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On Properties of $r_w$-Regular Graphs

Franklina Samani

East Tennessee State University

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On Properties of $r_w$-Regular Graphs

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by

Franklina Samani

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Debra Knisley, Ph.D., Chair

Teresa Haynes, Ph.D.

Robert Gardner, Ph.D.

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ABSTRACT

On Properties of $r_w$-Regular Graphs

by

Franklina Samani

If every vertex in a graph $G$ has the same degree, then the graph is called a regular graph. That is, if $\text{deg}(v) = r$ for all vertices in the graph, then it is denoted as an $r$-regular graph. A graph $G$ is said to be vertex-weighted if all of the vertices are assigned weights. A generalized definition for degree regularity for vertex-weighted graphs can be stated as follows: A vertex-weighted graph is said to be $r_w$-regular if the sum of the weights in the neighborhood of every vertex is $r_w$. If all vertices are assigned the unit weight of 1, then this is equivalent to the definition for $r$-regular graphs. In this thesis, we determine if a graph has a weighting scheme that makes it a weighted regular graph or prove no such scheme exists for a number of special classes of graphs such as paths, stars, caterpillars, spiders and wheels.
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DEDICATION

With love I dedicate this dissertation to the three most important people in my life.

Mr. Donald Samani

Miss Lucy Dabuo

Mr. Seth Bomangsaan Eledi.
ACKNOWLEDGMENTS

First and foremost I would like to praise and thank God, the Almighty, whose many blessings and favor have brought me this far and made me who I am today. His gift of wisdom, knowledge, understanding and good health enabled me to begin and complete this piece of work. I am eternally grateful for His guidance and protection, enabling me to overcome all the trials with determination and perseverance to make this study possible.

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1 INTRODUCTION

In this thesis we study a mathematical construction known as a graph. A graph $G$ consists of a finite set of vertices $V(G)$ called the vertex set, an edge set $E(G)$, and a relation that associates one edge with two vertices called the endpoints of the edge. Figure 1 is an example of a graph with 5 vertices and 6 edges.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{house_graph.png}
\caption{House graph}
\end{figure}

For the purposes of this thesis, we will assume that all graphs presented are simple, i.e. no loops (edges whose endpoints are equal), no multiple edges (more than one edge that has the same endpoints) [27]. An edge $uv$ in $G$ implies $u$ is adjacent to $v$ and the edge $uv$ is said to be incident to the vertices $u$ and $v$ ( $u$ and $v$ are the endpoints of the edge $uv$). The set of all vertices adjacent to $u$ is called the neighborhood of $u$. If every vertex in the graph has the same degree, then the graph is called a regular graph. That is, if $\text{deg}(v) = r$ for all vertices in the graph, then the graph is denoted as an $r$-regular graph. A large volume of work has been done in relation to regular graphs, some of which include [1, 7, 12, 15, 16, 25].
A graph $G$ is said to be weighted if all of the vertices are assigned non negative integers called weights. In this thesis, we define what we call a weighted-regular graph. First, we generalize the definition of the degree of a vertex in a vertex-weighted graph as follows: the weighted degree of a vertex $v$ is the sum of the weights of its neighbors and is denoted by $\deg_w(v) = \sum_{u \in N(v)} w(u)$ where $w(u)$ is the assigned weight of $u$ in $G$. If every vertex in the graph has the same weighted degree, then the graph is called a weighted-regular graph. That is, if $\deg_w(v) = a$ for all vertices in the graph, then it is denoted as an $a_w$-regular graph. Necessarily, all regular graphs can be considered weighted-regular graphs if each vertex has an understood weight of 1. In this thesis, we determine if a graph has a weighting scheme that makes it a weighted regular graph or prove no such scheme exists for a number of special classes of graphs.

For example, consider the graph in Figure 1. The graph in Figure 1 is not a regular graph since some of the vertices have degree 2 and some have degree 3. However, the vertex weighted graph in Figure 2 is a $8_w$-regular graph.

![Figure 2: An 8$_w$-regular House graph](image)

Not all graphs allow a weighting scheme that produces an $a_w$-regular graph. For
instance, the graph in Figure 3 does not allow a weighting scheme that produces an $a_w$-regular graph.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{graph.png}
\caption{A graph which is not weighted-regular}
\end{figure}

Observe that the two support vertices $u$ and $v$ are assigned weights $a$ each, for otherwise $\deg_w(l_1) \neq a$ and $\deg_w(l_2) \neq a$. However since $\deg_w(x) = w(u) + w(v)$, it implies $\deg_w(x) = a + a = 2a$ and since $a = \deg_w(l_1) = \deg_w(l_2) \neq \deg_w(x) = 2a$, the graph does not allow a weighting scheme which makes it weighted-regular.

To determine if an arbitrary graph allows a weighting scheme is beyond the scope of this thesis. In this thesis, we determine if a vertex weighting scheme that results in a weighted regular graph exists for selected families of graphs. For example, the graph in Figure 4 is a tree that is weighted regular.
Observe that the support vertices are assigned a weight of $a$, thus resulting in $\text{deg}_w(v) = a$ for all $v \in V$. This implies that there exist a non trivial weighting scheme which makes the graph weighted regular.

In order to continue our discussion, we now define the necessary terms that we use to prove our results on selected families of graphs.

A \textit{leaf} vertex is a vertex which has a degree of one and a \textit{support} vertex is defined as a vertex with a leaf in its neighborhood. A \textit{path} in $G$ is a walk in which no vertex is repeated. A graph $G$ is said to be \textit{connected} if for all $u, v \in V(G)$, there is a $uv$ path. Particularly, a path on $n$ vertices is denoted as $P_n$. A path $P_n$ is therefore a graph of order $n$ and size $n - 1$ with vertices denoted as $v_1, v_2, v_3, \ldots, v_n$ and edges $v_iv_{i+1}$ for $i = 1, 2, 3, \ldots, n - 1$.

The \textit{length} of a path with endpoints $u$ and $v$ is the number of edges between $u$ and $v$ on a particular path from $u$ to $v$. Figure 5 is an example of a graph of length 4. Thus, a path $P_n$ has length $n - 1$. 

Figure 4: An example of an $a_w - regular$ graph
A cycle is a $u - v$ path with $u = v$. A cycle on $n$ vertices is denoted as $C_n$. A tree denoted as $T$ is an acyclic connected graph, i.e. it has no cycles [10]. A caterpillar tree as defined by Gordon and Breach [21] is a tree with the property that the removal of its leaves (vertices of degree one) results in a path. The caterpillar is made up of two types of vertices, namely the leaves (vertices of degree one) and vertices which are in the neighborhood of the leaves which we will refer to as support vertices.

A star graph, $S_i$ (with $i$ being the order of the graph) is an acyclic graph in which one vertex has degree $i - 1$ and the rest have degree 1. A double star graph as defined in [11] is a graph consisting of the union of two stars $S_{1,m}$ and $S_{1,m}$ together with an edge joining their centers. A spider graph is a tree with at most one vertex of degree greater than one called the center. A lobster is a tree with the property that the removal of its leaves produces a caterpillar [5]. A wheel graph $W_n$ is a graph of order $n$ formed by connecting a vertex (sometimes called the central vertex) of degree $n - 1$ to every vertex on a cycle of length $n - 1$. A fan graph is a graph formed by connecting a vertex $v$ to all vertices on a path $P_n$. A complete bipartite graph $K_{m,n}$ is a bipartite graph with every vertex in one partite set connected to every vertex in the other partite set, and no two vertices in the same partite set are connected.
1.1 Background

Graph theory is the study of properties of graphs in several fields such as mathematics and computer science. Several problems of interest in the world today can be represented by graphs. In the year 1736, a Swiss mathematician Leonhard Euler (1707 - 1783) solved the famous Königsberg bridge and that was the birth of graph theory.

A lot of research has been done on edge weighted graphs, however very little research has been done on vertex weighted graphs and hence this has become a growing area of research in graph theory. There are examples such as [19] where vertex weights are studied. However, a thorough literature search did not reveal any work resembling weighted-regular graphs.

Graceful labeling is one of the areas of graph theory where vertices are assigned a numerical values known as vertex weight. Many researchers such as Brankovic [8] and Edwards [9] have done a lot of work on graceful graphs. Several other researchers such as [2, 3, 4, 6, 13, 14, 17, 18, 20, 22, 23, 24, 26, 28] have also done some work where numerical values are assigned to vertices.

In this thesis, we determine if a vertex weighting scheme that results in a weighted regular graph exists for selected families of graphs. In particular, we determine if selected trees such as paths, caterpillars and spiders have a weighting scheme which make them weighted-regular. We also answer this question with respect to cycles, complete bipartite graphs, wheels and fans.
RESULTS ON TREES

We will start by proving a very useful proposition that we use in determining whether or not a tree is weighted-regular.

**Proposition 2.1** If a graph $G$ contains two vertices of degree 1, $u$ and $v$, such that the distance $d(u, v) = 4$, then the graph does not have a nontrivial weighting scheme which makes it weighted regular.

**Proof** Assume the graph $G$ is an $a_w$-regular graph. We need to show that there exists a vertex $x$ in the graph where $\text{deg}_w(x) \neq a$. Let $u$ and $v$ be vertices that are leaves such that $d(u, v) = 4$. Let $s, y, z$ be the vertices between $u$ and $v$ as shown in Figure 6.

![Figure 6: A Tree with $d(u, v) = 4$](image)

Then since $u$ and $v$ are leaves, they have just one neighbor each. Let $a$ be the weight assigned to all the support vertices, then since $s$ and $z$ are support vertices; they will each be assigned a weight of $a$. Now as shown in Figure 6: vertex $y$ has at
least two neighbors, $s$ and $z$, that are support vertices. So the degree weight of $y$ will be the sum of the assigned weights of its neighbors which include the assigned weights of $s$ and $z$. This implies the degree weight of $y$ is at least twice the weight of a support vertex. Thus, $\text{deg}_w(y) \geq w(s) + w(z)$ and since $a \neq 0$, then $\text{deg}_w(y) \geq a + a$. So, $\text{deg}_w(y) \geq 2a > a$.

This is a contradiction since $a = \text{deg}_w(u) = \text{deg}_w(v) \neq \text{deg}_w(y) = 2a$. □

Proposition 2.1 is very useful in the determination of the weighted regularity of several other graphs such as a caterpillar and a lobster. Below are examples where Proposition 2.1 are used.

**Example 2.2** Let $G_9$ be a caterpillar with 9 vertices (Refer to Figure 7). Let the support vertices $(v_1, v_2, v_3, v_4)$ have weights 3 each. Thus, $w(v_i) = 3$ for $i = 1, 2, 3, 4$. This implies that $\text{deg}_w(v_i) = 3$ for $i = 5, 6, 7, 8, 9$ (the leaves). So for the graph to be weighted regular, the degree weight of each vertex must be the same. Thus $\text{deg}_w(v)$ has to be equal to 3 for all $v \in V$.

![Figure 7: Example of a Caterpillar with $k = 4$](image)

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Observe that there are two leaves, say $v_5$ and $v_7$, such that $d(v_5, v_7) = 4$. Then clearly by Proposition 2.1 this graph does not have a weighting scheme which makes it weighted regular. This is because $\deg_w(v_2) \geq w(v_1) + w(v_3) = 3 + 3 = 6 \neq \deg_w(v_5) = 3$.

**Example 2.3** Let $L$ be a lobster with two legs of length two, each connected to a vertex $v_2$ on the central path $(v_1, v_2, v_3, v_4)$ (Refer to Figure 8). Let the support vertices $(v_5, v_6, v_3, v_7, v_8)$ have weights 5 each. Thus, $w(v_i) = 5$ for $i = 3, 5, 6, 7, 8$. This implies that $\deg_w(v_i) = 5$ for $i = 9, 10, 11, 12, 13$ (the leaves). So for the graph to be weighted regular, the degree weight of each vertex must be the same. Thus $\deg_w(v)$ has to be equal to 5 for all $v \in V$.

![Figure 8: Example of a Lobster graph](image-url)
Observe that there are two leaves, say \( v_6 \) and \( v_8 \), such that \( d(v_6, v_8) = 4 \). Then clearly by Proposition 2.1 this graph does not have a weighting scheme which makes it weighted regular. This is because \( \deg_w(v_2) \geq w(v_6) + w(v_3) + w(v_8) = 5 + 5 + 5 = 15 \neq \deg_w(v_9) = 5 \).

Several other graphs with \( d(x, y) = 4 \) are also known not to have a nontrivial weighting scheme which makes it weighted regular as a result of Proposition 2.1.

**Theorem 2.4** For a path of order \( n \) greater than 1, there exists a weighting scheme that makes the path \( a_w - \) regular except for paths of order \( n \equiv 1 \) (mod 4).

**Proof** Let \( P_n \) be a path on \( n \) vertices. Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices on the path. Then \( w(v_2) = w(v_{n-1}) = a \) since \( v_2 \) and \( v_{n-1} \) are support vertices. This implies that \( \deg_w(v_1) = \deg_w(v_n) = a \). So since \( w(v_2) = a \), it implies that \( w(v_4) = 0 \), for otherwise \( \deg_w(v_3) \neq a \). This implies that \( w(v_6) = a \), for otherwise \( \deg_w(v_5) \neq a \).

So we have that:

\[
  w(v_i) = \begin{cases} 
    a & \text{if } i \equiv 2 \pmod{4} \\
    0 & \text{if } i \equiv 0 \pmod{4}
  \end{cases}
\]

**Case 1:** Paths of order \( n \equiv 1 \) (mod 4).

Since \( n \equiv 1 \) (mod 4), it implies that \( n - 1 \equiv 0 \) (mod 4). Therefore \( w(v_{n-1}) = 0 \) and this contradicts the hypothesis that \( w(v_{n-1}) = a \). Also \( w(v_{n-1}) \) can not be 0 since that implies that \( \deg_w(v_n) = 0 \neq \deg_w(v_1) = a \).

\[\square\]

**Case 2:** Paths of order \( n \equiv 2 \) (mod 4).
Since \( n \equiv 2 \pmod{4} \), it implies that \( w(n) = a \). Also since \( w(v_{n-1}) = a \), it implies that \( w(v_{n-3}) = 0 \), for otherwise \( \deg_w(v_{n-2}) \neq a \). So since \( n \equiv 2 \pmod{4} \implies n - 1 \equiv 1 \pmod{4} \implies n - 2 \equiv 0 \pmod{4} \implies n - 3 \equiv 3 \pmod{4} \). So we have:

\[
\begin{align*}
w(v_i) &= \begin{cases} 
a & \text{if } i \equiv 1 \pmod{4} \\
0 & \text{if } i \equiv 3 \pmod{4}
\end{cases}
\end{align*}
\]

Therefore \( \deg_w(v) = a \) for all \( v \in V \).

\[\square\]

**Example 2.5** Consider a path on 6 vertices (\( P_6 \)).

Let \( w(v_i) = \begin{cases} 5 & \text{if } i \equiv 2 \pmod{4} \\
0 & \text{if } i \equiv 0 \pmod{4} \\
5 & \text{if } i \equiv 1 \pmod{4} \\
0 & \text{if } i \equiv 3 \pmod{4}
\end{cases} \)

Then we have \( w(v_2) = w(v_6) = 5 = w(v_1) = w(v_5) \) and \( w(v_3) = w(v_4) = 0 \).

![Figure 9: A 5\(_w\)-regular \( P_6 \)](image)

*Therefore \( \deg_w(v) = 5 \) for all \( v \in V \). Hence the path is 5\(_w\)-regular.*

**Case 3:** Paths of order \( n \equiv 3 \pmod{4} \).

Since \( n \equiv 3 \pmod{4} \), it implies that \( n - 1 \equiv 2 \pmod{4} \). Therefore \( w(v_{n-1}) = a \). Now, \( w(v_1) \) can either be 0 or \( a \). Without loss of generality let \( w(v_1) = a \). This implies that \( w(v_3) = 0 \), for otherwise \( \deg_w(v_2) \neq a \). So we have

\[
\begin{align*}
w(v_i) &= \begin{cases} 
a & \text{if } i \equiv 1 \pmod{4} \\
0 & \text{if } i \equiv 3 \pmod{4}
\end{cases}
\end{align*}
\]

Therefore \( \deg_w(v) = a \) for all \( v \in V \).
Example 2.6  Consider a path on 7 vertices \((P_7)\).

Let \(w(v_i) = \begin{cases} 
3 & \text{if } i \equiv 2 \pmod{4} \\
0 & \text{if } i \equiv 0 \pmod{4} \\
3 & \text{if } i \equiv 1 \pmod{4} \\
0 & \text{if } i \equiv 3 \pmod{4}
\end{cases} \)

Then, \(w(v_2) = w(v_6) = 3 = w(v_1) = w(v_5)\) and \(w(v_4) = 0 = w(v_3) = w(v_7)\).

\[ 
\begin{array}{cccccccc}
V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \\
3 & 3 & 0 & 0 & 3 & 3 & 0 \\
\end{array} \]

Figure 10: A \(3_w\)-regular \(P_7\)

Therefore \(\deg_w(v) = 3\) for all \(v \in V\). Hence the path is \(3_w\)-regular.

Case 4: Paths of order \(n \equiv 0 \pmod{4}\).

Since \(n \equiv 0 \pmod{4}\), it implies that \(w(n) = 0\). Also since \(w(v_{n-1}) = a\), it implies that \(w(v_{n-3}) = 0\), for otherwise \(\deg_w(v_{n-2}) \neq a\). So since \(n \equiv 0 \pmod{4} \implies n - 1 \equiv 3 \pmod{4} \implies n - 2 \equiv 2 \pmod{4} \implies n - 3 \equiv 1 \pmod{4}\).

So we have \(w(v_i) = \begin{cases} 
a & \text{if } i \equiv 3 \pmod{4} \\
0 & \text{if } i \equiv 1 \pmod{4}
\end{cases} \)

Therefore \(\deg_w(v) = a\) for all \(v \in V\).

\[ 
\square
\]

Example 2.7  Consider a path on 8 vertices \((P_8)\).

Let \(w(v_i) = \begin{cases} 
6 & \text{if } i \equiv 2 \pmod{4} \\
0 & \text{if } i \equiv 0 \pmod{4} \\
6 & \text{if } i \equiv 3 \pmod{4} \\
0 & \text{if } i \equiv 1 \pmod{4}
\end{cases} \)

Then \(w(v_2) = w(v_6) = 6 = w(v_3) = w(v_7)\) and \(w(v_4) = w(v_8) = 0 = w(v_1) = w(v_5)\).
Therefore $\deg_w(v) = 6$ for all $v \in V$. Hence the path is $6_w$-regular.

**Proposition 2.8** Every star has a non trivial weighting scheme which makes it weighted regular and this is unique.

**Proof** Let $S_{n+1}$ be a star graph of order $n+1$. Let $v_{n+1}$ be the central vertex and $v_i$ for $i = 1, 2, \ldots, n$ be the vertices of degree 1. So since $v_{n+1}$ is adjacent to all the other vertices, its degree weight will be the sum of the weights of all the other vertices. Let $\deg_w(v_{n+1}) = \sum_{i=1}^{n} w(v_i) = a$. This implies that $\sum_{i=1}^{n} w(v_i) = w(v_{n+1})$, for otherwise $\deg_w(v_i) \neq a$ for $i = 1, 2, \ldots, n$. Therefore $\deg_w(v_i) = a$ for all $v \in V$. Hence $S_{n+1}$ is $a_w$-regular.

\[\square\]

**Example 2.9** Let $S_9$ be a star graph of order 9. Then $v_{n+1} = v_9$ for $n = 8$. Let $w(v_1) = 2, w(v_2) = 3, w(v_3) = 0, w(v_4) = 2, w(v_5) = 8, w(v_6) = 1, w(v_7) = 4, w(v_8) = 1$. Therefore $\deg_w(v_9) = \sum_{i=1}^{8} w(v_i) = 21$. This implies that $w(v_9) = 21$. Therefore $\deg_w(v_i) = 21$ for all $v \in V$. Hence the star graph $S_9$ is $21_w$-regular (Refer to Figure 12).
Theorem 2.10 Every double star has a non trivial weighting scheme which makes it weighted regular and this weighting scheme is unique.

Proof Let $T$ be a double star of order $n$. Let $u$ and $v$ be the central vertices of the two stars that are connected by an edge. Notice that $u$ and $v$ are support vertices and
the only two vertices with degree greater than 1, thus the remaining \( n - 2 \) vertices are leaves since they have degrees 1. Assign a weight of \( a \) to each of vertices \( u \) and \( v \) since they are support vertices. Also a weight of 0 is assigned to the remaining vertices (Refer to Figure 13). Observe that this nontrivial weighting scheme makes the double star weighted regular and hence we can conclude that there exist a weighting scheme which makes the double star weighted-regular.

\[ \square \]

We will now consider a particular type of spider with a fixed leg of length 2.

Let \( G_2 \) be a spider with a fixed leg of length 2 and the number of remaining legs equal to \( k \). We will call the central vertex \( x \) and the paths from \( x \), the legs of the spider. Also, let \( l \) be a path of length 2 with \( u_1 \) and \( u_2 \) the leaf and support vertices respectively on paths \( l \).

\[
\begin{align*}
    p &= \text{number of paths of length } \equiv 1 \pmod{4} \\
    q &= \text{number of paths of length } \equiv 2 \pmod{4} \\
    s &= \text{number of paths of length } \equiv 3 \pmod{4} \\
    t &= \text{number of paths of length } \equiv 0 \pmod{4}
\end{align*}
\]

This implies that \( p + q + s + t = k \).

**Proposition 2.11** \( G_2 \) has a nontrivial weighting scheme which makes it weighted-regular if \( k = t \), thus \( p = q = s = 0 \).

**Proof** Let \( w(u_2) = a \) where \( a \neq 0 \). This implies that \( \text{deg}_w(u_1) = a \). Thus graph \( G_2 \) must be \( a_w \)-regular. Let \( P = \{v_1, v_2, \ldots, v_n\} \) be a path of length \( h \) where \( h \equiv 0 \pmod{4} \). This implies that \( w(v_{h-1}) = a \) for all \( k \) legs. Note that this implies that \( w(v_{h-3}) = 0 \), for otherwise \( \text{deg}_w(v_{h-2}) \neq a \). So we have:
\[ w(v_{h-1}) = a \implies w(v_{h-3}) = 0 \implies w(v_{h-5}) = a \implies \cdots \implies w(v_1) = 0. \] Let \( u_2 = b \). Then this implies \( w(x) = a - b \), otherwise \( \deg_w(u_2) = w(x) + w(u_1) \neq a \).

Notice that the weights of \( v_1, v_2, v_3, v_4 \) are 0, \( b, a, a - b \) respectively. Therefore since the \( \deg_w(v) = a \) for all \( v \in V \), graph \( G_2 \) is \( a_w - regular \).

\[ \square \]

**Example**

Let the weight \( a = 4 \) and the weight \( b = 1 \).

![Graph](image)

**Figure 14: A 4\(_w\) - regular \( G_2 \) graph**

**Proposition 2.12** \( G_2 \) has a nontrivial weighting scheme which makes it weighted-regular if \( k = p \), thus \( q = s = t = 0 \).

**Proof** Let \( w(u_2) = a \) where \( a \neq 0 \). This implies that \( \deg_w(u_1) = a \). Thus graph \( G_2 \) must be \( a_w - regular \). Let \( P = \{v_1, v_2, ..., v_n\} \) be a path of length \( h + 1 \) where
$h + 1 \equiv 1 \pmod{4}$. This implies that $w(v_h) = a$ for all $k$ legs. Note that this implies that $w(v_{h-2}) = 0$, for otherwise $\deg_w(v_{h-1}) \neq a$. So we have:

$$w(v_h) = a \implies w(v_{h-2}) = 0 \implies w(v_{h-4}) = a \implies \cdots \implies w(v_2) = 0.$$ Then this implies $w(x) = a$, otherwise $\deg_w(u_1) = w(x) + w(u_2) \neq a$. And since $w(x) = a \implies w(u_1) = 0$, otherwise $\deg_w(u_2) \neq a$. Notice that the weights of $v_1, v_2, v_3, v_4$ are $a, b, b, a$ respectively, and this weighting pattern is repeated on all $p$ legs. Therefore since the $\deg_w(v) = a + b$ for $b = 0$. It implies that $\deg_w(v) = a$ for all $v \in V$, so graph $G_2$ is $a_w$-regular.

□

**Example 2.13** Let the weight $a = 3$ and $b = 0$.

![Figure 15: A 3$_w$-regular $G_2$ graph](image)
Proposition 2.14 There does not exist a nontrivial weighting scheme that makes $G_2$ regular if $k = q$, thus $p = s = t = 0$.

Proof Let $w(u_2) = a$ where $a \neq 0$. This implies that $deg_w(u_1) = a$. Thus graph $G_2$ must be $a_w$-regular. Let $P = \{v_1, v_2, ..., v_n\}$ be a path of length $h + 2$ where $h + 2 \equiv 2 \pmod{4}$. This implies that $w(v_{h+1}) = a$ for all $k$ legs. Note that this implies that $w(v_{h-1}) = 0$, for otherwise $deg_w(v_h) \neq a$. So we have:

$w(v_{h+1}) = a \implies w(v_{h-1}) = 0 \implies w(v_{h-3}) = a \implies \cdots \implies w(v_1) = a$. Now, since $u_2$ and $v_1$ are both neighbors of $x$, it implies that $deg_w(x) = w(u_2) + w(v_1) = a + a = 2a$. So $deg_w(x) \neq a$ and thus a contradiction. Therefore graph $G_2$ does not have a nontrivial weighting scheme which makes it weighted-regular when $k = q$.

□

Proposition 2.15 $G_2$ has a nontrivial weighting scheme which makes it weighted-regular if $k = s$, thus $p = q = t = 0$.

Proof Let $w(u_2) = a$ where $a \neq 0$. This implies that $deg_w(u_1) = a$. Thus graph $G_2$ must be $a_w$-regular. Let $P = \{v_1, v_2, ..., v_n\}$ be a path of length $h + 3$ where $h + 3 \equiv 3 \pmod{4}$. This implies that $w(v_{h+2}) = a$ for all $k$ legs. Note that this implies that $w(v_h) = 0$, for otherwise $deg_w(v_{h+1}) \neq a$. So we have:

$w(v_{h+2}) = a \implies w(v_h) = 0 \implies w(v_{h-2}) = a \implies \cdots \implies w(v_2) = a$. Then this implies $w(x) = 0$, otherwise $deg_w(u_1) = w(x) + w(u_2) \neq a$ and $deg_w(v_1) = w(x) + w(v_2) \neq a$. So since $w(x) = 0$, it implies that $w(u_1) = a$, otherwise $deg_w(u_2) \neq a$. Notice that the weights of $v_1, v_2, v_3, v_4$ are $b, b, a, a$ respectively, and this weighting pattern is repeated on all $s$ legs. Therefore since the $deg_w(v) = a + b$ for $b = 0$. It implies that $deg_w(v) = a$ for all $v \in V$, hence graph $G_2$ is $a_w$-regular.
Example 2.16 Let the weight $a = 5$ and the weight $b = 0$.

Proposition 2.17 $G_2$ has a nontrivial weighting scheme which makes it weighted-regular if $k = t + p$, thus $p = s = 0$.

Proof We know from Proposition 2.12 that $w(x) = a$ since $G_2$ has legs of length $1 \pmod{4}$ and we have $p$ of such legs. So using the weight assignment pattern in Propositions 2.11 and 2.12 for legs $t$ and $p$ respectively where $b = 0$: we have $\deg_w(v) = a$ for every $v \in V$, hence graph $G_2$ has a nontrivial weighting scheme which makes it $a_w$-regular. Therefore since the $\deg_w(v) = a + b$ for $b = 0$. It implies that $\deg_w(v) = a$ for all $v \in V$, hence graph $G_2$ is $a_w$-regular.
Example 2.18  Let the weight $a = 4$ and the weight $b = 0$.

Figure 17: A $4_w$-regular $G_2$ graph

Proposition 2.19  $G_2$ has a nontrivial weighting scheme which makes it weighted-
regular if $k = t + s$, thus $p = q = 0$.

Proof  We know from Proposition 2.14 that $w(x) = 0$ since $G_2$ has legs of length
$3(\text{mod } 4)$ and we have $s$ of such legs. Let $w(x) = 0 = b$, then $w(u_2) = a$ for
otherwise $\deg_w(u_1) \neq a$. Also since $w(x) = b$, let $w(u_1) = a - b$, for otherwise
$\deg_w(u_2) = w(x) + w(u_1) \neq a$. Notice that $v_1, v_2, v_3, v_4$ have weights $b, a - b, a, b$
where $b = 0$, on all $k$ legs. Thus, this is the repeated weighting pattern on all $k$ legs.
Note that $b = 0$, for otherwise $\deg_w(v_h + 3) \neq a$ since $w(v_h + 2) = a - b$ for all $s$ legs
(legs of length $\equiv 3 \pmod{4}$). Therefore $\deg_w(v) = a$ for all $v \in V$, hence $G_2$ has a
nontrivial weighting scheme which makes it weighted regular when $k = t + s$. 
Example 2.20 Let the weight $a = 4$ and the weight $b = 0$.

![Figure 18: A $3_w$ - regular $G_2$ graph](image)

**Proposition 2.21** $G_2$ does not have a nontrivial weighting scheme which makes it weighted-regular if $k = p + s$, thus $q = t = 0$.

**Proof** By Proposition 2.12, we know that the presence of legs of length $P \equiv 3 \pmod{4}$ in $G_2$ implies that $w(x) = 0$, however by Proposition 2.12 we have that $w(x) = a$ since $G_2$ also has legs of length $P \equiv 1 \pmod{4}$. So since $w(x)$ can not satisfy the weight requirements on both legs, there is a contradiction. Therefore we can conclude that there does not exist a nontrivial weighting scheme which makes $G_2$ weighted-regular when $k = p + s$. 

□

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In this chapter we are going to consider graphs either than trees. In particular, we will consider cycles, fan graphs, wheel graphs and bipartite graphs.

**Theorem 3.1** Let $C_n$ be a cycle on $n$ vertices. Then $C_n$ has a nontrivial weighting scheme which makes it weighted regular only if $n \equiv 0 \pmod{4}$.

**Proof** Let $C_n$ be a cycle with $n$ vertices. Also let $a, b, c$ be the assigned weights for $v_1, v_2, v_3$ respectively. Then $\deg_w(v_2) = w(v_1) + w(v_3) = a + c$. So,

- $w(v_5) = a$, for otherwise $\deg_w(v_4) \neq a + c$.
- $w(v_7) = c$, for otherwise $\deg_w(v_6) \neq a + c$.

\[ \vdots \]

This implies that $w(v_i) = \begin{cases} a & \text{if } i \equiv 1 \pmod{4} \\ c & \text{if } i \equiv 3 \pmod{4} \end{cases}$

**Case 1:** Consider $n \equiv 1 \pmod{4}$.

Since $n \equiv 1 \pmod{4}$, then $w(v_n) = a$. This implies that $b = c$, for otherwise $\deg_w(v_1) \neq a + c$. So $w(v_2) = c$, and this implies $w(v_4) = a$ for otherwise $\deg_w(v_3) \neq a + c$. Also, $w(v_6) = c$ for otherwise $\deg_w(v_5) \neq a + c$. So we have,

\[ w(v_i) = \begin{cases} a & \text{if } i \equiv 0 \pmod{4} \\ c & \text{if } i \equiv 2 \pmod{4} \end{cases} \]

Now, since $n \equiv 1 \pmod{4} \implies n-1 \equiv 0 \pmod{4} \implies n-2 \equiv 3 \pmod{4} \implies n-3 \equiv 2 \pmod{4}$. Therefore $w(v_{n-1}) = a$ since $i = n-1 \equiv 0 \pmod{4}$. This implies that $\deg_w(v_n) = w(v_1) + w(v_{n-1}) = a + a = 2a \neq a + c$. Hence $C_n$ for $n \equiv 1 \pmod{4}$ is not weighted regular.

\[ \square \]
Case 2: Consider $n \equiv 2 \pmod{4}$.

Since $n \equiv 2 \pmod{4} \implies n - 1 \equiv 1 \pmod{4}$. So, $w(v_{n-1}) = a$. Therefore $deg_w(v_n) = w(v_1) + w(v_{n-1}) = a + a = 2a \neq a + c$. Hence $C_n$ for $n \equiv 2 \pmod{4}$ is not weighted regular.

□

Case 3: Consider $n \equiv 3 \pmod{4}$.

Since $n \equiv 3 \pmod{4} \implies w(v_n) = c$. This implies that $b = c$, for otherwise $deg_w(v_1) \neq a + c$. Therefore $w(v_2) = c$. Then,

\begin{align*}
w(v_4) &= c, \text{ for otherwise } deg_w(v_3) \neq a + c. \\
w(v_6) &= a, \text{ for otherwise } deg_w(v_5) \neq a + c.
\end{align*}

So we have that $w(v_i) = \begin{cases} a & \text{if } i \equiv 2 \pmod{4} \\
c & \text{if } i \equiv 0 \pmod{4} \end{cases}$

Now, since $n \equiv 3 \pmod{4} \implies n - 1 \equiv 2 \pmod{4} \implies n - 2 \equiv 1 \pmod{4} \implies n - 3 \equiv 0 \pmod{4}$. So $w(v_{n-1}) = a$ since $i = n - 1 \equiv 2 \pmod{4}$ and therefore $deg_w(v_n) = w(v_1) + w(v_{n-1}) = a + a = 2a \neq a + c$. Hence $C_n$ for $n \equiv 3 \pmod{4}$ does not have a weighting scheme which makes it weighted regular.

□

Case 4: Consider $n \equiv 0 \pmod{4}$.

Since $n \equiv 0 \pmod{4} \implies n - 1 \equiv 3 \pmod{4}$, therefore $w(v_{n-1}) = c \implies w(v_{n-3}) = a$. The weight of $v_4$ has to be either $a$ or $c$, for otherwise $deg_w(v_3) \neq a + c$. Without loss of generality, let $w(v_4) = c$. This implies that $w(v_2) = b = a$. Also,
\[ w(v_6) = a \text{ for otherwise } \deg_w(v_5) \neq a + c. \]

\[ w(v_8) = c, \text{ for otherwise } \deg_w(v_7) \neq a + c. \]

\[ : \]

So we have \( w(v_i) = \begin{cases} a & \text{if } i \equiv 2 \pmod{4} \\ c & \text{if } i \equiv 0 \pmod{4} \end{cases} \)

Therefore \( \deg_w(v) = a + c \) for all \( v \in V \), hence we can conclude that there exist a nontrivial weighting scheme which makes \( C_n \) for \( n \equiv 0 \pmod{4} \) weighted regular.

\[ \square \]

**Theorem 3.2** Let \( F_4 \) be a fan graph on 4 vertices. Then there exist a nontrivial weighting scheme which makes \( F_4 \) weighting regular.

**Proof** Let \( v_1, v_2, v_3, v_4 \) be the 4 vertices with degrees 2, 3, 2, 3 respectively (Refer to Figure 19).

![Figure 19: A Fan graph on 4 vertices (\( F_4 \))](image)

Then, \( \deg_w(v_4) = w(v_1) + w(v_2) + w(v_3) \) and \( \deg_w(v_1) = w(v_2) + w(v_4) \). Since \( \deg_w(v_1) = \deg_w(v_4) \), it implies that \( w(v_4) = w(v_1) + w(v_3) \) \hspace{1cm} (1). \( \deg_w(v_2) = w(v_1) + w(v_3) + w(v_4) \). So since \( \deg_w(v_2) = \deg_w(v_1) \), it implies that \( w(v_2) = w(v_1) + w(v_3) \) \hspace{1cm} (2). So from equations (1) and
(2), we have \( w(v_2) = w(v_1) \). \( \text{deg}_w(v_3) = w(v_2) + w(v_4) \). So since \( \text{deg}_w(v_3) = \text{deg}_w(v_2) \), it implies that \( w(v_2) = w(v_1) + w(v_3) \). Therefore, \( w(v_1) = w(v_2) - w(v_3) \). So since \( \text{deg}_w(v) = 2w(v_4) \) for all \( v \in V \), then there exist a nontrivial weighting scheme which makes \( F_4 \) weighted regular.

\( \square \)

**Example 3.3** Let \( w(v_2) = w(v_4) = 7 \) and \( w(v_3) = 2 \), then \( w(v_1) = 7 - 2 = 5 \).

![Figure 20: A 14\( w \)-regular Fan Graph](image)

Clearly, \( \text{deg}_w(v) = 14 \) for all \( v \in V \) hence the graph is a 14\( w \)-regular graph.

**Theorem 3.4** Let \( F_5 \) be a fan graph on 5 vertices. Then there exists a nontrivial weighting scheme which makes \( F_5 \) weighting regular.

**Proof** Let \( v_1, v_2, v_3, v_4, v_5 \) be the 5 vertices with degrees 2, 3, 3, 2, 4 respectively (Refer to Figure 21).
The $\deg_w(v_5) = w(v_1) + w(v_2) + w(v_3) + w(v_4)$ and $\deg_w(v_1) = w(v_2) + w(v_5)$. Since $\deg_w(v_1) = \deg_w(v_5)$, it implies that,

$$w(v_5) = w(v_1) + w(v_3) + w(v_4) \quad (1) \quad \deg_w(v_2) =$$

$$w(v_1) + w(v_3) + w(v_5).$$

Since $\deg_w(v_2) = \deg_w(v_5)$, it implies that,

$$w(v_5) = w(v_2) + w(v_4) \quad (2) \quad \text{So from}$$

equations (1) and (2), we have $w(v_2) = w(v_1) + w(v_3)$

$$w(v_3) + w(v_5).$$

Since $\deg_w(v_4) = \deg_w(v_1)$, it implies that,

$$w(v_2) = w(v_3) \quad (3) \quad \text{So from}$$

(3) and (4), we have $w(v_1) = 0$. $\deg_w(v_3) = w(v_2) + w(v_4) + w(v_5)$. So since $\deg_w(v_3) = \deg_w(v_1)$, it implies that $w(v_4) = 0$. So from equation (1) and the fact that $w(v_1) = 0 = w(v_4)$, we have $w(v_5) = w(v_3)$

$$w(v_2) = w(v_1) \quad (5) \quad \text{So by}$$

transitivity, we have $w(v_2) = w(v_5)$ and this implies that $w(v_2) = w(v_3) = w(v_5)$. 

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Therefore since $\deg_w(v) = 2w(v_2)$ for all $v \in V$, then there exist a nontrivial weighting scheme which makes $F_5$ weighted regular.

□

Example 3.5 Let $w(v_2) = w(v_3) = w(v_5) = 3$ and $w(v_1) = w(v_2) = 0$.

![Figure 22: A 6_w-regular Fan Graph](image)

Clearly, $\deg_w(v) = 6$ for all $v \in V$ hence the graph is a $6_w$-regular graph.

Theorem 3.6 Let $F_6$ be a fan graph on 6 vertices. Then there does NOT exist a nontrivial weighting scheme which makes $F_6$ weighting regular.

Proof Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the 6 vertices with degrees 2, 3, 3, 3, 2, 5 respectively (Refer to Figure 26). We need to show that the degree weights are equal for all the vertices in the graph.
We have $\deg_w(v_6) = w(v_1) + w(v_2) + w(v_3) + w(v_4) + w(v_5)$ and $\deg_w(v_1) = w(v_2) + w(v_6)$. Since $\deg_w(v_1) = \deg_w(v_6)$, it implies that

$$w(v_6) = w(v_1) + w(v_3) + w(v_4) + w(v_5) \quad (1).$$

Since $\deg_w(v_1) = \deg_w(v_3)$, it implies that,

$$w(v_6) = w(v_2) + w(v_4) + w(v_5) \quad (2).$$

So from equations (1) and (2), we have $w(v_2) = w(v_1) + w(v_3)$

$$\deg_w(v_5) = w(v_1) + w(v_3) \quad (3).$$

Since $\deg_w(v_5) = \deg_w(v_1)$, it implies that

$$w(v_1) = w(v_2) \quad (4).$$

So, since $w(v_2) = w(v_4)$ from equations (2) and (4), we have $w(v_6) = w(v_5)$. $\deg_w(v_4) = w(v_3) + w(v_5) + w(v_6)$. However since $\deg_w(v_4) = \deg_w(v_2)$, it implies that $w(v_5) = w(v_1)$. 

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$\deg_w(v_3) = w(v_2) + w(v_4) + w(v_6)$. However since $\deg_w(v_3) = \deg_w(v_5)$, it implies $w(v_2) = 0$. Therefore $w(v_4) = 0$ by equation (4) and hence $w(v_3) = -w(v_1)$ by equation (3). Clearly, $w(v_5) = \deg_w(v_1) = \deg_w(v_2) = \deg_w(v_3) = \deg_w(v_5) = \deg_w(v_6) \neq \deg_w(v_4) = 2w(v_5) + w(v_3)$. Hence the graph $F_6$ does not have a nontrivial weighting scheme which makes it weighted regular.

\[ \square \]

**Theorem 3.7** Let $F_7$ be a fan graph on 7 vertices. Then there exist a nontrivial weighting scheme which makes $F_7$ weighting regular.

**Proof** Let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be the 7 vertices with degrees 2, 3, 3, 3, 3, 2, 6 respectively (Refer to Figure 24).

![Figure 24: A Fan graph on 7 vertices ($F_7$)](image)
\[ \deg_w(v_7) = w(v_1) + w(v_2) + w(v_3) + w(v_4) + w(v_5) + w(v_6) \text{ and } \deg_w(v_1) = w(v_2) + w(v_7). \] However since \( \deg_w(v_1) = \deg_w(v_7), \) it implies that \( w(v_7) = w(v_1) + w(v_3) + w(v_4) + w(v_5) + w(v_6). \) \( \deg_w(v_2) = w(v_1) + w(v_3) + w(v_7). \) So since \( \deg_w(v_2) = \deg_w(v_1), \) it implies that,

\[ w(v_2) = w(v_1) + w(v_3) \tag{1} \]

\( \deg_w(v_3) = w(v_2) + w(v_4) + w(v_7). \) So since \( \deg_w(v_3) = \deg_w(v_1), \) it implies \( w(v_4) = 0. \) \( \deg_w(v_6) = w(v_5) + w(v_7). \) So since \( \deg_w(v_6) = \deg_w(v_1), \) it implies \( w(v_5) = w(v_2). \) \( \deg_w(v_5) = w(v_4) + w(v_6) + w(v_7). \) So since \( \deg_w(v_5) = \deg_w(v_6), \) it implies \( w(v_5) = w(v_4) + w(v_6). \)

Therefore \( w(v_5) = w(v_6) \) since \( w(v_4) = 0. \) \( \deg_w(v_4) = w(v_3) + w(v_5) + w(v_7). \) So since \( \deg_w(v_4) = \deg_w(v_6), \) it implies,

\[ w(v_3) = 0 \tag{2} \]

So from equations (1) and (2), we have \( w(v_1) = w(v_2). \) This implies that \( w(v_1) = w(v_2) = w(v_5) = w(v_6). \) Therefore without loss of generality, \( \deg_w(v) = w(v_5) + w(v_6) \) for all \( v \in V. \) Hence there exist a nontrivial weighting scheme which makes \( F_7 \) weighted regular.

\[ \square \]

**Example 3.8** Let \( w(v_1) = w(v_2) = w(v_5) = w(v_6) = 2 \) and \( w(v_3) = w(v_4) = 0. \) This implies that \( w(v_7) = 6. \)
Clearly, $\deg_w(v) = 8$ for all $v \in V$ hence the graph is a $8_w$-regular graph.

OPEN CONJECTURE: A fan graph, $F_n$, has a nontrivial weighting scheme which makes it weighted regular when $n \equiv 0, 1, 3 \pmod{4}$, however for $n \equiv 2 \pmod{4}$, $F_n$ is not weighted regular.

**Theorem 3.9** A wheel graph, $W_{n+1}$, for $n \in \mathbb{N}$ has a nontrivial weighting scheme which makes them weighted regular.

**Proof** Let $W_{n+1}$ be a wheel graph with cardinality $n + 1$ and $v_{n+1}$ be the central vertex with degree $n$. This implies that there will be $n$ vertices with degree 3 each. Assign the degree 3 vertices $(v_1, v_2, v_3, v_4, ..., v_n)$ weights $a, b, a, b, ....$. This implies
that $\deg_w(v_{n+1}) = ia + jb$ where $i$ and $j$ are the number of assigned weights $a$ and $b$ respectively.

Note that $i + j = n$. So we need to show that $\deg_w(v) = ia + jb$ for all $v \in V$. Observe that $\deg_w(v_2) = w(v_1) + w(v_3) + w(v_{n+1}) = a + a + w(v_{n+1})$. However, since $\deg_w(v_2)$ has to be equal to the $\deg_w(v_{n+1})$, we have $a + a + w(v_{n+1}) = ia + jb$.

This implies that, $w(v_{n+1}) = (i - 2)a + jb$ (1). Therefore $\deg_w(v_2) = ia + jb$. Also, $\deg_w(v_3) = w(v_2) + w(v_4) + w(v_{n+1}) = b + b + w(v_{n+1}) = 2b + w(v_{n+1})$. However since $\deg_w(v_3)$ has to be equal to $\deg_w(v_{n+1}) = \deg_w(v_2)$, we have $2b + w(v_{n+1}) = ia + jb$. This implies that

\[
w(v_{n+1}) = ia + (j - 2)b \quad (2)
\]

So from equations (1) and (2), we have $(i - 2)a + jb = ia + (j - 2)b$. This implies $a = b$. Therefore without loss of generality, $w(v_{n+1}) = (i - 2 + j)a = (i + j - 2)a = (n - 2)a$.

This implies $\deg_w(v) = na = (i + j)a = ia + jb$ for all $v \in V$ and thus $W_{n+1}$ is weighted regular.

\[\square\]

**Example 3.10** Let $w(v_i) = 3$ for $i=1,2,3,4$. Then $w(v_5) = (4 - 2)3 = 6$. 

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Figure 26: A 12\textsubscript{w}-regular Wheel Graph (W\textsubscript{4+1})

Clearly, \( \deg_w(v) = 12 \) for all \( v \in V \) hence the graph is a 12\textsubscript{w}-regular graph.

**Theorem 3.11** A complete bipartite graph \( K_{m,n} \) has a nontrivial weighting scheme which makes it weighted regular.

**Proof** Let \( K_{m,n} \) be a complete bipartite graph with partite sets \( A \) and \( B \). Then \( |A| = m \) and \( |B| = n \) and thus \( |V| = m + n \). So for all \( v \in V \); \( \deg_w(v) = \sum w(v_i) \) for \( v_i \in B \) and \( i = 1, 2, ..., n \). Assign weights to \( u_i \in A \) for \( i = 1, 2, ..., m \) such that \( \sum_{u \in A} w(u_i) = \sum_{v \in B} w(v_i) \). Therefore clearly \( K_{m,n} \) is weighted regular.

\[ \square \]

Note that the above theorem is true for both equal\((m = n)\) and unequal\((m \neq n)\) partite sets.

**Example 3.12** Let the weights of \( v_i \in B \) for \( i = 1, 2, 3, 4, 5 \) be 7, 2, 5, 3, 0 respectively. Then the \( \deg_w(u_i) = 17 \) for all \( u_i \in A \). Also let the weights of \( u_i \in A \) for \( i = 1, 2, 3 \) be 8, 2, 7 respectively.
Figure 27: A $17_w$-regular Bipartite Graph

Clearly, $deg_w(v) = 17$ for all $v \in V$ hence the graph is a $17_w$-regular graph.
is a large volume of work on regular graphs in the field of graph theory. On the other hand, a review of the literature did not reveal a substantial amount of work on vertex weighted graphs. Only a few topics such as graceful labellings employed vertex weights. In this thesis, we utilize vertex weights to generalized regularity of graphs. All regular graphs are trivially vertex weighted regular graphs, however, a graph that is not necessarily regular can be vertex weighted regular.

Vertex weighted graphs have numerous application in contemporary science. It is therefore important to know if there exist a weighting scheme that will make a graph a vertex weighted regular graph. We address this question for a number of families of graphs such as paths, stars, spider and the wheel graph.

A problem for further study will be to “weight-regularize” other families of graphs such as bipartite graphs, friendship graphs etc.
BIBLIOGRAPHY


VITA
FRANKLINA SAMANI

Education: B.S. Statistics and Computer Science,
University Of Ghana,
Accra, Ghana 2009
M.S. Mathematical Sciences (Graph Theory),
East Tennessee State University.
Johnson City, Tennessee 2015

Professional Experience: Graduate Assistant, East Tennessee State University,
Johnson City, Tennessee, 2014 – present
Customer Service Representative, Liberty Walmart
Liberty, North Carolina, 2012–2013
Client Relationship Officer, uniBank Ghana Limited
Accra, Ghana, 2009–2012