The 2-Domination Number of a Caterpillar

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The 2-Domination Number of a Caterpillar

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In partial fulfillment

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Master of Science in Mathematical Sciences

by

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ABSTRACT

The 2-Domination Number of a Caterpillar

by

Presley Ugochukwu Chukwukere

A set $D$ of vertices in a graph $G$ is a 2-dominating set of $G$ if every vertex in $V - D$ has at least two neighbors in $D$. The 2-domination number of a graph $G$, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of $G$. In this thesis, we discuss the 2-domination number of a special family of trees, called caterpillars. A caterpillar is a graph denoted by $P_k(x_1, x_2, ..., x_k)$, where $x_i$ is the number of leaves attached to the $i^{th}$ vertex of the path $P_k$. First, we present the 2-domination number of some classes of caterpillars. Second, we consider several types of complete caterpillars. Finally, we consider classification of caterpillars with respect to their spine length and 2-domination number.
DEDICATION

With so much love, I would like to dedicate this thesis to my late father, Elvis Ephraim Idu, in appreciation for his love, encouragement and support during my undergraduate and to my lovely family for their love and patience when I was out working on my thesis.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>2</td>
</tr>
<tr>
<td>DEDICATION</td>
<td>4</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>5</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>8</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>9</td>
</tr>
<tr>
<td>1 INTRODUCTION</td>
<td>10</td>
</tr>
<tr>
<td>2 LITERATURE SURVEY</td>
<td>13</td>
</tr>
<tr>
<td>2.1 2-domination</td>
<td>13</td>
</tr>
<tr>
<td>3 RESULTS</td>
<td>16</td>
</tr>
<tr>
<td>3.1 2-domination of Paths</td>
<td>16</td>
</tr>
<tr>
<td>3.2 Caterpillar</td>
<td>17</td>
</tr>
<tr>
<td>3.3 Preliminary Results</td>
<td>18</td>
</tr>
<tr>
<td>3.4 Complete Caterpillars</td>
<td>19</td>
</tr>
<tr>
<td>3.5 Upper Bounds for all Complete Caterpillars</td>
<td>27</td>
</tr>
<tr>
<td>3.6 Upper Bounds for all General Caterpillars</td>
<td>29</td>
</tr>
<tr>
<td>3.7 Caterpillars of Small Length</td>
<td>30</td>
</tr>
<tr>
<td>4 CONCLUDING REMARKS</td>
<td>33</td>
</tr>
<tr>
<td>4.1 Open Problems</td>
<td>33</td>
</tr>
</tbody>
</table>
LIST OF TABLES

1  Caterpillars of length 2 ........................................ 31
2  Caterpillars of length 3 ........................................ 31
3  Caterpillars of length 4 ........................................ 31
4  Caterpillars of length 5 ........................................ 32
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Graph $G_1$</td>
</tr>
<tr>
<td>2</td>
<td>Graph $G_2 = G_1 - v$: removal of vertex $v$</td>
</tr>
<tr>
<td>3</td>
<td>2-domination of graph $G_3$</td>
</tr>
<tr>
<td>4</td>
<td>$T'$ is the subdivision of $T$</td>
</tr>
<tr>
<td>5</td>
<td>Examples of 2-domination of paths</td>
</tr>
<tr>
<td>6</td>
<td>A caterpillar $P_4(6, 1, 4, 3)$</td>
</tr>
<tr>
<td>7</td>
<td>A caterpillar $P_7(3, 0, 2, 0, 0, 0, 4)$</td>
</tr>
<tr>
<td>8</td>
<td>A caterpillar $P_7(3, 0, 0, 3, 0, 0, 3)$</td>
</tr>
<tr>
<td>9</td>
<td>A complete caterpillar $P_5(2, 3, 2, 2, 3)$</td>
</tr>
<tr>
<td>10</td>
<td>Complete caterpillars where $x_i = 1$</td>
</tr>
<tr>
<td>11</td>
<td>A complete caterpillar $P_6(1, 2, 1, 1, 3, 1)$</td>
</tr>
<tr>
<td>12</td>
<td>A complete caterpillar $P_{11}(1, 1, 1, 2, 1, 1, 2, 1, 1, 1, 1)$</td>
</tr>
<tr>
<td>13</td>
<td>A complete caterpillar $P_7(1, 2, 1, 1, 1, 3, 1)$</td>
</tr>
<tr>
<td>14</td>
<td>A caterpillar $P_{19}(1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1)$</td>
</tr>
<tr>
<td>15</td>
<td>A complete caterpillar $P_{15}(1, 1, 3, 1, 3, 1, 1, 3, 1, 3, 1, 2, 1, 1)$</td>
</tr>
<tr>
<td>16</td>
<td>A complete caterpillar $P_{14}(1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1)$</td>
</tr>
<tr>
<td>17</td>
<td>The caterpillar $P_{16}(1, 1, 1, 0, 0, 0, 2, 2, 0, 0, 0, 1, 2, 0, 1, 1, 1)$</td>
</tr>
</tbody>
</table>
1 INTRODUCTION

First we need to define some terminology and notation for the purpose of this thesis. Let $G$ be a finite, simple, and undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order of $G$, denoted $|V(G)| = n$, is the number of vertices in $G$. The size of $G$, denoted $|E(G)| = m$, is the number of edges in $G$. For any two vertices $x, y \in V(G)$, $x$ and $y$ are adjacent if the edge $xy \in E(G)$. The open neighborhood of $v$ in $V$ is the set $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood of $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The open neighborhood of a set $D \subseteq V$ is the set $N(D) = \bigcup_{v \in D} N(v)$, and the closed neighborhood of a set $D$ is the set $N[D] = N(D) \cup D$. The degree of $v$ is the cardinality of the open neighborhood of $v$, or $\text{deg}_G(v) = |N(v)|$. A vertex with exactly one neighbor is called a leaf and its neighbor is a support vertex. A support vertex with two or more leaf neighbors is called a strong support vertex [2]. The independence number of $G$, denoted $\beta(G)$, is the cardinality of the largest independent set of vertices in $G$. A path $P_k$ is a graph of order $k$ and size $k - 1$ with vertices denoted $v_1, v_2, ..., v_k$ and edges $v_iv_{i+1}$ for $i = 1, 2, ..., k - 1$. A subgraph $H$ of a graph $G$ is a graph contained in $G$, i.e., $V(H) \subseteq V(G), E(H) \subseteq E(G)$.

A dominating set of a graph $G$ is a nonempty subset $D$ of the vertex set $V$ such that for each $u \in V - D$, there exists a $v \in D$ adjacent to $u$. Equivalently, a subset $D$ of $V$ is a dominating set if for each $v \in V$, $|N[v] \cap D| \geq 1$ [4]. A dominating set having the smallest cardinality among all dominating sets in a given graph is called a minimum dominating set. The cardinality of a minimum dominating set in graph $G$ is called the domination number of $G$ and is denoted $\gamma(G)$. 

10
In [9], Fink and Jacobson introduced the concept of \(k\)-domination, of which 2-domination is a special case. It was shown in [9] that the dominating property of a minimum dominating set can be destroyed by removing at most two edges or vertices from the graph. In most cases, the removal of only one edge or vertex from a graph will leave some vertices undominated by what had been a minimum dominating set. For example, in Figure 1, a minimum dominating set \(D\) of graph \(G_1\) is \(D = \{v, c, u, d\}\). Figure 2 shows that the removal of vertex \(v\) in \(G_1\) will leave vertices \(a\) and \(b\) undominated in \(G_2\) by what had been a minimum dominating set in \(G_1\).

![Figure 1: Graph \(G_1\)](image)

![Figure 2: Graph \(G_2 = G_1 - v\): removal of vertex \(v\)](image)

As a result of this, the 2-domination number was introduced in [9] and mentioned in [3, 12]. For a graph \(G\), if \(D\) is a subset of \(V\) and \(u \in V - D\) is adjacent to at least two members of \(D\), we say that \(u\) is 2-dominated by \(D\). If every vertex in \(V - D\) is 2-dominated by \(D\), then \(D\) is called a 2-dominating set. Among all 2-dominating sets of the graph \(G\), if \(D\) has the smallest cardinality then \(D\) is a minimum 2-dominating set.
and its cardinality is the \textit{2-domination number} of $G$, denoted $\gamma_2(G)$. An example of 2-domination is given in Figure 3. $D = \{a, b, c, d, e, f, g, h, i, j, u, t\}$ is a 2-dominating set of minimum cardinality, so $\gamma_2(G_3) = 12$.

![Figure 3: 2-domination of graph $G_3$](image)

With this in mind, the removal of only one vertex or edge from a graph will leave the vertices still dominated.

A \textit{caterpillar} is a tree with the property that the removal of its leaves and incident edges results in a path, which we call the \textit{spine} of the caterpillar. We say a caterpillar is \textit{complete} if every vertex on the spine of the caterpillar has at least one leaf.

In this thesis, we study the 2-domination of a caterpillar. In Section 2, we will give a literature survey over some related work which is relevant to this thesis. In Section 3, we will discuss 2-domination of paths and by considering several simple cases of caterpillars with the number of leaves attached to the spine of the caterpillar. We will also consider upper bounds for all complete and general caterpillars. It is actually simple to figure out the 2-domination number of some classes of caterpillars, but making a generalization is a little more difficult. Thus we close with an open problem.
2 LITERATURE SURVEY

In this section, we review some background results to this thesis. These results present the motivation and origin of 2-domination. First, we review the basic results regarding this type of domination and then proceed with a simple observation of Fink and Jacobson.

2.1 2-domination

The concept of domination in graphs was known earlier, but Ore [13] was the first to use the term domination, who noticed that for every graph $G$, the relation $\gamma(G) \leq \beta(G)$ holds. Since the origin of Ore’s initial introduction of domination, a large amount of work has been done with dominating sets and the domination number. The reader is referred to [5, 6, 11]. In [9], 2-domination is motivated by the following theorem which says that the dominating property of a minimum dominating set can be destroyed by the removal of only one or two edges (or vertices) from the graph $G$.

Theorem 2.1. [9, 11] If $D$ is a minimum dominating set in a non-empty graph $G$, then at least one vertex in $V - D$ is dominated by at most two members of $D$.

As mentioned earlier, given any minimum dominating set $D$ of $G$, one can remove two edges from $G$ such that $D$ is no longer a dominating set of $G$. As a result of this process, in [9], a greater degree of assurance is introduced via 2-domination so that the removal of at most two edges or vertices from the graph $G$ will still retain the dominating property.

In [9, 11], a vertex $v \in V - D$ is $k$-dominated if it is dominated by at least $k$ vertices in $D$, that is $|N(v) \cap D| \geq k$. If every vertex in $V - D$ is $k$-dominated, then $D$ is
called a \textit{k-dominating set}. The minimum cardinality of a \textit{k}-dominating set is called the \textit{k-domination number} of \(G\), denoted \(\gamma_k(G)\).

**Proposition 2.2.** [9, 11] A \(k\)-dominating set \(D\) is minimal if and only if for every vertex \(v \in D\), either, (1) \(|N(v) \cap D| < k\) or (2) there exists a vertex \(u \in V - D\) such that \(|N(v) \cap D| = k\) and \(u \in N(v)\).

It was also noted in [9] that every \(k\)-dominating set is a dominating set and thus, for every graph \(G\) we have \(\gamma(G) \leq \gamma_k(G)\) for each \(k \geq 1\). Furthermore, if \(1 \leq j \leq k\), then every \(k\)-dominating set in \(G\) is also a \(j\)-dominating set and hence \(\gamma_j(G) \leq \gamma_k(G)\).

In [9], a lower bound for \(\gamma_k\) involving only the number of vertices and the number of edges was obtained. This leads to an extremal result for the 2-domination number of a tree.

**Theorem 2.3.** [9, 11] If \(G\) has \(n\) vertices and \(m\) edges, then \(\gamma_k(G) \geq n - (m/k)\).

The subdivision of an edge \(e\) with endpoints \(\{u, v\}\) yields a graph containing one new vertex \(w\) and with an edge set replacing \(e\) by two new edges, \(uw\) and \(wv\).

**Definition 2.4.** [14] A \textit{subdivision} of a graph \(G = (V, E)\) is a graph where each edge is subdivided exactly once in \(G\).

The path \(P_n\) can be subdivided into another path \(P_m\) by subdividing each edge of \(P_n\) exactly once, that is, \(\text{subdiv}(P_n) = P_m\) where \(m = 2n - 1\) is odd. For example, if \(n = 3\), then \(m = 2(3) - 1 = 5\). Thus, \(T' = P_5\) is the subdivision graph of \(T = P_3\), i.e., \(\text{subdiv}(P_3) = P_5\) as seen in Figure 4.

The edges \(uw\) and \(wv\) have been subdivided into four edges, \(ux, xw, wy,\) and \(yv\).
A tree is an undirected graph in which any two vertices are connected by exactly one simple path. Alternatively, a tree can be defined as a connected graph without a cycle subgraph. A caterpillar is a special type of tree. Theorem 2.3 yields the following bounds on the 2-domination number of a tree $T$ of order $n$.

Corollary 2.5. [9, 11] If $T$ is a tree with $n \geq 2$ vertices, then $\gamma_2(T) \geq \frac{n+1}{2}$.

Corollary 2.6. [9, 11] If $T$ is a tree with $n \geq 2$ vertices, then $\gamma_2(T) = \frac{n+1}{2}$ if and only if $T$ is a subdivision graph of a tree $T'$.

An important theorem involving the number of vertices that are of degree one was shown in [9].

Theorem 2.7. [7, 8] For a tree $T$, $\gamma_2(T) \leq \frac{n+\ell}{2}$, where $\ell$ refers to the number of vertices that are of degree one and $n$ is number vertices of $T$.

The vertices of degree one are included in any 2-dominating set $D$. 

Figure 4: $T'$ is the subdivision of $T$
3 RESULTS

3.1 2-domination of Paths

The path graph $P_n$ is a tree with two end vertices of degree 1, and the other $n-2$ vertices of degree 2. Examples are given in Figure 5.

$n = 3$, $\{1,3\}$ is a 2-dominating set and $\gamma_2(P_3) = 2$.

$n = 4$, $\{1,3,4\}$ is a 2-dominating set and $\gamma_2(P_4) = 3$.

$n = 5$, $\{1,3,5\}$ is a 2-dominating set and $\gamma_2(P_5) = 3$.

$n = 6$, $\{1,3,5,6\}$ is a 2-dominating set and $\gamma_2(P_6) = 4$.

Figure 5: Examples of 2-domination of paths

From the above examples, we have a pattern for odd and even vertices for any path graph, $P_n$. When $n$ is even, the 2-domination number of a path graph $P_n$ is given by $\gamma_2(P_n) = \frac{n+2}{2}$. When $n$ is odd, the 2-domination number of a path graph $P_n$ is given by $\gamma_2(P_n) = \frac{n+1}{2}$. From the patterns above, we have the following theorem.

Theorem 3.1. For a path $P_n$, the 2-domination number is given by $\gamma_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. Let $V$ be the vertex set and $D$ be a minimum 2-dominating set of $P_n$. For $n$ odd, the path $P_n$ is a subdivision graph of $P_{\frac{n+1}{2}}$. Thus, by Corollary 2.6, $\gamma_2(P_n) = \frac{n+1}{2}$.
For $n$ even, we show that $\gamma_2(P_n) = \frac{n+2}{2}$. First, we show that $\gamma_2(P_n) \geq \frac{n+2}{2}$. By Corollary 2.5 and 2.6, $\gamma_2(P_n) > \frac{n+1}{2}$, which implies that $\gamma_2(P_n) \geq \frac{n+2}{2}$.

Next, we show that $\gamma_2(P_n) \leq \frac{n+2}{2}$. Consider a 2-dominating set of a path $P_n$, where $D = \{v_1, v_3, ..., v_{n-1}, v_n\}$. Every vertex $u \in V - D - \{v_n\}$ is at least 2-dominated by two vertices of $D$. Since $v_n \in D$, then $\gamma_2(P_n) \leq |D| = \frac{n}{2} + 1 = \frac{n+2}{2}$. Hence, combining the results, we have $\gamma_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

3.2 Caterpillar

Recall, a caterpillar is a graph which can be obtained from the path on $k$ vertices by appending $x_i$ pendant vertices to the the $i^{th}$ vertex of the path, $P_k$. The caterpillar with parameters $k, x_1, ..., x_k$, where $x_1, x_k \neq 0$, will be denoted $P_k(x_1, x_2, ..., x_k)$ as in [1].

![Figure 6: A caterpillar $P_4(6, 1, 4, 3)$](image)

Note, this is a tree with the property that the removal of its leaves and incident edges results in a path $P_k$ called the spine of the caterpillar. Let $\ell$ denote the number of leaves, i.e., $\ell = \sum_{i=1}^{k} x_i$. We say a caterpillar is complete if every vertex on the spine of the caterpillar is adjacent to at least one leaf.
3.3 Preliminary Results

Let \( P_k(x_1, x_2, \ldots, x_k) \) be a caterpillar. We first consider the case of caterpillars where \( x_i \neq 1 \) for \( 1 \leq i \leq k \). An example is given in Figure 7.

Figure 7: A caterpillar \( P_7(3, 0, 2, 0, 0, 0, 4) \)

Note, the 2-domination number of the above caterpillar is

\[
\gamma_2(P_7(3, 0, 2, 0, 0, 0, 4)) = \ell + 3.
\]

First, we dissect our caterpillar into complete caterpillars and paths of maximal length. We denote the complete caterpillars with \( x_i > 1 \) as \( C_i \) and the paths as \( P_{k_i} \). From the above example in Figure 7, \( C_1 = P_1(3) \), \( C_2 = P_1(2) \), \( C_3 = P_1(4) \), \( k_1 = 1 \) and \( k_2 = 3 \).

**Proposition 3.2.** For a caterpillar, \( P_{k_i}(x_1, \ldots, x_k) \), where \( x_i \neq 1 \) for \( 1 \leq i \leq k \), then

\[
\gamma_2(P_{k_i}(x_1, \ldots, x_k)) \leq \ell + \sum_{i=1}^{r} \left\lceil \frac{k_i+1}{2} \right\rceil.
\]

**Proof.** Let \( D \) be a minimum 2-dominating set. Let \( k_i \) be the order of the path \( P_{k_i} \), where \( P_{k_i} \) is the set of vertices with \( x_i = 0 \) on the spine for \( i = 1, \ldots, r \). Since the leaves are in \( D \), we only need to 2-dominate each of the paths \( P_{k_i} \). Thus, \( D \) is the union of the leaves and a 2-dominating set for each path \( P_{k_i} \). Hence, \( |D| \leq \ell + \sum_{i=1}^{r} \left\lceil \frac{k_i+1}{2} \right\rceil \). \( \square \)

Thus, from Figure 7, we have

\[
\gamma_2(P_7(3, 0, 2, 0, 0, 0, 4)) = (3 + 2 + 4) + \left\lceil \frac{k_1+1}{2} \right\rceil + \left\lceil \frac{k_2+1}{2} \right\rceil
\]

\[
= \ell + \left\lceil \frac{1+1}{2} \right\rceil + \left\lceil \frac{3+1}{2} \right\rceil
\]

\[
= \ell + 3.
\]
This is an example where equality holds, but below is another example where equality fails.

![Figure 8: A caterpillar $P_7(3,0,0,3,0,0,3)$.](image)

From Figure 8, $C_1 = P_1(3)$, $C_2 = P_1(3)$, $C_3 = P_1(3)$, $k_1 = 2$ and $k_2 = 2$. Thus,

$$
\gamma_2(P_7(3,0,0,3,0,0,3)) < (3 + 3 + 3) + \left\lceil \frac{k_1 + 1}{2} \right\rceil + \left\lceil \frac{k_2 + 1}{2} \right\rceil
$$

$$
< \ell + \left\lceil \frac{2 + 1}{2} \right\rceil + \left\lceil \frac{2 + 1}{2} \right\rceil
$$

$$
< \ell + 4.
$$

### 3.4 Complete Caterpillars

Recall, a caterpillar, $P_k(x_1, x_2, ..., x_k)$, is complete if the $x_i > 0$ for $1 \leq i \leq k$. Note, since each leaf has degree one, each leaf must be in a 2-dominating set. First, let us consider the case of complete caterpillars with $x_i > 1$ for $1 \leq i \leq k$. An example of a complete caterpillar with $x_i > 1$ for $1 \leq i \leq k$ is given in Figure 8.

![Figure 9: A complete caterpillar $P_5(2,3,2,2,3)$.](image)

Proposition 3.3. For a complete caterpillar, $P_k(x_1, x_2, ..., x_k)$, where $x_i > 1$ for $1 \leq i \leq k$, then $\gamma_2(P_k(x_1, x_2, ..., x_k)) = \ell$. 

19
Proof. Let $D$ be a minimum 2-dominating set of $P_k(x_1, x_2, \ldots, x_k)$, where $x_k > 1$ for $1 \leq i \leq k$.

First, we show that $|D| \leq \ell$. Suppose $D$ is the set of leaves. Then each vertex on the spine, say $v_i$, has $x_i > 1$ neighbors in $D$. So, $D$ is a 2-dominating set. Thus $|D| \leq \ell$.

Next we show that $|D| \geq \ell$. Suppose to the contrary that $|D| < \ell$. Then, at least one leaf $v \notin D$. That is, $v$ has only one neighbor. So, $D$ is not a 2-dominating set. Thus, $|D| \geq \ell$. Hence, $|D| = \ell$ and we are done.

The next class of caterpillars that we consider will be complete caterpillars where $x_i = 1$ for $1 \leq i \leq k$, that is, having only one leaf attached to each vertex on the spine of the caterpillar. Examples are given in Figure 9.

![Figure 9: Complete caterpillars where $x_i = 1$.](image1)

Observe that the 2-domination number of the above caterpillars is the sum of the number of leaves and the domination number of path.

We need the following lemma to prove our result.

Lemma 3.4. [10] For $k \geq 3$, $\gamma_k = \lceil \frac{k}{3} \rceil$.
Now, we are ready to prove Proposition 3.5.

Proposition 3.5. For a complete caterpillar, $P_k(x_1, x_2, ..., x_k)$, where $x_i = 1$ for $1 \leq i \leq k$, then $\gamma_2(P_k(x_1, x_2, ..., x_k)) = \ell + \left\lceil \frac{k}{3} \right\rceil$.

Proof. Let $D$ be a minimum 2-dominating set. Since the leaves are of degree one, each leaf must be in $D$. The set of leaves dominate each vertex on the path exactly once. Thus, a minimum dominating set of the path $P_k$ unioned with the set of leaves $\ell$ is a minimum 2-dominating set of the caterpillar. Hence, $|D| = \ell + \left\lceil \frac{k}{3} \right\rceil$ and we are done.

Theorem 3.6. For a caterpillar, $P_k(x_1, x_2, ..., x_k)$, where $x_i \geq 1$ for $1 \leq i \leq k$, we have $\gamma_2(P_k(x_1, x_2, ..., x_k)) \leq \ell + \gamma(P_k) = \ell + \left\lceil \frac{k}{3} \right\rceil$.

Proof. Let $V$ be the vertex set and $D$ be a minimum 2-dominating set of $P_k(x_1, x_2, ..., x_k)$, where $x_i \geq 1$ for all positive integers $i$. Since the leaves are of degree one, then each leaf must be in $D$. Since $x_i \geq 1$ for all $i$, a 1-dominating set of the spine $P_k$ is sufficient to 2-dominate the caterpillar.

Our next result of interest gives both a descriptive and a constructive family of caterpillars denoted $\mathcal{T}$.

Let $\mathcal{T}$ be a family of caterpillars $P_{k_1}(x_1, ..., x_k)$ satisfying the following conditions:

- $x_1 = x_k = 1$ and
- if $x_i > 1$, then $x_{i+1} = 1$, for $i = 2, ..., k - 1$.

Complete caterpillars satisfying the above conditions for the family of caterpillars, $\mathcal{T}$, can be written in the form $C_1 - v_1 - C_2 - v_2 - C_3 - v_3 - ... - v_{r-1} - C_r$, where
$C_j = P_{k_j}(1, 1, \ldots, 1)$, $v_j$ is the vertex on the spine with more than one leaf and $|C_j|$ is the length of $P_{k_j}$ with $x_i = 1$ for $1 \leq i \leq k_j$. An example of a caterpillar satisfying the given conditions is given in Figure 10.

![Figure 11: A complete caterpillar $P_6(1, 2, 1, 3, 1)$.](image)

In Figure 11, the complete caterpillar $P_6(1, 2, 1, 3, 1)$ is of the form $C_1 - v_1 - C_2 - v_2 - C_3$ where $|C_1| = 1$, $|C_2| = 2$, $|C_3| = 1$, and $r = 3$. Note that $x_1 = x_k = 1$ and every $x_i > 1$ is followed by $x_{i+1} = 1$. The 2-domination number, $\gamma_2 = \ell + 2 = \ell + \left\lceil \frac{k}{3} \right\rceil$. Note, there are some cases when equality fails as well, i.e., $\gamma_2(T) < \ell + \left\lceil \frac{k}{3} \right\rceil$. What matters is the how much the vertex $v_j$ with $x_j > 1$ for $1 \leq j \leq r$ is “helping” both sides. An example where equality fails is given in Figure 12.

![Figure 12: A complete caterpillar $P_{11}(1, 1, 1, 2, 1, 1, 1, 1, 1, 1)$.](image)

In Figure 12, $\gamma_2 = \ell + 3 < \ell + \left\lceil \frac{k}{3} \right\rceil = \ell + 4$. We can dissect the complete caterpillar into $C_j = P_{k_j}(1, 1, \ldots, 1)$ and $v_j$ which is the vertex on the spine with $x_j > 1$ for $1 \leq j \leq k$. We have the following upper bounds since equality fails in some families of caterpillars $\mathcal{T}$.

**Proposition 3.7.** For a complete caterpillar in $\mathcal{T}$,

$$\gamma_2 \leq \ell + \sum_{j=1}^{r} \left\lfloor \frac{k_j}{3} \right\rfloor.$$  \hspace{1cm} (1)
Proof. Let $D$ be a minimum 2-dominating set and $k_j$ be the length of the spine with $x_i = 1$ for $1 \leq k_j \leq r$, $1 \leq j \leq r$. Since all leaves $\ell$ are of degree one, the leaves are in $D$, so $|D| = \ell = \sum_{i=1}^{r} x_i$. Next, we dominate the spine of each of the sub-caterpillars $k_j$ with $x_i = 1$ and thus, $\gamma_2 = |D| \leq \ell + \sum_{j=1}^{r} \left\lceil \frac{k_j}{3} \right\rceil$.

Proposition 3.8. For a complete caterpillar in $T$,

$$\gamma_2 \leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil.$$  \hfill (2)

Proof. Let $D$ be a minimum 2-dominating. Let $k_j$ be the length of the spine with $x_i = 1$ and $v_j$ the vertex with $x_j > 1$ for $1 \leq i \leq r$. First, all the leaves are in $D$ and, which 2-dominates each of the interior vertices $v_j$ with $x_j > 1$ for $1 \leq i \leq r$. Thus, we are dominating a path of length smaller than $k_1$ and $k_r$ by 1 and by 2 in the middle. Hence, $|D| \leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil$.\hfill \qed

We consider some cases to show that sometimes (1) is better than (2) and vice versa. We have the following:

Case 1: $\gamma_2 = \text{Upper Bound (1)} = \text{Upper Bound (2)}$.

Let us consider the caterpillar $P_7(1, 2, 1, 1, 1, 3, 1)$ where $\gamma_2 = \ell + 3$ as seen in Figure 13.

![Figure 13: A complete caterpillar $P_7(1, 2, 1, 1, 1, 3, 1)$](image)
For Upper Bound (1):

\[
\gamma_2(P_7(1, 2, 1, 1, 1, 3, 1)) \leq \ell + \sum_{i=1}^{r} \left\lceil \frac{k_i}{3} \right\rceil
\]

\[
= \ell + \left\lceil \frac{k_1}{3} \right\rceil + \left\lceil \frac{k_2}{3} \right\rceil + \left\lceil \frac{k_3}{3} \right\rceil
\]

\[
= \ell + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{3}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil
\]

\[
= \ell + 1 + 1 + 1
\]

\[
= \ell + 3.
\]

For Upper Bound (2):

\[
\gamma_2 \leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil
\]

\[
= \ell + \sum_{j=1}^{2} |v_j| + \sum_{j=2}^{2} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil
\]

\[
= \ell + 2(1) + \left\lceil \frac{3 - 2}{3} \right\rceil + \left\lceil \frac{1 - 1}{3} \right\rceil + \left\lceil \frac{1 - 1}{3} \right\rceil
\]

\[
= \ell + 2 + 1 + 0 + 0
\]

\[
= \ell + 3.
\]

Case 2: \( \gamma_2 < \text{Upper Bound (2)} < \text{Upper Bound (1)}. \)

Let us consider the caterpillar \( P_{19}(1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1) \) where \( \gamma_2 = \ell + 6 \) as seen in Figure 14.

\[
\begin{array}{c}
\ldots
\end{array}
\]

Figure 14: A caterpillar \( P_{19}(1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1) \).
For Upper Bound (1):

$$\gamma_2 \leq \ell + \sum_{j=1}^{r} \left\lceil \frac{k_j}{3} \right\rceil$$

$$= \ell + \left\lceil \frac{k_1}{3} \right\rceil + \left\lceil \frac{k_2}{3} \right\rceil + \left\lceil \frac{k_3}{3} \right\rceil + \left\lceil \frac{k_4}{3} \right\rceil$$

$$= \ell + \left\lceil \frac{4}{3} \right\rceil + \left\lceil \frac{4}{3} \right\rceil + \left\lceil \frac{4}{3} \right\rceil + \left\lceil \frac{4}{3} \right\rceil$$

$$= \ell + 2 + 2 + 2 + 2$$

$$= \ell + 8.$$

For Upper Bound (2):

$$\gamma_2 \leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil$$

$$= \ell + \sum_{j=1}^{3} |v_j| + \sum_{j=2}^{3} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_3 - 1}{3} \right\rceil$$

$$= \ell + 3(1) + \left\lceil \frac{4 - 2}{3} \right\rceil + \left\lceil \frac{4 - 2}{3} \right\rceil + \left\lceil \frac{4 - 1}{3} \right\rceil + \left\lceil \frac{4 - 1}{3} \right\rceil$$

$$= \ell + 3 + 1 + 1 + 1 + 1$$

$$= \ell + 7.$$

Case 3: $\gamma_2 < \text{Upper Bound (1)} < \text{Upper Bound (2)}.$

Let us consider the caterpillar $P_{15}(1, 1, 3, 1, 3, 1, 1, 1, 3, 1, 3, 1, 2, 1, 1)$ where $\gamma_2 = \ell + 5$ as seen in Figure 15.

![Figure 15: A complete caterpillar $P_{15}(1, 1, 3, 1, 3, 1, 1, 1, 3, 1, 3, 1, 2, 1, 1)$](image-url)
For Upper Bound (1):

\[
\gamma_2 \leq \ell + \sum_{j=1}^{r} \left\lceil \frac{k_j}{3} \right\rceil
\]

\[
= \ell + \left\lceil \frac{k_1}{3} \right\rceil + \left\lceil \frac{k_2}{3} \right\rceil + \left\lceil \frac{k_3}{3} \right\rceil + \left\lceil \frac{k_4}{3} \right\rceil + \left\lceil \frac{k_5}{3} \right\rceil + \left\lceil \frac{k_6}{3} \right\rceil
\]

\[
= \ell + \left\lceil \frac{2}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{3}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{2}{3} \right\rceil
\]

\[
= \ell + 1 + 1 + 1 + 1 + 1
\]

\[
= \ell + 6.
\]

For Upper Bound (2):

\[
\gamma_2 \leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil
\]

\[
= \ell + \sum_{j=1}^{5} |v_j| + \sum_{j=2}^{5} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_6 - 1}{3} \right\rceil
\]

\[
= \ell + 5(1) + \left\lceil \frac{1 - 2}{3} \right\rceil + \left\lceil \frac{3 - 2}{3} \right\rceil + \left\lceil \frac{1 - 2}{3} \right\rceil + \left\lceil \frac{1 - 2}{3} \right\rceil
\]

\[
+ \left\lceil \frac{2 - 1}{3} \right\rceil + \left\lceil \frac{2 - 1}{3} \right\rceil
\]

\[
= \ell + 5 + \left\lceil \frac{-1}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{-1}{3} \right\rceil + \left\lceil \frac{-1}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil
\]

\[
= \ell + 5 + 0 + 1 + 0 + 0 + 1 + 1
\]

\[
= \ell + 8.
\]

Observation: In Case 1, we found that the Upper Bound (1) equals Upper Bound (2) and both equal \(\gamma_2 = \ell + \left\lceil \frac{k}{3} \right\rceil\).

In Case 2, the Upper Bound (2) is better than the Upper Bound (1). This is an example where equality fails. That is \(\gamma_2 < \ell + \left\lceil \frac{k}{3} \right\rceil\).
In Case 3, the Upper Bound (1) is better than the Upper Bound (2). This is an example where equality fails. Hence, $\gamma_2 < \ell + \left\lceil \frac{k}{3} \right\rceil$.

From the cases above, we found out that in general neither of the upper bounds are better than the other. Thus, both serve as a useful upper bound.

3.5 Upper Bounds for all Complete Caterpillars

In this section, we construct a congruence class of caterpillars in $T$ for which we conjecture that equality holds in Theorem 3.6. That is $\gamma_2 = \ell + \left\lceil \frac{k}{3} \right\rceil$.

We let $C_j = P_{k_j}(1, 1, \ldots, 1)$. Let us consider the following congruence classes for $T$:

Class 1: Let $|C_j| \equiv 0 \pmod{3}$ for $1 \leq j \leq r$.

For $P_7(1, 1, 1, x_4, 1, 1, 1)$, $\gamma_2(P_7(1, 1, 1, x_4, 1, 1, 1)) = \ell + 2 \leq \ell + \left\lceil \frac{7}{3} \right\rceil$. So, equality fails with $x_4 > 1$.

Class 2: Let $|C_j| \equiv 1 \pmod{3}$ for $1 \leq j \leq r$.

For $P_5(1, x_2, 1)$, $\gamma_2(P_5(1, x_2, 1)) = \ell + 1 = \ell + \left\lceil \frac{5}{3} \right\rceil$.

For $P_7(1, x_2, 1, x_4, 1, x_6, 1)$, $\gamma_2(P_7(1, x_2, 1, x_4, 1, x_6, 1)) = \ell + 2 < \ell + \left\lceil \frac{7}{3} \right\rceil$. So, equality holds when $k = 3, 5$, but fails when $k = 7$ where $x_i > 1$ for $i = 2, 4, 6$.

Class 3: Let $|C_j| \equiv 2 \pmod{3}$ for $1 \leq j \leq r$.

For $P_8(1, 1, x_3, 1, 1, x_6, 1, 1)$,

$\gamma_2(P_8(1, 1, x_3, 1, 1, x_6, 1, 1)) = \ell + 3 = \ell + \left\lceil \frac{8}{3} \right\rceil$.

For $P_{11}(1, 1, x_3, 1, 1, x_6, 1, 1, x_9, 1, 1)$,

$\gamma_2(P_{11}(1, 1, x_3, 1, 1, x_6, 1, 1, x_9, 1, 1)) = \ell + 4 = \ell + \left\lceil \frac{11}{3} \right\rceil$. 27
For $P_{11}(1,1,1,1,1,x_6,1,1,1,1,1)$,
\[
\gamma_2(P_{11}(1,1,1,1,1,x_6,1,1,1,1,1)) = \ell + 4 = \ell + \left\lceil \frac{11}{3} \right\rceil.
\]
Thus, equality holds in the cases above.

![Figure 16: A complete caterpillar $P_{14}(1,1,1,2,1,1,2,1,1,1,2,1,1,2,1,1)$.](image)

In Figure 16, $\gamma_2(P_{14}(1,1,x_3,1,1,x_6,1,1,x_9,1,1,x_{12},1,1)) = \ell + 5 = \ell + \left\lceil \frac{14}{3} \right\rceil$,
where $x_i > 1$ for $i = 3, 6, 9, 12$.

We also have other classes of $T$ where equality fails when $C_j = P_{k_j}(1,1,...,1)$ for $1 \leq j \leq k$.

Class 4: Let $|C_j| \not\equiv |C_{j+1}| \pmod{3}$ for $1 \leq j \leq r$.

For $P_4(1,x_2,1,1) \equiv P_4(1,1,x_3,1)$, $\gamma_2(P_4(1,x_2,1,1)) = \ell + 2 = \ell + \left\lceil \frac{4}{3} \right\rceil$.

For $P_6(1,1,1,x_4,1,1) \equiv P_6(1,1,x_3,1,1,1)$,
\[
\gamma_2(P_6(1,1,1,x_4,1,1)) = \ell + 2 = \ell + \left\lceil \frac{6}{3} \right\rceil.
\]

For $P_{13}(1,x_2,1,1,x_5,1,1,1,x_9,1,1,1,1)$,
\[
\gamma_2(P_{13}(1,x_2,1,1,x_5,1,1,1,x_9,1,1,1,1)) = \ell + 5 = \ell + \left\lceil \frac{13}{3} \right\rceil.
\]

For $1 \leq j \leq r$, equality holds when $|C_j| \not\equiv |C_{j+1}| \pmod{3}$ in the examples above but fails for $P_{26}(1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1)$.

We calculate
\[
\gamma_2(P_{26}(1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1)) = \ell + 8
\]
\[
< \ell + \left\lceil \frac{26}{3} \right\rceil
\]
\[
= \ell + 9.
\]
Let $T_j$ be the family of caterpillars in $T$ satisfying the additional condition:

- $|C_j| \equiv 2 \pmod{3}$ for $1 \leq j \leq r$.

Conjecture 3.9. Let $T$ be a caterpillar, $P_k(x_1, x_2, ..., x_k)$, with $\ell$ leaves. If $T \in T_c$ then $\gamma_2(T) = \ell + \lceil \frac{k}{3} \rceil$.

3.6 Upper Bounds for all General Caterpillars

In this section, we obtain an upper bound for the 2-domination number of general caterpillars. We begin by dissecting a caterpillar $C$ into subgraphs of the following types

- $P_k$
- $P_k(1, 1, 1, ..., 1)$
- $P_k(x_1, ..., x_k), x_i > 1$.

Denote the first type subgraph by $P_{kj}$, the second type by $C_{\ell}$, and the third type by $CC_{m}$. An example is given in Figure 17.

![Figure 17: The caterpillar $P_{18}(1, 1, 1, 0, 0, 2, 2, 0, 0, 1, 1, 2, 0, 1, 1, 1)$.](image)

In Figure 17, the above caterpillar can be dissected into $C_1 = P_3(1, 1, 1)$, $C_2 = P_2(1, 1)$, $C_3 = P_3(1, 1, 1)$, $P_{k_1} = P_3$, $P_{k_2} = P_3$, $P_{k_3} = P_1$, $CC_1 = P_2(2, 2)$, and
Thus,

\[ \gamma_2(P_{18}(1, 1, 1, 0, 0, 0, 2, 2, 0, 0, 0, 1, 1, 2, 0, 1, 1, 1)) = \ell + 8 \]

\[ \leq \ell + \sum_{j=1}^{3} \gamma(C_j) + \sum_{j=1}^{3} \gamma_2(P_{k_j}). \]

Theorem 3.10. For a complete caterpillar, \( P_k(x_1, x_2, \ldots, x_k) \), dissected as above we have, \( \gamma_2 \leq \ell + \sum r_j \gamma(C_j) + \sum r_j \gamma_2(P_{k_j}). \)

Proof. Let \( D \) be a minimum 2-dominating set. First, all the leaves are in \( D \). Next, we dissect the caterpillar into \( C'_j \)s, \( P'_{k_j} \)s and \( CC'_j \)s. Then, dominate each of the spines of \( C_j \) and 2-dominate each of the paths \( P_{k_j} \). Hence, \( |D| \leq \ell + \sum_j \gamma(C_j) + \sum_j \gamma_2(P_{k_j}). \)

3.7 Caterpillars of Small Length

Our aim in this section is to determine the 2-domination number of caterpillars with small length. Let us consider a caterpillar, \( P_k(x_1, \ldots, x_k) \) for \( 1 < k \leq 5 \). The following tables below give the 2-domination number of the caterpillars with a spine of small length. In the tables below we consider the caterpillars up to isomorphism and order them lexicographically in \( x_i \). Also, we always choose the isomorphism class so that \( x_1 \geq x_k \). Furthermore, when we don’t specify the value of \( x_i \), we have \( x_i > 1 \) for \( 1 \leq i \leq k \).
\[
P_2(x_1, x_2) & \quad \gamma_2 \\
(1,1) & \quad \ell + 1 \\
(x_1,1) & \quad \ell + 1 \\
(x_1,x_2) & \quad \ell \\
\]

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<thead>
<tr>
<th>(P_2(x_1, x_2))</th>
<th>(\gamma_2)</th>
</tr>
</thead>
<tbody>
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<td>((1,1))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((x_1,1))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((x_1,x_2))</td>
<td>(\ell)</td>
</tr>
</tbody>
</table>

Table 1: Caterpillars of length 2

\[
P_3(x_1, x_2, x_3) & \quad \gamma_2 \\
(1,0,1) & \quad \ell + 1 \\
(1,1,1) & \quad \ell + 1 \\
(1,x_2,1) & \quad \ell + 1 \\
(x_1,0,1) & \quad \ell + 1 \\
(x_1,1,1) & \quad \ell + 1 \\
(x_1,1,x_3) & \quad \ell + 1 \\
(x_1,x_2,1) & \quad \ell + 1 \\
(x_1,x_2,x_3) & \quad \ell \\
\]

<table>
<thead>
<tr>
<th>(P_3(x_1, x_2, x_3))</th>
<th>(\gamma_2)</th>
</tr>
</thead>
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<tr>
<td>((1,0,1))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((1,1,1))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((1,x_2,1))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((x_1,0,1))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((x_1,1,1))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((x_1,1,x_3))</td>
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</tr>
<tr>
<td>((x_1,x_2,1))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((x_1,x_2,x_3))</td>
<td>(\ell)</td>
</tr>
</tbody>
</table>

Table 2: Caterpillars of length 3

\[
P_4(x_1, x_2, x_3, x_4) & \quad \gamma_2 \\
(1,0,0,1) & \quad \ell + 2 \\
(1,0,1,1) & \quad \ell + 2 \\
(1,0,x_3,1) & \quad \ell + 2 \\
(1,1,1,1) & \quad \ell + 2 \\
(1,1,x_3,1) & \quad \ell + 2 \\
(1,x_2,x_3,1) & \quad \ell + 2 \\
(x_1,0,0,1) & \quad \ell + 2 \\
(x_1,0,0,x_4) & \quad \ell + 2 \\
(x_1,0,1,1) & \quad \ell + 2 \\
(x_1,0,x_4) & \quad \ell + 2 \\
(x_1,x_2,x_3,1) & \quad \ell + 1 \\
(x_1,x_2,1,1) & \quad \ell + 1 \\
(x_1,x_2,1,x_4) & \quad \ell + 1 \\
(x_1,x_2,x_3,x_4) & \quad \ell + 1 \\
(x_1,x_2,x_4) & \quad \ell + 1 \\
(x_1,x_2,1,x_3) & \quad \ell + 1 \\
(x_1,x_2,1,x_4) & \quad \ell + 1 \\
(x_1,x_2,x_3,x_4) & \quad \ell \\
\]

<table>
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<th>(\gamma_2)</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
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<td>(\ell + 2)</td>
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<tr>
<td>((1,0,x_3,1))</td>
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</tr>
<tr>
<td>((1,1,x_3,1))</td>
<td>(\ell + 2)</td>
</tr>
<tr>
<td>((1,x_2,x_3,1))</td>
<td>(\ell + 2)</td>
</tr>
<tr>
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</tr>
<tr>
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<td>(\ell + 1)</td>
</tr>
<tr>
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<td>((x_1,x_2,x_4))</td>
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<td>((x_1,x_2,1,x_3))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((x_1,x_2,1,x_4))</td>
<td>(\ell + 1)</td>
</tr>
<tr>
<td>((x_1,x_2,x_3,x_4))</td>
<td>(\ell)</td>
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</table>

Table 3: Caterpillars of length 4
<table>
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<tr>
<th>$P_5(x_1,x_2,x_3,x_4,x_5)$</th>
<th>$\gamma_2$</th>
<th>$P_5(x_1,x_2,x_3,x_4,x_5)$</th>
<th>$\gamma_2$</th>
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<tr>
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<td>$(x_1,0,1,x_4,1)$</td>
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</tr>
<tr>
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<td>$(x_1,0,x_3,1,1)$</td>
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<td>$(x_1,0,x_3,0,1)$</td>
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</tr>
<tr>
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<td>$(x_1,1,0,0,1)$</td>
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<tr>
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<tr>
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<tr>
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<tr>
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<td>$(x_1,x_2,x_3,x_4,1)$</td>
<td>$\ell + 1$</td>
</tr>
</tbody>
</table>

Table 4: Caterpillars of length 5
4 CONCLUDING REMARKS

After many cases were discussed and many different caterpillars were examined, we obtained a general upper bound for the 2-domination number of caterpillars, \( \gamma_2(P_k(x_1, \ldots, x_k)) \leq \ell + \lceil \frac{k}{3} \rceil \). We discussed different cases of caterpillars with varying number of leaves on the spine of caterpillars.

In Section 3.5, we conjectured that which equality holds for the family of caterpillars \( \mathcal{T}_c \).

4.1 Open Problems

In this thesis, we have that for any caterpillar, \( \gamma_2 \leq \ell + \sum_{j=1}^{r} \gamma(C_j) + \sum_{j=1}^{r} \gamma_2(P_{k_j}) \).

- Can we characterize the caterpillar for which the bound is sharp?

- Can we determine the 2-domination number of caterpillars?
REFERENCES


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