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Vector Partitions

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ABSTRACT

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Integer partitions have been studied by many mathematicians over hundreds of years. Many identities exist between integer partitions, such as Euler's discovery that every number has the same amount of partitions into distinct parts as into odd parts. These identities can be proven using methods such as conjugation or generating functions. Over the years, mathematicians have worked to expand partition identities to vectors. In 1963, M. S. Cheema proved that every vector has the same number of partitions into distinct vectors as into vectors with at least one component odd. This parallels Euler's result for integer partitions. The primary purpose of this paper is to use generating functions to prove other vector partition identities that parallel results of integer partitions.

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1 INTRODUCTION

The study of integer partitions has been of interest to many mathematicians since Leibniz asked Bernoulli about the number of partitions of an integer n [6]. Over three hundred years of studies conducted on this subject has led to some significant results. In 1748, Euler made several significant discoveries related to integer partitions. One surprising discovery that Euler made was that the number of partitions of an integer into distinct parts is equal to the number of partitions into odd parts [2]. In 1913, S. Ramanujan and G. H. Hardy contributed to the area of integer partitions by proving several significant results [6]. An example of the modern approach to partitions can be found in *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series* by Ken Ono [5].

More recently, studies have been conducted on vector partitions. Vector partition functions are a natural generalization of integer partition functions that are closely related to plane partitions. Plane partitions are two-dimensional partitions where we consider rows of integers. The rows are left justified, and there is a non-increase along rows and columns. For information on plane partitions, see Chapter 10 of *Integer Partitions* by Andrews and Eriksson [2]. In 1963, Cheema provided a significant contribution to the area of vector partitions in his paper "Vector Partitions and Combinatorial Identities"[3]. In this paper, he proved that the number of partitions of a vector into distinct parts is equal to the number of partitions in which each part has at least one odd component. This parallels Euler's integer partition result regarding distinct parts and odd parts as a recursion relation for vector partitions. Cheema uses generating functions to prove his result. For a more detailed look at techniques involving generating functions, the interested reader is referred to [8].

The following chapters will explore integer and vector partitions. Chapter 2 will introduce integer partitions and generating functions for several types of partitions. Generating functions will be used to prove several known results related to integer partitions. Chapter 3 will extend these concepts to vector partitions. Generating functions for vector partitions will be used to prove several results that parallel the results of Chapter 2. Chapter 4 will discuss possible future directions for the study of vector partitions.

2 INTEGER PARTITIONS

This chapter will introduce integer partitions. Unless otherwise noted, all material from this chapter will reference *Integer Partitions* by Andrews and Eriksson [2].

2.1 Background

A partition is a way of writing a positive integer as a sum of positive integers where the order of the summands does not matter. The partition stays the same regardless of the order of the summands, so we may choose by convention to order the parts from largest to smallest. For $n \in \mathbb{N}$, we define the partition function, denoted $p(n)$, to be the number of partitions of n. Since the empty sum is the only partition of zero, we have $p(0) = 1$.

Example 2.1. *The following are all of the possible partitions of the number four:*

$$4 = 4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Therefore, we have $p(4) = 5$.

Often, we are interested in partitions of a number n that satisfy some condition. We denote the number of such partitions by $p(n \mid [\text{condition}]\ )$.

Example 2.2. *The following are all of the possible partitions of the number five:*

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

So the partitions of 5 with distinct parts are

$$5,\ 4+1,\ 3+2.$$

Therefore, $p(5\ |\textit{distinct parts}) = 3$.

2.2 Generating Functions

A generating function is a power series whose coefficients give a sequence of numbers. Using generating functions, we can keep track of the number of partitions that satisfy some condition. Convergence will not be an issue since each integer n can only be partitioned using numbers from the set $\{1, \ldots, n\}$. Suppose we want to illustrate all possible partitions with distinct elements from $S = \{1, 2, 3\}$. Define $p_S(n) = p(n|$ distinct parts in $S)$. Note, $p_S(n) = 0$ for $n \geq 7$, and by convention $p_S(0) = 1$. The following values can be easily calculated: $p_S(1) = 1,\ p_S(2) = 1,\ p_S(3) = 2,\ p_S(4) = 1,\ p_S(5) = 1,$ and $p_S(6) = 1$. Therefore, we have

$$\begin{aligned}
\sum_{n\geq 0} p_S(n)q^n &= 1 + q^1 + q^2 + 2q^3 + q^4 + q^5 + q^6 + 0q^7 + 0q^8 + \ldots \\
&= 1 + q^1 + q^2 + q^{2+1} + q^3 + q^{3+1} + q^{3+2} + q^{3+2+1} \\
&= (1 + q^1 + q^2 + q^{2+1})(1 + q^3) \\
&= (1 + q^1)(1 + q^2)(1 + q^3) \\
&= \prod_{k=1}^{3}(1 + q^k).
\end{aligned}$$

The coefficient of q^n in the first line is the number of partitions of n into distinct parts from $S = \{1, 2, 3\}$. For instance, since there are two such partitions of the number three, we have the term $2q^3$. Note that the exponents in the second line display all the partitions of this type. Thus, the generating function for these partitions is given by the following:

$$\sum_{n\geq 0} p(n|\text{ distinct parts in } \{1,2,3\}\ )q^n = \prod_{k=1}^{3}(1 + q^k). \tag{1}$$

If we allow distinct parts from $S = \{k_1, \dots, k_r\}$, we can extend the generating function given in (1) to the following:

$$\sum_{n\geq 0} p(n| \text{ distinct parts in } S)q^n = \prod_{i=1}^{r}(1+q^{k_i}) = \prod_{k\in S}(1+q^k). \tag{2}$$

It follows that if $S = \mathbb{N}$, we have

$$\sum_{n\geq 0} p(n| \text{ distinct parts })q^n = \prod_{k=1}^{\infty}(1+q^k). \tag{3}$$

Suppose that we wish to allow parts to repeat up to a certain amount of times. For instance, let $S = \{1, 2\}$ and suppose we allow parts to repeat up to three times. Define $p_S(n; 3) = p(n|$ parts in $S,$ no part repeated more than 3 times$)$. The following values can be easily calcuated: $p_S(0; 3) = 1$, $p_S(1; 3) = 1$, $p_S(2; 3) = 2$, $p_S(3; 3) = 2$, $p_S(4; 3) = 2$, $p_S(5; 3) = 2$, $p_S(6; 3) = 2$, $p_S(7; 3) = 2$, $p_S(8; 3) = 1$, $p_S(9; 3) = 1$, and $p_S(n; 3) = 0$ for $n \geq 10$. Therefore we have

$$\begin{aligned}
\sum_{n\geq 0} p_S(n;3)q^n =& 1 + q^1 + 2q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + 2q^7 + q^8 + q^9 + 0q^{10} + \dots \\
=& 1 + q^2 + q^{2+2} + q^{2+2+2} + q^1 + q^{2+1} + q^{2+2+1} + q^{2+2+2+1} \\
&+ q^{1+1} + q^{2+1+1} + q^{2+2+1+1} + q^{2+2+2+1+1} + q^{1+1+1} \\
&+ q^{2+1+1+1} + q^{2+2+1+1+1} + q^{2+2+2+1+1+1} \\
=& (1 + q^1 + q^{1+1} + q^{1+1+1})(1 + q^2 + q^{2+2} + q^{2+2+2}) \\
=& \prod_{k=1}^{2}(1 + q^k + q^{k+k} + q^{k+k+k}) \\
=& \prod_{k=1}^{2}(1 + q^k + q^{2k} + q^{3k}).
\end{aligned}$$

The exponents of the polynomial beginning on the second line display all partitions with elements from $S = \{1, 2\}$ where the elements are allowed to repeat up to three times, and the coefficient of q^n in the first line is the number of such partitions of n. Therefore, we have the following generating function:

$$\begin{aligned}&\sum_{n\geq 0} p(n| \text{ parts in } \{1{,}2\}, \text{ no part repeated more than 3 times })q^n \\ &= \prod_{k=1}^{2}(1 + q^k + q^{2k} + q^{3k}).\end{aligned} \tag{4}$$

For the set $S = \{k_1, \ldots, k_r\}$ with parts allowed to repeat up to d times, the generating series in (4) is extended to the following:

$$\begin{aligned}&\sum_{n\geq 0} p(n| \text{ parts in } S, \text{ none repeated more than } d \text{ times })q^n \\ &= \prod_{i=1}^{r}(1 + q^{k_i} + q^{2k_i} + \cdots + q^{dk_i}) \\ &= \prod_{k\in S}(1 + q^k + q^{2k} + \cdots + q^{dk}).\end{aligned} \tag{5}$$

It follows that if $S = \mathbb{N}$, we have

$$\begin{aligned}&\sum_{n\geq 0} p(n| \text{ no part repeated more than } d \text{ times })q^n \\ &= \prod_{k=1}^{\infty}(1 + q^k + \cdots + q^{dk}).\end{aligned} \tag{6}$$

By using the formula of a geometric series, we obtain the following generating function:

$$\sum_{n\geq 0} p(n|\text{ parts in } S)q^n = \prod_{i=1}^{r}(1+q^{k_i}+q^{2k_i}+q^{3k_i}+\dots)$$
$$= \prod_{i=1}^{r}\frac{1}{1-q^{k_i}} \tag{7}$$
$$= \prod_{k\in S}\frac{1}{1-q^k}.$$

If we take the generating function given in (7) and let $S = \mathbb{N}$, we obtain the standard generating function as follows:

$$\sum_{n\geq 0} p(n)q^n = \prod_{k=0}^{\infty}\frac{1}{1-q^k}. \tag{8}$$

The preceding generating functions are useful for proving certain results related to integer partitions.

2.3 Integer Partition Identities

When every number has the same amount of integer partitions of one type as another type, we say this is a partition identity. Many partition identities exist. One such identity says that every number has the same amount of integer partitions into distinct parts as into odd parts. This identity was first proved by Leonhard Euler in 1748 [2]. Table 1 illustrates this identity for positive integers up to six. Notice for each positive integer n in this table, there are always the same amount of partitions in each column.

Table 1: Examples illustrating Euler's Identity.

n	Odd Parts	Distinct Parts
1	1	1
2	1+1	2
3	3 1+1+1	3 2+1
4	3+1 1+1+1+1	4 3+1
5	5 3+1+1 1+1+1+1+1	5 4+1 3+2
6	5+1 3+3 3+1+1+1 1+1+1+1+1+1	6 5+1 4+2 3+2+1

Theorem 2.3 (Euler's Partition Identity [2])**.** *The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.*

Proof. Consider the following generating functions which result from the equations given in (3) and (7), respectively:

$$\sum_{n=0}^{\infty} p(n|\text{ parts distinct})q^n = \prod_{k=1}^{\infty}(1+q^k) \ ,$$

$$\sum_{n=0}^{\infty} p(n|\text{ parts all odd})q^n = \prod_{k \text{ odd}} \frac{1}{1-q^k} \ .$$

Then we have

$$\begin{aligned}\prod_{k=1}^{\infty}(1+q^k) =&(1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)(1+q^6)\dots\\ =&(\frac{1-q^2}{1-q})(\frac{1-q^4}{1-q^2})(\frac{1-q^6}{1-q^3})(\frac{1-q^8}{1-q^4})(\frac{1-q^{10}}{1-q^5})(\frac{1-q^{12}}{1-q^6})\dots\\ =&\frac{1}{(1-q)(1-q^3)(1-q^5)\dots}\\ =&\prod_{k \text{ odd}}\frac{1}{1-q^k}.\end{aligned}$$

Therefore, the generating functions are equal. Hence, for every positive integer n, we have $p(n|\text{ parts distinct}) = p(n|\text{ parts all odd})$. $\square$

Euler's Partition Identity can be generalized to give us another partition identity. For any integer $r \geq 2$, every number has the same amount of partitions in which no part is divisible by r as partitions in which there are less than r copies of each part. Notice, if we let $r = 2$, then we have every number has the same amount of partitions in which no part is divisible by two as partitions in which there are less than two copies of each part. This statement is equivalent to Euler's Theorem. Table 2 illustrates this identity for integers up to six and $r = 3$.

Table 2: Examples illustrating Theorem 2.4 for r=3

n	No Part Divisible by 3	Less Than 3 Copies of Each Part
1	1	1
2	2 2+1	2 2+1
3	2+1 1+1+1	3 2+1
4	4 2+2 2+1+1 1+1+1+1	4 3+1 2+2 2+1+1
5	5 4+1 2+2+1 2+1+1+1 1+1+1+1+1	5 4+1 3+2 3+1+1 2+2+1
6	5+1 4+2 4+1+1 2+2+2 2+2+1+1 2+1+1+1+1 1+1+1+1+1+1	6 5+1 4+2 4+1+1 3+3 3+2+1 2+2+1+1

Theorem 2.4. [2] *The number of partitions of n in which no part is divisible by r is equal to the number of partitions of n in which there are less than r copies of each part.*

Proof. The proof is similar to that of Euler's Partition Identity. Consider the following generating functions which result from the equations given in (6) and (7), respectively:

$$\sum_{n=0}^{\infty} p(n| \text{ less than } r \text{ copies of each part})q^n = \prod_{k=1}^{\infty}(1 + q^k + q^{2k} + \cdots + q^{(r-1)k}) \ ,$$

$$\sum_{n=0}^{\infty} p(n|\text{ no part divisible by } r)q^n = \prod_{k \text{ not divisible by } r} \frac{1}{1-q^k}\ .$$

Then for $r \geq 2$ we have that $\prod_{k=1}^{\infty}(1+q^k+q^{2k}+\cdots+q^{(r-1)k})$ is equal to

$$\begin{aligned}
&(1+q+\cdots+q^{r-1})(1+q^2+\cdots+q^{2(r-1)})(1+q^3+\cdots+q^{3(r-1)})\ldots \\
&= \left(\frac{1-q^r}{1-q}\right)\left(\frac{1-q^{2r}}{1-q^2}\right)\left(\frac{1-q^{3r}}{1-q^3}\right)\left(\frac{1-q^{4r}}{1-q^4}\right)\cdots \\
&= \prod_{k \text{ not divisible by } r} \frac{1}{1-q^k}.
\end{aligned}$$

Therefore, the generating functions are equal. Hence, for every positive integer n, we have $p(n|$ no part divisible by $r) = p(n|$ less than r copies of each part). $\square$

Another partition identity gives us that every number has the same amount of partitions into even parts as into parts that are repeated an even number of times. Table 3 illustrates this identity for positive even integers up to six. Note for odd integers, there are no such partitions of either kind. This identity can also be generalized to partitions in which all parts are multiples of a positive integer r and partitions in which all parts are repeated a multiple of r times. Table 4 illustrates this for integers up to nine and $r = 3$. As with even parts, note that for numbers that are not a multiple of 3, there are no such partitions of either kind.

Table 3: Examples Illustrating Theorem 2.5 for $r = 2$

n	All Parts Even	All Parts Repeated an Even Number of Times
2	2	1+1
4	4	2+2
	2+2	1+1+1+1
6	6	3+3
	4+2	2+2+1+1
	2+2+2	1+1+1+1+1+1
8	8	4+4
	6+2	3+3
	4+4	2+2+2+2
	4+2+2	2+2+1+1+1+1
	2+2+2+2	1+1+1+1+1+1+1+1

Table 4: Examples Illustrating Theorem 2.5 for $r = 3$

n	All Parts a Multiple of 3	All Parts Repeated a Multiple of 3 Times
3	3	1+1+1
6	6	2+2+2
	3+3	1+1+1+1+1+1
9	9	3+3+3
	6+3	2+2+2+1+1+1
	3+3+3	1+1+1+1+1+1+1+1+1

Theorem 2.5. [2] *The number of partitions of n in which all parts are a multiple of r is equal to the number of partitions of n in which all parts are repeated a multiple of r times.*

Proof. Consider the following generating functions which result from the equations given in (6) and (7), respectively:

$$\sum_{n=0}^{\infty} p(n|\text{ parts repeated a multiple of } r \text{ times})q^n = \prod_{k=1}^{\infty}(1+q^{rk}+q^{2rk}+q^{3rk}+q^{4rk}+\dots),$$

$$\sum_{n=0}^{\infty} p(n| \text{ all parts a multiple of } r) q^n = \prod_{k \text{ a multiple } r} \frac{1}{1-q^k} \ .$$

Then we have

$$\prod_{k=1}^{\infty}(1 + q^{rk} + q^{2rk} + q^{3rk} + q^{4rk} + \dots) = \prod_{k=1}^{\infty} \frac{1}{1-q^{rk}}$$
$$= \prod_{j \text{ a multiple } r} \frac{1}{1-q^j}.$$

Therefore, the generating functions are equal. Hence, for every positive integer n, we have $p(n|$ parts repeated a multiple of r times$) = p(n|$ all parts a multiple of $r)$. $\square$

In order to prove the next partition identity, we must introduce Ferrers graphs and conjugate partitions. A Ferrers graph is a way of representing a partition of a number. Each part in the partition is represented by a row of dots. The Ferrers graph is arranged so that the parts are ordered from largest to smallest, and the left side of the graph is aligned vertically. For example, Figure 1 shows the Ferrers graph for the partition $11 = 5 + 4 + 2$.

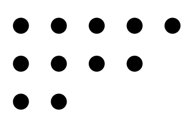

Figure 1: Example of a Ferrers graph.

Suppose we take a Ferrers graph and rearrange it by taking its rows and making them into columns. This process is called conjugation. Figure 2 shows the Ferrers graph from Figure 1 along with its conjugate partition. The resulting partition is

$11 = 3 + 3 + 2 + 2 + 1$. Since conjugation preserves the number of dots in a Ferrers graph, it is useful in proving certain partition identities. If two types of partitions are conjugate, then there is a partition identity between them.

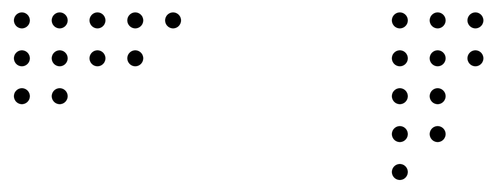

Figure 2: Example of conjugation.

For more information on Ferrers graphs and conjugate partitions, refer to Chapter 3 of *Integer Partitions* by Andrews [2]. The following theorem was proved by MacMahon in the $r = 1$ case [4], and by Andrews for general r [1].

Theorem 2.6. [1] *Let $A_r(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + \cdots + b_s$, where $b_i \geq b_{i+1}$, all odd parts are greater than or equal to $2r+1$, and if $b_i - b_{i+1}$ is odd then $b_i - b_{i+1} \geq 2r+1$. Let $B_r(n)$ be the number of partitions of n into parts which are even or else congruent to $2r+1$ (mod $4r+2$). Then $A_r(n) = B_r(n)$.*

Proof. The partitions which are conjugate to those enumerated by $A_r(n)$ are just those partitions of n in which any part appearing an odd number of times appears at least $2r+1$ times. Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} A_r(n)q^n &= \prod_{k=1}^{\infty}(1+q^{2k}+q^{4k}+\cdots+q^{(2r-2)k}+q^{2rk}+q^{(2r+1)k}+q^{(2r+2)k}+\dots) \\
&= \prod_{k=1}^{\infty}\left(\frac{1-q^{2k(r+1)}}{1-q^{2k}}+\frac{q^{(2r+1)k}}{1-q^k}\right) \\
&= \prod_{k=1}^{\infty}\left(\frac{1-q^{(2r+2)k}}{1-q^{2k}}+\frac{q^{(2r+1)k}+q^{(2r+2)k}}{1-q^{2k}}\right) \\
&= \prod_{k=1}^{\infty}\left(\frac{1+q^{(2r+1)k}}{1-q^{2k}}\right).
\end{aligned}$$

Note by Euler's Theorem, we have

$$\prod_{k=1}^{\infty}(1+q^{(2r+1)k}) = \prod_{k=1}^{\infty}\left(\frac{1}{1-q^{(2r+1)(2k-1)}}\right).$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} A_r(n)q^n &= \prod_{k=1}^{\infty}\left(\frac{1+q^{(2r+1)k}}{1-q^{2k}}\right) \\
&= \prod_{k=1}^{\infty}\left(\frac{1}{1-q^{(2r+1)(2k-1)}}\right)\left(\frac{1}{1-q^{2k}}\right) \\
&= \prod_{k=1}^{\infty}\left(\frac{1}{1-q^{(4r-2)k-(2r+1)}}\right)\left(\frac{1}{1-q^{2k}}\right) \\
&= \sum_{n=0}^{\infty} B_r(n)q^n.
\end{aligned}$$

$\square$

3 VECTOR PARTITIONS

The previous chapter was an introduction to partitions of positive integers. The focus of this chapter will be to extend this concept to vectors of integers. Sage will be used to define functions to verify all results from this chapter [7].

3.1 Background

A vector partition is a way of writing a vector as a sum of other vectors where the order of summands does not matter. Define $\mathbb{N}$ to be the nonnegative integers. Unless otherwise noted, all vectors will be of the form $\mathbf{n}=\begin{bmatrix} n_1 \\ \vdots \\ n_s \end{bmatrix}$, where $n_1, \ldots n_s \in \mathbb{N}$ with at least one nonzero n_i. We define the partition function, denoted $p(\mathbf{n})$, to be the number of partitions of vector $\mathbf{n}$. By convention, for the zero vector, $\mathbf{0}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, we define $p(\mathbf{0}) = 1$. If we are interested in the number of partitions of a vector that satisfy some condition, we denote this by $p(\mathbf{n}|\ [\text{condition}])$.

Example 3.1. *The following are all of the possible partitions of the vector* $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$*:*

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore $p\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = 9$. *Since the only partitions of* $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ *with all components even are* $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, *we have* $p\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \middle| \text{all components even}\right) = 2$.

3.2 Generating Functions for Vector Partitions

Much like we did with integers, we will use generating functions to keep track of the number of partitions of a vector that satisfy some condition. For example, suppose we want to illustrate all possible partitions with distinct vectors from the set $S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Let $\mathbf{n}=\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ and define $p_S(\mathbf{n}) = p(\mathbf{n}|$ distinct vectors in $S)$. By convention $p_S\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 1$. The following values can be calculated by writing out the partitions: $p_S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 1$, $p_S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1$, $p_S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2$, $p_S\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 1$, $p_S\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 1$, $p_S\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = 1$, otherwise $p_S(\mathbf{n}) = 0$. Therefore, we have

$$\begin{aligned}
\sum p_S(\mathbf{n}) x^{n_1} y^{n_2} =& 1 + x^0y^1 + x^1y^0 + 2x^1y^1 + x^1y^2 + x^2y^1 + x^2y^2 \\
=& 1 + x^0y^1 + x^1y^0 + (x^1y^0)(x^0y^1) + x^1y^1 + (x^1y^1)(x^0y^1) \\
& + (x^1y^1)(x^1y^0) + (x^1y^1)(x^1y^0)(x^0y^1) \\
=& \left[1 + x^0y^1 + x^1y^0 + (x^1y^0)(x^0y^1)\right]\left[1 + x^1y^1\right] \\
=& (1 + x^0y^1)(1 + x^1y^0)(1 + x^1y^1).
\end{aligned}$$

Note, the exponents in the polynomial starting on the second line illustrate all possible partitions using distinct vectors from $S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, and the coefficient of $x^{n_1}y^{n_2}$ in the first line gives the number of such partitions of $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$. Thus, the generating function for partitions into distinct vectors from S is given by

$$\sum p(\mathbf{n}| \text{ distinct vectors in } S) x^{n_1} y^{n_2} = \prod_{\mathbf{k} \in S} (1 + x^{k_1} y^{k_2}), \tag{9}$$

where $\mathbf{k}=\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

If we allow our set S to be an arbitrary set of vectors of the form $\mathbf{k}=\begin{bmatrix} k_1 \\ \vdots \\ k_s \end{bmatrix}$, we can extend the generating function given in (9) to the following:

$$\sum p(\mathbf{n}|\text{ distinct vectors in } S)x_1^{n_1}x_2^{n_2}\dots x_s^{n_s} = \prod_{\mathbf{k}\in S}(1+x_1^{k_1}x_2^{k_2}\dots x_s^{k_s}). \tag{10}$$

Now suppose that we wish to allow each vector to repeat up to a certain amount of times. For example, say $S = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}\right\}$, and we wish to allow vectors to repeat up to two times. Define $p_S(\mathbf{n};2) = p(\mathbf{n}|\text{ vectors in } S, \text{ none repeated more than 2 times})$ where $\mathbf{n}=\begin{bmatrix}n_1\\n_2\end{bmatrix}$. By convention, $p_S\left(\begin{bmatrix}0\\0\end{bmatrix};2\right) = 1$. The following values can be calculated by writing out the partitions: $p_S\left(\begin{bmatrix}2\\1\end{bmatrix};2\right) = 1$, $p_S\left(\begin{bmatrix}4\\2\end{bmatrix};2\right) = 2$, $p_S\left(\begin{bmatrix}6\\3\end{bmatrix};2\right) = 1$, $p_S\left(\begin{bmatrix}8\\4\end{bmatrix};2\right) = 2$, $p_S\left(\begin{bmatrix}10\\5\end{bmatrix};2\right) = 1$, $p_S\left(\begin{bmatrix}12\\6\end{bmatrix};2\right) = 1$, $p_S(\mathbf{n};2) = 0$ otherwise. Therefore, we have

$$\begin{aligned}
\sum p_S(\mathbf{n};2)x^{n_1}y^{n_2} =&1 + x^2y^1 + 2x^4y^2 + x^6y^3 + 2x^8y^4 + x^{10}y^5 + x^{12}y^6 \\
=&1 + x^2y^1 + (x^2y^1)(x^2y^1) + x^4y^2 + (x^4y^2)(x^2y^1) \\
&+ (x^4y^2)(x^2y^1)(x^2y^1) + (x^4y^2)(x^4y^2) \\
&+ (x^4y^2)(x^4y^2)(x^2y^1) + (x^4y^2)(x^4y^2)(x^2y^1)(x^2y^1) \\
&\Big[1 + x^2y^1 + (x^2y^1)(x^2y^1)\Big]\Big[1 + x^4y^2 + (x^4y^2)(x^4y^2)\Big].
\end{aligned}$$

Thus, the generating function for partitions into vectors from $S = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}\right\}$ where each vector is allowed to repeat up to two times is given by:

$$\begin{aligned}
&\sum p(\mathbf{n}|\text{ vectors in } S,\ \text{none repeated more than two times })x^{n_1}y^{n_2} \\
&= \prod_{\mathbf{k}\in S}\left[1 + x^{k_1}y^{k_2} + (x^{k_1}y^{k_2})(x^{k_1}y^{k_2})\right] \\
&\prod_{\mathbf{k}\in S}(1 + x^{k_1}y^{k_2} + x^{2k_1}y^{2k_2}),
\end{aligned} \tag{11}$$

where $\mathbf{k}=\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

Suppose we allow vectors to repeat up to d times. If we allow our set S to be an arbitrary set of vectors of the form $\mathbf{k}=\begin{bmatrix} k_1 \\ \vdots \\ k_s \end{bmatrix}$, we can extend the generating function given in (11) to the following:

$$\begin{aligned}
&\sum p(\mathbf{n}|\text{ vectors in } S,\ \text{none repeated more than } d \text{ times })x_1^{n_1}x_2^{n_2}\dots x_s^{n_s} \\
&= \prod_{\mathbf{k}\in S}(1 + x_1^{k_1}x_2^{k_2}\dots x_s^{k_s} + x_1^{2k_1}x_2^{2k_2}\dots x_s^{2k_s} + \dots + x_1^{dk_1}x_2^{dk_2}\dots x_s^{dk_s}).
\end{aligned} \tag{12}$$

Now suppose we allow parts to repeat arbitrarily many times. We can require $|x_1x_2\dots x_s| < 1$ since we only use $x_1, x_2, \dots, x_s$ to keep track of the partitions and not for their values. Then, by using the formula of a geometric series, we obtain the following generating function:

$$\begin{aligned}
&\sum p(\mathbf{n}|\text{ vectors in } S)x_1^{n_1}x_2^{n_2}\dots x_s^{n_s} \\
&= \prod_{\mathbf{k}\in S}(1 + x_1^{k_1}x_2^{k_2}\dots x_s^{k_s} + x_1^{2k_1}x_2^{2k_2}\dots x_s^{2k_s} + x_1^{3k_1}x_2^{3k_2}\dots x_s^{3k_s} + \dots) \\
&= \prod_{\mathbf{k}\in S}[1 + x_1^{k_1}x_2^{k_2}\dots x_s^{k_s} + (x_1^{k_1}x_2^{k_2}\dots x_s^{k_s})^2 + (x_1^{k_1}x_2^{k_2}\dots x_s^{k_s})^3 + \dots] \\
&= \prod_{\mathbf{k}\in S}\frac{1}{1 - x_1^{k_1}x_2^{k_2}\dots x_s^{k_s}}.
\end{aligned} \tag{13}$$

If we take the generating function given in (13) and let S be the set of all vectors of the form $\mathbf{k}=\begin{bmatrix} k_1 \\ \vdots \\ k_s \end{bmatrix}$, we obtain the standard generating function for vector partitions as follows:

$$\sum p(\mathbf{n}) x_1^{n_1} x_2^{n_2} \dots x_s^{n_s} = \prod_{k_i \geq 0} \frac{1}{1 - x_1^{k_1} x_2^{k_2} \dots x_s^{k_s}}. \tag{14}$$

The preceding generating functions are useful for proving certain results related to vector partitions.

3.3 Vector Partition Identities

A vector partition identity exists when every vector has the same amount of partitions of one type as another type. One such identity says that every vector has the same amount of partitions into distinct vectors as into vectors with at least one component odd. This parallels Euler's Theorem for integer partitions and was proven by Cheema[3]. Table 5 gives one example of this identity. The Sage code in Figure 3 and Figure 4 can be used to verify this identity for other vectors of the form $\mathbf{n} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. This code can easily be extended to vectors with more than two components.

```
def atleastoneodd(x,y):
    E=[]
    L=VectorPartitions([x,y])
    for p in L:
        if all(v[0]%2!=0 or v[1]%2!=0 for v in p):
            E.append(p)
    return(E)
```

Figure 3: Sage code for the number of partitions with at least one odd component in each vector.

```
def distinct(x,y):
    L=VectorPartitions([x,y])
    i=0
    for p in L:
        l=list(p)
        if all(l.count(part)<2 for part in l):
            i+=1
    return i
```

Figure 4: Sage code for the number of partitions into distinct parts.

Table 5: Examples illustrating Theorem 3.2.

n	Distinct Vectors	Vectors With At Least One Component Odd
$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$
	$\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$
	$\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}$
	$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$
	$\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$

Theorem 3.2 (Cheema's Theorem [3])**.** *The number of partitions of* $\boldsymbol{n}=\begin{bmatrix}n_1\\ \vdots\\ n_s\end{bmatrix}$ *into vectors with at least one component odd is equal to the number of partitions of* $\boldsymbol{n}$ *into distinct parts.*

Proof. By the generating function given in (10), we have

$$\begin{aligned}\sum p(\mathbf{n}|\text{ distinct vectors })x_1^{n_1}x_2^{n_2}\dots x_s^{n_s} &= \prod_{k_i\geq 0}(1+x_1^{k_1}x_2^{k_2}\dots x_s^{k_s})\\ &= \prod_{k_i\geq 0}\frac{1-(x_1^{k_1}x_2^{k_2}\dots x_s^{k_s})^2}{1-x_1^{k_1}x_2^{k_2}\dots x_s^{k_s}}\\ &= \prod_{k_i\geq 0}\frac{1}{1-x_1^{k_1}x_2^{k_2}\dots x_s^{k_s}},\end{aligned}$$

where at least one k_i is odd. This proves the result, since by the generating function given in (13) we have

$$\sum p(\mathbf{n}| \text{ vectors with at least one component odd}) x_1^{n_1} x_2^{n_2} \dots x_s^{n_s}$$
$$= \prod_{k_i \geq 0} \frac{1}{1 - x_1^{k_1} x_2^{k_2} \dots x_s^{k_s}},$$

where at least one k_i is odd. □

A generalization of Cheema's Theorem gives us another vector partition identity. Every vector has the same amount of partitions in which there are less than r copies of each vector as partitions into vectors in which at least one component is not divisible by r. Note, if $r = 2$, then this is equivalent to Cheema's Theorem. This parallels Theorem 2.4. The Sage code in Figure 5 and Figure 6 can be used to verify this identity for vectors of the form $\mathbf{n} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. It can also be extended to vectors with more than two components. Although, there are too many partitions of each type to list, Table 6 gives some values of these Sage functions.

```
def notdivisiblebyr(x,y,r):

    E=[]

    L=VectorPartitions([x,y])

    for p in L:

        if all(v[0]%r!=0 or v[1]%r!=0 for v in p):

            E.append(p)

    return(E)
```

Figure 5: Sage code for the number of partitions with at least one component of each vector not divisible by r.

```
def lessthanrcopies(x,y,r):

    L=VectorPartitions([x,y])

    i=0

    for p in L:

        l=list(p)

        if all(l.count(part)<r for part in l):

            i+=1

    return i
```

Figure 6: Sage code for the number of partitions in which there are less than r copies of each vector.

Table 6: Examples calculated by Sage illustrating Theorem 3.3 .

$\mathbf{n}$,r	Less Than r Copies	At Least One Component Not Divisible by r
$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $r = 3$	14	14
$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $r = 3$	45	45
$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $r = 4$	27	27
$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$, $r = 4$	90	90

Theorem 3.3. *The number of partitions of* $\boldsymbol{n}$=$\begin{bmatrix} n_1 \\ \vdots \\ n_s \end{bmatrix}$ *into vectors with at least one component not divisible by* r *is equal to the number of partitions of* $\boldsymbol{n}$ *in which there are less than* r *copies of each part.*

Proof. By the generating function given in (12), we have

$$\sum p(\mathbf{n}| \text{ less than } r \text{ copies of each vector}) x_1^{n_1} x_2^{n_2} \ldots x_s^{n_s}$$

$$= \prod_{k_i \geq 0} (1 + x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s} + x_1^{2k_1} x_2^{2k_2} \ldots x_s^{2k_s} + \cdots + x_1^{(r-1)k_1} x_2^{(r-1)k_2} \ldots x_s^{(r-1)k_s})$$

$$= \prod_{k_i \geq 0} (1 + x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s} + (x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s})^2 + \cdots + (x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s})^{r-1})$$

$$= \prod_{k_i \geq 0} \frac{1 - (x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s})^r}{1 - x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s}}$$

$$= \prod_{k_i \geq 0} \frac{1}{1 - x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s}},$$

where at least one k_i is not divisible by r. This proves the result, since by the generating function given in (13) we have

$$\sum p(\mathbf{n}| \text{ vectors with at least one component not divisible by } r) x_1^{n_1} x_2^{n_2} \ldots x_s^{n_s}$$

$$= \prod_{k_i \geq 0} \frac{1}{1 - x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s}},$$

where at least one k_i is not divisible by r. □

Another partition identity gives us that every vector has the same amount of partitions into vectors where every component is even as partitions in which each vector is repeated an even number of times. This can also be generalized to partitions into vectors where every component is a multiple of r and partitions in which each vector is repeated a multiple of r times. This identity parallels Theorem 2.5. Table 7 gives an example of this identity for the even case. The Sage code in Figure 7 and Figure 8 can be used to verify this identity for other vectors and r values.

```
def componentsmultipleofr(x,y,r):

    E=[]

    L=VectorPartitions([x,y])

    for p in L:

        if all(v[0]%r==0 and v[1]%r==0 for v in p):

            E.append(p)

    return(E)
```

Figure 7: Sage code for the number of partitions in which every component is a multiple of r.

```
def repeatedmultipleofr(x,y,r):

    L=VectorPartitions([x,y])

    i=0

    for p in L:

        l=list(p)

        if all(l.count(part)%r==0 for part in l):

            i+=1

    return i
```

Figure 8: Sage code for the number of partitions in which each vector is repeated a multiple of r times.

Table 7: Examples illustrating Theorem 3.4 for r=2.

n	Even Components	Vectors Repeated Even Number of Times
$\begin{bmatrix}4\\2\end{bmatrix}$	$\begin{bmatrix}4\\2\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}$
	$\begin{bmatrix}4\\0\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}$	$\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$
	$\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}2\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}$
	$\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$

Theorem 3.4. *The number of partitions of* $\boldsymbol{n}=\begin{bmatrix}n_1\\\vdots\\n_s\end{bmatrix}$ *into vectors with all components a multiple of* r *is equal to the number of partitions of* $\boldsymbol{n}$ *in which each part is repeated a multiple of* r *times.*

Proof. By the generating function given in (12) we have

$$\sum p(\mathbf{n}|\text{ vectors repeated a multiple of } r \text{ times })x_1^{n_1}x_2^{n_2}\dots x_s^{n_s}$$

$$=\prod_{k_i\geq 0}(1+x_1^{rk_1}x_2^{rk_2}\dots x_s^{rk_s}+x_1^{2rk_1}x_2^{2rk_2}\dots x_s^{2rk_s}+\dots)$$

$$=\prod_{k_i\geq 0}(1+x_1^{rk_1}x_2^{rk_2}\dots x_s^{rk_s}+(x_1^{rk_1}x_2^{rk_2}\dots x_s^{rk_s})^2+\dots)$$

$$=\prod_{k_i\geq 0}\frac{1}{1-x_1^{rk_1}x_2^{rk_2}\dots x_s^{rk_s}}$$

$$=\sum p(\mathbf{n}|\text{ vectors with all components a multiple of } r)x_1^{n_1}x_2^{n_2}\dots x_s^{n_s},$$

where the last line follows from the generating function given in (13). This proves the result. $\square$

The partitions in the statement of the next theorem parallel the partitions from the second half of the proof of Theorem 2.6. The first half of the proof has not yet been proven for vectors. In order to complete the proof, one would need to determine how to measure the size of vectors and a well defined notion of conjugation for vectors.

Theorem 3.5. *Let $A_r(\boldsymbol{n})$ denote the number of partitions of $\boldsymbol{n}=\begin{bmatrix} n_1 \\ \vdots \\ n_s \end{bmatrix}$ in which any vector appearing an odd number of times appears at least $2r+1$ times. Let $B_r(\boldsymbol{n})$ be the number of partitions of $\boldsymbol{n}$ into vectors of one of the following forms: all components are even, or at least one component is congruent to $2r+1$ (mod $4r+2$) with all other components congruent to 0 (mod $4r+2$). Then $A_r(\boldsymbol{n}) = B_r(\boldsymbol{n})$.*

Proof. By the generating function given in (12), we have $\sum A_r(\mathbf{n}) x_1^{n_1} x_2^{n_2} \dots x_s^{n_s}$

$$
\begin{aligned}
&= \prod_{k_i \geq 0} [1 + (x_1^{k_1} \dots x_s^{k_s})^2 + \dots + (x_1^{k_1} \dots x_s^{k_s})^{2r} + (x_1^{k_1} \dots x_s^{k_s})^{2r+1} + (x_1^{k_1} \dots x_s^{k_s})^{2r+2} + \dots] \\
&= \prod_{k_i \geq 0} [1 + x_1^{2k_1} \dots x_s^{2k_s} + \dots + (x_1^{2k_1} \dots x_s^{2k_s})^{r} + (x_1^{k_1} \dots x_s^{k_s})^{2r+1} + (x_1^{k_1} \dots x_s^{k_s})^{2r+2} + \dots] \\
&= \prod_{k_i \geq 0} \frac{1 - (x_1^{2k_1} \dots x_s^{2k_s})^{r+1}}{1 - x_1^{2k_1} \dots x_s^{2k_s}} + \frac{(x_1^{k_1} \dots x_s^{k_s})^{2r+1}}{1 - x_1^{k_1} \dots x_s^{k_s}} \\
&= \prod_{k_i \geq 0} \frac{1 - (x_1^{k_1} \dots x_s^{k_s})^{2r+2}}{1 - x_1^{2k_1} \dots x_s^{2k_s}} + \frac{(x_1^{k_1} \dots x_s^{k_s})^{2r+1} + (x_1^{k_1} \dots x_s^{k_s})^{2r+2}}{1 - x_1^{2k_1} \dots x_s^{2k_s}} \\
&= \prod_{k_i \geq 0} \frac{1 + (x_1^{k_1} \dots x_s^{k_s})^{2r+1}}{1 - x_1^{2k_1} \dots x_s^{2k_s}}.
\end{aligned}
$$

Note by Cheema's Theorem (Theorem 3.2), we have

$$\prod_{k_i\geq 0}(1+(x_1^{k_1}\dots x_s^{k_s})^{2r+1}) = \prod_{k_i\geq 0}\frac{1}{1-(x_1^{k_1}\dots x_s^{k_s})^{2r+1}},$$

where at least one k_i is odd. That is, $\prod_{k_i\geq 0}(1+(x_1^{k_1}\dots x_s^{k_s})^{2r+1})$ represents vectors with at least one component of the form $(2j+1)(2r+1) = (4r+2)j+(2r+1)$, and other components are of the form $2j(2r+1) = (4r+2)j$. Thus,

$$\prod_{k_i\geq 0}\frac{1+(x_1^{k_1}\dots x_s^{k_s})^{2r+1}}{1-x_1^{2k_1}\dots x_s^{2k_s}} = \sum B_r(\mathbf{n})x_1^{n_1}x_2^{n_2}\dots x_s^{n_s}$$

This proves the result. $\square$

The Sage code in Figure 9 and Figure 10 can be used to verify the partition identity in Theorem 3.5 for vectors of the form $\mathbf{n}=\begin{bmatrix}n_1\\n_2\end{bmatrix}$.

```
def a(x,y,r):

    L=VectorPartitions([x,y])

    i=0

    for p in L:

        l=list(p)

        if all((l.count(part)%2==0) or (l.count(part)>=(2r+1)) for part in l):

            i+=1

    return i
```

Figure 9: Sage code for partitions enumerated by $A_r(\mathbf{n})$.

```
def b(x,y,r):

    E=[]

    L=VectorPartitions([x,y])

    for p in L:

        if all((v[0]%2==0 and v[1]%2==0)

        or (v[0]%(4r+2)==(2r+1) and v[1]%(4r+2)==(2r+1))

        or (v[0]%(4r+2)==(2r+1) and v[1]%(4r+2)==0)

        or (v[0]%(4r+2)==0 and v[1]%(4r+2)==(2r+1)) for v in p):

            E.append(p)

    return(E)
```

Figure 10: Sage code for partitions enumerated by $B_r(\mathbf{n})$.

4 FUTURE WORK

There is much more research that can be done in the area of vector partitions. One possibility for future work would be to prove identities similar to the ones in the previous chapters. Many integer partition identities are known. These may be able to be expanded to vectors using generating series arguments like the ones in Chapter 3. Another possibility for future work would be to attempt to come up with a well-defined notion of conjugation of vector partitions in order to complete Theorem 3.5 to parallel Andrew's theorem given in Theorem 2.6. Conjugation of vector partitions could also lead to the discovery of other vector partition identities. Conjugation of integer partitions gives us a way to prove identities involving the size of the parts or the size of the partitions. If one could come up with a way to conjugate vector partitions, this could lead to identities parallel to those of integer partitions.

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