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# Distribution of a Sum of Random Variables when the Sample Size is a Poisson Distribution

Mark Pfister

# Distribution of a Sum of Random Variables when the Sample Size is a Poisson Distribution

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by

Mark Pfister

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Robert Price, Ph.D., Chair

Nicole Lewis, Ph.D.

JeanMarie Hendrickson, Ph.D.

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ABSTRACT

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A probability distribution is a statistical function that describes the probability of possible outcomes in an experiment or occurrence. There are many different probability distributions that give the probability of an event happening, given some sample size $n$. An important question in statistics is to determine the distribution of the sum of independent random variables when the sample size n is fixed. For example, it is known that the sum of $n$ independent Bernoulli random variables with success probability $p$ is a Binomial distribution with parameters $n$ and $p$. However, this is not true when the sample size is not fixed but a random variable. The goal of this thesis is to determine the distribution of the sum of independent random variables when the sample size is randomly distributed as a Poisson distribution. We will also discuss the mean and the variance of this unconditional distribution.

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# 1 INTRODUCTION

To explain how values of a random variable is distributed, a probability distribution is used. A *probability distribution* is a mathematical function that gives the probability that an event occurs. Probability distributions can be simple to complex, from determining the probability of rolling a 2 with a dice, to determining the success of a particular stock in the stock market. There are two different kinds of probability distributions, discrete probability distributions and continuous probability distributions. This thesis will cover the sum of independent discrete random variables when the sample size is random.

**Definition 1.1** *A discrete probability distribution is a distribution where the set of possible outcomes is discrete and is defined by a probability mass function. Two things must be true for discrete probability distributions:*

1. *The probability of any individual event must be between 0 and 1.*
2. *The sum of the probabilities for the events must equal 1.*

Depending on the situation, different probability distributions may be used. Looking at discrete probability distributions, some examples include Bernoulli, binomial, geometric, negative binomial, or Poisson. Problems in probability and statistics, we are often interested in the sum of n independent random variables.

**Definition 1.2** *Let $Y_1, ..., Y_n$ be independently distributed discrete random variables. Let $Y = Y_1 + ... + Y_n$ where n is fixed. Here are some examples of sums of independent and identically distributed discrete random variables:*

1. *If $Y_i$ follows a Bernoulli $(p)$, then $Y$ will be distributed as a binomial $(n, p)$*

2. *If $Y_i$ follows a geometric $(p)$, then $Y$ will be distributed as a negative binomial $(n, p)$*

3. *If $Y_i$ follows a Poisson $(\mu)$, then $Y$ will be distributed as a Poisson $(n\mu)$*

The previous examples assumed that the sample size $n$ was fixed. Consider the following hierarchical model:

$$Y|N = Y_1 + Y_2 + ... + Y_N$$

where

$$N \sim Poisson(\lambda).$$

Hierarchical models are useful in developing probability distributions to better match the data. A *hierarchical model* is formed by taking different random variables in a single probability distribution and giving them their own probability distribution in a hierarchy.

A discrete probability distribution that expresses the probability of a number of events occurring within a given time period is a *Poisson distribution*. The Poisson distribution assumes that there is a constant rate of occurrence and that the events occur independently of each other. Applications of the Poisson distribution would be the number of car wrecks in a city from 8 AM to 9 AM, or the number of customers walking into a store from 12 PM to 1 PM. There would be prior information giving the average number of wrecks happening or the number of customers walking into a store, which is required for the Poisson distribution. The average number of events that occurs per time interval is $\lambda$. Below are the probability mass function, mean, and variance of the Poisson distribution

$$P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!}, \ n = 0, 1, 2, ..., \lambda > 0$$

$$E(N) = \lambda,$$

and

$$Var(N) = \lambda,$$

respectively.

If one assumes that $P(N = 0) = 0$, then the Poisson distribution needs to be adjusted. A zero-truncated Poisson distribution is very similar to a Poisson distribution but assumes $P(N = 0) = 0$. An example of a zero-truncated Poisson distribution would be the number of items in a person's cart at a grocery store. If a person is waiting in line to checkout with a cart, it would be safe to assume there is at least one item in the cart. The probability mass function, mean, and variance of the zero-truncated Poisson distribution are

$$P(N_T = n) = \frac{e^{-\lambda}\lambda^n}{n!(1 - e^{-\lambda})}, n = 1, 2, ..., \ \lambda > 0$$

$$E(N_T) = \frac{\lambda}{1 - e^{-\lambda}},$$

and

$$Var(N_T) = \frac{\lambda + \lambda^2}{1 - e^{-\lambda}} - \frac{\lambda^2}{(1 - e^{-\lambda})^2},$$

respectively.

In the different distributions covered, its probability mass function will be discussed, then it will be assumed that $N$ is random, and a hierarchical model will be made with that new probability distribution. In general, the distribution equation used is the definition of conditional probability, this equation will be used to find the unconditional distribution of $Y$:

$$P(Y = y|\lambda) = \sum_{n=y}^{\infty} P(Y = y|N = n, \lambda) \cdot P(N = n|\lambda)$$

The mean and variance of Y can be found using the following equations:

$$E(Y) = E(E(Y|N))$$

and

$$Var(Y) = Var(E(Y|N)) + E(Var(Y|N)),$$

respectively.

# 2 MOTIVATIONAL WORK

Solomon (1983) details the following biological model. Suppose that each of a random number, N, or insects lays $y_i$ eggs, where $y_i$'s are independent, identically distributed random variables. the total number of eggs laid is $Y = Y_1 + ... + Y_N$. What is the distribution of $Y$? It is common to assume that $N \sim Poisson(\lambda)$. Furthermore, if we assume that each $Y_i$ has the logarithmic series distribution with success probability $p$, we have the hierarchical model:

$$Y|N = Y_1 + ... + Y_N$$

$$P(Y_i = y) = \frac{-1}{ln(p)} \frac{(1-p)^y}{y}, \; y = 0, 1, ...$$

$$N \sim Poisson(\lambda).$$

It will be shown in the next section that the marginal distribution of $Y$ is negative binomial$(r, p)$, where $r = \frac{-\lambda}{ln(p)}$.

### 2.1 Proof of Motivational Work

We can show this by computing the moment generating function (mgf) of $Y$. It is denoted by the following:

$$\begin{aligned} M(t) &= E(e^{Yt}) \\ &= E[E(e^{Yt}|N)] \\ &= E[E(e^{(Y_1+Y_2+...+Y_N)t}|N)] \\ &= E[E(e^{Y_1 t}|N)]^N. \end{aligned}$$

This fact is due to the mgf of the sum of the independent variables is equal to the product of the individual moment generating functions. Hence, we can find the mgf of $Y_1$ in particular:

$$\begin{aligned} E(e^{Y_1 t}) &= \sum_{y_1=1}^{\infty} e^{y_1 t} \frac{-1}{ln(p)} \frac{(1-p)^{y_1}}{y_1} \\ &= \frac{-1}{ln(p)} \sum_{y_1=1}^{\infty} \frac{((1-p)e^t)^{y_1}}{y_1} \\ &= \frac{-1}{ln(p)}(-ln(1-e^t(1-p))). \end{aligned}$$

The conclusion above is a result of using the Taylor Series of $ln(x)$. Rearranging the equation above gives us the mgf of $Y_1$:

$$M(t) = \frac{ln(1-e^t(1-p))}{ln(p)}.$$

Since we have the mgf of $Y_1$, we can obtain the mgf of $Y$, since all $Y_i$'s are independent identically distributed. Hence, the mgf of $Y$ is:

$$\begin{aligned} E(e^{Yt}) &= E[E(e^{Y_1 t}|N)]^N \\ &= E\Big[\Big(\frac{ln(1-e^t(1-p))}{ln(p)}\Big)^N\Big] \\ &= \sum_{n=0}^{\infty}\Big(\frac{ln(1-e^t(1-p))}{ln(p)}\Big)^n\Big(\frac{e^{-\lambda}\lambda^n}{n!}\Big). \end{aligned}$$

The above equation resembles a Poisson distribution. We can pull out some terms from inside the sum to get a probability mass function (pmf) that is equivalent to a pmf of a Poisson distribution. Hence, we have:

$$e^{-\lambda}e^{\frac{\lambda ln(1-e^t(1-p))}{ln(p)}}\sum_{n=0}^{\infty}\frac{e^{\frac{-\lambda ln(1-e^t(1-p))}{ln(p)}}\Big(\frac{\lambda ln(1-e^t(1-p))}{ln(p)}\Big)^n}{n!}.$$

Inside the sum we have a pmf of the form $\frac{\lambda^k e^{-\lambda}}{k!}$, which is the pmf of a Poisson distribution. The sum is equal to 1, since it's the sum of a Poisson probability random variable. Hence, the mgf of $Y$ is:

$$\begin{aligned} E(e^{Yt}) &= e^{-\lambda+\frac{\lambda ln(1-e^t(1-p))}{ln(p)}} \\ &= e^{\frac{-\lambda}{ln(p)}\Big(ln\Big(\frac{p}{1-e^t(1-p)}\Big)\Big)} \\ &= \Big(\frac{p}{1-e^t(1-p)}\Big)^{\frac{-\lambda}{ln(p)}},\ t < -ln(1-p). \end{aligned}$$

This is the mgf of a negative binomial $(r,p)$ distribution with $r = \frac{-\lambda}{ln(p)}$.

# 3 BINOMIAL

A discrete probability distribution where a single trial is conducted and has two possible outcomes, such as success or failure, is known as a *Bernoulli distribution*. An example of a Bernoulli trial is flipping a coin once, there are two possible outcomes. When the sample size is greater than 1, $n > 1$, the distribution of the sum of Bernoulli trials is known as the *binomial distribution*. The binomial distribution models the total number of successes of a fixed number of $n$ independent random trials with the same probability of success $p$. The probability distribution of getting exactly y successes in n trials is given by:

$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y},\ y = 0, 1, ..., n$$

The mean and the variance of a binomial distribution with parameters n and p are:

$$E(Y) = np$$

and

$$Var(Y) = np(1-p),$$

respectively.

In the next section, we assume that the number of trials, $N$, is randomly distributed as a Poisson. Given an unknown random sample $N$ and unknown number of successes, we will calculate the unconditional probability distribution of $Y$.

## 3.1 Binomial-Poisson Mixture

Let $Y_i$ have a Bernoulli distribution with success probability $p$. Hence, the hierarchical model is the following:

If

$$Y_i \sim Bernoulli(1, p), Y_i = 0, 1$$

then

$$Y|N \sim Binomial(N, p)$$

$$N \sim Poisson(\lambda).$$

The following derives the unconditional probability distribution of $Y$:

$$\begin{aligned} P(Y = y|\lambda) &= \sum_{n=y}^{\infty} P(Y = y|N = n, \lambda) \cdot P(N = n|\lambda) \\ &= \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \sum_{n=y}^{\infty} \Big(\frac{n!}{y!(n-y)!}\Big) p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \sum_{n=y}^{\infty} \Big(\frac{n!}{y!(n-y)!}\Big) p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^y\lambda^{n-y}}{n!}. \end{aligned}$$

Rearranging terms, we get:

$$P(Y = y|\lambda) = \sum_{n=y}^{\infty} \Big(\frac{n!}{y!(n-y)!}\Big) p^y (\lambda(1-p))^{n-y} \frac{e^{-\lambda}\lambda^y}{n!}.$$

The series begins with $n = y$, not $n = 0$. For example, $P(Y = 3|N = 2) = 0$. For example, this would be the probability that the coin lands on heads 3 times, yet the coin is only tossed 2 times. Thus, it starts at $n = y$. The series can have terms not involving the n sum. Doing this, the series is put into a form that represents a power series for $e^x = \sum_{y=0}^{\infty} \frac{x^y}{y!}$. Hence,

$$
\begin{aligned}
P(Y = y|\lambda) &= \frac{e^{-\lambda}(\lambda p)^y}{y!} \sum_{n=y}^{\infty} \frac{(\lambda(1-p))^{n-y}}{(n-y)!} \\
&= \frac{e^{-\lambda}(\lambda p)^y}{y!} e^{\lambda(1-p)} \\
&= \frac{e^{-\lambda p}(\lambda p)^y}{y!}, y = 0, 1, ...
\end{aligned}
$$

This is the binomial-Poisson mixture distribution of $Y$. This is in the form of the Poisson distribution probability mass function, $\frac{e^{-\lambda}\lambda^n}{n!}$. Below is the function computed in R, as well as figures, showing the difference in the probability distribution as $\lambda$ or $p$ changes.

```
# P(Y = y) where Y | N ~ Binomial(N,p) and N ~ Poisson(lambda)
prob = dpois(y,lambda * p)
```

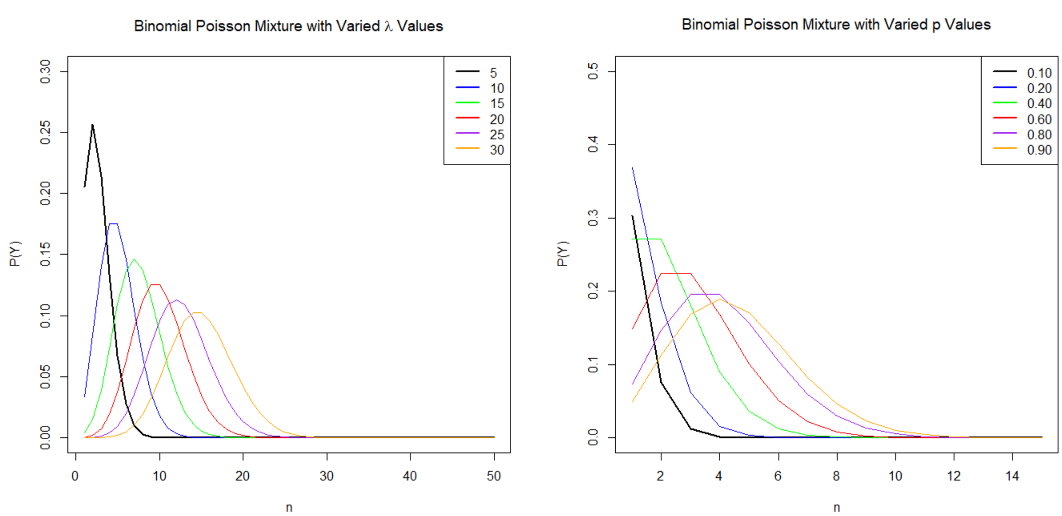


Figure 1: Binomial-Poisson Mixture

The mean and variance of $Y$ are:

$$E(Y) = E(E(Y|N))$$
$$= E(Np)$$
$$= \lambda p$$

and

$$Var(Y) = Var(E(Y|N)) + E(Var(Y|N))$$
$$= Var(Np) + E(Np(1-p))$$
$$= \lambda p^2 + \lambda p(1-p)$$
$$= \lambda p,$$

respectively.

### 3.2 Binomial-Zero Truncated Poisson Mixture

In this section, we will derive the distribution, the mean, and the variance of $Y$ when N is distributed as a zero-truncated Poisson distribution. In the zero-truncated Poisson distribution, there is a different probability mass function, since there must be at least one success in this conditional distribution. Using the probability mass function of the binomial distribution and the zero-truncated Poisson distribution, the binomial-zero truncated Poisson mixture distribution of $Y$ can be solved.

Recall the probability distribution of a zero-truncated Poisson ($\lambda$) is

$$P(N_T = n) = \frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})}, n = 1, 2, ...$$

with mean

$$E(N_T) = \frac{\lambda}{1 - e^{-\lambda}}$$

and variance

$$Var(N_T) = \frac{\lambda + \lambda^2}{1 - e^{-\lambda}} - \frac{\lambda^2}{(1 - e^{-\lambda})^2}.$$

The following derives unconditional probability distribution of $Y$:

$$\begin{aligned}
P(Y = y|\lambda) &= \sum_{n=y}^{\infty} P(Y = y|N_T = n, \lambda) \cdot P(N_T = n|\lambda) \\
&= \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!(1 - e^{-\lambda})} \\
&= \sum_{n=y}^{\infty} \Big(\frac{n!}{y!(n-y)!}\Big) p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!(1 - e^{-\lambda})} \\
&= \sum_{n=y}^{\infty} \Big(\frac{n!}{y!(n-y)!}\Big) p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^y\lambda^{n-y}}{n!(1 - e^{-\lambda})}.
\end{aligned}$$

Similar to the binomial-Poisson distribution example, this is $n = y$ in the series, not $n = 0$. For example, $P(Y = 3|N = 2) = 0$. Thus, it must start at $n = y$. The series can have terms taken outside the sum. Doing this, the series is put into a form that represents a power series for $e^x$. Rearranging terms gives us

$$P(Y = y|\lambda) = \frac{e^{-\lambda}(\lambda p)^y}{y!(1 - e^{-\lambda})} \sum_{n=y}^{\infty} \frac{(\lambda(1-p))^{n-y}}{(n-y)!}.$$

This series represents a power series for $e^x$. Representing this series as $e^{\lambda(1-p)}$, the mixture distribution can be solved. We get the following:

$$P(Y = 0|\lambda) = \frac{e^{-\lambda p} - e^{-\lambda}}{(1 - e^{-\lambda})}$$

and

$$P(Y = y|\lambda) = \frac{e^{-\lambda p}(\lambda p)^y}{y!(1 - e^{-\lambda})}, y = 1, 2, ...$$

This is the binomial-zero truncated Poisson unconditional distribution of $Y$. Below is the function computed in R, as well as figures, showing the difference in the probability distribution as $\lambda$ or $p$ changes.

```
# P(Y = y) where Y | N ~ Binomial(N,p) and N ~ Trunc-Poisson(lambda)

TOL = 1.0e-300

prob = function(y,lambda,mu){

tot = 0

n = y

repeat {

next_term <- exp(-lambda + n*log(lambda) - lfactorial(n)) *

exp(lfactorial(n) - lfactorial(y) - lfactorial(n-y)) *

exp(y*log(p) + (n-y)*log(1-p)) / (1 - exp(-lambda))

if (next_term < TOL) break

tot <- tot + next_term

n <- n + 1}

return(tot)}
```

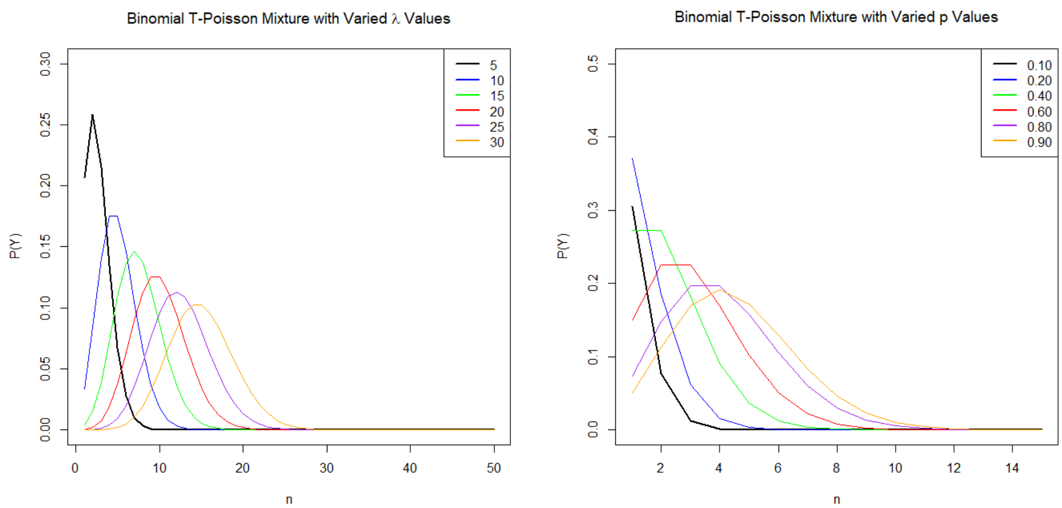


Figure 2: Binomial-Truncated Poisson Mixture

The mean and variance of $Y$ are:

$$
\begin{aligned}
E(Y) &= E(E(Y|N)) \\
&= E(Np) \\
&= \frac{\lambda p}{1-e^{-\lambda}}
\end{aligned}
$$

and

$$
\begin{aligned}
Var(Y) &= Var(E(Y|N)) + E(Var(Y|N)) \\
&= Var(Np) + E(Np(1-p)) \\
&= \Big(\frac{\lambda+\lambda^2}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2}\Big)p^2 + \Big(\frac{\lambda}{1-e^{-\lambda}}\Big)p(1-p) \\
&= \frac{\lambda p+\lambda^2 p^2}{1-e^{-\lambda}} - \frac{\lambda^2 p}{(1-e^{-\lambda})^2},
\end{aligned}
$$

respectively.

# 4 NEGATIVE BINOMIAL

In a *negative binomial distribution*, the random variable is the number of trials performed $X$ until the $i^{th}$ success. Thus, the negative binomial counts the number of failures until a fixed number of successes. An example of a negative binomial distribution would be to continue to draw a card out of a deck with replacement until an ace is drawn, counting the number of attempts. Another example would be a traffic stop, where the police may be checking for drivers under the influence. The police may know, based on prior traffic stops, the probability that a driver is under the influence. The failure would be that the driver is not under the influence. The police will keep testing the drivers until there is a success, or the police find someone under the influence. Below are the probability mass function, mean, and variance of the negative binomial distribution:

$$P(Y = y) = \binom{y+n+1}{y} p^n (1-p)^y, y = 0, 1, ...$$

$$E(Y) = \frac{pn}{1-p},$$

and

$$Var(Y) = \frac{pn}{(1-p)^2},$$

respectively.

Now assume that the number of trials, $N$ is unknown or random. In the example above, it is unknown how many cards are drawn out of the deck. Given an unknown sampling $N$ and unknown number of failures, we can calculate the probability distribution.

### 4.1 Negative Binomial-Poisson Mixture

Let $Y_i$ have a geometric distribution with success probability $p$. Hence, the hierarchical model is the following:

If

$$Y_i \sim Geometric(1, p), Y_i = 0, 1,$$

then

$$Y|N \sim Negative\ Binomial(N, p)$$

$$N \sim Poisson(\lambda).$$

The following derives the unconditional probability distribution of $Y$:

$$\begin{aligned} P(Y = y|\lambda) &= \sum_{n=y}^{\infty} P(Y = y|N = n, \lambda) \cdot P(N = n|\lambda) \\ &= \sum_{n=y}^{\infty} \binom{y+n-1}{y} p^n (1-p)^y \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \sum_{n=y}^{\infty} \frac{(y+n-1)!}{y!(n-1)!} p^n (1-p)^y \frac{e^{-\lambda}\lambda^n}{n!}. \end{aligned}$$

This series can have terms taken out of the sum. The last equation can be rewritten as

$$\begin{aligned} P(Y = y|\lambda) &= \frac{e^{-\lambda}(1-p)^y}{y!} \sum_{n=y}^{\infty} \frac{(y+n-1)!}{(n-1)!} \frac{(\lambda p)^n}{n!} \\ &= \frac{e^{-\lambda}(1-p)^y}{\Gamma(y+1)} \sum_{n=y}^{\infty} \frac{\Gamma(y+n)}{\Gamma(n)} \frac{(\lambda p)^n}{\Gamma(n+1)}. \end{aligned}$$

At this point, there is not a clear way to simplify this equation into a closed form, like in the binomial-Poisson mixture distribution. However, similar to the binomial-Poisson mixture distribution, we can compute the probabilities of the equation and draw conclusions about the distribution. Below is the function in R, as well as figures, showing the difference in the probability distribution as $\lambda$ or $p$ changes.

```
# P(Y = y) where Y | N ~ NegBin(N,p) and N ~ Poisson(lambda)
TOL = 1.0e-300
prob = function(y,lambda,p){
tot = 0
n = 1
repeat {
next_term <- exp(-lambda + n*log(lambda) - lfactorial(n)) *
exp(lfactorial(y+n-1) - lfactorial(y) - lfactorial(n-1)) *
exp(n*log(p) + y*log(1-p))
if (next_term < TOL) break
tot <- tot + next_term
n <- n + 1}
return(tot)}
```

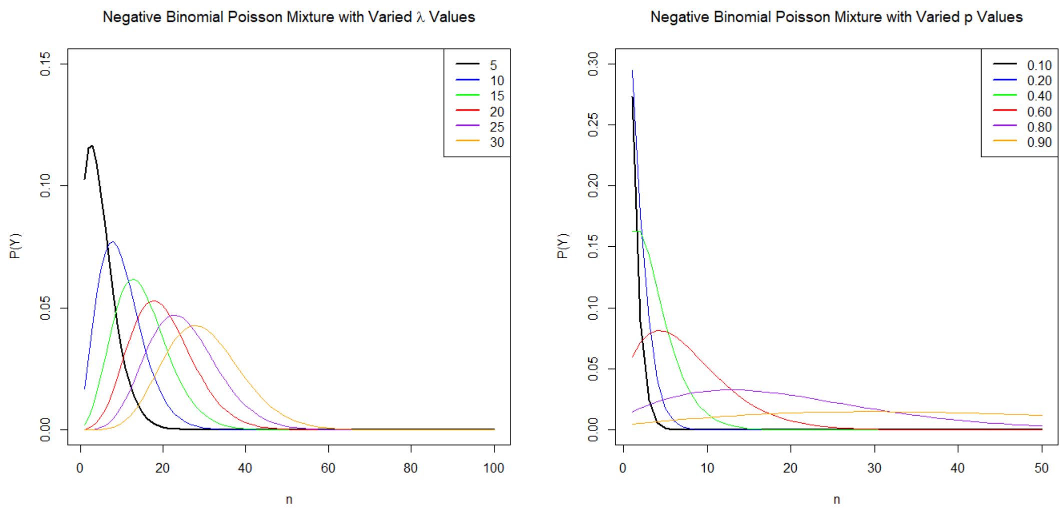


Figure 3: Negative Binomial-Poisson Mixture

The mean and variance of $Y$ are:

$$\begin{aligned} E(Y) &= E(E(Y|N)) \\ &= E\Big(\frac{Np}{1-p}\Big) \\ &= \frac{\lambda p}{1-p} \end{aligned}$$

and

$$\begin{aligned} Var(Y) &= Var(E(Y|N)) + E(Var(Y|N)) \\ &= Var\Big(\frac{Np}{1-p}\Big) + E\Big(\frac{Np}{(1-p)^2}\Big) \\ &= \frac{\lambda p^2}{(1-p)^2} + \frac{\lambda p}{(1-p)^2} \\ &= \frac{\lambda p^2 + \lambda p}{(1-p)^2}, \end{aligned}$$

respectively.

## 4.2 Negative Binomial-Zero Truncated Poisson Mixture

In this section, we will derive the distribution, the mean, and the variance of $Y$ when $N$ is distributed as a zero-truncated Poisson distribution. In the zero-truncated Poisson distribution, there is a different probability mass function, since there must be at least one success in this conditional distribution. Using the probability mass function of the negative binomial distribution and the zero-truncated Poisson distribution, the negative binomial-zero truncated Poisson mixture distribution can be solved.

Recall the probability distribution of a zero-truncated Poisson($\lambda$) is

$$P(N_T = n) = \frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})}, n = 1, 2, ...$$

with mean

$$E(N_T) = \frac{\lambda}{1-e^{-\lambda}}$$

and variance

$$Var(N_T) = \frac{\lambda + \lambda^2}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2}.$$

The following derives unconditional probability distribution of $Y$:

$$\begin{aligned} P(Y = y|\lambda) &= \sum_{n=y}^{\infty} P(Y = y|N_T = n, \lambda) \cdot P(N_T = n|\lambda) \\ &= \sum_{n=y}^{\infty} \binom{y+n-1}{y} p^n (1-p)^y \frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})} \\ &= \sum_{n=y}^{\infty} \frac{(y+n-1)!}{y!(n-1)!} p^n (1-p)^y \frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})}. \end{aligned}$$

This series can have terms taken out of the sum. The last equation can be rewritten as

$$\begin{aligned}
P(Y = y|\lambda) &= \frac{e^{-\lambda}(1-p)^y}{y!(1-e^{-\lambda})} \sum_{n=y}^{\infty} \frac{(y+n-1)!}{(n-1)!} \frac{(\lambda p)^n}{n!} \\
&= \frac{e^{-\lambda}(1-p)^y}{\Gamma(y+1)(1-e^{-\lambda})} \sum_{n=y}^{\infty} \frac{\Gamma(y+n)}{\Gamma(n)} \frac{(\lambda p)^n}{\Gamma(n+1)}, y \geq 1.
\end{aligned}$$

If $y = 0$, then the probability will be:

$$P(Y = y|\lambda) = \frac{e^{-\lambda}}{(1-e^{-\lambda})}\Big(e^{\lambda p} - 1\Big).$$

Similarly to the negative binomial-Poisson mixture distribution, there is not a clear way to simplify this equation into a closed form, like in the binomial-Poisson mixture distribution. However, similar to the binomial-Poisson mixture distribution, we can compute the probabilities of the equation and draw conclusions about the distribution. Below is the function in R, as well as figures, showing the difference in the probability distribution as $\lambda$ or $p$ changes.

```
# P(Y = y) where Y | N ~ NegBin(N,p) and N ~ Trunc-Poisson(lambda)

TOL = 1.0e-300

prob = function(y,lambda,mu){

tot = 0

n = 1

repeat {

next_term <- exp(-lambda + n*log(lambda) - lfactorial(n)) *

exp(lfactorial(y+n-1) - lfactorial(y) - lfactorial(n-1)) *

exp(n*log(p) + (y)*log(1-p)) / (1 - exp(-lambda))
```

```
if (next_term < TOL) break

tot <- tot + next_term

n <- n + 1}

return(tot)}
```

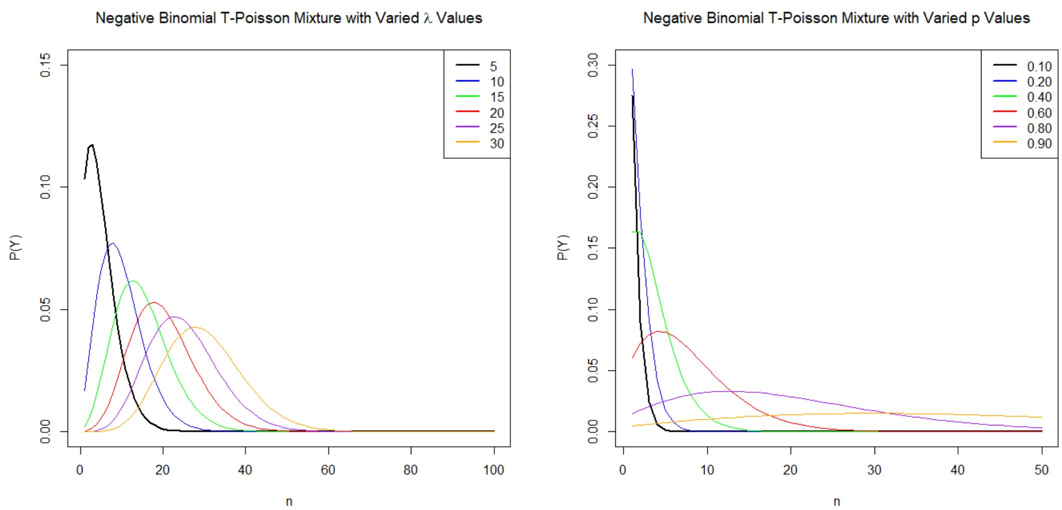


Figure 4: Negative Binomial-Truncated Poisson Mixture

The mean and variance of $Y$ are:

$$
\begin{aligned}
E(Y) &= E(E(Y|N)) \\
&= E\left(\frac{Np}{1-p}\right) \\
&= \frac{\lambda p}{(1-p)(1-e^{-\lambda})}
\end{aligned}
$$

and

$$
\begin{aligned}
Var(Y) &= Var(E(Y|N)) + E(Var(Y|N)) \\
&= Var\Big(\frac{Np}{1-p}\Big) + E\Big(\frac{Np}{(1-p)^2}\Big) \\
&= \frac{\Big(\frac{\lambda+\lambda^2}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2}\Big)p^2}{(1-p)^2} + \frac{\lambda p}{(1-p)^2(1-e^{-\lambda})} \\
&= \frac{\lambda^2 p^2 + \lambda p^2 + \lambda p}{(1-e^{-\lambda})(1-p)^2} - \frac{\lambda^2 p}{(1-e^{-\lambda})^2(1-p)},
\end{aligned}
$$

respectively.

# 5 POISSON

A discrete probability distribution that expresses the probability of a number of events occurring within a given time period is a *Poisson distribution*. As an example, consider the number of items that a store clerk rings up in each customer's cart in an hour. The number of items in each customer's cart could resemble a Poisson distribution. However, now assume that the number of events that occur, $N$, is random. In the example, it is random how many customers use that particular check out line in the hour. Given an unknown random sampling $N$, we will calculate the unconditional probability distribution of $Y$.

## 5.1 Poisson-Poisson Mixture

Let $Y_i$ have a Poisson distribution with success probability $N\mu$. Hence, the hierarchical model is the following:

$$Y|N \sim Poisson(N\mu)$$

$$N \sim Poisson(\lambda).$$

The following derives the unconditional probability distribution of $Y$:

$$\begin{aligned} P(Y = y|\lambda) &= \sum_{n=0}^{\infty} P(Y = y|N = n, \lambda) \cdot P(N = n|\lambda) \\ &= \sum_{n=0}^{\infty} \Big(\frac{e^{-n\mu}(n\mu)^y}{y!}\Big)\Big(\frac{e^{-\lambda}\lambda^n}{n!}\Big) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda-n\mu}(n\mu)^y\lambda^n}{y!n!}. \end{aligned}$$

Unlike the binomial-Poisson mixture distribution, this series does not represent a power series for $e^x$ or anything familiar to get the equation into a simple, closed form.

However, we can compute the probabilities of the equation and draw conclusions about the distribution. Below is the function in R, as well as figures, showing the difference in the probabilities as $\lambda$ or $\mu$ changes.

```
# P(Y = y) where Y | N ~ Poisson(N*mu) and N ~ Poisson(lambda)

TOL = 1.0e-300

prob = function(y,lambda,mu){

tot = 0

n = 1

repeat {

next_term <- exp(-lambda + n*log(lambda) -lfactorial(n)) *

exp(-n*mu + y*log(n) + y*log(mu) - lfactorial(y))

if (next_term < TOL) break

tot <- tot + next_term

n <- n + 1}

return(tot)}
```

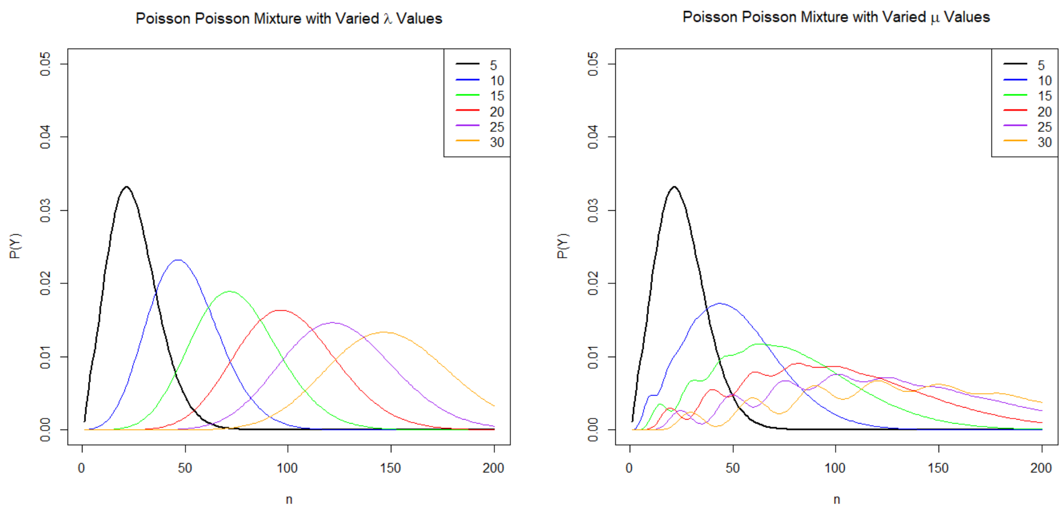


Figure 5: Poisson-Poisson Mixture

The mean and variance of $Y$ are:

$$
\begin{aligned}
E(Y) &= E(E(Y|N)) \\
&= E(N\mu) \\
&= \lambda\mu
\end{aligned}
$$

and

$$
\begin{aligned}
Var(Y) &= Var(E(Y|N)) + E(Var(Y|N)) \\
&= Var(N\mu) + E(N\mu) \\
&= \lambda\mu^2 + \lambda\mu \\
&= \lambda\mu(1+\mu),
\end{aligned}
$$

respectively.

## 5.2 Poisson-Zero Truncated Poisson Mixture

In this section, we will derive the distribution, the mean, and the variance of $Y$ when $N$ is distributed as a zero-truncated Poisson distribution. In the zero-truncated Poisson distribution, there is a different probability mass function, since there must be at least one success in this conditional distribution. Using the probability mass function of the Poisson distribution an the zero-truncated Poisson distribution, the Poisson-zero truncated Poisson mixture distribution of $Y$ can be solved.

Recall the probability distribution of a zero-truncated Poisson ($\lambda$) is

$$P(N_T = n) = \frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})}, n = 1, 2, ...$$

with mean

$$E(N_T) = \frac{\lambda}{1-e^{-\lambda}}$$

and variance

$$Var(N_T) = \frac{\lambda+\lambda^2}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2}.$$

The following derives unconditional probability distribution of $Y$:

$$\begin{aligned} P(Y = y|\lambda) &= \sum_{n=1}^{\infty} P(Y = y|N_T = n, \lambda) \cdot P(N_T = n|\lambda) \\ &= \sum_{n=1}^{\infty} \Big(\frac{e^{-n\mu}(n\mu)^y}{y!}\Big)\Big(\frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})}\Big) \\ &= \sum_{n=1}^{\infty} \frac{e^{-\lambda-n\mu}(n\mu)^y\lambda^n}{y!n!(1-e^{-\lambda})}, y \geq 1 \end{aligned}$$

This series does not represent a power series for $e^x$ or anything familiar to get the equation into a simple, closed form. However, we can compute the probabilities of the equation and draw conclusions about the distribution. Below is the function in R, as well as figures, showing the difference in the probabilities as $\lambda$ or $\mu$ changes.

```
# P(Y = y) where Y | N ~ Poisson(N*mu) and N ~ Trunc-Poisson(lambda)

TOL = 1.0e-300

prob = function(y,lambda,mu){

tot = 0

n = 1

repeat {

next_term <- exp(-lambda + n*log(lambda) -lfactorial(n)) *

exp(-n*mu + y*log(n) + y*log(mu) - lfactorial(y)) / (1 - exp(-lambda))

if (next_term < TOL) break

tot <- tot + next_term

n <- n + 1}

return(tot)}
```

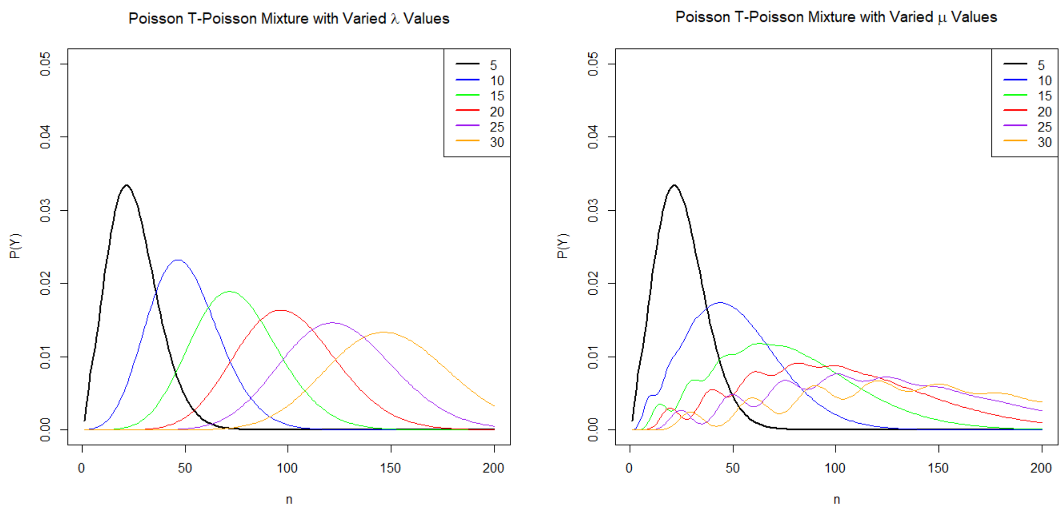


Figure 6: Poisson-Truncated Poisson Mixture

The mean and variance of $Y$ are:

$$\begin{aligned}E(Y) &= E(E(Y|N)) \\ &= E(N\mu)) \\ &= \frac{\lambda\mu}{1-e^{-\lambda}}\end{aligned}$$

and

$$\begin{aligned}Var(Y) &= Var(E(Y|N)) + E(Var(Y|N)) \\ &= Var(N\mu) + E(N\mu) \\ &= \Big(\frac{\lambda+\lambda^2}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2}\Big)\mu^2 + \frac{\lambda\mu}{1-e^{-\lambda}} \\ &= \frac{\lambda\mu+\lambda\mu^2+\lambda^2\mu^2}{1-e^{-\lambda}} - \frac{\lambda^2\mu}{(1-e^{-\lambda})^2},\end{aligned}$$

respectively.

# 6 CONCLUSION

The goal of the thesis was to find the probability distribution of a sum of discrete random variables when the sample size follows a Poisson distribution. In the case of the binomial-Poisson and binomial-truncated Poisson distributions, a closed form of the hierarchical models were found. However, not all distributions are as closely related and linked to the Poisson distribution as the binomial distribution. In addition, the mean and variance for the unconditional distributions were given for all the cases that were considered in this thesis.

For the other cases, there was no closed form in the distribution but we provided R programs that compute these probabilities. When producing the graphs for the different probability distributions, the graphs have a similar distribution. When $\lambda$ is small, such as $\lambda = 5$, the graph has a high probability at the beginning (with the peak of the curve at $n = \lambda$), then decreases rapidly. As $\lambda$ gets larger, the distribution has a smaller peak probability values at $n = \lambda$, but the probability values are larger over a larger period of $n$. Below is an example of the Poisson distribution with the varied $\lambda$ values.

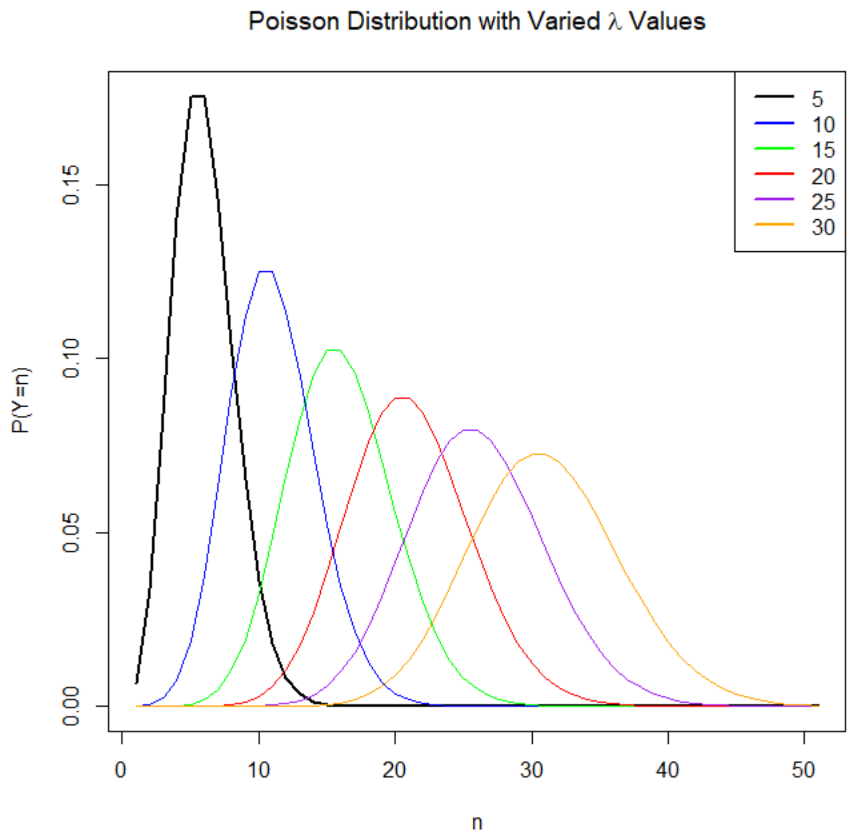


Figure 7: Poisson Distribution

The same pattern is true for the graphs of the varied $p$ values (or $\mu$ values for the Poisson-Poisson distributions). As the $p$ or $\mu$ values get larger, the graph has a smaller peak probability value, but the probability values are larger over a larger period on $n$, as shown in the graph above. Thus, for the various distribution mixtures we have covered, there is a strong resemblance to the original Poisson distribution in all of the cases.

Knowing that when there is a hierarchical model where $N$ is random, the distribution resembles a Poisson distribution, we could draw further inference about these distributions, since these hierarchical models are not uncommon. For example, suppose an insurance agency wanted to find the distribution of the average time it takes to process an insurance claim. If there was a set sample size and it is not random, it would be a simple distribution. However, the sample from day to day is random. For example, factors for how many insurance claims are filed in a day could be the condition of the roads, day of the week, etc. Thus, this probability distribution could resemble the same form as a Poisson distribution, based on the conclusions we have made earlier. This is one example of a distribution where the sample size is not fixed. Oftentimes, samples are collected without a set sample size, leading to using these hierarchical models more often. Thus, inference of these distributions we have covered could be made into many different areas of study, leading to more accurate probability distributions.

BIBLIOGRAPHY

[1] Casella, G., & Berger, R. L. (2008). Statistical Inference (2nd ed.). Belmont, CA: Brooks/Cole Cengage Learning.

[2] Hogg, R. V., McKean, J. W., & Craig, A. T. (2012). Introduction to Mathematical Statistics (7th ed.). Boston: Pearson.

[3] Hogg, R. V., & Tanis, E. A. (1983). Probability and Statistical Inference (2nd ed.). New York, NY: Macmillan Publishing.

[4] Papoulis, A., & Pillai, S. U. (2002). Probability, Random Variables, and Stochastic Processes (4th ed.). Boston, Mass.: McGraw-Hill.

[5] Sarguta, R. J. (2012). On the Construction of Mixed Poisson Distributions (Master's thesis, University of Nairobi, 2012). Nairobi, Kenya: University of Nairobi

[6] Weiss, N. A. (2006). A Course in Probability (1st ed.). Reading, MA: Addison-Wesley.

VITA

MARK PFISTER

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|---|---|
| Education: | B.S. Mathematics, University of Kentucky,<br>Lexington, Kentucky 2013<br>M.S. Mathematical Sciences, East Tennessee State University<br>Johnson City, Tennessee 2018 |
| Professional Experience: | Graduate Assistant, East Tennessee State University<br>Johnson City, Tennessee, 2014–2016 |