\}$, let $V_i = \{v \in V \mid f(v) = i \text{ for some } i \in I\}$. Since this partitions $V(G)$ into three distinct vertex sets and determines f, we write $f = (V_0, V_1, V_2)$. The *weight* of a Roman dominating function is the value $f(V) = \sum_{v \in V(G)} f(v)$, or equivalently, $f(V) = |V_1| + 2|V_2|$. The minimum weight of a RDF on G is called the *Roman domination number* of G, denoted $\gamma_R(G)$. Roman domination was motivated by Stewart in [28], and a Roman dominating function was first formally defined in [7]. Since then, Roman domination has been studied in a number of papers. See for example [1, 2, 4, 10, 12, 20, 21, 29, 31, 32].

An *Italian dominating function*, or IDF, on a graph G is a function $f : V \to \{0, 1, 2\}$ such that for every $v \in V(G)$ such that $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$. In a manner similar to Roman domination, an IDF partitions G into three V_i sets for $i \in \{0, 1, 2\}$, such that $f = (V_0, V_1, V_2)$. The weight of an IDF is $\sum_{v \in V(G)} f(v)$, or equivalently, $f(V) = |V_1| + 2|V_2|$. As with previous types of domination, the minimum weight among all Italian dominating functions of G is called the *Italian domination number*, denoted $\gamma_I(G)$. Italian domination was introduced in [6] as Roman {2}-domination in 2016. The concept was further examined in a number of papers, such as [18, 23, 26]. Two examples of Italian dominating functions are given in Figure 3, where the vertex labels represent the Italian dominating function.



(a) (b)

Figure 3: Italian domination examples

Finally, it is necessary to discuss some related terminology which was given by [23]. A graph is defined to be an *I1* graph if every minimum weight Italian dominating function uses only elements of the set $\{0, 1\}$. Similarly, a graph is defined to be an *I2* graph if every minimum weight Italian dominating function uses only elements the set $\{0, 2\}$. Finally, a graph is an *I1a* graph if the range of some minimum weight

Italian dominating function has range $\{0, 1\}$.

As previously stated, in this thesis we will discuss Italian domination in ladder graphs and related products of various graphs together with K_2. First, we will conduct a survey of known results relevant to this thesis. Then, we will begin our discussion with Italian domination on a ladder L_n, various cartesian products of the form $G \Box L_n$, selected categorical products of the form $G \times L_n$, and lexicographic products of the form $G \cdot L_n$. Finally, we will conclude with open problems.

2 LITERATURE SURVEY

In this section, we enumerate some known results relevant to this research. These results were the motivation behind the results proven in this thesis.

The following results and theorems are not an exhaustive list of known results related to Italian domination in graphs, but is rather a list of known results relevant to the results in this research. For a more complete overview of known results regarding Italian domination, the reader is referred to [6, 18, 23]. Though more broadly known today as Italian domination, this concept was introduced in [6] as Roman $\{2\}$-domination, denoted in that paper as $\gamma_{\{R2\}}(G)$. In the interest of consistency, all the results taken from [6] have been restated using our notation, $\gamma_I(G)$, for the Italian domination number. To begin with, let us state several bounds on the Italian domination number.

Proposition 2.1. *[6] For every graph G, $\gamma(G) \leq \gamma_I(G) \leq \gamma_R(G)$.*

Observation 2.2. *[6] For a graph G, $\gamma(G) < \gamma_I(G) < \gamma_R(G)$ is possible, even for paths.*

Theorem 2.3. *[6, 23] For every graph G, $\gamma_I(G) \leq 2\gamma(G)$.*

Proposition 2.4. *[6] For every graph G, $\gamma_I(G) \leq \gamma_2(G)$.*

The bound given in this proposition is sharp in the next result.

Corollary 2.5. *[6] For every graph G with $\Delta(G) \leq 2, \gamma_I(G) = \gamma_2(G)$.*

Using this Corollary, the Italian domination numbers for two major families of graphs is given by the following result.

Corollary 2.6. *[6, 23] For paths P_n and cycles C_n, $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$, and $\gamma_I(C_n) = \lceil \frac{n}{2} \rceil$.*

We may further characterize the bound given by Proposition 2.1.

Proposition 2.7. *[6, 23] For all G, $\gamma_I(G) = \gamma_2(G)$ if and only if G is $I1a$.*

Theorem 2.8. *[23] For all connected graphs G on $n \geq 3$ vertices, $\gamma_I(G) \leq \frac{3n}{4}$.*

Theorem 2.9. *[23] Let G be a graph with $n \geq 3$ vertices and $\delta(G) \geq 2$. Then, $\gamma_I(G) \leq \frac{2n}{3}$.*

Theorem 2.10. *[23] Let G be a graph on n vertices with $\delta(G) \geq 3$. Then, $\gamma_I(G) \leq \frac{n}{2}$.*

Then, we state a result given in [23] characterizing the *I1a* graphs.

Proposition 2.11. *[23] For all G, $\gamma_I(G) = \gamma_2(G)$ if and only if G is I1a.*

Now, we state some related results for Italian domination in complementary prisms.

Theorem 2.12. *[26] For any graph G:*

i. $\gamma_I(G\overline{G}) = 2$ *if and only if* $G = K_1$.

ii. $\gamma_I(G\overline{G}) = 3$ *if and only if* $G = K_2$.

iii. *If* $\gamma_I(G) = 3$ *and G has an isolated vertex, then* $\gamma_I(G\overline{G}) = 4$.

iv. *If G is a star graph with order $n \geq 3$, then* $\gamma_I(G\overline{G}) = 4$.

v. *If* $G = C_4$, *then* $\gamma_I(G\overline{G}) = 4$.

The above result is given in [26] as five distinct results, but we combine them here for brevity.

Finally, some bounds on the Roman domination number in lexicographic products are given in [29].

Corollary 2.13. *[29] Let G and H be nontrivial connected graphs. Then,* $2\gamma(G) \leq \gamma_R(G \cdot H)$.

Proposition 2.14. *[29] Let G be a nontrivial connected graph and G a connected graph with* $\gamma_R(H) = 2$. *Then,* $\gamma_R(G \cdot H) = 2\gamma(G)$.

3 RESULTS

3.1 Italian Domination on Ladders

Recall that a ladder graph L_n is the cartesian product $P_n \square K_2$, with two copies of P_n labeled P_n and P'_n where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(P'_n) = \{v'_1, v'_2, \ldots, v'_n\}$. So, L_n has order $2n$ and size $3n-2$. In addition, we call an edge $v_i v'_i$ where $v_i \in V(P_n)$ and $v'_i \in V(P'_n)$ a *rung* $r_i \in E(L_n)$.

Note that by Corollary 2.5, $\gamma_I(L_n) \leq 2(\gamma_I(P_n))$, and by Corollary 2.6, we have that $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$. Thus, $\gamma_I(L_n) \leq n+1$, but this bound can be improved. We show that $\gamma_I(L_n) = n$. We give three lemmas before our result.

Lemma 3.1. *Let L_n be a ladder on n rungs. It follows that $\gamma_i(L_n) \leq n$.*

Proof. Let $f = (V_0, V_1, V_2)$ be an Italian dominating function on L_n. Let each rung of L_n be constructed of corresponding vertices v_i, v'_i where $v_i \in V(P_n)$ and $v'_i \in V(P'_n)$.

Let $f(v_i) = 1$ if i is even, $f(v'_j) = 1$ if j is odd, and $f(x) = 0$ otherwise. Then, f is an Italian dominating function of weight n, so we have that $\gamma_I(L_n) \leq n$. □

To show equality, we first consider the following lemmas.

Lemma 3.2. *If G is a connected graph with $\Delta(G) = 3$, then there exists an $f = (V_1, V_2, V_3)$ on G such that the set V_2 is independent.*

Proof. Let G be a connected graph with $\Delta(G) = 3$. Among all γ_I-functions, let $f = (V_0, V_1, V_2)$ be one that minimizes the number of edges in the induced subgraph $G[V_2]$. Suppose to the contrary that V_2 is not independent. Then, there are two vertices $u, v \in V_2$ such that the edge $uv \in E(G)$. We consider the following cases.

Case 1. $deg(v) = 1$. If v has degree one, then v is a pendant vertex whose only neighbor is $u \in V_2$. Let g be an Italian dominating function such that $g(v) = 0$, $g(x) = f(x)$ for all $x \neq v$. Now, g is an Italian dominating function of G with total weight less than f, a contradiction.

Case 2. $deg(v) = 2$. Since v has degree two, v has two neighbors, namely $u \in V_2$ and another neighbor w. We consider two further subcases.

Case 2a. $w \in V_1 \cup V_2$. In this case, let g be the function such that $g(v) = 0$, and $g(x) = f(x)$ for all $x \neq v$. Then, G is an Italian dominating function of G having total weight less than f, a contradiction.

Case 2b. $w \in V_0$. In this case, let g be the function such that $g(v) = 0$, $g(w) = 1$ $g(x) = f(x)$ for all $x \notin \{v, w\}$. Then, G is an Italian dominating function of G having total weight less than f, a contradiction.

Case 3. $deg(v) = 3$. Then v has three neighbors, namely $u \in V_2$, and two other neighbors w and y.

Notice first that if $w, y \in V_1 \cup V_2$ the function g such that $g(v) = 0$ and $g(x) = f(x)$ for all $x \neq v$, is an Italian dominating function on G with total weight less than f, a contradiction. Hence, we may assume that at least one of w and y is in V_0.

Suppose first that $w \in V_0$ and $y \in V_0$. Then let g be the function such that $g(v) = 0$, $g(w) = g(y) = 1$ and $g(x) = f(x)$ for all $x \notin \{v, w, y\}$. Now, $g = (V_0', V_1', V_2')$ is a function where $G[V_2']$ has fewer edges than $G[V_2]$, contradicting our choice of f.

Therefore, without loss of generality, we may assume that $w \in V_0$ and $y \in V_1 \cup V_2$.

But now, the function g such that $f(v) = 0$ and $f(w) = 1$, and $g(x) = f(x)$ for all $x \notin \{v, w\}$ is an Italian dominating function of G with total weight less than f, a contradiction.

Thus, if $\Delta(G) = 3$, then there exists a γ_I-function $f = (V_0, V_1, V_2)$ on G such that V_2 is independent. □

For two sets of vertices X and Y, let $[X, Y]$ denote the set of edges having an endpoint in X and an endpoint in Y. We then consider the following lemma.

Lemma 3.3. *If G is a connected graph with $\Delta(G) = 3$, then there exists a γ_I-function $f = (V_0, V_1, V_2)$ on G such that V_2 is independent and $[V_1, V_2] = \emptyset$.*

Proof. By Lemma 3.2, there exists a γ_I-function $f = (V_0, V_1, V_2)$ such that V_2 is independent. Among all such γ_I-functions, select $f = (V_0, V_1, V_2)$ to minimize the edges in $[V_1, V_2]$. If $[V_1, V_2] = \emptyset$, then we are finished.

Suppose, to the contrary, that $[V_1, V_2] \neq \emptyset$. That is, there is an edge $uv \in E(G)$ where $u \in V_1$ and $v \in V_2$. We consider the following cases.

Case 1. $deg(v) = 1$. If v has degree one, then v is a pendant vertex whose only neighbor is $u \in V_1$. Let g be a function such that $g(v) = 1$, and $g(x) = f(x)$ for all $x \neq v$. Then g is an Italian dominating function of G with total weight less than f, a contradiction.

Case 2. $deg(v) = 2$. Since v has degree two, v has two neighbors, namely $u \in V_1$ and another neighbor w. Since V_2 is independent, $w \notin V_2$. If $w \in V_1$, then let g be a function such that $g(v) = 0$, and $g(x) = f(x)$ for all $x \neq v$. This produces an IDF with total weight less than f, a contradiction.

Hence, we may assume that $w \in V_0$. In this case, let g be the function such that $g(v) = 0$, $g(w) = 1$, and $g(x) = f(x)$ for all $x \notin \{v, w\}$. Then g is an Italian dominating function with total weight less than f, a contradiction.

Case 3. $deg(v) = 3$. If v has degree 3, then v has three neighbors, namely $u \in V_1$, and two other neighbors w, y. Notice first since V_2 is independent, neither w nor y is in V_2. If $w \in V_1$ and $y \in V_1$, then we can immediately find an Italian dominating function of G, say g, such that $g(v) = 0$ and $g(x) = f(x)$ for all $x \neq v$. In this case, g is an Italian dominating function on G with total weight less than f. Thus, at least one of w and y is in V_0.

Suppose that $w \in V_0$ and $y \in V_0$. If neither w nor y has a neighbor in $V_2 \setminus \{v\}$, then let g be a function such that $g(v) = 0$, $g(w) = 1$, $g(y) = 1$ and $g(x) = f(x)$ for all $x \notin \{v, w\}$. Then, $g = (V_0', V_1', V_2')$ is a γ_I-function of G that has fewer edges in $[V_1', V_2']$ than in $[V_1, V_2]$, contradicting our choice of f.

Hence, at least one of w and y has a neighbor in $V_2 \setminus \{v\}$. If both w and y have neighbors in $V_2 \setminus \{v\}$, then let g be the function such that $g(v) = 1$ and $g(x) = f(x)$ for all $x \neq v$. Then, g is an Italian dominating function with total weight less than $\gamma_I(G)$, a contradiction.

Thus, without loss of generality, we may assume that w has a neighbor in $V_2 \setminus \{v\}$ and y does not. In this case, the function g such that $g(v) = 0$, $g(y) = 1$, and $g(x) = f(x)$ for all $x \notin \{v, y\}$ is an Italian dominating function of G with weight less than $\gamma_I(G)$, a contradiction.

Hence, exactly one of w and y is in V_0. Then, without loss of generality, let $y \in V_0$ and let $w \in V_1$. Then, the function g where $g(v) = 0$, $g(y) = 1$, and $g(x) = f(x)$ for all $x \notin \{v, y\}$, is an Italian dominating function of G with total weight less than $\gamma_I(G)$, a contradiction.

Thus, if $\Delta(G) = 3$, then there exists a γ_I-function $f = (V_0, V_1, V_2)$ on G such that

V_2 is independent and $[V_1, V_2]$ is empty. □

These lemmas are significant because they provide some very useful conditions on Italian dominating functions for any graph (not only graph products) with $\Delta(G) = 3$. In particular, this includes all of the cubic graphs (which are 3-regular), a rich area of study for all forms of domination, Italian domination in particular. We will apply these results to ladders and related prism graphs.

We use these results to prove the following theorem regarding the Italian domination number of a ladder L_n with n rungs. We define the weight of a rung to be the total weight from an Italian dominating function assigned to any corresponding pair of vertices v_i and v_i'. In other words, if both vertices in a rung are assigned a zero, that rung has a weight of zero. We call this a *zero rung*. If one vertex is assigned a one and one is assigned a zero, then that rung has weight one. A weight of two can be achieved by assigning v_i a two and v_i' a zero, or vice-versa, or by assigning both v_i and v_i' a one. Let r_j denote the j^{th} rung, that is, the rung connecting v_j and v_j'. Additionally, we call the rungs r_1 and r_n *end rungs*.

Theorem 3.4. *Let L_n be a ladder of the form $P_n \Box K_2$ for $n \geq 3$. Then, $\gamma_I(L_n) = n$.*

Proof. We select a γ_I-function $f = (V_0, V_1, V_2)$ on L_n as follows. Note first that since L_n has $\Delta(G) = 3$, then by Lemma 3.3 we can choose f such that the set V_2 is independent, and $[V_1, V_2] = \emptyset$ (1). Moreover, subject to (1), select f such that the first zero rung r_i has the largest possible index i.

Now, suppose, to the contrary, that $\gamma_I(L_n) \leq n - 1$. Then, there must be at least one zero rung in L_n. Notice immediately that if either of the end rungs are zero-rungs, then since V_2 is independent, the vertices of the end rung are not Italian dominated. So, we must have that $2 \leq i \leq n-1$.

Then, in order to Italian dominate this zero rung, f must assign a total weight of at least four to the rungs r_{i-1} and r_{i+1} (that is, the rungs immediately preceding and following r_i). Additionally, since r_i is the first zero rung, then all rungs r_k such that $k < i$ have weight at least one.

Since the set V_2 is independent and $[V_1, V_2] = \emptyset$, the only possibilities are that for all vertices $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1}$, without loss of generality $v_{i-1}, v'_{i+1} \in V_2$ and $v_{i+1}, v'_{i-1} \in V_0$, or that $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1} \in V_1$. We consider these two cases:

Case 1. $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1} \in V_1$. Suppose that the rung r_{i-1} is an end rung. Then, the function g such that $g(v_{i-1}) = 0, g(v_i) = 1$, and $g(x) = f(x)$ for all $x \notin \{v_{i-1}, v_i\}$, and r_i is not the first zero rung, contradicting our choice of f. Hence, r_{i-1} is not an end rung.

Thus, the rung r_{i-2} exists, and one of its vertices, say v_{i-2}, has weight at least one. Furthermore, since $[V_1, V_2] = \emptyset$, $v_{i-2}, v'_{i-2} \notin V_2$. Thus, $f(v_{i-2}) = 1$. Now, let g be a function such that $g(v_{i-1}) = 0$, $g(v_i) = 1$, and $g(x) = f(x)$ for all $x \notin \{v_{i-1}, v_i\}$. Thus, g is a γ_I-function of L_n where r_i is not the first zero rung, contradicting our choice of f.

Case 2. $f(v_{i-1}) = 2$ and $f(v'_{i+1}) = 2$. Since V_2 is independent and $[V_1, V_2] = \emptyset$, if $i = 2$, then let the function g such that $g(v'_{i-1}) = g(v_i) = 1$, $g(v_{i-1}) = 0$, and $g(x) = f(x)$ for all $x \notin \{v_i.v_{i-1}, v'_{i-1}\}$. Hence, g is a γ_i-function of L_n satisfying (1) such that v_i is not the first zero rung, a contradiction.

Hence, we may assume that $f(v_{i-2}) = 0$. Then, since r_i is the first zero rung, r_{i-2}

must have total weight at least 1, implying that $f(v'_{i-2}) \geq 1$. Let g be the function such that $g(v_{i-1}) = 1$, $g(v_i) = 1$, and $g(x) = f(x)$ for all $x \notin \{v_{i-1}, v_i\}$. Thus, g is a γ_I-function of L_n satisfying (1) where r_i is not the first zero rung, contradicting our choice of f.

Therefore, $\gamma_I(L_n) \geq n$, and so $\gamma_I(L_n) = n$. $\square$

3.2 Characterizing the $G\Box K_2$ with $\gamma_I(G\Box K_2) = 4$

We begin with some observations regarding the graphs with $\Delta(G) = n - 1$, where n is the order of G.

Observation 3.5. *If G is a graph with $\Delta(G) = n - 1$ with $n \geq 3$, then $\gamma_I(G) = 2$.*

For example, consider the star $K_{1,n}$ for $n \geq 3$. It is not difficult to see that $\gamma_I(K_{1,n}) = 2$ where the vertex $v \in V(K_{1,n})$ is the center of the star is assigned a two by the γ_I-function of $K_{1,n}$.

Note that a graph $G\Box K_2$ is composed of two copies of G labeled G and G'. Let $f = (V_0, V_1, V_2)$ be an Italian dominating function on G. Applying f similarly to corresponding vertices in G' will Italian dominate $G\Box K_2$, so $\gamma_I(G\Box K_2) \leq 2w(f)$ where $w(f)$ denotes the total weight assigned by f to G. Our next observation follows directly.

Observation 3.6. *Let G be a graph with Italian domination number $\gamma_I(G)$. Then $\gamma_I(G\Box K_2) \leq 2\gamma_I(G)$.*

It follows from Observations 3.5 and 3.6 that for any graph G of order $n > 2$ and $\Delta(G) = n - 1$, we have $\gamma_I(G\Box K_2) \leq 4$. We next show that equality holds.

Proposition 3.7. *Let G be a graph of order $n \geq 4$ and $\Delta(G) = n - 1$. Then,* $\gamma_I(G \Box K_2) = 4$.

Proof. Observations 3.5 and 3.6 imply that $\gamma_I(G \Box K_2) \leq 4$. Suppose, to the contrary, that $\gamma_I(G \Box K_2) \leq 3$. Label the two copies of G in $G \Box K_2$ as G and G'.

Let $f = (V_0, V_1, V_2)$ be a γ_I-function on $G \Box K_2$.

Since $\gamma_I(G) \leq 3$, without loss of generality, we may assume that f assigns a total weight of at least two to G and a total weight of at most one to G'.

Case 1. $\sum_{v' \in V(G')} f(v') = 0$. Then, since $n \geq 4$, at least one $v' \in V(G')$ is adjacent to a neighbor $v \in V(G)$ such that $f(v) = 0$, and so the graph is not Italian dominated, a contradiction.

Case 2. $\sum_{v' \in V(G')} f(v') = 1$. Then there is some $v' \in V(G')$, such that $f(v') = 1$.

Suppose that $deg(v') = n - 1$. Then, there is at least one vertex $u' \in N(v')$ with corresponding vertex $u \in G$ such that $f(u') = f(u) = 0$, so $G \Box K_2$ is not Italian dominated, a contradiction.

Suppose that $deg(v') < n - 1$.

Then, there is a vertex $w' \in V(G'), w' \notin N(v')$ such that $f(w') = 0$. So, its corresponding vertex $w \in V(G)$ must have $f(w) \geq 2$.

Since $\Delta(G) = n-1$, there is a vertex $z' \in G'$ with $deg(z') = n-1$, and $f(z') = 0$. Note that $z' \neq w'$ and $z' \neq v'$. Similarly, its corresponding vertex $z \in V(G)$ must have $f(z) \geq 1$. But then, since the total weight of G is at most 2 and $f(w) \geq 2$, it follows that $f(z) = 0$, a contradiction.

In any case, we arrive at a contradiction, thus $\gamma_I(G) = 4$, as desired. $\square$

Proposition 3.8. *If G is a graph of order $n \geq 4$ with a pair of non-adjacent vertices* *u and v with* $N[u] = N[v] = V(G) \setminus \{u, v\}$*, then* $\gamma_I(G\Box K_2) = 4$.

Proof. Let $G\Box K_2$ be composed of two copies of G, labeled G and G'. Let u and v be non-adjacent vertices of G, each with $N[u] = N[v] = V(G) \setminus \{u, v\}$, so $deg(u) = deg(v) = n - 2$.

First, note that a function that assigns $f(u) = f(v') = 2$ and $f(x) = 0$ for $x \in V(G\Box K_2) \setminus \{u, v'\}$ is an Italian dominating function of $G\Box K_2$, so $\gamma_I(G\Box K_2) \leq 4$.

Let $f = (V_0, V_1, V_2)$ be a γ_I-function of $G\Box K_2$ and suppose to the contrary that $\gamma_I(G) \leq 3$.

Then, without loss of generality, we may assume that $0 \leq \sum_{v\in V(G)} f(v) \leq 1$ and $2 \leq \sum_{v'\in V(G')} f(v') \leq 3$.

If no vertex of G is assigned one, then every vertex of G must be adjacent to a vertex assigned a two in G. But since $n \geq 4$ and $\sum_{v'\in V(G')} f(v') = 2$, we have a contradiction.

Hence, we may assume that $\sum_{v\in V(G)} f(v) = 1$ and so $\sum_{v'\in V(G')} f(v') = 2$.

Since $\Delta(G) = n - 2$, there exists a $z \in V(G)$ such that w is not adjacent to z. Hence, the total weight assigned to the vertices of $N[z] \cap V(G)$ is 0.

Thus, $f(z) = 2$ in order to Italian dominate z'. But then, every other vertex in G' must be assigned a zero by f. It follows that w' is not adjacent to z', and $f(w') = 0$. Furthermore, the only vertex in $N(w')$ with positive weight is w with weight of one, and so w' is not Italian dominated by f, a contradiction.

Thus, it must be that $\gamma_I(G\Box K_2) \geq 4$, and so $\gamma_I(G\Box K_2) = 4$, as desired. □

Proposition 3.9. *If* G *is a graph of order* $n \geq 4$ *with a pair of non-adjacent vertices* u *and* v *with* $N(u) = N(w) = V(G) \setminus \{v\}$ *and* $N(v) = V(G) \setminus \{u, w\}$ *for some*

$w \in V(G)$, then $\gamma_I(G\Box K_2) = 4$.

Proof. Let $G\Box K_2$ be composed of two copies of G as defined, labeled G and G'. Let u and v be non-adjacent vertices of G, with $N(u) = V(G) \setminus \{v\}$ and $N(v) = V(G) \setminus \{u, w\}$ for some $w \in V(G)$.

First, note that a function that assigns $f(u) = f(v) = f(v') = f(w') = 1$ and $f(x) = 0$ for $x \in V(G\Box K_2) \setminus \{u, v, v', w'\}$ is an Italian dominating function of $G\Box K_2$, so $\gamma_I(G\Box K_2) \leq 4$.

Let $f = (V_0, V_1, V_2)$ be a γ_I-function of $G\Box K_2$ and suppose to the contrary that $\gamma_I(G) \leq 3$. Then, without loss of generality, we may assume that $0 \leq \sum_{v\in V(G)} f(v) \leq 1$ and $2 \leq \sum_{v'\in V(G')} f(v') \leq 3$.

If $\sum_{v\in V(G)} f(v) = 0$, then every vertex in $V(G)$ must be adjacent to a vertex $v' \in V(G')$ assigned a two. Since $n \geq 4$ and f assigns a total weight of at most three to the vertices in $V(G')$, we have a contradiction.

Hence, we must assume that $\sum_{v\in V(G)} f(v) = 1$ and so $\sum_{v'\in V(G')} f(v') = 2$.

Assume that $f(x) = 1$ for some $x \in V(G) \setminus \{u, v, w\}$. Since $x \in N(u) \cap N(v)$, we must have that $f(u') = f(v') \geq 1$. However, in in order for w to be Italian dominated by f, we must have $f(w') \geq 1$, and f assigns a total weight of at least four to $G\Box K_2$, a contradiction.

Hence, we must assume that either $f(u) = 1$, $f(v) = 1$, or $f(w) = 1$.

Assume that $f(u) = 1$. Then, it must be that $f(v') = 2$ in order for v to be Italian dominated by f. But then, $f(w') = 0$ and so w is not Italian dominated, a contradiction.

Assume that $f(v) = 1$. Then, $f(w) = 0$, and since $w \notin N(v)$, we must have that

$f(w') = 2$ in order for w to be Italian dominated. Further, since $f(u) = 0$, it must be that $f(u') \geq 2$ in order for u to be Italian dominated, a contradiction.

Finally, assume that $f(w) = 1$. Then, $f(v) = 0$, and so $f(v') = 2$ in order for v to be Italian dominated. Furthermore, $f(u') \geq 1$ in order for u to be Italian dominated, and so f assigns a total weight of at least four to $G\Box K_2$, a contradiction.

In any case, we arrive at a contradiction. Thus $\gamma_I(G\Box K_2) \geq 4$, and so $\gamma_I(G\Box K_2) = 4$, as desired.

$\Box$

Proposition 3.10. *If G is a graph of order $n \geq 4$ with a pair of non-adjacent vertices u and v such that either $N[u] = N[w] = V(G) \setminus \{v, z\}$ and $N[v] = N[z] = V(G) \setminus \{u, w\}$, or $N[u] = V(G) \setminus \{v, z\}, N[w] = V(G) \setminus \{v\}, N[v] = V(G) \setminus \{u, w\}$, and $N[z] = V(G) \setminus \{u\}$ for some $w, z \in V(G)$.*

Proof. Let $G\Box K_2$ be composed of two copies of G, labeled G and G'. Let u and v be non-adjacent vertices of G, with $N[u] = V(G) \setminus \{v, z\}$ for some $z \in V(G)$ and $N[v] = V(G) \setminus \{u, w\}$ for some $w \in V(G)$.

First, note that if $w = z$, then the result holds by Proposition 3.8.

Next, note that a function that assigns $f(u) = f(v) = f(w') = f(z') = 1$ and $f(x) = 0$ for $x \in V(G\Box K_2) \setminus \{u, v, v', w'\}$ is an Italian dominating function of $G\Box K_2$, so $\gamma_I(G\Box K_2) \leq 4$.

Let $f = (V_0, V_1, V_2)$ be a γ_I-function of $G\Box K_2$ and suppose to the contrary that $\gamma_I(G) \leq 3$. Then, without loss of generality, we may assume that $0 \leq \sum_{v\in V(G)} f(v) \leq 1$ and $2 \leq \sum_{v'\in V(G')} f(v) \leq 3$.

If $\sum_{v\in V(G)} f(v) = 0$, then every vertex in $V(G)$ must be adjacent to a vertex

$v' \in V(G')$ assigned a two. Since $n \geq 4$ and f assigns a total weight of at most three to the vertices in $V(G')$, we have a contradiction.

Hence, we must assume that $\sum_{v\in V(G)} f(v) = 1$ and so $\sum_{v'\in V(G')} f(v') = 2$.

Assume that $f(x) = 1$ for some $x \in V(G) \setminus \{u, v, w, z\}$. Since $x \in N[u] \cap N[v]$, we must have that $f(u') = f(v') \geq 1$. However, in order for w to be Italian dominated by f, we must have $f(w') \geq 1$, and f assigns a total weight of at least four to $G\Box K_2$, a contradiction.

Hence, we must assume that either $f(u) = 1$, $f(v) = 1$, $f(w) = 1$, or $f(z) = 1$.

Without loss of generality, assume that $f(u) = 1$. Then, it must be that $f(v') \geq 2$ in order for v to be Italian dominated by f. Also, $f(z) = 0$, and since $z \notin N[u]$ we have $f(z') \geq 2$ in order for z to be Italian dominated, and so f assigns a total weight of at least 4 to $G\Box K_2$, a contradiction.

Thus, $\gamma_I(G\Box K_2) \geq 4$, and so $\gamma_I(G\Box K_2) = 4$, as desired.

A similar argument holds for the second condition □

Proposition 3.11. *If G is an isolate-free graph of order $n = 4$, then $\gamma_I(G\Box K_2) = 4$.*

Proof. The isolate-free graphs of order $n = 4$ are given in $\Delta(G) = n - 1$ and by Proposition 3.7, $\gamma_I(G\Box K_2) = 4$.

If $\Delta(G) = 2$, then either $G = C_4$ or $G = P_4$.

If $G = C_4$, then by Proposition 3.8, $\gamma_I(G) = 4$.

If $G = P_4$, then $G\Box P_4 = L_4$ and by Theorem 3.4, $\gamma_I(G) = 4$.

If $\Delta(G) = 1$, then $G = 2P_2$, and by Proposition 3.10, $\gamma_I(G) = 4$.

Thus, if G is an isolate-free graph of order $n = 4$, we have that$\gamma_I(G\Box K_2) = 4$, as desired. □

We may now characterize the graphs of the form $G\Box K_2$ where $\gamma_I(G) = 4$.

Theorem 3.12. *Let $G\Box K_2$ where G is a graph of order $n \geq 4$. Then, $\gamma_I(G) = 4$ if and only if one of the following is true:*

i. $\Delta(G) = n - 1$.

ii. G can be constructed from two non-adjacent vertices u and v such that one of the following holds:

a. $N(u) = N(v) = V(G) \setminus \{u, v\}$,

b. $N[u] = N[w] = V(G) \setminus \{v\}$ *and* $N[v] = V(G) \setminus \{u, w\}$ *for some* $w \in V(G)$,

or

c. $N[u] = N[w] = V(G) \setminus \{v, z\}$ *and* $N[v] = N[z] = V(G) \setminus \{u, w\}$,

or

$N[u] = V(G) \setminus \{v, z\}, N[w] = V(G) \setminus \{v\}, N[v] = V(G) \setminus \{u, w\}$, *and* $N[z] = V(G) \setminus \{u\}$ *for some* $w, z \in V(G)$.

Proof. Let $G\Box K_2$ be composed of two copies of G labeled G and G'. Let $f = (V_0, V_1, V_2)$ be an γ_I-function of $G\Box K_2$. Let $\gamma_I(G\Box K_2) = 4$.

Since $n \geq 4$, if G (respectively, G') is assigned a total weight of zero by f, then every vertex of G' (respectively, G) is assigned at least two by f to Italian dominate the corresponding vertex. But then, the total weight of f is at least $2n \geq 8$, a contradiction.

Thus, we may assume that f assigns a total weight of at least one and at most three to each of G and G'. Without loss of generality, we consider the following two cases.

Case 1. $\sum_{v\in V(G)} f(v) = 3$ and $\sum_{v'\in V(G')} f(v') = 1$. If $n \geq 5$, then there is at least one vertex in G' not Italian dominated by f, a contradiction.

Hence, we may assume that $n = 4$. Since the total weight assigned to G' is one, let x' be the vertex in $V(G')$ with $f(x') = 1$. Then, there exist three vertices u', v', w' with $f(u') = f(v') = f(w') = 0$. Thus, each of $f(u)$, $f(v)$, $f(w)$ is at least one, so it must be that $f(u) = f(v) = f(w) = 1$, and each of u', v', w' is adjacent to x', implying that $\Delta(G') = \Delta(G) = n - 1$, satisfying (i).

Case 2. $\sum_{v\in V(G)} f(v) = \sum_{v'\in V(G')} f(v') = 2$. We consider the following three subcases.

a. $f(u) = f(v') = 2$ for some $u \in V(G), v' \in V(G')$. If $u = v$, then u is adjacent to every vertex in G, thus $\Delta(G) = n - 1$, and (i) is satisfied.

Thus, we must assume that $u \neq v$. Then, u must dominate $V(G) \setminus \{u, v\}$, so $deg(u) \geq n - 2$ in G. Additionally, if u is adjacent to v, then $deg(u) = n - 1$, and (i) is satisfied.

Hence, we must assume that u and v are not adjacent. Then $deg(u) = n - 2$ in G. Similarly, we can show that $deg(v') = n - 2$ in G', and so $deg(v) = n - 2$ in G, thus $N(u) = N(v) = V(G) \setminus \{u, v\}$ and $(ii.a)$ is satisfied.

b. $f(u) = 2$ for some $u \in V(G)$ and $f(v') = f(w') = 1$ for some $v', w' \in V(G')$. First, note that if $\Delta(G) = n - 1$ then (i) is satisfied, and we are finished. Hence, assume that $\Delta(G) \leq n - 2$. Thus, there is a vertex $x \in V(G)$ that is not adjacent to u. Then, $N[x]$ is assigned a total weight of at most one by f, and so x is not Italian dominated, a contradiction.

c. $f(u) = f(v) = f(w) = f(z) = 1$. Assume that $n \geq 5$. Notice that if

$\Delta(G) = n - 1$, then (i) holds, and we are finished. Similarly, if there are two non-adjacent vertices $v, u \in V(G)$ such that $N(u) = N(v) = V(G) \setminus \{u, v\}$, then $(ii.a)$ is satisfied, and we are finished. Thus, there must be some vertex $x \in V(G)$ such that x is adjacent to at most one of u, v.

Assume that x is adjacent to neither of u, v. Then x is adjacent only to x' in $G \Box K_2$, and x is adjacent to at most a weight of one, and x is not Italian dominated, a contradiction.

Thus, it must be that x is adjacent to exactly one of u, v. Without loss of generality, assume x is adjacent to u. Then $f(x') = 1$ and so x must be either w or z. Without loss of generality, assume $x = w$.

Now, $f(z') = 1$ by hypothesis, and so either $f(z) = 0$, or otherwise $z = u$ or $z = v$.

Assume that $z = u$.

Since $n \geq 5$, there must be a vertex y such that $y \notin \{u, v, w\}$. By hypothesis and the above, $f(u') = f(w') = 1$, and so y' must be adjacent to both u' and w'. Additionally, y must be adjacent to u and v in order to be Italian dominated, and so $u, v, w \in N(y)$ for all $y \notin \{u, v, w\}$.

Notice that if u and v are adjacent, then $deg(u) = n - 1$ and (i) is satisfied. Thus, we must assume that u and v are not adjacent.

Then, since $f(v) = f(w') = 1$, we have that v' must be adjacent to w', and so v must be adjacent to w, and we have a contradiction, since w is not adjacent to v.

Thus, we must assume that $z \neq u$.

Assume that $z = v$.

Since $n \geq 5$, there must be a vertex y such that $y \notin \{u, v, w\}$. By hypothesis and the above, $f(v') = f(w') = 1$, and so y' must be adjacent to both u' and w'. Additionally, y must be adjacent to u and v in order to be Italian dominated, and so every vertex $y \in V(G) \setminus \{u, v, w\}$ is adjacent to each of u, v, w.

Once again, notice that if u and v are adjacent, then $deg(u) = n - 1$ and (i) is satisfied. Thus, we must assume that u and v are not adjacent.

Then, we have that $N[u] = N[w] = V(G) \setminus \{v\}$ and $N[v] = V(G) \setminus \{u, w\}$, and $(ii.b)$ is satisfied.

Hence, we must assume that $z \neq v$.

First, notice that if z is adjacent to neither u nor v, then $N[z]$ is assigned a total weight of one by f, and z is not Italian dominated. Hence, z must be adjacent to at least one of u and v. Similarly, v' must be adjacent to at least one of w' and z'.

As with the previous arguments, every vertex in $V(G) \setminus \{u, v, w, z\}$ is adjacent to each of u, v, w, and z.

If the only edges in $G[\{u, v, w, z\}]$ are uw and vz, then $ii.c$ holds.

If u and v are adjacent to both w and z, then either i or $ii.a$ holds.

Thus, we may assume that without loss of generality, u is not adjacent to z, and so $vz \in E(G)$, and it follows that either $ii.a$, $ii.b$, or $ii.c$ holds.

Therefore, the result holds for $n \geq 5$.

Hence, we may assume that $n = 4$. Assume, for the purpose of contradiction, that G has an isolated vertex. Then, $G \square K_2$ will have a K_2 component requiring a total weight of 2 assigned by f. Label this component P. Then, f assigns a total weight of two to the vertices in $Q = (G \square K_2) - P$.

By hypothesis, we have that $\Delta(G) \leq n - 2 = 2$, and so we must have that

$\Delta(G\Box K_2) = 3$.

Suppose that $f(v_m) = 2$ for some $v_m \in V(Q)$. Then, in order for Q to be Italian dominated, $N[v_m] = Q$, and so v_m has degree 5, a contradiction.

Hence, we may assume that there are two vertices v_p and v_q such that $f(v_p) = f(v_q) = 1$. Then, $N(v_p) = N(v_q)$ necessarily and so $deg(v_p) = deg(v_q) = 4$, again a contradiction.

Thus, we may assume that G is isolate-free. The only possible isolate-free graphs of order 4 with $\Delta(G) \leq 2$ are C_4, P_4, and $2P_2$ (that is, the graph consisting of two copies of a P_2 graph). If $G = C_4$, this satisfies $(ii.a)$. If $G = P_4$ or $2P_2$, this satisfies $(ii.c)$.

The converse statements are shown by Propositions 3.7, 3.8, 3.9, and 3.10, and so the result holds. $\Box$

3.3 Prisms and related products

Consider the prism $C_n \Box K_2 = \Pi_n$. By way of a construction similar to that of L_n above, we define corresponding vertices v_i, v_i', so one "copy" of C_n has vertices $v_0, v_1, \ldots, v_{n-1}$, and the other copy contains corresponding vertices $v_0', v_1', \ldots, v_{n-1}'$.

Notice that we can construct such a prism by constructing a ladder L_n, and adding edges $v_1 v_n$ and $v_1' v_n'$. As such, we may still define "rungs" constructed of corresponding vertices v_i, v_i' as we did in the case of L_n.

Let $f(v_i) = 1$ for $i \equiv 0 \pmod 2$, $f(v_j') = 1$ for $j \equiv 1 \pmod 2$, and $f(x) = 0$ otherwise. This produces an IDF of weight n, so we have the upper bound $\gamma_I(\Pi_n) \leq n$.

Theorem 3.13. *If Π_n is a prism of the form $C_n \Box K_2$ for $n \geq 3$, then $\gamma_I(\Pi_n) = n$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_I-function on Π_n. Note first that since Π_n has $\Delta(G) = 3$, then by Lemma 3.3 we can choose f such that the set V_2 is independent, and $[V_1, V_2] = \emptyset$ (1). Moreover, subject to (1), select f such that the first zero rung r_i has the largest possible index i.

Now, suppose, to the contrary, that $\gamma_I(\Pi_n) \leq n-1$. Then, there must be at least one zero rung in Π_n.

Then, in order to Italian dominate this zero rung, f must assign a total weight of at least 4 to the rungs r_{i-1} and r_{i+1} (that is, all computations on the indices are done modulo n). Additionally, since r_i is the first zero rung, then all rungs r_k such that $k < i$ have weight at least 1.

Since the set V_2 is independent, and $[V_1, V_2] = \emptyset$, the only possibilities are that for all vertices $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1}$, without loss of generality $v_{i-1}, v'_{i+1} \in V_2$ and $v_{i+1}, v'_{i-1} \in V_0$, or that $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1} \in V_1$. We consider these two cases:

Case 1. $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1} \in V_1$. Since Π_n has no end rungs, the rung r_{i-2} exists (modulo n), and one of its vertices, say v_{i-2}, has weight at least one. Furthermore, since $[V_1, V_2] = \emptyset$, $v_{i-2}, v'_{i-2} \notin V_2$. Thus, $f(v_{i-2}) = 1$. Now, let g be a function such that $g(v_{i-1}) = 0$, $g(v_i) = 1$, and $g(x) = f(x)$ for all $x \notin \{v_{i-1}, v_i\}$. Thus, g is a γ_I-function of Π_n where r_i is not the first zero rung, contradicting our choice of f.

Case 2. $f(v_{i-1}) = 2$ and $f(v'_{i+1}) = 2$. By our choice of f, we must have that $f(v_{i+1}) = .f(v'_{i-1}) = 0$. Moreover, if $i \geq 3$, then $f(v_{i-2}) = 0$ Then, since r_i is the first zero rung, r_{i-2} must have total weight at least one, implying that $f(v'_{i-2}) \geq 1$. Let g be the function such that $g(v_{i-1}) = 1$, $g(v'_i) = 1$, and $g(x) = f(x)$ for all $x \notin \{v_{i-1}, v'_i\}$. Thus, g is a γ_I-function of Π_n satisfying (1) where r_i is not the first zero rung, contradicting our choice of f.

Therefore, $\gamma_I(\Pi_n) \geq n-1$, and so $\gamma_I(\Pi_n) = n$. $\square$

Notice that we can construct such a Möbius ladder by constructing a prism Π_n of the form $C_n \Box K_2$, omitting a pair of edges $v_i v_{i+1}$ and $v'_i v'_{i+1}$, and adding edges $v_i v'_{i+1}$ and $v'_i v_{i+1}$ to form a 'twist' in the ladder structure. Furthermore, since we can label the rungs our ladder arbitrarily, we may place the "twist" between any pair of rungs we wish.

Corollary 3.14. *Let M_m be a Möbius ladder of order $m = 2n$ Then, $\gamma_I(M_m) = n$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_I function on M_m.

Let $f = (V_0, V_1, V_2)$ be a γ_I-function on Π_n. Note first that since M_m has $\Delta(G) = 3$, then as before, by Lemma 3.3 we can choose f such that the set V_2 is independent, and $[V_1, V_2] = \emptyset$ (1). Moreover, subject to (1), select f such that the first zero rung r_i has the largest possible index i.

Now, suppose to the contrary that $\gamma_I(M_m) = n-1$. Then there must be at least one zero rung in M_m. Then, in order to Italian dominate this zero rung, f must assign a total weight of at least four to the rungs r_{i-1} and r_{i+1} (that is, the rungs immediately preceding and following r_i). Notably, this is true regardless of whether the twist is between r_{i-1}, r_i or between r_i, r_{i+1}. Additionally, since r_i is the first zero rung, then all rungs r_k such that $k < i$ have weight at least one.

Once a rung r_i is fixed, we may label the vertices as v_i or v'_i such that if the twist is located between r_j and r_{j+1}, the vertices v'_j and v'_{j+1} are adjacent, and correspondingly, the vertices v_j and v_{j+1} are adjacent. In other words, all vertices v'_i where $j < i \leq n$ are located on the opposite "side" of M_m as those where $1 \leq i \leq j$.

Since the set V_2 is independent, and $[V_1, V_2] = \emptyset$, the only possibilities are that for all vertices $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1}$, without loss of generality $v_{i-1}, v'_{i+1} \in V_2$ and $v_{i+1}, v'_{i-1} \in V_0$, or that $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1} \in V_1$. We consider these two cases:

Case 1. $v_{i-1}, v'_{i-1}, v_{i+1}, v'_{i+1} \in V_1$. Since M_m has no end rungs, the rung r_{i-2} exists (relabeling if necessary), and one of its vertices, say v_{i-2}, has weight at least one. Furthermore, since $[V_1, V_2] = \emptyset$, $v_{i-2}, v'_{i-2} \notin V_2$. Thus, $f(v_{i-2}) = 1$. Now, let g be a function such that $g(v_{i-1}) = 0$, $g(v_i) = 1$, and $g(x) = f(x)$ for all $x \notin \{v_{i-1}, v_i\}$. Thus, g is a γ_I-function of M_m where r_i is not the first zero rung, contradicting our choice of f.

Case 2. $f(v_{i-1}) = 2$ and $f(v'_{i+1}) = 2$. By our choice of f, we must have that $f(v_{i+1}) = .f(v'_{i-1}) = 0$. Moreover, if $i \geq 3$, then $f(v_{i-2}) = 0$. Then, since r_i is the first zero rung, r_{i-2} must have total weight at least one, implying that $f(v'_{i-2}) \geq 1$. Let g be the function such that $g(v_{i-1}) = 1$, $g(v'_i) = 1$, and $g(x) = f(x)$ for all $x \notin \{v_{i-1}, v'_i\}$. Thus, g is a γ_I-function of M_m satisfying (1) where r_i is not the first zero rung, contradicting our choice of f.

Therefore, $\gamma_I(M_m) \geq n - 1$, and so $\gamma_I(M_m) = n$. □

3.4 Categorical Products

Recall that the categorical product $G \times K_2$ is equivalent to the bipartite double graph of G, also known as a Kronecker cover or bipartite double cover. This graph is constructed by making two copies of the vertices of G (no edges), labeled G and G' and constructing edges uv' and $u'v$ for every edge $uv \in E(G)$.

Proposition 3.15. *Let $G = P_n$. Then $\gamma_I(G \times K_2) = 2\lceil \frac{n+1}{2} \rceil$.*

Proof. Notice that the graph $P_n \times K_2 \cong 2P_n$, where $2P_n$ is a graph composed of two disjoint copies of P_n. By [6], $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$, and so $\gamma_I(G \times K_2) = 2\lceil \frac{n+1}{2} \rceil$. □

Proposition 3.16. *Let* $G = C_n$. *Then* $\gamma_I(C_n) = n$.

Proof. Notice that the graph $C_n \times K_2 \cong C_{2n}$ for n odd, and $C_n \times K_2 \cong 2C_n$ for n even. By [6], $\gamma_I(C_n) = \lceil \frac{n}{2} \rceil$, and so $\gamma_I(G \times K_2) = n$ for n odd, and for n even, $\gamma_I(G \times K_2) = 2\lceil \frac{n}{2} \rceil = n$. □

Proposition 3.17. *Let* $G = K_n$. *Then* $\gamma_I(G \times K_2) = 4$.

Proof. Notice that the graph $K_n \times K_2 \cong K_{n,n}$. Since $K_{n,n}$ is a complete bipartite graph, it is composed of two disjoint independent vertex sets A and B, where every $v \in A$ has all of $B \in N(v)$ (and vice versa).

Without loss of generality, assigning $v \in A$ a two or assigning two $v_1, v_2 \in A$ with one will Italian dominate all of B. Thus, $\gamma_I(K_{n,n}) \leq 4$.

Let $f = (V_0, V_1, V_2)$ be a γ_I-function on $K_{n,n}$. Assume, to the contrary, that $\gamma_I(K_{n,n}) = 3$. Then one of the partite sets is assigned a total weight of zero or one by f. Again, without loss of generality, if $A \cap V_0 = A$, then B is not Italian dominated. Similarly, if only one $v \in A$ has $f(v) = 1$, and $A \cap V_0 = A \setminus \{v\}$, B is not Italian dominated. In either case, f is not a γ_I function, and so $\gamma_I(K_{n,n}) \geq 4$, thus $\gamma_I(K_{n,n}) = 4$. □

Proposition 3.18. *Let* $G = L_n$. *Then* $\gamma_I(G \times K_2) = 2n$.

Proof. Notice that the graph $L_n \times K_2 \cong 2L_n$. Then, $\gamma_I(L_n \times K_2) = 2\gamma_I(L_n) = 2n$. □

Proposition 3.19. *Let* $G = \Pi_n$. *Then* $\gamma_I(G \times K_2) = 2n$.

Proof. Notice that the graph $\Pi_n \times K_2 \cong \Pi_{2n}$ for n odd, and $\Pi_n \times K_2 \cong 2\Pi_n$ for n even. In either case, $\gamma_I(\Pi_n \times K_2) = 2n$. $\square$

3.5 Lexicographic Products

Recall that the lexicographic product $G \cdot K_2$ is equivalent to the double graph of G, constructed from two copies of G and adding edges uv' and $u'v$ for every edge $uv \in E(G)$.

As an observation, let G be a graph and let D be a dominating set of G. Consider the lexicographic product $G \cdot K_2$, resulting in two copies of G, labeled G and G'. Furthermore, in each of these we can identify a copy of the dominating set, say D and D'. Let g be an Italian dominating function such that $g(v) = 1$ for all $v \in D \cup D'$, and $g(v) = 0$ otherwise. Since this function results in a dominating set on each of G, G', $D \cup D'$ is an Italian dominating set on $G \cdot K_2$. As expected, $\gamma_I(G \cdot K_2) \leq 2\gamma(G)$.

We will further explore this concept to arrive at a value for the Italian domination number of $P_n \cdot K_2$. In such a graph, we call a non-adjacent pair of vertices v, v' a *row* (as opposed to a rung, since the edge vv' does not exist). First, we consider the following lemma.

Lemma 3.20. *Let P_n be a path graph. Consider the lexicographic product $P_n \cdot K_2$ containing two copies of P_n labeled P_n and P'_n. Let $v_i \in V(P_n)$ and $v'_i \in V(P'_n)$. Then, we may choose a γ_I-function f of $P_n \cdot K_2$ such that if $f(v_i) = 0$, then $f(v'_i) = 0$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_I function on $P_n \cdot K_2$. Suppose, to the contrary, that we must have $f(v_i) = 0$ and $f(v'_i) \geq 1$ for some i. We call this property P. Choose f

such that (1) the number of rows with property $\mathcal{P}$ is minimized, and (2) subject to (1), the index i is maximized for the first row with property $\mathcal{P}$.

Consider the case if $f(v_i) = 0$ and $f(v_i') = 2$. Since $N(v_i) = N(v_i')$, then let g be the function such that $g(v_i) = 1$ and $g(v_i') = 1$. All of $N(v_i) = N(v_i')$ is still Italian dominated, so we may assume that V_2 is empty. Hence, we may assume, without loss of generality, that $f(v_i) = 0$, and $f(v_i') = 1$.

Now, consider the vertices v_1, v_1'. Note that if $f(v_1) = f(v_1') = 0$, then it is necessary that $f(v_2) = f(v_2') = 1$.

Further, if $f(v_1) = 1$ and $f(v_1') = 0$, then it is still necessary that $f(v_2) = f(v_2') = 1$, and we may define a function g such that $g(v_1) = 0$, and $g(x) = f(x)$ for $x \notin \{v_1\}$. Then, g has total weight less than $\gamma_I(P_n \cdot K_2)$, a contradiction. Thus, we may assume that either $f(v_1) = f(v'1) = 1$, or $f(v_2) = f(v_2') = 1$ and $f(v_1) = f(v_2) = 0$.

That is, there is a pair of corresponding vertices v_k, v_k' such that $f(v_k) = f(v_k') = 1$, and so, $i \geq 2$. For $i - 1$, we consider the following three cases.

Case 1. $f(v_{i-1}) = f(v_{i-1}') = 1$. Since i is the largest index of a row having property $\mathcal{P}$, then either $f(v_{i+1}) = f(v_{i+1}') = 0$ or $f(v_{i+1}) = f(v_{i+1}') = 1$.

If $f(v_{i+1}) = f(v_{i+1}') = 0$, then let g be the function such that $g(v_i') = 0, g(v_{i+1}') = 1$, and $g(x) = f(x)$ for all $x \notin \{v_i', v_{i+1}'\}$ is a γ_I-function on $P_n \cdot K_2$ with a larger index i for a row with property $\mathcal{P}$, contradicting our choice of f.

If $f(v_{i+1}) = f(v_{i+1}') = 1$, then let g be the function such that $g(v_i') = 0$ and $g(x) = f(x)$ for all $x \notin \{v_i'\}$. Then, g is an IDF with total weight less than $\gamma_I(P_n \cdot K_2)$, a contradiction.

Case 2. $f(v_{i-1}) = f(v_{i-1}') = 0$. Then, it is necessary that $f(v_{i+1}) = f(v_{i+1}') = 1$

in order to Italian dominate v_i and not be a row with property $\mathcal{P}$ and index greater than i. Moreover, at least one of v_{i-2} and v'_{i-2} is assigned a one by f in order to Italian dominate v_{i-1} and v'_{i-1}.

If $f(v_{i-2}) = f(v'_{i-2}) = 1$, then let g be the function such that $f(v'_i) = 0$. Then, g is an IDF with total weight less than $\gamma_I(P_n \cdot K_2)$, a contradiction.

Hence, we must assume that exactly one of v_{i-2} and v'_{i-2} is assigned a one by f. Without loss of generality, assume that $f(v_{i-2}) = 1$ and $f(v'_{i-2}) = 0$. Then, we must have that $f(v_{i-3}) = f(v_{i-3})' = 1$ in order to Italian dominate v'_{i-2}. Let g be the function such that $g(v_{i-2}) = 0$, $g(v_i) = 1$, and $g(x) = f(x)$ for all $x \notin \{v_{i-2}, v_i\}$. Then, g is a γ_I-function with fewer rows having property $\mathcal{P}$, contradicting our choice of f.

Case 3. Without loss of generality, $f(v_{i-1}) = 1$, and $f(v'_{i-1}) = 0$. Now, at least one of v_{i+1} and v'_{i+1} must be assigned a one by f in order to Italian dominate v_i. Further, since i is the largest index for a row with property $\mathcal{P}$, it follows that $f(v_{i+1}) = f(v'_{i+1}) = 1$. Let g be the function such that $g(v'_i) = 0$, $g(v'_{i-1}) = 1$. Then, g is a γ_I-function with fewer rows having property $\mathcal{P}$, contradicting our choice of f.

Thus, we may choose a γ_I-function f of $P_n \cdot K_2$ such that if $f(v_i) = 0$, then $f(v'_i) = 0$, as desired. □

Importantly, while this lemma states a useful result for $P_n \cdot K_2$, this is not generally true for $G \cdot K_n$.

We will use the preceding lemma to show equality of our previous upper bound for $\gamma_I(P_n \cdot K_2)$.

Theorem 3.21. *For any path graph P_n, we have that $\gamma_I(P_n \cdot K_2) = 2(\lfloor \frac{n+2}{3} \rfloor)$.*

Proof. Note that by [5], $\gamma(P_n) = \lfloor \frac{n+2}{3} \rfloor$.

Using our previous notation for the constituent parts of $G \cdot K_2$, let D be a γ-set of P_n, and let D' be the corresponding γ-set of P_n'. Then, the function $f = (V_0, V_1, V_2)$ such that $V_2 = \emptyset$, $V_1 = D \cup D'$, and $V_0 = V(P_n \cdot K_2) \setminus V_1$ is an Italian dominating function on $P_n \cdot K_2$. Thus, $\gamma_I(P_n \cdot K_2) \leq 2\gamma(P_n) = 2(\lceil \frac{n+1}{2} \rceil)$.

To show that $\gamma_I(P_n \cdot K_2) \geq 2\gamma(P_n)$, suppose to the contrary that $\gamma_I(P_n \cdot K_2) \leq 2\gamma(P_n) - 1$.

Let $g = (V_0, V_1, V_2)$ be a γ_I-function on $P_n \cdot K_2$. By Lemma 3.20, we may choose g such that if $g(v_i) = 0$, then $g(v_i') = 0$.

Since the total weight of g is at most $2\gamma(P_n) - 1$, then, without loss of generality, the total weight assigned to G is at most $\gamma(P_n) - 1$. That is, the set $(V_1 \cup V_2) \cap V(G)$ does not dominate P_n. That is, there exists some vertex $v \in V(P_n)$ such that $g(v) = 0$ and $g(x) = 0$ for all $x \in N(v) \cap V(P_n)$.

However, since g is a γ_I-function of $P_n \cdot K_2$, there must be some vertex $x' \in N(v) \cap V(P_n')$ such that $f(x') \geq 1$, contradicting our choice of g.

Therefore, $\gamma_I(P_n \cdot K_2) \geq 2\gamma(G)$, and so $\gamma_I(P_n \cdot K_2) = 2\gamma(L_n) = 2(\lfloor \frac{n+2}{3} \rfloor)$, as desired. $\square$

4 CONCLUDING REMARKS

Research into the parameters of Italian domination in graph products is ongoing, and a rich area of study. We conclude by presenting several open questions suggested by this research.

1. Certain prisms and Möbius ladders are also circulant graphs. Further explore the Italian domination numbers of circulant graphs.

2. A graph of the form $G \Box P_n$ is called a *generalized prism* graph. Further explore the Italian domination numbers of generalized prism graphs.

3. Use Lemmas 3.2 and 3.3 to explore the parameters of Italian domination in the cubic graphs.

4. Characterize the prisms for which $\gamma_I(G) = \gamma_R(G)$.

5. Explore Italian domination in graph products $G \Box H$, $G \times H$, and $G \cdot H$ where $H \neq K_2$.

6. Further refine the upper bound $\gamma_I(G \cdot L_n) \leq 2\gamma(G)$.

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