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The Number of Zeros of a Polynomial in a Disk as a Consequence of Coefficient Inequalities with Multiple Reversals

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A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

Derek Bryant

December 2015

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Keywords: complex analysis, polynomial, number of zeros

ABSTRACT

The Number of Zeros of a Polynomial in a Disk as a Consequence of Coefficient Inequalities with Multiple Reversals

by

Derek Bryant

In this thesis, we explore the effect of restricting the coefficients of polynomials on the bounds for the number of zeros in a given region. The results presented herein build on a body of work, culminating in the generalization of bounds among three classes of polynomials. The hypotheses of monotonicity on each class of polynomials were further subdivided into sections concerning r reversals among the moduli, real parts, and both real and imaginary parts of the coefficients.

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DEDICATION

I would like to dedicate this thesis to my parents, who brought me to this earth; my friends, who kept me sane; and Red Bull® energy drink. It may not have given me wings, but it did fuel my fifty-seven-hour marathon sessions of typesetting, so that counts for something.

ACKNOWLEDGMENTS

I would like to acknowledge my committee for their gift time, careful consideration, and thoughtful input. To Dr. Beeler, your editing skills are unmatched. To Dr. Haynes, a more caring, considerate mathematician has never lived. To Dr. Gardner, your research interests have sparked in me a love and appreciation for a vast, teeming field of Mathematics. Thank you all.

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1 INTRODUCTION

Both the theoretical and applied branches of the mathematical community benefit greatly from research in the field of zeros of polynomials. In particular, the problem of counting zeros has been the subject of study for many mathematicians, and its difficulty has led to the creation of several other fields. In practice, a common technique is to apply restrictions on the coefficients to lessen the difficulty of finding zeros to a manageable degree. Like several fields of mathematics, this field owes much of its early history to Gauss, with his body of work leading to Gauss becoming known by contemporaries as 'Analysis Incarnate' (see page 440 of [3].)

While Gauss' and Cauchy's work developed in the early 19th century [8], it was not until the early 20th century that a Japanese mathematician by the name of Soichi Kakeya developed a result simultaneously with the Swedish mathematician Gustaf Hjalmar Eneström [13] that placed a bound on the location of the zeros of a polynomial having real, positive, and monotonically increasing coefficients. The Eneström-Kakeya Theorem, as it came to be known, concerns the location of zeros of a polynomial with monotonically increasing real coefficients. This theorem is the first we explore, but first we will state a few definitions.

For $z \in \mathbb{C}$, if $z = a+ib$ with $a, b \in \mathbb{R}$, the *modulus* of z is denoted $|z|$ and is defined as $|z| = \sqrt{a^2 + b^2}$. Further, we denote the *argument* of z as $\arg(z)$, and it represents the angle between the positive real axis to the line joining the point z to the origin. In addition, we denote the *real part* of z as $\mathrm{Re}(z) = a$ and the imaginary part of z as Im(z) = b. Finally, we say a function $f : G \to \mathbb{C}$, where G is an open connected subset of $\mathbb{C}$, is *analytic* if f is continuously differentiable on G.

Theorem 1.1. (Eneström-Kakaya) [16] For polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$, if the coefficients satisfy $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| \leq 1$.

Like many, Aziz and Mohammad [2] worked with the locations of zeros of an analytic function $f(z) = \sum_{j=0}^{\infty} a_j z^j$, where $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$. They imposed the condition $0 < \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots$, along with a similar condition on the β_j's [2]. Shields denoted this restriction on the coefficients as a "flip at k," where the monotonicity of the coefficients changes from increasing to decreasing [17]. In Chapter 2, we impose a generalization of this reversal condition on the coefficients of polynomials in order to count the number of zeros in a prescribed region.

As did Shields [17], we now put forth the idea of counting zeros of a polynomial which Titchmarsh's Number of Zeros Theorem used to obtain a bound on the number of zeros in a certain region.

Theorem 1.2. (Titchmarsh's Number of Zeros Theorem) [18]
Let f be analytic in $|z| < R$. Let $|f(z)| \leq M$ in the disk $|z| \leq R$ and suppose $f(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of $f(z)$ in the disk $|z| \leq \delta R$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}.$$

A proof of Theorem 1.2 is given by Shields on pages 9-10 of [17]. This theorem forms the basis of our research, since we are always able to relate back to Titchmarsh's result to obtain the number of zeros in a given region. Given a specific hypothesis on these coefficients, our work is to seek a specific value of M so that we obtain a better bound on the number of zeros of the polynomials with given coefficients. Note that our proofs will rely on Theorem 1.2, but they also rely on the Maximum Modulus Theorem.

Theorem 1.3. Maximum Modulus Theorem (page 165 of [18]) If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

Before we begin, there should be a discussion on what others have researched concerning the number of zeros of a polynomial using Titchmarsh-type results. Research in the field of mathematics concerning the counting of the number of zeros in a specific region is still very much active, with papers being published as recently as 2013 [9].

Mohammad used a special case of Theorem 1.2 by putting a restriction on the coefficients of a polynomial similar to that of Theorem 1.1 in order to prove the following:

Theorem 1.4. [14] Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be such that $0 < a + 0 \leq a_1 \leq \cdots \leq a_n$. Then the number of zeros in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \left(\frac{a_n}{a_0} \right).$$

Dewan weakened the hypothesis of Theorem 1.4 in her dissertation work, and proved the following two results for polynomials with complex coefficents:

Theorem 1.5. [7] Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be such that $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for all $1 \leq j \leq n$ and some real α and β, and $0 < |a_0| \leq |a_1| \leq \cdots \leq |a_n|$. Then the number of zeros of p in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Theorem 1.6. [7] Let $p(z) = \sum_{j=0}^{n} a_j z^j$ where $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$ for all j and $0 < \alpha_0 \leq \alpha_1 \leq \cdots \leq a_{n-1} \leq a_n$, then the number of zeros of p in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{\alpha_0}.$$

Theorems 1.5 and 1.6 were generalized by Pukhta, who found the number of zeros in $|z| \leq \delta$, for some $0 < \delta < 1$ [15]. The next theorem, due to Pukhta, concerns a monotonicity condition on the moduli of the coefficients.

Theorem 1.7. [15] Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be such that $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for all $1 \leq j \leq n$ and some real α and β, and $0 < |a_0| \leq |a_1| \leq \cdots \leq |a_n|$. Then the number of zeros of p in $|z| \leq \delta$, $0 < \delta < 1$, does not exceed

$$\frac{1}{\log 1/\delta} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Pukhta also proved a result involving a monotonicity condition on only the real part of the coefficients [15]. As noted by Shields [17], there was a slight typographical error in the statement of the result as it appeared in print, though the proof was correct. The correct statement of the theorem is as follows:

Theorem 1.8. [15] Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be such that $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for all $1 \leq j \leq n$ and some real α and β, and $0 < \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$. Then the number of zeros of p in $|z| \leq \delta$, $0 < \delta < 1$, does not exceed

$$\frac{1}{\log 1/\delta} \log \frac{2\left(\alpha_n + \sum_{j=0}^{n} |\beta_j|\right)}{|a_0|}.$$

Aziz and Zargar [1] together introduced the idea of imposing an inequality on both the even and odd indices for the coefficients of a polynomial separately. Cao and Gardner [4] impose conditions on the real parts of the coefficients and gave a result restriction the location of the zeros of a polynomial. In the same paper, Cao and Gardner gave a hypothesis with restriction on the real and imaginary parts of the coefficients, splitting them into even and odd indices.

We will reproduce the three major results of [17] in their entirety here, leaving it up to the reader to look up those cited in future sections. Note that these results on moduli restrictions are cases of those we will explore throughout this thesis; we are using the hypotheses in these results and generalizing them.

Theorem 1.9. [9] Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where for some $t > 0$ and some $0 \leq k \leq n$,

$$0 < |a_0| \leq |a_1|\, t^1 \leq |a_2|\, t^2 \leq \cdots \leq |a_k|\, t^k \geq |a_{k+1}|\, t^{k+1} \geq \cdots \geq |a_{n-1}|\, t^{n-1} \geq |a_n|\, t^n$$

and $|\arg(a_j) - \beta| \leq \alpha \leq \frac{\pi}{2}$ for $1 \leq j \leq n$ for some $\alpha, \beta \in \mathbb{R}$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{l}{\log 1/\delta} \log \frac{M}{|a_0|},$$

where

$$\begin{aligned} M \;&=\; |a_0|t(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha \\ &\quad +|a_n|t^{n+1}(1 + \sin\alpha - \cos\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|t^{j+1}. \end{aligned}$$

Theorem 1.10. [10] Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where for some $t > 0$ and some nonnegative integers k and ℓ,

$$0 \neq |a_0| \leq |a_2|\, t^2 \leq |a_4|\, t^4 \leq \cdots \leq |a_{2k}|\, t^{2k} \geq |a_{2k+2}|\, t^{2k+2} \geq \cdots \left|a_{2\lfloor n/2 \rfloor}\right| t^{2\lfloor n/2 \rfloor}$$

$$|a_1| \leq |a_3 t^2| \leq |a_5 t^4| \leq \cdots \leq |a_{2\ell-1} t^{2\ell-2}| \geq |a_{2\ell+1} t^{2\ell}| \geq \cdots |a_{2\lfloor (n+1)/2 \rfloor -1} t^{2\lfloor n/2 \rfloor}|$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{l}{\log 1/\delta} \log \frac{M}{|a_0|},$$

where

$$\begin{aligned} M &= (|a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2})(1 - \cos\alpha - \sin\alpha) \\ &\quad +2\cos\alpha(|a_{2k}|t^{2k+2} + |a_{2\ell-1}|t^{2\ell+1}) + 2\sin\alpha \sum_{j=0}^{n} |a_j|t^{j+2}. \end{aligned}$$

Theorem 1.11. [11] Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ where $|a_0| \neq 0$ and for some $t > 0$ and some k with $\mu \leq k \leq n$,

$$|a_\mu|\, t^\mu \leq \cdots \leq |a_{k-1}|\, t^{k-1} \leq |a_k|\, t^k \geq |a_{k+1}|\, t^{k+1} \geq \cdots \geq |a_{n-1}|\, t^{n-1} \geq |a_n|\, t^n$$

and $|\arg(a_j) - \beta| \leq \alpha \leq \frac{\pi}{2}$ for $1 \leq j \leq n$ for $\mu \leq j \leq n$ and for some real α and β. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{l}{\log 1/\delta} \log \frac{M}{|a_0|},$$

where

$$\begin{aligned} M &= 2|a_0|t + |a_\mu|t^{\mu+1}(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha \\ &\quad +|a_n|t^{n+1}(1 - \sin\alpha - \cos\alpha) + 2\sin\alpha \sum_{j=\mu}^{n} |a_j|t^{j+1}. \end{aligned}$$

Notice that in Theorem 1.9, Theorem 1.10, and Theorem 1.11, there is only one reversal, with such a reversal occurring at k, k and s, and k respectively. Chattopadhyay, Das, Jain, and Konwar introduced the concept of multiple reversals [6]. In particular, observe Theorem 2 from [6], which introduced the concept upon which this thesis builds, here reproduced nearly exactly as it appeared in print.

Theorem 1.12. (Theorem 2 from [6]) Let $p(z) = \sum_{j=0}^{n} a_n z^j$, be a polynomial of degree n. If $\operatorname{Re}(a_j) = \alpha_j$, $\operatorname{Im}(a_j) = \beta_j$, for $j = 0, 1, 2, \ldots, n$ and for certain non-negative integers $k_1, k_2, \ldots, k_p$; $r_1, r_2, \ldots, r_q$ and for certain $t > 0$

$$\alpha_0 \leq \alpha_1 t^1 \leq \ldots \leq \alpha_{k_1} t^{k_1} \geq \alpha_{k_1+1} t^{k_1+1} \geq \ldots \geq \alpha_{k_2} t^{k_2} \leq \alpha_{k_2+1} t^{k_2+1} \leq \ldots$$

$$\beta_0 \leq \beta_1 t^1 \leq \ldots \leq \beta_{k_1} t^{k_1} \geq \beta_{k_1+1} t^{k_1+1} \geq \ldots \geq \beta_{k_2} t^{k_2} \leq \beta_{k_2+1} t^{k_2+1} \leq \ldots$$

(with inequalities getting reversed at p indices $k_1, k_2, \ldots, k_p$ in the first inequality and $\alpha_n t^n$ being the last term in the first inequality, and similarly, inequalities getting reversed at p indices $r_1, r_2, \ldots, r_p$ in the second inequality and $\beta_n t^n$ being the last term in the second inequality), then all the zeros of $p(z)$ lie in

$$R_1 \leq |z| \leq R_2,$$

where

$$
\begin{aligned}
R_1 &= \min\left(\frac{t^2|a_0|}{M_1}, t\right) = \frac{t^2|a_0|}{M_1} = \frac{t^2|a_0|}{M_1'} \\
R_2 &= \max\left(\frac{M_2}{|a_n|}, \frac{1}{t}\right), \\
M_1 &= tM_1' \\
M_1' &= -\left\{\alpha_0 + (-1)^{p+1}\alpha_n t^n + \sum_{u=1}^{p}(-1)^u \alpha_{k_u} t^{k_u}\right\} \\
&\quad -\left\{\beta_0 + (-1)^{p+1}\beta_n t^n + \sum_{u=1}^{p}(-1)^u \beta_{k_u} t^{k_u}\right\} + |a_n|\, t^n, \\
M_2 &= -\alpha_0 t^{n-1} + (-1)^{p+1}\alpha_n t + (t^2+1)\sum_{u=1}^{p}(-1)^u \alpha_{k_u} t^{n-k_u-1} \\
&\quad +(t^2-1)\sum_{u=0}^{p}\left\{(-1)^{u+1}\sum_{m=k_u+1}^{k_{u+1}-1}\alpha_m t^{n-m-1}\right\} \\
&\quad -\beta_0 t^{n-1} + (-1)^{p+1}\beta_n t + (t^2+1)\sum_{s=1}^{q}(-1)^s \beta_{r_s} t^{n-r_s-1} \\
&\quad +(t^2-1)\sum_{s=0}^{q}\left\{(-1)^{s+1}\sum_{v=r_s+1}^{r_{s+1}-1}\beta_v t^{n-v-1}\right\} + |a_0|t^{n+1}, \\
&\quad k_0 = r_0 = 0, \\
&\quad k_{p+1} - r_{q+1} = n.
\end{aligned}
$$

It is Theorem 1.12 that provided the motivation for this thesis, allowing us to apply the idea of multiple reversals to each of the theorems explored by Shields in [17]. Namely, applying such an idea to Theorem 1.9 gave rise to our Theorem 2.1, making Theorem 1.9 a corollary.

In Chapter 3, we extend the hypotheses to count the number of zeros of a polynomial by considering the moduli, real, as well as real and imaginary restrictions of the even and odd indexed coefficients, given some numbers of reversals on the different coefficients.

In Chapter 4, we study a class of polynomials with a gap between the leading coefficient and the following nonzero coefficient, denoting the class of all such polynomials as $\mathcal{P}_{n,\mu}$, where each polynomial is of the form $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$. During their study of Bernstein-type inequalities, Chan and Malik [5], introduced this class of polynomials. It should be noted that $\mathcal{P}_{n,1} = \mathcal{P}_n$, the class of all polynomials of degree n.

2 A MONOTONICITY CONDITION ON ALL OF THE COEFFICIENTS WITH A NUMBER OF REVERSALS

In this chapter, we investigate the effect of placing a monotonicity condition on all the coefficients, assuming $1 \leq r < n$ reversals among the coefficients. In Section 2.1, we impose the condition on the moduli of the coefficients, in the manner of Dewan for locations of zeros [7]. In Section 2.2, we split the coefficients into the real and imaginary parts, placing a monotonicity restriction on only the real part, in the manner of Pukhta's generalization of Theorem 1.6 [15]. In Section 2.3, we consider the monotonicity restriction on both the real and imaginary parts of the coefficients.

2.1 Restrictions on the Moduli of the Coefficients Given r Reversals

In this section, we impose the condition on the moduli of the coefficients, assuming r reversals. These polynomials are related to results like Theorem 1.7 with monotonicity flips at each k_j, where $1 \leq j \leq r$. Note that we consider only those coefficients in the sector $|\arg(a_j) - \beta| \leq \alpha \leq \frac{\pi}{2}$ for $1 \leq j \leq n$ for some $\alpha, \beta \in \mathbb{R}$. The visualization of this is displayed in Figure 1, with each a_j represented by a yellow dot, and α and β are as shown. Please note that this is only an example of one arrangement of the α and β values.

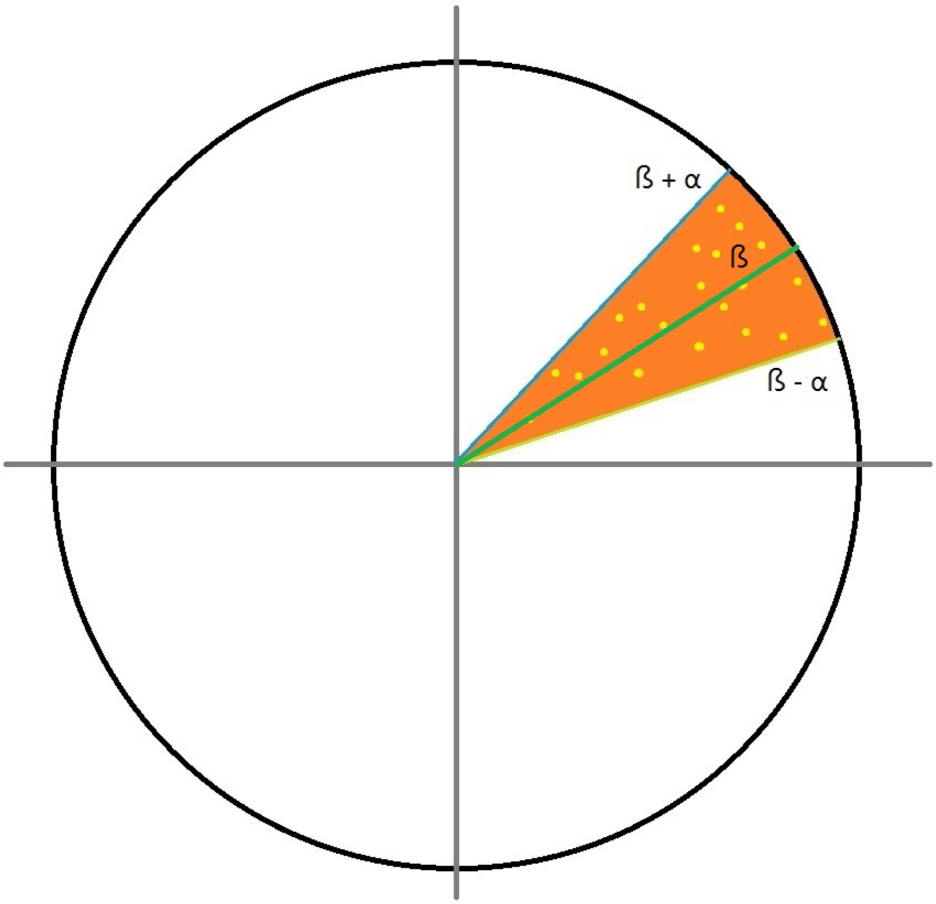

Figure 1: View of a specific sector.

Theorem 2.1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where for some $t > 0$ and some $0 < k_1 < k_2 < \cdots < k_r < n$,

$$0 < |a_0| \leq |a_1| t^1 \leq |a_2| t^2 \leq \cdots \leq |a_{k_1}| t^{k_1} \geq |a_{k_1+1}| t^{k_1+1} \geq \cdots \geq |a_{k_2}| t^{k_2} \leq \cdots$$

and $|\arg(a_j) - \beta| \leq \alpha \leq \frac{\pi}{2}$ for $1 \leq j \leq n$ for some $\alpha, \beta \in \mathbb{R}$ with $r \in \mathbb{N}, 1 \leq r < n$ the number of reversals. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk

$|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{|a_0|},$$

where

$$\begin{aligned} M_r &= |a_0|t(1-\cos\alpha-\sin\alpha)+2\cos\alpha\sum_{h=1}^{r}(-1)^{h+1}\left|a_{k_h}\right|t^{k_h+1} \\ &\quad +|a_n|t^{n+1}(1+\sin\alpha+(-1)^r\cos\alpha)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|t^{j+1}. \end{aligned}$$

Notice that when $r = 1$ in Theorem 2.1, it reduces to Theorem 1.9, a main result in [9]. Further, with $t = 1$ in Theorem 2.1, we obtain:

Corollary 2.1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ with $0 < k_1 \leq k_2 \leq \cdots \leq k_r < n$,

$$0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{k_1}| \geq |a_{k_1+1}| \geq \cdots \geq |a_{k_2}| \leq \cdots$$

and $|\arg(a_j) - \beta| \leq \alpha \leq \frac{\pi}{2}$ for $1 \leq j \leq n$ for some $\alpha, \beta \in \mathbb{R}$ with $r \in \mathbb{N}, 1 \leq r \leq n$ the number of reversals. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{|a_0|},$$

where

$$\begin{aligned} M_r &= |a_0|(1-\cos\alpha-\sin\alpha)+2\cos\alpha\sum_{h=1}^{r}(-1)^{h+1}\left|a_{k_h}\right| \\ &\quad +|a_n|(1+\sin\alpha+(-1)^r\cos\alpha)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|. \end{aligned}$$

For $r = 2$ and if each a_j is real and positive (that is, $\alpha = 0$) then Corollary 2.1 reduces to the following:

Corollary 2.2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ with $0 < k_1 \leq k_2 < n$,

$$0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{k_1}| \geq |a_{k_1+1}| \geq \cdots \geq |a_{k_2}| \leq |a_{k_2+1}| \leq \cdots \leq |a_n|$$

and $|\arg(a_j) - \beta| \leq \alpha = 0$ for $1 \leq j \leq n$ for some $\beta \in \mathbb{R}$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log\left(2\frac{|a_{k_1}| - |a_{k_2}| + |a_n|}{|a_0|}\right).$$

Example 2.1. Consider the polynomial $P(z) = 1 + 50z + 45z^2 + 1000z^3$. Since we have $a_0 = 1, a_{k_1} = 50$, and $a_{k_2} = 45$, there are two reversals. Note that $P(z) = 0$ when $z \approx -0.0202024$ and $z \approx 0.0123988 \pm 0.222138i$. With $\delta = 0.021$, Corollary 2.2 implies that the number of zeros in $|z| \leq \delta = 0.021$ is less than $\frac{1}{\log(1/0.021)} \log \frac{2(50-45+1000)}{1} \approx 1.97$, which implies that P has at most one zero in $|z| \leq 0.021$, and P has exactly one zero in this region. Therefore, this example shows that Corollary 2.2 is sharp (that is, best possible) for certain examples.

Example 2.2. Consider the polynomial $P(z) = 1 + 50z + 45z^2 + 10000z^3$. Since we have $a_0 = 1, a_{k_1} = 50$, and $a_{k_2} = 45$, there are two reversals. Note that $P(z) = 0$ when $z \approx -0.0189603$ and $z \approx 0.00723016 \pm 0.0722627i$. With $\delta = 0.0726235$, Corollary 2.2 implies that the number of zeros in $|z| \leq \delta = 0.073$ is less than $\frac{1}{\log(1/0.073)} \log \frac{2(50-45+10000)}{1} \approx 3.784$, which implies that P has at most three zeros in $|z| \leq 0.073$, and P has exactly three zeros in this region. In this example it has been shown that in particular cases our results can be used to locate zeros.

Before we begin the proof of Theorem 2.1, we need a lemma previously shown by Govil and Rahman as well as a statement of the Maximum Modulus Theorem.

Lemma 2.1. [12] Let $z, z' \in \mathbb{C}$ with $|z| \geq |z'|$. Suppose that $|\arg(z^*) - \beta| \leq \alpha \leq \pi/2$ for $z^* \in \{z, z'\}$ and for some real α and β. Then

$$|z - z'| \leq (|z| - |z'|)\cos\alpha + (|z| + |z'|)\sin\alpha.$$

Proof of Theorem 2.1. Consider

$$\begin{aligned} F(z) &= (t-z)P(z) = (t-z)\sum_{j=0}^{n} a_j z^j \\ &= \sum_{j=0}^{n}(a_j t z^j - a_j z^{j+1}) = \sum_{j=0}^{n} a_j t z^j - \sum_{j=0}^{n} a_j z^{j+1} \\ &= a_0 t + \sum_{j=1}^{n} a_j t z^j - \sum_{j=1}^{n}(a_{j-1} z^j) - a_n z^{n+1} \\ &= a_0 t + \sum_{j=1}^{n}[(a_j t - a_{j-1}) z^j] - a_n z^{n+1}. \end{aligned}$$

For $|z| = t$ we have:

$$\begin{aligned} |F(z)| &\leq |a_0|t + \sum_{j=1}^{n} |a_j t - a_{j-1}| t^j + |a_n| t^{n+1} \\ &\leq |a_0|t + \sum_{j=1}^{k_1} |(-1)^0 (a_j t - a_{j-1})| t^j + \sum_{j=k_1+1}^{k_2} |(-1)^1 (a_j t - a_{j-1})| t^j + \cdots \\ &\quad + \sum_{j=k_r+1}^{n} |(-1)^r (a_j t - a_{j-1})| t^j + |a_n| t^{n+1} \\ &= |a_0|t + \sum_{j=1}^{k_1} |(-1)^0 (a_j t - a_{j-1})| t^j + \sum_{h=1}^{r-1} \left(\sum_{j=k_h+1}^{k_{h+1}} |(-1)^h (a_j t - a_{j-1})| t^j \right) \\ &\quad + \sum_{j=k_r+1}^{n} |(-1)^r (a_j t - a_{j-1})| t^j + |a_n| t^{n+1} := S. \end{aligned}$$

Then by Lemma 2.1 with $z = a_j t$ and $z' = a_{j-1}$ when

$$\begin{array}{cl} 1 \leq j \leq k_1 & \\ k_2 + 1 \leq j \leq k_3 & \\ k_4 + 1 \leq j \leq k_5 & \\ \vdots & \\ k_r + 1 \leq j \leq n & \text{if } r \text{ is even} \\ k_{r-1} + 1 \leq j \leq k_r & \text{if } r \text{ is odd} \end{array}$$

and with $z = a_{j-1}$ and $z' = a^j t$ when

$$\begin{array}{cl} k_1+1 \le j \le k_2 & \\ k_3+1 \le j \le k_4 & \\ k_5+1 \le j \le k_6 & \\ \vdots & \\ k_{r-1}+1 \le j \le k_r & \text{if } r \text{ is even} \\ k_r+1 \le j \le n & \text{if } r \text{ is odd,} \end{array}$$

we have for $r \in \mathbb{N}$ with $2 \le r \le n$,

$$\begin{aligned} S \quad \le \quad & |a_0|t + \sum_{j=1}^{k_1} \left\{ |(-1)^0(|a_j t| - |a_{j-1}|) \cos\alpha + (|a_{j-1}| + |a_j t|) \sin\alpha \right\} t^j \\ & + \sum_{h=1}^{r-1} \left(\sum_{j=k_h+1}^{k_{h+1}} \left\{ |(-1)^h(|a_j t| - |a_{j-1}|) \cos\alpha + (|a_{j-1}| + |a_j t|) \sin\alpha \right\} t^j \right) \\ & + \sum_{j=k_r+1}^{n} \{ |(-1)^r(|a_j t| - |a_{j-1}|) \cos\alpha + (|a_{j-1}| + |a_j t|) \sin\alpha \} t^j + |a_n| t^{n+1} \\ = \quad & |a_0|t + |a_n|t^{n+1} + \sum_{j=1}^{k_1} (-1)^0 (|a_j| t^{j+1} - |a_{j-1}| t^j) \cos\alpha \\ & + \sum_{h=1}^{r-1} \left(\sum_{j=k_h+1}^{k_{h+1}} (-1)^h (|a_j| t^{j+1} - |a_{j-1}| t^j) \cos\alpha \right) \\ & + \sum_{j=k_r+1}^{n} (-1)^r (|a_j| t^{j+1} - |a_{j-1}| t^j) \cos\alpha + \sum_{j=1}^{k_1} (|a_{j-1}| t^j + |a_j| t^{j+1}) \sin\alpha \\ & + \sum_{h=1}^{r-1} \left(\sum_{j=k_h+1}^{k_{h+1}} (|a_{j-1}| t^j + |a_j| t^{j+1}) \sin\alpha \right) + \sum_{j=k_r+1}^{n} (|a_{j-1}| t^j + |a_j| t^{j+1}) \sin\alpha \\ := \quad & S' \end{aligned}$$

Remark 2.1. Note the summations including a $\sin\alpha$ term may be collected as

$$\mathcal{S} = \sum_{j=1}^{n} (|a_{j-1}| t^j + |a_j| t^{j+1}) \sin\alpha.$$

Then we have

$$\begin{aligned}
\mathcal{S} &= (|a_0|t + |a_1|t^2)\sin\alpha + (|a_1|t^2 + |a_2|t^3)\sin\alpha + (|a_2|t^3 + |a_3|t^4)\sin\alpha \\
&\quad + \cdots + (|a_{n-2}|t^{n-1} + |a_{n-1}|t^n)\sin\alpha + (|a_{n-1}|t^n + |a_n|t^{n+1})\sin\alpha \\
&= |a_0|t\sin\alpha + (|a_1|t^2 + |a_1|t^2)\sin\alpha + (|a_2|t^3 + |a_2|t^3)\sin\alpha + (|a_3|t^4 + |a_3|t^4)\sin\alpha \\
&\quad + \cdots + (|a_{n-1}|t^n + |a_{n-1}|t^n)\sin\alpha + |a_n|t^{n+1}\sin\alpha \\
&= |a_0|t\sin\alpha + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|t^{j+1} + |a_n|t^{n+1}\sin\alpha.
\end{aligned}$$

Remark 2.2. Now consider

$$C_0 = \sum_{j=1}^{k_1}(-1)^0(|a_j|t^{j+1} - |a_{j-1}|t^j)\cos\alpha = \left(|a_{k_1}|t^{k_1+1} - |a_0|t^1\right)\cos\alpha,$$

since it is clearly a telescoping sum. By the same logic, for $1 \le h \le r-1$, each

$$C_h = \sum_{j=k_h+1}^{k_{h+1}}(-1)^h(|a_j|t^{j+1} - |a_{j-1}|t^j)\cos\alpha = (-1)^h\left(|a_{k_{h+1}}|t^{k_{h+1}+1} - |a_{k_h}|t^{k_h+1}\right)\cos\alpha.$$

Finally, we have

$$C_r = \sum_{j=k_r+1}^{n}(-1)^r(|a_j|t^{j+1} - |a_{j-1}|t^j)\cos\alpha = (-1)^r\left(|a_n|t^{n+1} - |a_{k_r}|t^{k_r+1}\right)\cos\alpha.$$

Adding each of these $r+1$ sums together gives

$$\mathcal{C} = C_0 + \sum_{h=1}^{r-1}C_h + C_r$$

so that we have

$$\begin{aligned}
\mathcal{C} = {} & \left(|a_{k_1}|t^{k_1+1} - |a_0|t^1\right)\cos\alpha \\
& + (-1)^1\left(|a_{k_2}|t^{k_2+1} - |a_{k_1}|t^{k_1+1}\right)\cos\alpha \\
& + (-1)^2\left(|a_{k_3}|t^{k_3+1} - |a_{k_2}|t^{k_2+1}\right)\cos\alpha \\
& + (-1)^3\left(|a_{k_4}|t^{k_4+1} - |a_{k_3}|t^{k_3+1}\right)\cos\alpha + \cdots \\
& + (-1)^r\left(|a_n|t^{n+1} - |a_{k_r}|t^{k_r+1}\right)\cos\alpha.
\end{aligned}$$

Observe that the $|a_0|t$ and $|a_n|t^{k_n+1}$ terms have nothing to pair with, but each other term appears twice in the sum. Further, terms with odd indices have positive sign, whereas terms with even indices have negative sign. Then we have that

$$\mathcal{C} = (-1)^r |a_n| t^{k_n+1} \cos\alpha + 2\cos\alpha \left(\sum_{h=1}^{r} (-1)^{h+1} |a_{k_h}| t^{k_h+1} \right) - |a_0| t \cos\alpha.$$

Then using Remarks 2.1 and 2.2, we may rewrite S' as

$$\begin{aligned} S' &= |a_0|t + \mathcal{C} + \mathcal{S} + |a_n|t^{n+1} \\ &= |a_0|t + (-1)^r |a_n| t^{k_n+1} \cos\alpha + 2\cos\alpha \left(\sum_{h=1}^{r} (-1)^{h+1} |a_{k_h}| t^{k_h+1} \right) - |a_0| t \cos\alpha \\ &\quad + |a_0| t \sin\alpha + 2\sin\alpha \sum_{j=1}^{n-1} (|a_j| t^{j+1}) + |a_n| t^{n+1} \sin\alpha + |a_n| t^{n+1} \\ &= |a_0| t (1 - \cos\alpha - \sin\alpha) + 2\cos\alpha \sum_{h=1}^{r} (-1)^{h+1} \, |a_{k_h}| \, t^{k_h+1} \\ &+ \quad |a_n| t^{n+1} (1 + \sin\alpha + (-1)^r \cos\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| t^{j+1} \\ &= M_r. \end{aligned}$$

Now $F(z)$ is analytic in $|z| \leq t$ and $|F(z)| \leq M_r$ for $|z| = t$. So by Theorem 1.2 and the Maximum Modulus Theorem, the number of zeros of F (and hence of P) in $|z| \leq \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{|a_0|}.$$

The theorem follows. □

2.2 Restrictions on the Real Part of the Coefficients Given r Reversals

In this section, we force a restriction of the monotonicity only on the real part of the coefficients, along with a t condition and reversals at each $k_1, k_2, \ldots, k_r$. With our number of zeros result in mind, we again seek a different M_r value. First, we show Theorem 2.2:

Theorem 2.2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $0 \leq j \leq n$. Suppose that for some $t > 0$ and some $0 < k_1 \leq k_2 \leq \cdots \leq k_r < n$,

$$0 < \alpha_0 \leq \alpha_1 t^1 \leq \alpha_2 t^2 \leq \cdots \leq \alpha_{k_1} t^{k_1} \geq \alpha_{k_1+1} t^{k_1+1} \geq \cdots \geq \alpha_{k_2} t^{k_2} \leq \cdots$$

with $r \in \mathbb{N}, 1 \leq r < n$ the number of reversals. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{|a_0|},$$

where

$$M_r = (|\alpha_0| - \alpha_0)t + 2\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j} t^{k_j+1} + (|\alpha_n| + (-1)^r \alpha_n)t^{n+1} + 2\sum_{j=0}^{n} |\beta_j| t^{j+1}.$$

Note that for $r = 1$, we have a result of Gardner and Shields, which appeared in [9]. Now for $t = 1$ in Theorem 2.2, we obtain:

Corollary 2.3. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $0 \leq j \leq n$. Suppose that for some $0 < k_1 < k_2 < \cdots < k_r < n$,

$$0 < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{k_1} \geq \alpha_{k_1+1} \geq \cdots \geq \alpha_{k_2} \leq \cdots$$

with $r \in \mathbb{N}, 1 \leq r < n$ the number of reversals. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{|a_0|},$$

where

$$M_r = (|\alpha_0| - \alpha_0) + 2\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j} + (|\alpha_n| + (-1)^r \alpha_n) + 2\sum_{j=0}^{n} |\beta_j|.$$

Further, let $r = 2$ and $\beta_j = 0$ for all $0 \le j \le n$ in Corollary 2.3 to obtain:

Corollary 2.4. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $0 \le j \le n$. Suppose that for some $0 < k_1 \le k_2 < n$, we have

$$0 < \alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{k_1} \ge \alpha_{k_1+1} \ge \cdots \ge \alpha_{k_2} \le \cdots \le \alpha_n$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \le \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{|a_0|},$$

where

$$M_r = (|\alpha_0| - \alpha_0) + 2(\alpha_{k_1} - \alpha_{k_2}) + (|\alpha_n| + \alpha_n).$$

Proof of Theorem 2.2. As in the proof of Theorem 2.1,

$$\begin{aligned} F(z) &= (t-z)P(z) \\ &= (|\alpha_0| + i|\beta_0|) + \sum_{j=1}^{n} [(\alpha_j + t\beta_j)t - (\alpha_{j-1} + i\beta_{j-1})]z^j \\ &\quad -(\alpha_n + i\beta_n)z_{n+1} \\ &= (|\alpha_0| + i|\beta_0|) + \sum_{j=1}^{n} (\alpha_j t - \alpha_{j-1})z^j + i\sum_{j=1}^{n} (\beta_j t - \beta_{j-1})z^j \\ &\quad -(\alpha_n + i\beta_n)z_{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$\begin{aligned} |F(z)| \le\ & (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{n} |\alpha_j t - \alpha_{j-1}|t^j \\ & + \sum_{j=1}^{n} |\beta_j t - \beta_{j-1}|t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \end{aligned}$$

$$
\begin{aligned}
&= (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{k_1}((-1)^0(\alpha_j t - \alpha_{j-1}))t^j \\
&\quad + \sum_{j=k_1+1}^{k_2}((-1)^1(\alpha_j t - \alpha_{j-1}))t^j + \cdots \\
&\quad + \sum_{j=k_r+1}^{n}((-1)^r(\alpha_j t - \alpha_{j-1}))t^j + \sum_{j=1}^{n-1}|\beta_j|t^{j+1} \\
&\quad +|\beta_n|t^{n+1} + |\beta_0|t + \sum_{j=1}^{n-1}|\beta_j t^{j+1} + (|\alpha_n| + |\beta_n|)t_{n+1} \\
&= |\alpha_0|t + \sum_{j=1}^{k_1}((-1)^0(\alpha_j t - \alpha_{j-1}))t^j \\
&\quad + \sum_{j=k_1+1}^{k_2}((-1)^1(\alpha_j t - \alpha_{j-1}))t^j + \cdots \\
&\quad + \sum_{j=k_r+1}^{n}((-1)^r(\alpha_j t - \alpha_{j-1}))t^j + 2\sum_{j=0}^{n}|\beta_j|t^{j+1} + |\alpha_n|t_{n+1} \\
&= S.
\end{aligned}
$$

Note now that each summation involving α has its terms cancel and pair as in Remark 2.2 in Theorem 2.1. Then we have

$$
\begin{aligned}
S &= |\alpha_0|t + (-1)^r\alpha_n t^{n+1} + 2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}t^{k_j+1}\right) - \alpha_0 t \\
&\quad +2\sum_{j=0}^{n}|\beta_j|t^{j+1} + |\alpha_n|t_{n+1} \\
&= (|\alpha_0| - \alpha_0)t + 2\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}t^{k_j+1} + (|\alpha_n| + (-1)^r\alpha_n)t^{n+1} + 2\sum_{j=0}^{n}|\beta_j|t^{j+1} \\
&= M_r.
\end{aligned}
$$

The result now follows as in the proof of Theorem 2.1. $\square$

Note that Theorem 2.2 does not impose any condition on the imaginary parts of the coefficients. However, if we have a monotonicity condition on the imaginary part as well, we can further refine it, as we do in the next section.

2.3 Restrictions on the Real and Imaginary Parts of the Coefficients Given r, ρ Reversals

In this section, we consider results stemming from the placement of restrictions on both the real and imaginary parts of the coefficients. The real and imaginary parts have reversals at k_j and l_m respectively, where $1 \leq j \leq r$ and $1 \leq m \leq \rho$. Note that r is the number of reversals among the real parts of the coefficients and ρ is the number of reversals among the imaginary parts of the coefficients. To begin, observe Theorem 2.3:

Theorem 2.3. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $0 \leq j \leq n$. Suppose that for some $t > 0$ and some $0 < k_1 \leq k_2 \leq \cdots \leq k_r < n$,

$$0 < \alpha_0 \leq \alpha_1 t^1 \leq \alpha_2 t^2 \leq \cdots \leq \alpha_{k_1} t^{k_1} \geq \alpha_{k_1+1} t^{k_1+1} \geq \cdots \geq \alpha_{k_2} t^{k_2} \leq \cdots$$

and for some $0 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_r \leq n$,

$$0 < \beta_0 \leq \beta_1 t^1 \leq \beta_2 t^2 \leq \cdots \leq \beta_{\ell_1} t^{\ell_1} \geq \beta_{\ell_1+1} t^{\ell_1+1} \geq \cdots \geq \beta_{\ell_2} t^{\ell_2} \leq \cdots$$

with $r \in \mathbb{N}, 1 \leq r < n$ the number of reversals for the real part of a_j and with $\rho \in \mathbb{N}, 1 \leq \rho < n$ the number of reversals for the imaginary part of a_j. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_{[r,\rho]}}{|a_0|},$$

where

$$
\begin{aligned}
M_{[r,\rho]} &= (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0)t + 2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}t^{k_j+1}\right) \\
&\quad +2\left(\sum_{j=1}^{\rho}(-1)^{j+1}\beta_{\ell_j}t^{\ell_j+1}\right) + (|\alpha_n| + (-1)^r\alpha_n + |\beta_n| + (-1)^\rho\beta_n)t^{n+1}.
\end{aligned}
$$

Proof of Theorem 2.3. As in the proof of Theorem 2.2,

$$
\begin{aligned}
F(z) &= (t-z)P(z) \\
&= (|\alpha_0| + i|\beta_0| + \sum_{j=1}^{n}(\alpha_j t - \alpha_{j-1})z^j + i\sum_{j=1}^{n}(\beta_j t - \beta_{j-1})z^j \\
&\quad -(\alpha_n + i\beta_n)z_{n+1}
\end{aligned}
$$

For $|z| = t$ we have

$$
\begin{aligned}
|F(z)| &\leq (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{n}|\alpha_j t - \alpha_{j-1}|t^j \\
&\quad + \sum_{j=1}^{n}|\beta_j t - \beta_{j-1}|t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{k_1}((-1)^0(\alpha_j t - \alpha_{j-1}))t^j + \sum_{j=k_1+1}^{k_2}((-1)^1(\alpha_j t - \alpha_{j-1}))t^j \\
&\quad + \cdots + \sum_{j=k_r+1}^{n}((-1)^r(\alpha_j t - \alpha_{j-1}))t^j + \sum_{j=1}^{\ell_1}((-1)^0(\beta_j t - \beta_{j-1}))t^j \\
&\quad + \sum_{j=\ell_1+1}^{\ell_2}((-1)^1(\beta_j t - \beta_{j-1}))t^j + \cdots + \sum_{j=\ell_\rho+1}^{n}((-1)^\rho(\beta_j t - \beta_{j-1}))t^j \\
&\quad +(|\alpha_n| + |\beta_n|)t_{n+1} \\
&= S.
\end{aligned}
$$

Note now that each summation involving α has its terms cancel and pair as in Remark 2.2 in Theorem 2.1, with the same being said for each summation involving β. Then we have

$$\begin{aligned}
S &= (|\alpha_0|+|\beta_0|)t + (-1)^r\alpha_n t^{n+1} + 2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}t^{k_j+1}\right) - \alpha_0 t \\
&\quad +(-1)^\rho\beta_n t^{n+1} + 2\left(\sum_{j=1}^{\rho}(-1)^{j+1}\beta_{\ell_j}t^{\ell_j+1}\right) - \beta_0 t + (|\alpha_n|+|\beta_n|)t^{n+1} \\
&= (|\alpha_0|-\alpha_0+|\beta_0|-\beta_0)t + 2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}t^{k_j+1}\right) - \alpha_0 t \\
&\quad +2\left(\sum_{j=1}^{\rho}(-1)^{j+1}\beta_{\ell_j}t^{\ell_j+1}\right) + (|\alpha_n|+(-1)^r\alpha_n+|\beta_n|+(-1)^\rho\beta_n)t^{n+1} \\
&= M_{[r,\rho]}.
\end{aligned}$$

The result now follows as in the proof of Theorem 2.1. □

When $r = \rho = 1$, Theorem 2.3 reduces to Theorem 2 in [9]. With $t = 1$ in Theorem 2.3, we have the following:

Corollary 2.5. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $0 \leq j \leq n$. Suppose that for some $0 < k_1 \leq k_2 \leq \cdots \leq k_r < n$,

$$0 < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{k_1} \geq \alpha_{k_1+1} \geq \cdots \geq \alpha_{k_2} \leq \cdots$$

and for some $0 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_r \leq n$,

$$0 < \beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{\ell_1} \geq \beta_{\ell_1+1} \geq \cdots \geq \beta_{\ell_2} \leq \cdots$$

with $r \in \mathbb{N}, 1 \leq r < n$ the number of reversals for the real part of a_j and with $\rho \in \mathbb{N}, 1 \leq \rho < n$ the number of reversals for the imaginary part of a_j. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta}\log\frac{M_{[r,\rho]}}{|a_0|},$$

where

$$
\begin{aligned}
M_{[r,\rho]} &= (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0) + 2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}\right) \\
&\quad +2\left(\sum_{j=1}^{\rho}(-1)^{j+1}\beta_{\ell_j}\right) + (|\alpha_n| + (-1)^r\alpha_n + |\beta_n| + (-1)^\rho\beta_n).
\end{aligned}
$$

In this chapter, we investigated the effect of placing a monotonicity condition on all the coefficients, assuming $1 \leq r < n$ reversals among the coefficients. In Section 2.1, we imposed the condition on the moduli of the coefficients, in the manner of Dewan for locations of zeros [7]. In Section 2.2, we split the coefficients into the real and imaginary parts, placing a monotonicity restriction on only the real part, in the manner of Pukhta's generalization of Theorem 1.6 [15]. In Section 2.3, we considered the monotonicity restriction on both the real and imaginary parts of the coefficients.

3 A MONOTONICITY CONDITION ON THE COEFFICIENTS OF EVEN POWERS AND COEFFICIENTS OF ODD POWERS OF THE VARIABLE WITH A NUMBER OF REVERSALS

In this chapter, we explore similar types of restrictions of the coefficients of the polynomial as before, but in addition we impose the monotonicity condition on the even and odd indexed coefficients separately, as did Cao and Gardner [4] for the locations of zeros. There are several possibilities for corollaries in this chapter, so most are omitted with a note that they can all be obtained through standard algebra.

3.1 Restrictions on the Moduli of the Coefficients Given r_1, r_2 Reversals

In this section, we consider the moduli of the coefficients with reversals considered separately for both the even and odd indices. As usual, we seek an M value so that we may use Theorem 1.2. As a notational choice, we use a superscript e to denote even coefficients and a superscript o to denote those coefficients that are odd.

Theorem 3.1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where for some $t > 0$, for some $0 = 2k_0^e < 2k_1^e < \cdots < 2k_{r_1}^e < 2k_{r_1+1}^e = 2\lfloor n/2 \rfloor$ we have

$$0 < |a_0| \leq t^2|a_2| \leq t^4|a_4| \leq \cdots \leq t^{2k_1^e}|a_{2k_1^e}| \geq t^{2k_1^e+1}|a_{2k_1^e+1}| \geq \cdots$$

$$\geq t^{2k_2^e}|a_{2k_2^e}| \leq t^{2k_2^e+1}|a_{2k_2^e+1}| \leq \cdots \leq t^{2k_3^e}|a_{2k_3^e}| \geq \cdots$$

(with inequalities reversed at indices $2k_1^e, 2k_2^e, \ldots, 2k_{r_1}^e$ and $t^{2\lfloor n/2 \rfloor}|a_{2\lfloor n/2 \rfloor}|$ is the last term in the inequality), and for some $1 = 2k_0^o - 1 < 2k_1^o - 1 < \cdots < 2k_{r_2}^o - 1 < 2k_{r_2+1}^o - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ we have

$$|a_1| \leq t^2|a_3| \leq t^4|a_5| \leq \cdots \leq t^{2k_1^o-2}|a_{2k_1^o-1}| \geq t^{2k_1^o}|a_{2k_1^o+1}| \geq \cdots$$

$$\geq t^{2k_2^o-2}|a_{2k_2^o-1}| \leq t^{2k_2^p}|a_{2k_2^o+1}| \leq \cdots \leq t^{2k_3^o-2}|a_{2k_3^o-1}| \geq \cdots$$

(with inequalities reversed at indices $2k_1^o - 1, 2k_2^o - 1, \ldots, 2k_{r_2}^o - 1$ and $t^{2\lfloor (n+1)/2 \rfloor -2}|a_{2\lfloor (n+1)/2 \rfloor -1}|$ is the last term in the inequality). Also suppose $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for $1 \leq j \leq n$ and some real α and β. Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log(1/\delta)} \log\left(\frac{M}{|a_0|}\right)$$

where

$$\begin{aligned}
M &= (|a_0|t^2 + |a_1|t^3)(1 - \sin\alpha - \cos\alpha) + (|a_{n-1}|t^{n+1} + |a_n|t^{n+2})(1 - \sin\alpha) \\
&\quad +(-1)^{r_1}|a_{2\lfloor n/2 \rfloor}|t^{2\lfloor n/2 \rfloor+2}\cos\alpha + (-1)^{r_2}|a_{2\lfloor (n+1)/2 \rfloor -1}|t^{2\lfloor (n+1)/2 \rfloor +1}\cos\alpha \\
&\quad +2\sin\alpha \sum_{j=0}^{n} |a_j|t^{j+2} + 2\cos\alpha \sum_{j=1}^{r_1} (-1)^{j+1}|a_{2k_j^e}|t^{k_j^e+2} \\
&\quad +2\cos\alpha \sum_{j=1}^{r_2} (-1)^{j+1}|a_{2k_j^o-1}|t^{k_j^o+1}.
\end{aligned}$$

When $r_1 = r_2 = 1$, Theorem 3.1 reduces to Theorem 1.10, which is a main result in [10]. Let $t = 1$ to obtain the following:

Corollary 3.1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where for some $0 = 2k_0^e < 2k_1^e < \cdots < 2k_{r_1}^e < 2k_{r_1+1}^e = 2\lfloor n/2 \rfloor$ we have

$$0 < |a_0| \leq t^2|a_2| \leq t^4|a_4| \leq \cdots \leq t^{2k_1^e}|a_{2k_1^e}| \geq t^{2k_1^e+1}|a_{2k_1^e+1}| \geq \cdots$$

$$\geq t^{2k_2^e}|a_{2k_2^e}| \leq t^{2k_2^e+1}|a_{2k_2^e+1}| \leq \cdots \leq t^{2k_3^e}|a_{2k_3^e}| \geq \cdots$$

(with inequalities reversed at indices $2k_1^e, 2k_2^e, \ldots, 2k_{r_1}^e$ and $t^{2\lfloor n/2 \rfloor}|a_{2\lfloor n/2 \rfloor}|$ is the last term in the inequality), and for some $1 = 2k_0^o - 1 < 2k_1^o - 1 < \cdots < 2k_{r_2}^o - 1 < 2k_{r_2+1}^o - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ we have

$$|a_1| \leq t^2|a_3| \leq t^4|a_5| \leq \cdots \leq t^{2k_1^o-2}|a_{2k_1^o-1}| \geq t^{2k_1^o}|a_{2k_1^o+1}| \geq \cdots$$

$$\geq t^{2k_2^o-2}|a_{2k_2^o-1}| \leq t^{2k_2^p}|a_{2k_2^o+1}| \leq \cdots \leq t^{2k_3^o-2}|a_{2k_3^o-1}| \geq \cdots$$

(with inequalities reversed at indices $2k_1^o - 1, 2k_2^o - 1, \ldots, 2k_{r_2}^o - 1$ and $t^{2\lfloor (n+1)/2\rfloor -2}|a_{2\lfloor (n+1)/2\rfloor -1}|$ is the last term in the inequality). Also suppose $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for $1 \leq j \leq n$ and some real β. Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log(1/\delta)} \log\left(\frac{M}{|a_0|}\right)$$

where

$$\begin{aligned} M &= (|a_0| + |a_1|)(1 - \sin\alpha - \cos\alpha) + (|a_{n-1}| + |a_n|)(1 - \sin\alpha) \\ &\quad +(-1)^{r_1}|a_{2\lfloor n/2\rfloor}|\cos\alpha + (-1)^{r_2}|a_{2\lfloor (n+1)/2\rfloor -1}|\cos\alpha \\ &\quad +2\sin\alpha \sum_{j=0}^{n}|a_j| + 2\cos\alpha\left[\sum_{j=1}^{r_1}(-1)^{j+1}|a_{2k_j^e}| + \sum_{j=1}^{r_2}(-1)^{j+1}|a_{2k_j^o-1}|\right]. \end{aligned}$$

Further, consider the case when the coefficients are real and positive, we may take $\alpha = 0$ in Corollary 3.1 to obtain:

Corollary 3.2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ have real positive coefficients where for some $0 = 2k_0^e < 2k_1^e < \cdots < 2k_{r_1}^e < 2k_{r_1+1}^e = 2\lfloor n/2\rfloor$ we have

$$0 < a_0 \leq t^2 a_2 \leq t^4 a_4 \leq \cdots \leq t^{2k_1^e} a_{2k_1^e} \geq t^{2k_1^e+1} a_{2k_1^e+1} \geq \cdots$$

$$\geq t^{2k_2^e} a_{2k_2^e} \leq t^{2k_2^e+1} a_{2k_2^e+1} \leq \cdots \leq t^{2k_3^e} a_{2k_3^e} \geq \cdots$$

(with inequalities reversed at indices $2k_1^e, 2k_2^e, \ldots, 2k_{r_1}^e$ and $t^{2\lfloor n/2\rfloor}|a_{2\lfloor n/2\rfloor}|$ is the last term in the inequality), and for some $1 = 2k_0^o - 1 < 2k_1^o - 1 < \cdots < 2k_{r_2}^o - 1 < 2k_{r_2+1}^o - 1 = 2\lfloor (n+1)/2\rfloor - 1$ we have

$$a_1 \leq t^2 a_3 \leq t^4 a_5 \leq \cdots \leq t^{2k_1^o-2} a_{2k_1^o-1} \geq t^{2k_1^o} a_{2k_1^o+1} \geq \cdots$$

$$\geq t^{2k_2^o-2}a_{2k_2^o-1} \leq t^{2k_2^p}a_{2k_2^o+1} \leq \cdots \leq t^{2k_3^o-2}a_{2k_3^o-1} \geq \cdots$$

(with inequalities reversed at indices $2k_1^o - 1, 2k_2^o - 1, \ldots, 2k_{r_2}^o - 1$ and $t^{2\lfloor(n+1)/2\rfloor-2}|a_{2\lfloor(n+1)/2\rfloor-1}|$ is the last term in the inequality). Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log(1/\delta)} \log\left(\frac{M}{a_0}\right)$$

where

$$\begin{aligned} M &= (a_{n-1}+a_n) + (-1)^{r_1}a_{2\lfloor n/2\rfloor} + (-1)^{r_2}a_{2\lfloor(n+1)/2\rfloor-1} \\ &\quad +2\left[\sum_{j=1}^{r_1}(-1)^{j+1}a_{2k_j^e} + \sum_{j=1}^{r_2}(-1)^{j+1}a_{2k_j^o-1}\right]. \end{aligned}$$

Example 3.1. Consider the polynomial $P(z) = 1 + 1z + 2z^2 + 2z^3 + 1z^4 + 1z^5 + 1000z^6 + 2z^7$. Since we have $a_0 = 1, a_{2k_1^o-1} = 1, a_{2k_1^e} = 2, a_{2k_2^o-1} = 2, a_{2k_2^e} = 1, a_{n-2} = 1, a_{n-1} = 1000$, and $a_n = 2$, we may let $t = 1$, $r_1 = 2$, and $r_2 = 2$ in Corollary 3.2 to see that $M = (1000 + 2) + 1000 + 2 + 2((2-1) + (2-1)) = 2008$. Then the number of zeros of $P(z)$ in the disk $|z| \leq 0.336$ is less than $\frac{1}{\log 1/0.336} \log \frac{2008}{1} \approx 6.98$. Since the roots of P are $z \approx -499.999$, $z \approx -0.26664 \pm 0.151029i$, $z \approx -0.0144517 \pm 0.307263i$, and $z \approx 0.280592 \pm 0.183883i$, it is obvious that the bound is sharp for this example, since six of the roots of P lie within $|z| \leq 0.336$.

Proof of Theorem 3.1. Consider

$$G(z) = (t^2 - z^2)P(z) = a_0t^2 + a_1t^2z + \sum_{j=2}^{n}(a_jt^2 - a_{j-2})z^j - a_{n-1}z^{n+1} - a_nz^{n+2}.$$

For $|z| = t$, we have

$$\begin{aligned}
|G(z)| &\leq |a_0|t^2 + |a_1|t^3 + \sum_{j=2}^{n} |a_j t^2 - a_{j-2}|t^j + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
&= |a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k_1^e} |a_j t^2 - a_{j-2}|t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k_1^o - 1} |a_j t^2 - a_{j-2}|t^j \\
&\quad + \sum_{\substack{j=2k_1^e+2 \\ j \text{ even}}}^{2k_2^e} |a_j t^2 - a_{j-2}|t^j + \sum_{\substack{j=2k_1^o+1 \\ j \text{ odd}}}^{2k_2^o - 1} |a_j t^2 - a_{j-2}|t^j + \cdots \\
&\quad + \sum_{\substack{j=2k_{r_1}^e+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |a_j t^2 - a_{j-2}|t^j + \sum_{\substack{j=2k_{r_2}^o+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} |a_j t^2 - a_{j-2}|t^j \\
&= S.
\end{aligned}$$

Then by Lemma 2.1,

$$\begin{aligned}
S &\leq |a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k_1^e} \{(-1)^0(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_j|t^2 + |a_{j-2}|)\sin\alpha)\}t^j \\
&\quad + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k_1^o-1} \{(-1)^0(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_j|t^2 + |a_{j-2}|)\sin\alpha)\}t^j \\
&\quad + \sum_{\substack{j=2k_1^e+2 \\ j \text{ even}}}^{2k_2^e} \{(-1)^1(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_j|t^2 + |a_{j-2}|)\sin\alpha)\}t^j \\
&\quad + \sum_{\substack{j=2k_1^o+1 \\ j \text{ odd}}}^{2k_2^o-1} \{(-1)^1(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_j|t^2 + |a_{j-2}|)\sin\alpha)\}t^j + \cdots \\
&\quad + \sum_{\substack{j=2k_{r_1}^e+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \{(-1)^{r_1}(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_j|t^2 + |a_{j-2}|)\sin\alpha)\}t^j \\
&\quad + \sum_{\substack{j=2k_{r_2}^o+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} \{(-1)^{r_2}(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_j|t^2 + |a_{j-2}|)\sin\alpha)\}t^j \\
&= S'.
\end{aligned}$$

So then we have that

$$
\begin{aligned}
S' \leq\ & |a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
& + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k_1^e} (-1)^0(|a_j|t^{j+2} - |a_{j-2}|t^j)\cos\alpha + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k_1^e} (|a_j|t^{j+2} + |a_{j-2}|t^j)\sin\alpha \\
& + \sum_{\substack{j=2k_1^e+2 \\ j \text{ even}}}^{2k_2^e} (-1)^1(|a_j|t^{j+2} - |a_{j-2}|t^j)\cos\alpha + \sum_{\substack{j=2k_1^e+2 \\ j \text{ even}}}^{2k_2^e} (|a_j|t^{j+2} + |a_{j-2}|t^j)\sin\alpha + \cdots \\
& + \sum_{\substack{j=2k_{r_1}^e+2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} (-1)^{r_1}(|a_j|t^{j+2} - |a_{j-2}|t^j)\cos\alpha + \sum_{\substack{j=2k_{r_1}^e+2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} (|a_j|t^{j+2} + |a_{j-2}|t^j)\sin\alpha \\
& + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k_1^o-1} (-1)^0(|a_j|t^{j+2} - |a_{j-2}|t^j)\cos\alpha + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k_1^o-1} (|a_j|t^{j+2} + |a_{j-2}|t^j)\sin\alpha \\
& + \sum_{\substack{j=2k_1^o+1 \\ j \text{ odd}}}^{2k_2^o-1} (-1)^1(|a_j|t^{j+2} - |a_{j-2}|t^j)\cos\alpha + \sum_{\substack{j=2k_1^o+1 \\ j \text{ odd}}}^{2k_2^o-1} (|a_j|t^{j+2} + |a_{j-2}|t^j)\sin\alpha + \cdots \\
& + \sum_{\substack{j=2k_{r_2}^o+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-1} (-1)^{r_1}(|a_j|t^{j+2} - |a_{j-2}|t^j)\cos\alpha + \sum_{\substack{j=2k_{r_2}^o+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-1} (|a_j|t^{j+2} + |a_{j-2}|t^j)\sin\alpha \\
=\ & S''.
\end{aligned}
$$

Then by Remarks 1 and 2 in the proof of Theorem 2.1, we have that

$$
\begin{aligned}
S'' =\ & |a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} + (|a_{2k_1^e}|t^{2k_1^e+2} - |a_0|t^2)\cos\alpha \\
& +(|a_{2k_1^e}|t^{2k_1^e+2} - |a_{2k_2^e}|t^{2k_2^e+2})\cos\alpha + \cdots \\
& +(-1)^{r_1}(|a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2} - |a_{2k_{r_1}^e}|t^{2k_{r_1}^e+2})\cos\alpha + (|a_{2k_1^o-1}|t^{2k_1^o+1} - |a_1|t^3)\cos\alpha \\
& +(|a_{2k_1^o-1}|t^{2k_1^o+1} - |a_{2k_2^o-1}|t^{2k_2^o+1})\cos\alpha + \cdots \\
& +(-1)^{r_2}(|a_{2\lfloor (n+1)/2\rfloor-1}|t^{2\lfloor (n+1)/2\rfloor+1} - |a_{2k_{r_2}^o-1}|t^{2k_{r_2}^o+1}) \\
& +|a_0|t^2\sin\alpha + |a_n|t^{n+2}\sin\alpha + |a_1|t^3\sin\alpha + |a_{n-1}|t^{n+1}\sin\alpha + 2\sin\alpha\sum_{j=2}^{n-2}|a_j|t^{j+2} \\
=\ & (|a_0|t^2 + |a_1|t^3)(1-\sin\alpha-\cos\alpha) + (|a_{n-1}|t^{n+1}| + |a_n|t^{n+2})(1-\sin\alpha) \\
& +(-1)^{r_1}|a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}\cos\alpha + (-1)^{r_2}|a_{2\lfloor (n+1)/2\rfloor-1}|t^{2\lfloor (n+1)/2\rfloor+1}\cos\alpha \\
& +2\sin\alpha\sum_{j=0}^{n}|a_j|t^{j+2} + 2\cos\alpha\left[\sum_{j=1}^{r_1}(-1)^{j+1}|a_{2k_j^e}|t^{k_j^e+2} + \sum_{j=1}^{r_2}(-1)^{j+1}|a_{2k_j^o-1}|t^{k_j^o+1}\right] \\
=\ & M.
\end{aligned}
$$

The theorem follows as in the proof of Theorem 2.1. □

3.2 Restrictions on the Real Part of the Coefficients Given r_1, r_2 Reversals

In this section, we put the restriction on the real part of the coefficients only as in Chapter 2, adding also the even and odd restriction on the coefficients.

Theorem 3.2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ for $0 \le j \le n$. Suppose that for some $t > 0$, and some $0 = 2k_0^e < 2k_1^e < \cdots < 2k_{r_1}^e < 2k_{r_1+1}^e = 2\lfloor n/2 \rfloor$, we have

$$0 < \alpha_0 \le t^2\alpha_2 \le t^4\alpha_4 \le \cdots \le t^{2k_1^e}\alpha_{2k_1^e} \ge t^{2k_1^e+1}\alpha_{2k_1^e+1} \ge \cdots$$

$$\ge t^{2k_2^e}\alpha_{2k_2^e} \le t^{2k_2^e+1}\alpha_{2k_2^e+1} \le \cdots \le t^{2k_3^e}\alpha_{2k_3^e} \ge \cdots$$

(with inequalities reversed at indices $2k_1^e, 2k_2^e, \ldots, 2k_{r_1}^e$ and $t^{2\lfloor n/2 \rfloor}\alpha_{2\lfloor n/2 \rfloor}$ is the last term in the inequality), and for some $1 = 2k_0^o - 1 < 2k_1^o - 1 < \cdots < 2k_{r_2}^o - 1 < 2k_{r_2+1}^o - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ we have

$$\alpha_1 \le t^2\alpha_3 \le t^4\alpha_5 \le \cdots \le t^{2k_1^o-2}\alpha_{2k_1^o-1} \ge t^{2k_1^o}\alpha_{2k_1^o+1} \ge \cdots$$

$$\ge t^{2k_2^o-2}\alpha_{2k_2^o-1} \le t^{2k_2^p}\alpha_{2k_2^o+1} \le \cdots \le t^{2k_3^o-2}\alpha_{2k_3^o-1} \ge \cdots$$

(with inequalities reversed at indices $2k_1^o - 1, 2k_2^o - 1, \ldots, 2k_{r_2}^o - 1$ and $t^{2\lfloor (n+1)/2 \rfloor - 2}\alpha_{2\lfloor (n+1)/2 \rfloor - 1}$ is the last term in the inequality). Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \le \delta t$ is less than

$$\frac{1}{\log(1/\delta)} \log\left(\frac{M}{|a_0|}\right)$$

where

$$\begin{aligned} M \;=\; & (|\alpha_0| + 2|\beta_0| - \alpha_0)t^2 + (|\alpha_2| - \alpha_1 + 2|\beta_1|)t^3 + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} \\ & + (|\alpha_n| + |\beta_n|)t^{n+2} + (-1)^{r_1}\alpha_{2\lfloor n/2 \rfloor}t^{2\lfloor n/2 \rfloor+2} + (-1)^{r_2}\alpha_{2\lfloor (n+1)/2 \rfloor-1}t^{2\lfloor (n+1)/2 \rfloor+1} \\ & + 2\left(\sum_{j=1}^{r_1}(-1)^{j+1}\alpha_{2k_j^e}t^{2k_j^e+2}\right) + 2\left(\sum_{j=1}^{r_2}(-1)^{j+1}\alpha_{2k_j^o-1}t^{2k_j^o+1}\right) + 2\left(\sum_{j=2}^{n}|\beta_j|t^{j+2}\right). \end{aligned}$$

When $r_1 = r_2 = 1$, Theorem 3.2 reduces to Gardener and Shields Theorem 2.3 in [10] Now for $t = 1$ in Theorem 3.2,

Corollary 3.3. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ for $0 \le j \le n$. Suppose that for some $0 = 2k_0^e < 2k_1^e < \cdots < 2k_{r_1}^e < 2k_{r_1+1}^e = 2\lfloor n/2 \rfloor$, we have

$$0 < \alpha_0 \le \alpha_2 \le \alpha_4 \le \cdots \le \alpha_{2k_1^e} \ge \alpha_{2k_1^e+1} \ge \cdots \ge \alpha_{2k_2^e} \le \alpha_{2k_2^e+1} \le \cdots \le \alpha_{2k_3^e} \ge \cdots$$

(with inequalities reversed at indices $2k_1^e, 2k_2^e, \ldots, 2k_{r_1}^e$ and $t^{2\lfloor n/2 \rfloor}\alpha_{2\lfloor n/2 \rfloor}$ is the last term in the inequality), and for some $1 = 2k_0^o - 1 < 2k_1^o - 1 < \cdots < 2k_{r_2}^o - 1 < 2k_{r_2+1}^o - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ we have

$$\alpha_1 \le \alpha_3 \le \alpha_5 \le \cdots \le \alpha_{2k_1^o-1} \ge \alpha_{2k_1^o+1} \ge \cdots \ge \alpha_{2k_2^o-1} \le \alpha_{2k_2^o+1} \le \cdots \le \alpha_{2k_3^o-1} \ge \cdots$$

(with inequalities reversed at indices $2k_1^o - 1, 2k_2^o - 1, \ldots, 2k_{r_2}^o - 1$ and $t^{2\lfloor (n+1)/2 \rfloor - 2}\alpha_{2\lfloor (n+1)/2 \rfloor - 1}$ is the last term in the inequality). Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \le \delta t$ is less than

$$\frac{1}{\log(1/\delta)} \log\left(\frac{M}{|a_0|}\right)$$

where

$$\begin{aligned} M \;=\; & (|\alpha_0| + 2|\beta_0| - \alpha_0) + (|\alpha_2| - \alpha_1 + 2|\beta_1|) + (|\alpha_{n-1}| + |\beta_{n-1}|) \\ & + (|\alpha_n| + |\beta_n|) + (-1)^{r_1}\alpha_{2\lfloor n/2 \rfloor} + (-1)^{r_2}\alpha_{2\lfloor (n+1)/2 \rfloor - 1} \\ & + 2\left(\sum_{j=1}^{r_1} (-1)^{j+1}\alpha_{2k_j^e}\right) + 2\left(\sum_{j=1}^{r_2} (-1)^{j+1}\alpha_{2k_j^o-1}\right) + 2\left(\sum_{j=2}^{n} |\beta_j|\right). \end{aligned}$$

Further, for $0 \le j \le n$ let each $\beta_j = 0$ and let $r_1 = r_2 = 2$ in Corollary 3.3 to obtain:

Corollary 3.4. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j = 0$ for $0 \leq j \leq n$. Suppose that for some $0 = 2k_0^e < 2k_1^e < 2k_2^e < 2k_3^e = 2\lfloor n/2 \rfloor$, we have

$$0 < \alpha_0 \leq \alpha_2 \leq \alpha_4 \leq \cdots \leq \alpha_{2k_1^e} \geq \alpha_{2k_1^e+1} \geq \cdots \geq \alpha_{2k_2^e} \leq \alpha_{2k_2^e+1} \leq \cdots \leq \alpha_{2\lfloor n/2 \rfloor}$$

and for some $1 = 2k_0^o - 1 < 2k_1^o - 1 < 2k_2^o - 1 < 2k_3^o - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ we have

$$\alpha_1 \leq \alpha_3 \leq \alpha_5 \leq \cdots \leq \alpha_{2k_1^o-1} \geq \alpha_{2k_1^o+1} \geq \cdots \geq \alpha_{2k_2^o-1} \leq \alpha_{2k_2^o+1} \leq \alpha_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log(1/\delta)} \log\left(\frac{M}{|a_0|}\right)$$

where

$$\begin{aligned} M &= (|\alpha_0| - \alpha_0) + (|\alpha_2| - \alpha_1) + |\alpha_{n-1}| \\ &\quad + |\alpha_n| + \alpha_{2\lfloor n/2 \rfloor} + \alpha_{2\lfloor (n+1)/2 \rfloor - 1} \\ &\quad + 2(\alpha_{2k_1^e} - \alpha_{2k_2^e}) + 2(\alpha_{2k_1^o-1} - \alpha_{2k_2^o-1}). \end{aligned}$$

Proof of Theorem 3.2. Consider

$$G(z) = (t^2 - z^2)P(z) = a_0 t^2 + a_1 t^2 z + \sum_{j=2}^{n} (a_j t^2 - a_{j-2}) z^j - a_{n-1} z^{n+1} - a_n z^{n+2}.$$

For $|z| = t$, we have

$$\begin{aligned} G(z) &\leq (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 + \sum_{j=2}^{n} |\alpha_j t^2 - \alpha_{j-2}| t^j \\ &\quad + \sum_{j=2}^{n} (|\beta_j| t^2 + |\beta_{j-2}|) t^j + (|\alpha_{n-1}| + |\beta_{n-1}|) t^{n+1} \\ &\quad + (|\alpha_n| + |\beta_n|) t^{n+2} \end{aligned}$$

$$
\begin{aligned}
= \quad & (|\alpha_0|+|\beta_0|)t^2 + (|\alpha_2|+|\beta_1|)t^3 + (|\alpha_{n-1}|+|\beta_{n-1}|)t^{n+1} + (|\alpha_n|+|\beta_n|)t^{n+2} \\
& + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} |\alpha_j t^2 - \alpha_{j-2}|t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor} |\alpha_j t^2 - \alpha_{j-2}|t^j \\
& + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} (|\beta_j|t^2 + |\beta_{j-2}|t^j) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor \frac{n+1}{2}\rfloor} (|\beta_j|t^2 + |\beta_{j-2}|t^j) \\
= \quad & (|\alpha_0|+|\beta_0|)t^2 + (|\alpha_2|+|\beta_1|)t^3 + (|\alpha_{n-1}|+|\beta_{n-1}|)t^{n+1} + (|\alpha_n|+|\beta_n|)t^{n+2} \\
& + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k_1^e} (-1)^0(\alpha_j t^{j+2} - \alpha_{j-2}t^j) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k_1^o-1} (-1)^0(\alpha_j t^{j+2} - \alpha_{j-2}t^j) \\
& + \sum_{\substack{j=2k_1^e+2 \\ j \text{ even}}}^{2k_2^e} (-1)^1(\alpha_j t^{j+2} - \alpha_{j-2}t^j) + \sum_{\substack{j=2k_1^o+1 \\ j \text{ odd}}}^{2k_2^o-1} (-1)^1(\alpha_j t^{j+2} - \alpha_{j-2}t^j) + \cdots \\
& + \sum_{\substack{j=2k_{r_1}^e+2 \\ j \text{ even}}}^{2\lfloor \frac{n}{2}\rfloor} (-1)^{r_1}(\alpha_j t^{j+2} - \alpha_{j-2}t^j) + \sum_{\substack{j=2k_{r_2}^o+1 \\ j \text{ odd}}}^{2\lfloor \frac{n+1}{2}\rfloor-1} (-1)^{r_2}(\alpha_j t^{j+2} - \alpha_{j-2}t^j) \\
& + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} |\beta_j|t^2 + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} |\beta_{j-2}|t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor \frac{n+1}{2}\rfloor} |\beta_j|t^2 + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor \frac{n+1}{2}\rfloor} |\beta_{j-2}|t^j \\
= \quad & (|\alpha_0|+|\beta_0|)t^2 + (|\alpha_2|+|\beta_1|)t^3 + (|\alpha_{n-1}|+|\beta_{n-1}|)t^{n+1} + (|\alpha_n|+|\beta_n|)t^{n+2} \\
& +(\alpha_{2k_1^e}t^{2k_1^e+2} - \alpha_0 t^2) + (\alpha_{2k_1^o-1}t^{2k_1^o+1} - \alpha_1 t^3) \\
& +(\alpha_{2k_1^e}t^{2k_1^e+2} - \alpha_{2k_2^e}t^{2k_2^e+2}) + (\alpha_{2k_1^o-1}t^{2k_1^o+1} - \alpha_{2k_2^o-1}t^{2k_2^o+1}) + \cdots \\
& +(-1)^{r_1}(\alpha_{2\lfloor \frac{n}{2}\rfloor}t^{2\lfloor \frac{n}{2}\rfloor+2} - \alpha_{2k_{r_1}^e}t^{2k_{r_1}^e+2}) + (-1)^{r_2}(\alpha_{2\lfloor \frac{n+1}{2}\rfloor-1}t^{2\lfloor \frac{n+1}{2}\rfloor+1} - \alpha_{2k_{r_2}^o}t^{2k_{r_2}^o+1}) \\
& +|\beta_0|t^2 + |\beta_1|t^3 - |\beta_{2\lfloor \frac{n}{2}\rfloor}|t^{2\lfloor \frac{n}{2}\rfloor+2} - |\beta_{2\lfloor \frac{n+1}{2}\rfloor-1}|t^{2\lfloor \frac{n+1}{2}\rfloor+1} \\
= \quad & (|\alpha_0| + 2|\beta_0| - \alpha_0)t^2 + (|\alpha_2| - \alpha_1 + 2|\beta_1|)t^3 + (|\alpha_{n-1}|+|\beta_{n-1}|)t^{n+1} \\
& +(|\alpha_n|+|\beta_n|)t^{n+2} + (-1)^{r_1}\alpha_{2\lfloor n/2\rfloor}t^{\lfloor n/2\rfloor+2} + (-1)^{r_2}\alpha_{2\lfloor (n+1)/2\rfloor-1}t^{\lfloor (n+1)/2\rfloor+1} \\
& +2\left(\sum_{j=1}^{r_1}(-1)^{j+1}\alpha_{2k_j^e}t^{2k_j^e+2}\right) + 2\left(\sum_{j=1}^{r_2}(-1)^{j+1}\alpha_{2k_j^o-1}t^{2k_j^o+1}\right) + 2\left(\sum_{j=2}^{n}|\beta_j|t^{j+2}\right) \\
= \quad & M.
\end{aligned}
$$

The theorem follows as in the proof of Theorem 2.1. □

3.3 Restrictions on the Real and Imaginary Parts of the Coefficients Given r_1, r_2, r_3, r_4 Reversals

In this section, we place restrictions on the real and imaginary part of the coefficients as we did in Section 2.3 , yet we also impose the restriction of even and odd indices on the coefficients. This gives four restrictions in the hypotheses: on even indexed and real coefficients, on even indexed and imaginary coefficients, on odd indexed and real coefficients, as well as on odd indexed and imaginary coefficients. Due to the restrictions, this section produces several corollaries. We do not list them all, but again we note how we can easily obtain the remaining corollaries with standard algebra.

Theorem 3.3. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ for $0 \leq j \leq n$. Suppose that for some $t > 0$, and some $0 = 2k_0^{er} < 2k_1^{er} < \cdots < 2k_{r_1}^{er} < 2k^{er}2_{r_1+1} = 2\lfloor n/2 \rfloor$ and some $0 = 2k_0^{ei} < 2k_1^{ei} < \cdots < 2k_{r_1}^{ei} < 2k_{r_1+1}^{ei} = 2\lfloor n/2 \rfloor$, we have

$$0 < \alpha_0 \leq t^2\alpha_2 \leq t^4\alpha_4 \leq \cdots \leq t^{2k_1^{er}}\alpha_{2k_1^{er}} \geq t^{2k_1^{er}+1}\alpha_{2k_1^{er}+1} \geq \cdots$$

$$\geq t^{2k_2^{er}}\alpha_{2k_2^{er}} \leq t^{2k_2^{er}+1}\alpha_{2k_2^{er}+1} \leq \cdots \leq t^{2k_3^{er}}\alpha_{2k_3^{er}} \geq \cdots$$

and

$$0 < \beta_0 \leq t^2\beta_2 \leq t^4\beta_4 \leq \cdots \leq t^{2k_1^{ei}}\beta_{2k_1^{ei}} \geq t^{2k_1^{ei}+1}\beta_{2k_1^{ei}+1} \geq \cdots$$

$$\geq t^{2k_2^{ei}}\beta_{2k_2^{ei}} \leq t^{2k_2^{ei}+1}\beta_{2k_2^{ei}+1} \leq \cdots \leq t^{2k_3^{ei}}\beta_{2k_3^{ei}} \geq \cdots$$

(with inequalities reversed at indices $2k_1^e, 2k_2^e, \ldots, 2k_{r_1}^e$ and $t^{2\lfloor n/2 \rfloor}\alpha_{2\lfloor n/2 \rfloor}$ or $t^{2\lfloor n/2 \rfloor}\beta_{2\lfloor n/2 \rfloor}$ is the last term in the inequality,) and for some $1 = 2k_0^{or} - 1 < 2k_1^{or} - 1 < \cdots < 2k_{r_2}^{or} - 1 < 2k_{r_2+1}^{or} - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ and some $1 = 2k_0^{oi} - 1 < 2k_1^{oi} - 1 < \cdots <$

$2k^{oi}_{r_2} - 1 < 2k^{oi}_{r_2+1} - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ we have

$$\alpha_1 \leq t^2\alpha_3 \leq t^4\alpha_5 \leq \cdots \leq t^{2k_1^{or}-2}\alpha_{2k_1^{or}-1} \geq t^{2k_1^{or}}\alpha_{2k_1^{or}+1} \geq \cdots$$

$$\geq t^{2k_2^{or}-2}\alpha_{2k_2^{or}-1} \leq t^{2k_2^{or}}\alpha_{2k_2^{or}+1} \leq \cdots \leq t^{2k_3^{or}-2}\alpha_{2k_3^{or}-1} \geq \cdots$$

and

$$\beta_1 \leq t^2\beta_3 \leq t^4\beta_5 \leq \cdots \leq t^{2k_1^{oi}-2}\beta_{2k_1^{oi}-1} \geq t^{2k_1^{oi}}\beta_{2k_1^{oi}+1} \geq \cdots$$

$$\geq t^{2k_2^{oi}-2}\beta_{2k_2^{oi}-1} \leq t^{2k_2^{oi}}\beta_{2k_2^{oi}+1} \leq \cdots \leq t^{2k_3^{oi}-2}\beta_{2k_3^{oi}-1} \geq \cdots$$

(with inequalities reversed at indices $2k_1^o - 1, 2k_2^o - 1, \ldots, 2k_{r_2}^o - 1$ and $t^{2\lfloor (n+1)/2\rfloor-2}\alpha_{2\lfloor (n+1)/2\rfloor-1}$ or $t^{2\lfloor (n+1)/2\rfloor-2}\beta_{2\lfloor (n+1)/2\rfloor-1}$ is the last term in the inequality). Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log(1/\delta)} \log\left(\frac{M}{|a_0|}\right)$$

where

$$\begin{aligned} M \;=\; & (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0)t^2 + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1)t^3 + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} \\ & +(|\alpha_n| + |\beta_n|)t^{n+2} + (-1)^{r_1}\alpha_{2\lfloor n/2\rfloor}t^{\lfloor n/2\rfloor+2} + (-1)^{r_2}\alpha_{2\lfloor (n+1)/2\rfloor-1}t^{\lfloor (n+1)/2\rfloor+1} \\ & +(-1)^{r_3}\beta_{2\lfloor n/2\rfloor}t^{\lfloor n/2\rfloor+2} + (-1)^{r_4}\beta_{2\lfloor (n+1)/2\rfloor-1}t^{\lfloor (n+1)/2\rfloor+1} \\ & +2\left(\sum_{j=1}^{r_1}(-1)^{j+1}\alpha_{2k_j^{er}}t^{2k_j^{er}+2}\right) + 2\left(\sum_{j=1}^{r_2}(-1)^{j+1}\alpha_{2k_j^{or}-1}t^{2k_j^{or}+1}\right) \\ & +2\left(\sum_{j=1}^{r_3}(-1)^{j+1}\beta_{2k_j^{ei}}t^{2k_j^{ei}+2}\right) + 2\left(\sum_{j=1}^{r_4}(-1)^{j+1}\beta_{2k_j^{oi}-1}t^{2k_j^{oi}+1}\right). \end{aligned}$$

Proof of Theorem 3.3. Consider

$$G(z) \;=\; (t^2 - z^2)P(z) = a_0t^2 + a_1t^2z + \sum_{j=2}^{n}(a_jt^2 - a_{j-2})z^j - a_{n-1}z^{n+1} - a_nz^{n+2}.$$

For $|z| = t$, we have

$$
\begin{aligned}
G(z) \;\leq\;& (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_1| + |\beta_1|)t^3 + \sum_{j=2}^{n} |\alpha_j t^2 - \alpha_{j-2}|t^j \\
&+ \sum_{j=2}^{n} (|\beta_j|t^2 + |\beta_{j-2}|)t^j + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} \\
&+ (|\alpha_n| + |\beta_n|)t^{n+2} \\
=\;& (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_1| + |\beta_1|)t^3 + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&+ \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |\alpha_j t^2 - \alpha_{j-2}|t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor} |\alpha_j t^2 - \alpha_{j-2}|t^j \\
&+ \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} (|\beta_j|t^2 + |\beta_{j-2}|t^j) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor \frac{n+1}{2} \rfloor} (|\beta_j|t^2 + |\beta_{j-2}|t^j) \\
=\;& (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_1| + |\beta_1|)t^3 + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&+ \sum_{\substack{j=2 \\ j \text{ even}}}^{2k_1^{er}} (-1)^0(\alpha_j t^{j+2} - \alpha_{j-2}t^j) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k_1^{or}-1} (-1)^0(\alpha_j t^{j+2} - \alpha_{j-2}t^j) \\
&+ \sum_{\substack{j=2k_1^{er}+2 \\ j \text{ even}}}^{2k_2^{er}} (-1)^1(\alpha_j t^{j+2} - \alpha_{j-2}t^j) + \sum_{\substack{j=2k_1^{or}+1 \\ j \text{ odd}}}^{2k_2^{or}-1} (-1)^1(\alpha_j t^{j+2} - \alpha_{j-2}t^j) + \cdots \\
&+ \sum_{\substack{j=2k_{r_1}^{er}+2 \\ j \text{ even}}}^{2\lfloor \frac{n}{2} \rfloor} (-1)^{r_1}(\alpha_j t^{j+2} - \alpha_{j-2}t^j) + \sum_{\substack{j=2k_{r_2}^{or}+1 \\ j \text{ odd}}}^{2\lfloor \frac{n+1}{2} \rfloor-1} (-1)^{r_2}(\alpha_j t^{j+2} - \alpha_{j-2}t^j) \\
&+ \sum_{\substack{j=2 \\ j \text{ even}}}^{2k_1^{ei}} (-1)^0(\beta_j t^{j+2} - \beta_{j-2}t^j) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k_1^{oi}-1} (-1)^0(\beta_j t^{j+2} - \beta_{j-2}t^j) \\
&+ \sum_{\substack{j=2k_1^{ei}+2 \\ j \text{ even}}}^{2k_2^{ei}} (-1)^1(\beta_j t^{j+2} - \beta_{j-2}t^j) + \sum_{\substack{j=2k_1^{oi}+1 \\ j \text{ odd}}}^{2k_2^{oi}-1} (-1)^1(\beta_j t^{j+2} - \beta_{j-2}t^j) + \cdots \\
&+ \sum_{\substack{j=2k_{r_3}^{ei}+2 \\ j \text{ even}}}^{2\lfloor \frac{n}{2} \rfloor} (-1)^{r_1}(\beta_j t^{j+2} - \beta_{j-2}t^j) + \sum_{\substack{j=2k_{r_4}^{oi}+1 \\ j \text{ odd}}}^{2\lfloor \frac{n+1}{2} \rfloor-1} (-1)^{r_2}(\beta_j t^{j+2} - \beta_{j-2}t^j)
\end{aligned}
$$

$$
\begin{aligned}
&= (|\alpha_0|+|\beta_0|)t^2+(|\alpha_1|+|\beta_1|)t^3+(|\alpha_{n-1}|+|\beta_{n-1}|)t^{n+1}+(|\alpha_n|+|\beta_n|)t^{n+2}\\
&\quad +(\alpha_{2k_1^{er}}t^{2k_1^{er}+2}-\alpha_0t^2)+(\alpha_{2k_1^{or}-1}t^{2k_1^{or}+1}-\alpha_1t^3)\\
&\quad +(\alpha_{2k_1^{er}}t^{2k_1^{er}+2}-\alpha_{2k_2^{er}}t^{2k_2^{er}+2})+(\alpha_{2k_1^{or}-1}t^{2k_1^{or}+1}-\alpha_{2k_2^{or}-1}t^{2k_2^{or}+1})+\cdots\\
&\quad +(-1)^{r_1}(\alpha_{2\lfloor\frac{n}{2}\rfloor}t^{\lfloor\frac{n}{2}\rfloor+2}-\alpha_{2k_{r_1}^{er}}t^{2k_{r_1}^{er}+2})+(-1)^{r_2}(\alpha_{2\lfloor\frac{n+1}{2}\rfloor-1}t^{\lfloor\frac{n+1}{2}\rfloor+1}-\alpha_{2k_{r_2}^{or}}t^{2k_{r_2}^{or}+1})\\
&\quad +(\beta_{2k_1^{ei}}t^{2k_1^{ei}+2}-\beta_0t^2)+(\beta_{2k_1^{oi}-1}t^{2k_1^{oi}+1}-\beta_1t^3)\\
&\quad +(\beta_{2k_1^{ei}}t^{2k_1^{ei}+2}-\beta_{2k_2^{ei}}t^{2k_2^{e}+2})+(\beta_{2k_1^{oi}-1}t^{2k_1^{oi}+1}-\beta_{2k_2^{oi}-1}t^{2k_2^{oi}+1})+\cdots\\
&\quad +(-1)^{r_3}(\beta_{2\lfloor\frac{n}{2}\rfloor}t^{\lfloor\frac{n}{2}\rfloor+2}-\beta_{2k_{r_3}^{ei}}t^{2k_{r_3}^{ei}+2})+(-1)^{r_4}(\beta_{2\lfloor\frac{n+1}{2}\rfloor-1}t^{\lfloor\frac{n+1}{2}\rfloor+1}-\beta_{2k_{r_4}^{oi}}t^{2k_{r_4}^{oi}+1})\\
&= (|\alpha_0|-\alpha_0+|\beta_0|-\beta_0)t^2+(|\alpha_1|-\alpha_1+|\beta_1|-\beta_1)t^3+(|\alpha_{n-1}|+|\beta_{n-1}|)t^{n+1}\\
&\quad +(|\alpha_n|+|\beta_n|)t^{n+2}+(-1)^{r_1}\alpha_{2\lfloor n/2\rfloor}t^{2\lfloor n/2\rfloor+2}+(-1)^{r_2}\alpha_{2\lfloor(n+1)/2\rfloor-1}t^{2\lfloor(n+1)/2\rfloor+1}\\
&\quad +(-1)^{r_3}\beta_{2\lfloor n/2\rfloor}t^{\lfloor n/2\rfloor+2}+(-1)^{r_4}\beta_{2\lfloor(n+1)/2\rfloor-1}t^{\lfloor(n+1)/2\rfloor+1}\\
&\quad +2\left(\sum_{j=1}^{r_1}(-1)^{j+1}\alpha_{2k_j^{er}}t^{2k_j^{er}+2}\right)+2\left(\sum_{j=1}^{r_2}(-1)^{j+1}\alpha_{2k_j^{or}-1}t^{2k_j^{or}+1}\right)\\
&\quad +2\left(\sum_{j=1}^{r_3}(-1)^{j+1}\beta_{2k_j^{ei}}t^{2k_j^{ei}+2}\right)+2\left(\sum_{j=1}^{r_4}(-1)^{j+1}\beta_{2k_j^{oi}-1}t^{2k_j^{oi}+1}\right)\\
&= M.
\end{aligned}
$$

The theorem follows as in the proof of Theorem 2.1. □

Note that if $r_1 = r_2 = 1$, then Theorem 3.3 reduces to Gardner and Shields Theorem 2.7 in [10].

In this chapter, we explored similar types of restrictions of the coefficients of the polynomial as before, but in addition we imposed the monotonicity condition on the even and odd indexed coefficients separately, as did Cao and Gardner [4] for the locations of zeros.

4 A MONOTONICITY CONDITION ON THE COEFFICIENTS OF POLYNOMIALS WITH A GAP WITH A NUMBER OF REVERSALS

In this chapter, we again consider the same three types of hypotheses on the coefficients of a polynomial as before: namely those concerning the monotonicity of the moduli, real parts, as well as real and imaginary parts of the coefficients. Unlike before, we now consider such restrictions on the class of polynomials denoted $\mathcal{P}_{n,\mu}$. Polynomials of this class have a gap between the leading coefficient and the preceding nonzero coefficient, which has an index of μ. These polynomials are typically studied for their connection with Bernstein-type inequalities [5], and we obtain a number of new results for this class of polynomials. Symbolically, we say a polynomial $P \in \mathcal{P}_{n,\mu}$ if it is of the form

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$$

with $\mu \in \mathbb{N}$, $\mu \geq 2$, $z \in \mathbb{C}$, and for $k \in \mathbb{N}$, each a_k is a coefficient of the polynomial P with $a_1 = a_2 = \cdots = a_{\mu-1} = 0$.

4.1 Restrictions on the Moduli of the Coefficients Given r Reversals

In this section, we consider the $\mathcal{P}_{n,\mu}$ class of polynomials with the same restriction as in section 2.1. As before, we initially only restrict the real part of the coefficients, and we are seeking a bound on M_r to count the number of zeros using Theorem 1.2.

Theorem 4.1. Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ where for some $t > 0$ and some $0 < k_1 < k_2 < \cdots < k_r < n$,

$$|a_\mu| \, t^\mu \leq \cdots \leq |a_{k_1}| \, t^{k_1} \geq |a_{k_1+1}| \, t^{k_1+1} \geq \cdots \geq |a_{k_2}| \, t^{k_2} \leq \cdots$$

and $|\arg(a_j) - \beta| \leq \alpha \leq \frac{\pi}{2}$ for $\mu \leq j \leq n$ for some $\alpha, \beta \in \mathbb{R}$ with $r \in \mathbb{N}, 1 \leq r < n$ the number of reversals with $a_0 \neq 0$ and

$$|a_1| = |a_2| = \cdots = |a_{\mu-1}| = 0.$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{|a_0|},$$

where

$$\begin{aligned} M_r &= 2|a_0|t + |a_\mu|t^{\mu+1}(1 + \sin\alpha - \cos\alpha) + 2\sin\alpha \left(\sum_{h=\mu+1}^{n-1} |a_j|t^{j+1} \right) \\ &\quad + |a_n|t^{n+1}(1 + \sin\alpha + (-1)^r \cos\alpha) + 2\cos\alpha \left(\sum_{h=1}^{r} (-1)^{h+1} |a_{k_h}| t^{k_h+1} \right). \end{aligned}$$

Proof of Theorem 4.1. Consider

$$\begin{aligned} F(z) &= (t-z)P(z) \\ &= (t-z)\left(a_0 + \sum_{j=\mu}^{n} a_j z^j \right) \\ &= a_0(t-z) + \sum_{j=\mu}^{n} a_j t z^j - \sum_{j=\mu}^{n} a_j z^{j+1} \\ &= a_0(t-z) + \sum_{j=\mu}^{n} a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j \\ &= a_0(t-z) + a_\mu t z^\mu + \sum_{j=\mu+1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$|F(z)| \leq 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{n} |a_j t - a_{j-1}| t^j - |a_n| t^{n+1}$$

$$
\begin{aligned}
&= 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k_1} |(-1)^0(a_jt - a_{j-1})|t^j \\
&\quad + \sum_{j=k_1+1}^{k_2} |(-1)^1(a_jt - a_{j-1})|t^j + \cdots \\
&\quad + \sum_{j=k_r+1}^{n} |(-1)^r(a_jt - a_{j-1})|t^j + |a_n|t^{n+1} \\
&= S.
\end{aligned}
$$

Then by Lemma 2.1 with $z = a_jt$ and $z' = a_{j-1}$ when

$$
\begin{gathered}
1 \leq j \leq k_1 \\
k_2 + 1 \leq j \leq k_3 \\
k_4 + 1 \leq j \leq k_5 \\
\vdots \\
\begin{cases} k_r + 1 \leq j \leq n & \text{if } r \text{ is even} \\ \\ k_{r-1} + 1 \leq j \leq k_r & \text{if } r \text{ is odd} \end{cases}
\end{gathered}
$$

and with $z = a_{j-1}$ and $z' = a^jt$ when

$$
\begin{gathered}
k_1 + 1 \leq j \leq k_2 \\
k_3 + 1 \leq j \leq k_4 \\
k_5 + 1 \leq j \leq k_6 \\
\vdots \\
\begin{cases} k_r + 1 \leq j \leq n & \text{if } r \text{ is odd} \\ \\ k_{r-1} + 1 \leq j \leq k_r & \text{if } r \text{ is even} \end{cases}
\end{gathered}
$$

we have for $r \in \mathbb{N}$ with $2 \leq r < n$,

$$
\begin{aligned}
S \;\leq\;& 2|a_0|t+|a_\mu|t^{\mu+1}+\sum_{j=\mu+1}^{k_1}\left\{|(-1)^0(|a_jt|-|a_{j-1}|)\cos\alpha+(|a_{j-1}|+|a_jt|)\sin\alpha\right\}t^j\\
&+\sum_{h=1}^{r-1}\left(\sum_{j=k_h+1}^{k_{h+1}}\left\{|(-1)^h(|a_jt|-|a_{j-1}|)\cos\alpha+(|a_{j-1}|+|a_jt|)\sin\alpha\right\}t^j\right)\\
&+\sum_{j=k_r+1}^{n}\{|(-1)^r(|a_jt|-|a_{j-1}|)\cos\alpha+(|a_{j-1}|+|a_jt|)\sin\alpha\}\,t^j+|a_n|t^{n+1}\\
=\;& 2|a_0|t+|a_\mu|t^{\mu+1}+\sum_{j=\mu+1}^{k_1}(-1)^0(|a_j|t^{j+1}-|a_{j-1}|t^j)\cos\alpha\\
&+\sum_{h=1}^{r-1}\left(\sum_{j=k_h+1}^{k_{h+1}}(-1)^h(|a_j|t^{j+1}-|a_{j-1}|t^j)\cos\alpha\right)\\
&+\sum_{j=k_r+1}^{n}(-1)^r(|a_j|t^{j+1}-|a_{j-1}|t^j)\cos\alpha+\sum_{j=1}^{k_1}(|a_{j-1}|t^j+|a_j|t^{j+1})\sin\alpha\\
&+\sum_{h=1}^{r-1}\left(\sum_{j=k_h+1}^{k_{h+1}}(|a_{j-1}|t^j+|a_j|t^{j+1})\sin\alpha\right)+\sum_{j=k_r+1}^{n}(|a_{j-1}|t^j+|a_j|t^{j+1})\sin\alpha\\
&+|a_n|t^{n+1}\\
:=\;& S'.
\end{aligned}
$$

So we note that by Remark 1.2 from the proof of Theorem 2.1 that the sum $\mathcal{C}$ of the $\cos\alpha$ terms is

$$\mathcal{C}=(-1)^r|a_n|t^{n+1}\cos\alpha+2\cos\alpha\left(\sum_{h=1}^{r}(-1)^{h+1}|a_{k_h}|t^{k_h+1}\right)-|a_\mu|t^{\mu+1}\cos\alpha.$$

By Remark 1.1, the sum $\mathcal{S}$ of the $\sin\alpha$ terms is

$$\mathcal{S}=|a_n|t^{n+1}\sin\alpha+2\sin\alpha\left(\sum_{h=\mu+1}^{n-1}|a_j|t^{j+1}\right)+|a_\mu|t^{\mu+1}\sin\alpha.$$

Then

$$
\begin{aligned}
S' &= 2|a_0|t + |a_\mu|t^{\mu+1} + \mathcal{C} + \mathcal{S} + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + (-1)^r|a_n|t^{n+1}\cos\alpha + 2\cos\alpha\left(\sum_{h=1}^{r}(-1)^{h+1}|a_{k_h}|t^{k_h+1}\right) \\
&\quad -|a_\mu|t^{\mu+1}\cos\alpha + |a_n|t^{n+1}\sin\alpha + 2\sin\alpha\left(\sum_{h=\mu+1}^{n-1}|a_j|t^{j+1}\right) + |a_\mu|t^{\mu+1}\sin\alpha \\
&\quad +|a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1}(1+\sin\alpha-\cos\alpha) + 2\sin\alpha\left(\sum_{h=\mu+1}^{n-1}|a_j|t^{j+1}\right) \\
&\quad +|a_n|t^{n+1}(1+\sin\alpha+(-1)^r\cos\alpha) + 2\cos\alpha\left(\sum_{h=1}^{r}(-1)^{h+1}|a_{k_h}|t^{k_h+1}\right) \\
&= M_r.
\end{aligned}
$$

The theorem follows as in the proof of Theorem 2.1. $\square$

When $r = 1$, this reduces to Corollary 1.11 in [11]. Further, let $t = 1$ and let $\alpha = 0$, thereby considering only the real, positive parts of the coefficients. We obtain the following corollary:

Corollary 4.1. Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ have real nonnegative coefficients where for some $0 < k_1 < k_2 < \cdots < k_r < n$,

$$\alpha_\mu \le \cdots \le \alpha_{k_1} \ge \alpha_{k_1+1} \ge \cdots \ge \alpha_{k_2} \le \cdots$$

and with $r \in \mathbb{N}, 1 \le r < n$ the number of reversals, $a_0 \ne 0$, and

$$a_1 = a_2 = \cdots = a_{\mu-1} = 0.$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \le \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{a_0},$$

where

$$M_r = 2a_0 + a_n(1 + (-1)^r) + 2\left(\sum_{h=1}^{r}(-1)^{h+1}a_{k_h}\right).$$

Example 4.1. Consider the polynomial $P(z) = 1 + 1z^2 + 2z^3 + z^4 + 100z^5 + z^6$. Note that we may apply Corollary 4.3 with $\mu = 2$ and $r = 3$. First, note that we have $a_0 = 1, a_\mu = 1, a_{k_1} = 2, a_{k_2} = 1, a_{k_3} = 100$, with $a_n = 1$. Then we may find that $M_3 = 2(1) + (1)(0) + 2((2) - (1) + (100)) = 204$. Then the number of zeros in the disk $|z| \leq \delta = 0.403$ is less than $\frac{1}{\log 1/0.403} \log \frac{204}{1} \approx 5.851$, implying that there are less than or equal to five zeros in the region. This bound is sharp, since the six zeros of $P(z)$ are $z \approx -99.9902$, $z \approx -0.402938$, $z \approx -0.11113 \pm 0.380903i$, and $z \approx 0.307698 \pm 0.250946i$, five of which lie in $|z| \leq 0.403$.

4.2 Restrictions on the Real Part of the Coefficients Given r Reversals

In this section, we place a monotonicity restriction on the real part of the coefficients only for the polynomials in the class $\mathcal{P}_{n,\mu}$.

Theorem 4.2. Let $P(z) = a_0 + \sum_{j=\mu+1}^{n} a_j z^j$ where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some $t > 0$ and some $0 < k_1 < k_2 < \cdots < k_r < n$,

$$\alpha_\mu t^\mu \leq \cdots \leq \alpha_{k_1} t^{k_1} \geq \alpha_{k_1+1} t^{k_1+1} \geq \cdots \geq \alpha_{k_2} t^{k_2} \leq \cdots$$

with $r \in \mathbb{N}, 1 \leq r < n$ the number of reversals and $\alpha_1 = \alpha_2 = \cdots = \alpha_{\mu-1} = 0$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_r}{|a_0|},$$

where

$$\begin{aligned} M_r &= 2|\alpha_0+\beta_0|t+(|\alpha_\mu|-\alpha_\mu)t^{\mu+1}+2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}t^{k_j+1}\right) \\ &\quad +(|\alpha_n|+(-1)^r\alpha_n)t^{n+1}+2\sum_{j=\mu+1}^{n}|\beta_j|t^{j+1} \end{aligned}$$

Note that with $r=1$ in Theorem 4.2, we obtain Corollary 1.11, which appears in [11]. For $t=1$ in Theorem 4.2, we obtain:

Corollary 4.2. Let $P(z)=a_0+\sum_{j=\mu+1}^{n}a_jz^j$ where $\operatorname{Re}a_j=\alpha_j$ and $\operatorname{Im}a_j=\beta_j$ for $\mu\le j\le n$. Suppose that for some $0<k_1<k_2<\cdots<k_r<n$,

$$\alpha_\mu\le\cdots\le\alpha_{k_1}\ge\alpha_{k_1+1}\ge\cdots\ge\alpha_{k_2}\le\cdots$$

with $r\in\mathbb{N}, 1\le r<n$ the number of reversals and $\alpha_1=\alpha_2=\cdots=\alpha_{\mu-1}=0$. Then for $0<\delta<1$, the number of zeros of $P(z)$ in the disk $|z|\le\delta t$ is less than

$$\frac{1}{\log 1/\delta}\log\frac{M_r}{|a_0|},$$

where

$$\begin{aligned} M_r &= 2|\alpha_0+\beta_0|+(|\alpha_\mu|-\alpha_\mu)+(|\alpha_n|+(-1)^r\alpha_n) \\ &\quad +2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}\right)+2\sum_{j=\mu+1}^{n}|\beta_j| \end{aligned}$$

Further, let each $\beta_j=0$ where $0\le j\le n$ with $r=2$ in Corollary 4.3 to obtain:

Corollary 4.3. Let $P(z)=a_0+\sum_{j=\mu+1}^{n}a_jz^j$ where $\operatorname{Re}a_j=\alpha_j$ and $\operatorname{Im}a_j=\beta_j=0$ for $\mu\le j\le n$. Suppose that for some $0<k_1<k_2<n$,

$$\alpha_\mu\le\cdots\le\alpha_{k_1}\ge\alpha_{k_1+1}\ge\cdots\ge\alpha_{k_2}\le\cdots\le\alpha_n$$

with $\alpha_1=\alpha_2=\cdots=\alpha_{\mu-1}=0$. Then for $0<\delta<1$, the number of zeros of $P(z)$ in the disk $|z|\le\delta t$ is less than

$$\frac{1}{\log 1/\delta}\log\frac{M_2}{|a_0|},$$

where

$$M_2 = 2|\alpha_0| + (|\alpha_\mu| - \alpha_\mu) + (|\alpha_n| + \alpha_n) + 2(\alpha_{k_1} - \alpha_{k_2}).$$

Proof of Theorem 4.2. As in the proof of Theorem 4.1,

$$\begin{aligned} F(z) = (t-z)P(z) &= (|\alpha_0| + i|\beta_0|)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu \\ &\quad + \sum_{j=\mu+1}^{n} [(\alpha_j + t\beta_j)t - (\alpha_{j-1} + i\beta_{j-1})]z^j - (\alpha_n + i\beta_n)z^{n+1} \\ &= (|\alpha_0| + i|\beta_0|)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu) + \sum_{j=\mu+1}^{n} (\alpha_j t - \alpha_{j-1})z^j \\ &\quad + i\sum_{j=\mu+1}^{n} (\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$\begin{aligned} |F(z)| &\leq 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} \sum_{j=\mu+1}^{n} |\alpha_j t - \alpha_{j-1}|t^j \\ &\quad + \sum_{j=\mu+1}^{n} |\beta_j t - \beta_{j-1}|t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} \\ &\quad + \sum_{j=\mu+1}^{k_1} ((-1)^0(\alpha_j t - \alpha_{j-1}))t^j \\ &\quad + \sum_{j=k_1+1}^{k_2} ((-1)^1(\alpha_j t - \alpha_{j-1}))t^j + \cdots \\ &\quad + \sum_{j=k_r+1}^{n} ((-1)^r(\alpha_j t - \alpha_{j-1}))t^j \\ &\quad + \left(\sum_{j=\mu+1}^{n} |\beta_j|t + |\beta_{j-1}| \right) t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \end{aligned}$$

$$
\begin{aligned}
&= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{k_1} ((-1)^0(\alpha_j t - \alpha_{j-1}))t^j \\
&\quad + \sum_{j=k_1+1}^{k_2} ((-1)^1(\alpha_j t - \alpha_{j-1}))t^j + \cdots \\
&\quad + \sum_{j=k_r+1}^{n} ((-1)^r(\alpha_j t - \alpha_{j-1}))t^j \\
&\quad + \left(\sum_{j=\mu+1}^{n} |\beta_j|t + |\beta_{j-1}| \right) t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= S.
\end{aligned}
$$

Note now that each summation involving α has its terms cancel and pair as in Remark 2.2 in Theorem 2.1. In addition, the terms involving β pair as in Remark 2.1 in Theorem 2.1 Then we have

$$
\begin{aligned}
S &= 2|\alpha_0 + \beta_0|t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + (-1)^r \alpha_n t^{n+1} + 2\left(\sum_{j=1}^{r} (-1)^{j+1} \alpha_{k_j} t^{k_j+1} \right) \\
&\quad - \alpha_\mu t^{\mu+1} + |\beta_n| t^{n+1} + 2 \sum_{j=\mu+2}^{n-1} |\beta_j| t^{j+1} + |\beta_\mu| t^{\mu+1} + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= 2|\alpha_0 + \beta_0|t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\left(\sum_{j=1}^{r} (-1)^{j+1} \alpha_{k_j} t^{k_j+1} \right) \\
&\quad + (|\alpha_n| + (-1)^r \alpha_n)t^{n+1} + 2 \sum_{j=\mu+1}^{n} |\beta_j| t^{j+1} \\
&= M_r.
\end{aligned}
$$

The result now follows as in the proof of Theorem 2.1. □

4.3 Restrictions on the Real and Imaginary Parts of the Coefficients Given r, ρ Reversals

In this section, we place the monotonicity restriction on both the real and imaginary parts of the coefficients for polynomials of the class $\mathcal{P}_{n,\mu}$.

Theorem 4.3. Let $P(z) = a_0 + \sum_{j=\mu+1}^{n} a_j z^j$ where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $\mu \le j \le n$. Suppose that for some $t > 0$ and some $0 < k_1 < k_2 < \cdots < k_r < n$, we have $a_0 \ne 0$ and

$$\alpha_\mu t^\mu \le \cdots \le \alpha_{k_1} t^{k_1} \ge \alpha_{k_1+1} t^{k_1+1} \ge \cdots \ge \alpha_{k_2} t^{k_2} \le \cdots$$

and for some $0 < \ell_1 < \ell_2 < \cdots < \ell_r < n$,

$$\beta_\mu t^\mu \le \cdots \le \beta_{\ell_1} t^{\ell_1} \ge \beta_{\ell_1+1} t^{\ell_1+1} \ge \cdots \ge \beta_{\ell_2} t^{\ell_2} \le \cdots$$

with $r \in \mathbb{N}, 1 \le r < n$ the number of reversals for the real part of a_j and with $\rho \in \mathbb{N}, 1 \le \rho < n$ the number of reversals for the imaginary part of a_j and $\alpha_1 = \alpha_2 = \cdots = \alpha_{\mu-1} = 0$ and $\beta_1 = \beta_2 = \cdots = \beta_{\mu-1} = 0$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \le \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_{[r,\rho]}}{|a_0|},$$

where

$$\begin{aligned} M_{[r,\rho]} &= 2|\alpha_0 + \beta_0|t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j} t^{k_j+1}\right) \\ &\quad +(|\alpha_n| + (-1)^r\alpha_n + |\beta_n| + (-1)^\rho \beta_n)t^{n+1} + 2\left(\sum_{j=1}^{\rho}(-1)^{j+1}\beta_{\ell_j} t^{\ell_j+1}\right). \end{aligned}$$

Proof of Theorem 4.3. As in the proof of Theorem 4.2,

$$\begin{aligned} F(z) &= (t-z)P(z) = (|\alpha_0| + i|\beta_0|)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu \\ &\quad + \sum_{j=\mu+1}^{n} [(\alpha_j + t\beta_j)t - (\alpha_{j-1} + i\beta_{j-1})]z^j - (\alpha_n + i\beta_n)z^{n+1} \\ &= (|\alpha_0| + i|\beta_0|)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^{n} (\alpha_j t - \alpha_{j-1})z^j \\ &\quad + i\sum_{j=\mu+1}^{n} (\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$
\begin{aligned}
|F(z)| &\leq 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{n} |\alpha_j t - \alpha_{j-1}|t^j \\
&\quad + \sum_{j=\mu+1}^{n} |\beta_j t - \beta_{j-1}|t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&\quad + \sum_{j=\mu+1}^{k_1} ((-1)^0(\alpha_j t - \alpha_{j-1}))t^j + \sum_{j=k_1+1}^{k_2} ((-1)^1(\alpha_j t - \alpha_{j-1}))t^j + \cdots \\
&\quad + \sum_{j=k_r+1}^{n} ((-1)^r(\alpha_j t - \alpha_{j-1}))t^j + \left(\sum_{j=\mu+1}^{n} |\beta_j|t + |\beta_{j-1}|\right) t^j \\
&= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&\quad + \sum_{j=\mu+1}^{k_1} ((-1)^0(\alpha_j t - \alpha_{j-1}))t^j + \sum_{j=k_1+1}^{k_2} ((-1)^1(\alpha_j t - \alpha_{j-1}))t^j + \cdots \\
&\quad + \sum_{j=k_r+1}^{n} ((-1)^r(\alpha_j t - \alpha_{j-1}))t^j + \sum_{j=\mu+1}^{\ell_1} ((-1)^0(\beta_j t - \beta_{j-1}))t^j \\
&\quad + \sum_{j=\ell_1+1}^{\ell_2} ((-1)^1(\beta_j t - \beta_{j-1}))t^j + \cdots + \sum_{j=\ell_r+1}^{n} ((-1)^r(\beta_j t - \beta_{j-1}))t^j \\
&= S.
\end{aligned}
$$

Note now that each summation involving α or β has its terms cancel and pair as in Remark 2.2 in Theorem 2.1. Then we have

$$
\begin{aligned}
S &= 2|\alpha_0 + \beta_0|t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + (-1)^r\alpha_n t^{n+1} + 2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}t^{k_j+1}\right) \\
&\quad -\alpha_\mu t^{\mu+1} + (-1)^\rho\beta_n t^{n+1} + 2\left(\sum_{j=1}^{\rho}(-1)^{j+1}\beta_{\ell_j}t^{\ell_j+1}\right) - \beta_\mu t^{\mu+1} + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= 2|\alpha_0 + \beta_0|t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2\left(\sum_{j=1}^{r}(-1)^{j+1}\alpha_{k_j}t^{k_j+1}\right) \\
&\quad +(|\alpha_n| + (-1)^r\alpha_n + |\beta_n| + (-1)^\rho\beta_n)t^{n+1} + 2\left(\sum_{j=1}^{\rho}(-1)^{j+1}\beta_{\ell_j}t^{\ell_j+1}\right) \\
&= M_{[r,\rho]}.
\end{aligned}
$$

The result now follows as in the proof of Theorem 2.1. $\square$

If $r = \rho = 1$ we obtain Gardner and Shields Theorem 2.5 in [10]. If $t = 1$ in Theorem 4.3, we obtain:

Corollary 4.4. Let $P(z) = a_0 + \sum_{j=\mu+1}^{n} a_j z^j$ where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $\mu \le j \le n$. Suppose that for some $0 < k_1 < k_2 < \cdots < k_r < n$,

$$\alpha_\mu \le \cdots \le \alpha_{k_1} \ge \alpha_{k_1+1} \ge \cdots \ge \alpha_{k_2} \le \cdots$$

and for some $0 < \ell_1 < \ell_2 < \cdots < \ell_r < n$,

$$\beta_\mu \le \cdots \le \beta_{\ell_1} \ge \beta_{\ell_1+1} \ge \cdots \ge \beta_{\ell_2} \le \cdots$$

with $r \in \mathbb{N}, 1 \le r < n$ the number of reversals for the real part of a_j and with $\rho \in \mathbb{N}, 1 \le \rho < n$ the number of reversals for the imaginary part of a_j and $\alpha_1 = \alpha_2 = \cdots = \alpha_{\mu-1} = 0$ and $\beta_1 = \beta_2 = \cdots = \beta_{\mu-1} = 0$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in the disk $|z| \le \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M_{[r,\rho]}}{|a_0|},$$

where

$$\begin{aligned} M_{[r,\rho]} &= 2|\alpha_0 + \beta_0| + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + 2\left(\sum_{j=1}^{r} (-1)^{j+1} \alpha_{k_j}\right) \\ &\quad + (|\alpha_n| + (-1)^r \alpha_n + |\beta_n| + (-1)^\rho \beta_n) + 2\left(\sum_{j=1}^{\rho} (-1)^{j+1} \beta_{\ell_j}\right) \end{aligned}$$

In this chapter, we again considered the same three types of hypotheses on the coefficients of a polynomial as before: namely those concerning the monotonicity of the moduli, real parts, as well as real and imaginary parts of the coefficients. Unlike before, we consider in this chapter such restrictions on the class of polynomials denoted $\mathcal{P}_{n,\mu}$, which was defined in the introduction.

5 CONCLUSION

In this thesis, we considered three different classes of polynomials in the same manner as Shields [17]. Where his results were restricted only to one reversal in each instance, our results were generalized to allow up to $n-1$ reversals among the coefficients. Each of our restrictions on the coefficients were placed on our polynomials in order to count the number of zeros in a prescribed region. We used Titchmarsh's result in the conclusion of each proof, which allowed us access to formulæ that rely only on a subset of the given coefficients. In each of the classes of polynomials, namely those with a monotonicity condition on all the coefficients, those with a monotonicity condition on the coefficients of even powers and coefficients of odd powers of the variable, and those with a monotonicity condition on the coefficients of polynomials with a gap. We have given an example showing our results can yield best possible bounds on the number of zeros of a polynomial in a certain region.

In conclusion, we mention that a similar proof technique could be applied to analytic functions, provided the coefficients in the power series of the analytic function satisfy the appropriate hypothesis. This could be the focus of future research.

BIBLIOGRAPHY

[1] A. Aziz and B. A. Zargar, Some Extentions of Eneström-Kakeya, *Glasnik Matematički*, **31**(51), 239-244 (1996).

[2] A. Aziz and Q. G. Mohammad, On the Zeros of a Certain Class of Polynomials and Related Entire Functions, *Journal of Mathematical Anaysis and Applications*, **75** (1980), 495-502.

[3] C. B. Boyer, U. C. Merzbach, *A History of Mathematics*, Second Edition, Wiley, (1991)

[4] J. Cao and R. Gardner, Restrictions on the Zeros of a Polynomial as a Consequence of Conditions on the Coefficients of Even Powers and Odd Powers of the Variable, *Journal of Computational and Applied Mathematics*, **155**, 153-163 (2003).

[5] T. Chan and M. Malik, On the Erös-Lax Theorem, *Proceeding of the Indian Academy of Sciences*, **92**, (1983) 191-193.

[6] A. Chattopadhyay, S. Das, V. K. Jain, and H. Konwar, Certain Generalizations of the Eneström-Kakeya Theorem, *Journal of the Indian Mathematical Society*, **72**(1), 147-156 (2005).

[7] K.K. Dewan, Extremal Properties and Coefficient Estimates for Polynomials with Restricted Zeros and on Location of Zeros of Polynomials, Ph.D. Thesis, Indian Institutes of Technology, Delhi, 1980.

[8] R. Gardner and N. K. Govil, The Eneström-Kakeya Theorem and Some of Its Generalizations, *Current Topics in Pure and Computational Complex Analysis*, Springer-Verlag, 171-199 (2014).

[9] R. Gardner and B. Shields, The Number of Zeros of a Polynomial in a Disk, *Journal of Classical Analysis,* **3**(2) (2013), 167-176

[10] R. Gardner and B. Shields, The Number of Zeros of a Polynomial in a Disk as a Consequence of Restrictions on the Coefficients, *Acta et Commentationes Universitatis Tartuensis de Mathematica*, to appear.

[11] R. Gardner and B. Shields, The Number of Zeros in a Disk for a Certain Class of Polynomials, in preparation.

[12] N. K. Govil and Q. I. Rahman, On the Eneström-Kakeya Theorem, *Tôhoku Mathematics Journal*, **20** (1968), 126-136.

[13] S. Kakeya, Some Problems on Maximum and Minimum Regarding Ovals, *Tôhoku Science Reports*, 71-88 (1917).

[14] Q. G. Mohammad, On the Zeros of the Polynomials, *American Mathematical Monthly*, **72**(6) (1965), 631-633.

[15] M. S. Pukhta, On the Zeros of a Polynomial, *Applied Mathematics,* **2** (2001), 1356-1358.

[16] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* Oxford: Clarendon Press, 2002.

[17] B. Shields, The Number of Zeros of a Polynomial in a Disk as a Consequence of Restrictions on the Coefficients, M.S. Thesis, East Tennessee State University, Johnson City, TN, 2014.

[18] E. C. Titchmarsh, *The Theory of Functions*, 2nd Edition, Oxford University Press, London, 1939.

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