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The Apprentices' Tower of Hanoi

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ABSTRACT

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The Apprentices' Tower of Hanoi is introduced in this thesis. Several bounds are found in regards to optimal algorithms which solve the puzzle. Graph theoretic properties of the associated state graphs are explored. A brief summary of other Tower of Hanoi variants is also presented.

DEDICATION

I dedicate this thesis to my late grandfather Brady Stevenson, who taught me from a very young age how to read, how to write, and how to do arithmetic, and also guided me in the development my faith, my character, and my morality.

ACKNOWLEDGMENTS

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To my mother, Ronda, and my dad, Greg, thank you for providing for me for so many years. I would like to thank my brother Adam for never staying angry with me, despite the number of fights we've had. Thank you to my grandparents for always helping me. I am eternally grateful to the Lord and Saviour Jesus Christ.

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TABLE OF CONTENTS

7

LIST OF TABLES

LIST OF FIGURES

1 CLASSICAL TOWER OF HANOI

In this thesis we intend to study a variant of the Tower of Hanoi. The Tower of Hanoi was first published under the original French title La Tour D'Hanoï in 1883 by Professor N. Claus. This was a pen name of French number theorist François Édouard Anatole Lucas [16].

1.1 The Legend

Upon creation of the Tower of Hanoi, Lucas penned a legend to describe his puzzle. Later, the legend was further illustrated by Henri de Parville, whose story was then translated by W. W. R. Ball as given below [13].

> It was said that in the Indian City of Benares, beneath a dome which marked the centre of the world, there was to be found a brass plate in which were set three diamond needles, "each a cubit high and as thick as the body of a bee." It was also said that God had placed sixty-four discs of pure gold on one of these needles at the time of Creation. Each disc was said to be of different size, and each was said to have been placed so that it rested on top of another disc of greater size, with the largest resting on the brass plate at the bottom and the smallest at the top. This was known as the Tower of Brahma.
>
> Within the temple there were said to be priests whose job it was to transfer all the gold discs from their original needle to one of the others, without ever moving more than one disc at a time. No priest could ever place any

disc on top of a smaller one, or anywhere else except on one of the needles.

When the task had been completed, and all sixty-four discs had been successfully transferred to another needle, it was suggested that the tower, temple, and Brahmins alike will crumble into dust, and with a thunder-clap the world will vanish [1].

1.2   Rules

Supposing the legend were true we may like to know how long it will take for the universe to end. Let us now examine the rules. The Tower of Hanoi is a mathematical puzzle consisting of three pegs and n discs with distinct diameter. In the initial state the puzzle has the n discs placed on an origin peg in descending diametric order. In other words, the disc with largest diameter is on the bottom and the diameters strictly decrease so that the smallest is on top.

The objective is to move the n discs to a destination peg while at the same time obeying three rules. The rules are as follows:

1. The player may move only one disc at a time. When carried out, this is considered to be one move.

2. Only the top disc of a given peg may be moved.

3. (The Divine Rule) No disc shall be placed atop a disc of smaller diameter.

We also need the definition of auxiliary peg and the definition of solved. An *auxiliary peg* is any peg which is neither the origin peg nor the destination peg. The puzzle is said to be *solved* when all discs are placed on the destination peg.

1.3 An Optimal Algorithm

Upon release of the puzzle, Lucas included a conjecture about the number of moves required in an optimal algorithm for the Tower of Hanoi with sixty-four discs [16].

Claim 1. [16] *An optimal algorithm for the Tower of Hanoi with sixty-four discs requires 18446744073709551615 moves. Supposing each move takes one second it will take more than five billion centuries to solve the puzzle.*

Let us examine this claim. First we must define optimal algorithm. In terms of the Tower of Hanoi, an *optimal algorithm* is one in which the least number of moves are required to solve the puzzle. Denote the number of moves in an optimal algorithm by M_n.

Most people who have attempted the puzzle on a small number of discs notice a recursive algorithm by which the puzzle may be solved. This consists of moving the top $n - 1$ discs to the auxiliary peg, then moving the largest disc to the destination peg, and finally moving the remaining $n - 1$ discs to the destination peg. This recursive algorithm has been shown to be optimal by D. Wood in 1981 [23]. The key to showing the optimality is showing the largest disc may be moved only once in an optimal algorithm.

Lemma 1.1. [23] *The largest disc is moved only once in an optimal algorithm for the Tower of Hanoi.*

Theorem 1.2. [23] *An optimal solution to The Tower of Hanoi with n discs consists of twice the number of moves for $n - 1$ discs plus one additional move. That is,*

$M_n = 2M_{n-1} + 1.$

While this is an optimal algorithm it would be difficult to examine Lucas' claim by recursion. It is with this in mind that we consider the following.

Corollary 1.3. $M_n = 2^n - 1.$

Proof. From Theorem 1.2, $M_n = 2M_{n-1} + 1$. Also, it is clear that $M_1 = 1$. Let us proceed by induction on n. $M_1 = 1 = 2^1 - 1$. So, the base case holds. Now let us assume the claim holds for n. That is, $M_n = 2M_{n-1} + 1 = 2^n - 1$. Now consider $M_{n+1} = 2M_n + 1$. By hypothesis,

$$M_{n+1} = 2M_n + 1 = 2(2M_{n-1} + 1) + 1 = 2(2^n - 1) + 1 = 2(2^n) - 2 + 1 = 2^{n+1} - 1.$$

Hence, the claim follows by induction. Q.E.D.

We may now consider Lucas' claim. We will use the optimal algorithm.

$$M_n = 2^n - 1 \implies M_{64} = 2^{64} - 1 = 18446744073709551615. \tag{1}$$

So, the number of moves is correct. If, as the legend says, we assume one move is made every second, then we can conclude it would take more than five billion centuries, as Lucas claimed.

1.4 Hanoi Graphs

Of great interest to those who study Tower of Hanoi problems are the corresponding graphs. *Hanoi Graphs*, denoted H_3^n, are graphs whose vertex set consists of all possible states of the puzzle, and two vertices are adjacent in H_3^n if and only if one state could be obtained from another in a single move [19].

Let us suppose the discs are distinctly labeled $1, 2, \ldots, n$ with larger discs being assigned greater numbers. When listing an arrangement, we shall list discs left to right to represent top to bottom. Also a vertical bar will separate the pegs.

For example, if we have one disc, then there are three possible states. The first, $1||$, where the disc is on the first peg. Another, $|1|$, where the disc is on the second peg. Finally, $||1$, where the disc is on the destination peg. In general, the initial state will be $12\ldots n||$. Other states include but are not limited to $n|12\ldots n-1|$ and $|12\ldots n-1|n$. The final state is $||12\ldots n$.

Example 1.4. *The Hanoi Graph for the Tower of Hanoi with 2 discs has vertex set* $V(H_3^2) = \{12||, 1|2|, 1||2, 2|1|, 2||1, |12|, |1|2, |2|1, ||12\}$. *The edge set is*

$$E(H_3^2) = \{(12||)(1|2|), (12||)(1||2), (1|2|)(1||2), (1|2|)(2|1|), (1|2|)(|12|), (1||2)(|1|2),$$

$$(1||2)(||12), (2|1|)(2||1), (2|1|)(|1|2), (2||1)(|2|1), (|12|)(|2|1), (1|2)(||12)\}.$$

The graph can be seen in Figure 1.
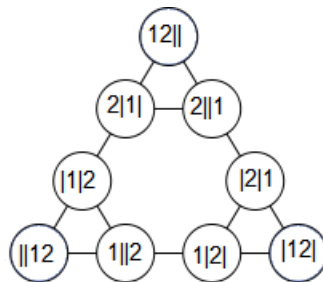
Figure 1: H_3^2

A *subgraph* of a graph G is a graph comprised of only vertices and edges from the

14

edge set of G [2]. An *isomorphism* is a one to one and onto mapping that preserves the structure of the set, i.e., a bijective homomorphism [8]. A *complete graph*, denoted K_n, has n vertices and all possible adjacencies between these vertices[2].

Proposition 1.5. *Each H_3^n, where $n \geq 1$, contains a K_3 subgraph.*

Proof. There are only three possible states for the Tower of Hanoi with one disc. Each of which can be attained by a single move from the other. Hence, $H_3^1 \cong K_3$. Moreover, H_3^1 is clearly a subgraph of H_3^n. Q.E.D.

A *walk* of length n in a graph is an alternating sequence of vertices and edges, $v_0, e_0, v_1, e_1, v_2, \ldots, v_{n-1}, e_{n-1}, v_n$, which begins and ends with vertices. A *trail* is a walk with no repeated edges. A *path* is a trail with no repeated vertices. A $u - v$ *path* is a path that begins at vertex u and ends at vertex v [11].

Now we define connected for vertices and graphs. Two vertices u and v in a graph G are *connected* if G contains a $u - v$ path. A graph G is *connected* if every two vertices of G are connected [2].

Proposition 1.6. [13] *H_3^n is connected.*

Next we will define graph distance and geodesic. The *distance* from a vertex u to a vertex v in a connected graph is the minimum of the lengths of the $u - v$ paths in the graph, denote this $d(u, v)$ [2].

Proposition 1.7. [13] *The distance from the vertex representing the initial state, v_i, to the vertex representing the final state, v_f, is the number of moves required in an optimal algorithm to solve the Tower of Hanoi.*

Corollary 1.8. [13] $d(v_i, v_f) = 2^n - 1 = M_n = 2M_{n-1} + 1.$

Let us now define coloring, chromatic number, and edge chromatic number. A *proper coloring* (or simply, a *coloring*) of a graph is an assignment of colors to the vertices, such that adjacent vertices have different colors [3]. The smallest number of colors in any coloring of a graph G is called the *chromatic number* of G, denote this $\chi(G)$ [3]. The *edge chromatic number*, or chromatic index, is the least number of colors needed to color the edges of a graph G so that no two adjacent edges are assigned the same color. This is denoted $\chi_1(G)$ or $\chi'(G)$ [10].

Proposition 1.9. [13] *For* $n \geq 1$, $\chi(H_3^n) = 3 = \chi'(H_3^n)$.

We now consider domination in Hanoi Graphs. A vertex is said to *dominate* itself and all adjacent vertices [2]. Furthermore, a set of vertices is a *dominating set* if every vertex of a graph is dominated by at least one vertex in the set [2]. Finally, the *domination number*, denoted $\gamma(G)$, is the minimum cardinality among the dominating sets of G [2].

Proposition 1.10. [13] *The domination number of a Hanoi Graph is*

$$\gamma(H_3^n) = \frac{1}{4}(3^n + 2 + (-1)^n).$$

Let us now consider the symmetries of Hanoi Graphs. An *automorphism* is an isomorphism mapping a set to itself [14]. Also, the *automorphism group*, denoted $Aut(G)$, of a group G is the set of all automorphisms of G [9]. Lastly, if A is the set of all natural numbers less than or equal to n, then the set of all permutations of A is called the symmetric group of degree n and is denoted by S_n [9].

Proposition 1.11. [13] *For* $n \in \mathbb{N}$, $Aut(H_3^n) \cong S_3 \cong S_T$, *where* S_T *is the symmetric group formed by the vertex set,* T^n.

The symmetries can best be seen by considering the fractal structure which the graphs take. This structure is known as The Sierpiński Triangle or Sierpiński's Gasket which is shown in Figure 2 [17].
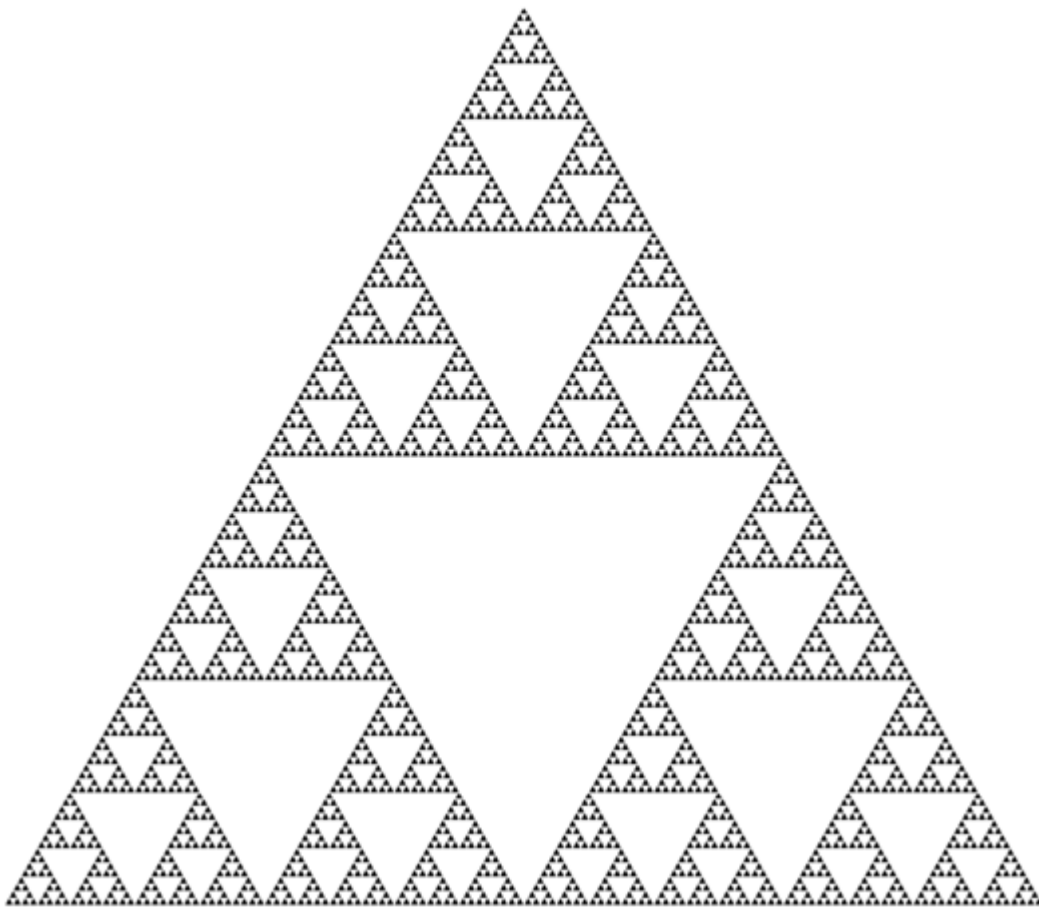
Figure 2: The Sierpiński Triangle[5]

2 TOWER OF HANOI VARIANTS

Over the years several variations on the classical Tower of Hanoi have been considered. Most such variants involve either breaking the Divine Rule or considering additional pegs. A *Tower of Hanoi Variant* is any variant that consists of discs and pegs, such that discs may be placed on pegs. Additionally the following rules must be obeyed:

1. Pegs are distinguishable.

2. Discs are distinguishable.

3. Discs are on pegs all the time except for moves.

4. One or more discs can be moved from only the top of a stack.

5. Given an initial distribution of discs among pegs and a goal distribution among pegs, find a shortest sequence of moves that transfers discs from the initial state to the final state obeying any added rules [13].

2.1 The Reve's Puzzle

Figure 3: The Reve's Puzzle[7]

The first and perhaps most famous variant is The Reve's Puzzle which is illustrated in Figure 3. As given by Henry Dudeney in *The Canterbuy Puzzles And Other Curious Problems*:

The Reve was a wily man and something of a scholar. As Chaucer tells us, "There was no auditor could of him win," and "there could no man bring him in arrear." The poet also noticed that "ever he rode the hindermost of the route." This he did that he might the better, without interruption, work out the fanciful problems and ideas that passed through his active brain. When the pilgrims were stopping at a wayside tavern, a number of cheeses of varying sizes caught his alert eye; and calling for four stools, he told the company that he would show them a puzzle of his own that would keep them amused during their rest. He then placed eight cheeses of graduating sizes on one of the end stools, the smallest cheese being at the top, as clearly shown in the illustration. "This is a riddle," quoth he, "that I did once set before my fellow townsmen at Baldeswell, that is in Norfolk, and, by Saint Joce, there was no man among them that could rede it aright. And yet it is withal full easy, for all that I do desire is that, by the moving of one cheese at a time from one stool unto another, ye shall remove all the cheeses to the stool at the other end without ever putting any cheese on one that is smaller than itself. To him that will perform this feat in the least number of moves that be possible will I give a draught of the best that our good host can provide." To solve this puzzle in the fewest possible moves, first with 8, then with 10, and afterwards

19

with 21 cheeses, is an interesting recreation [7].

Proposition 2.1. *The Reve's Puzzle is a Tower of Hanoi variant with four pegs.*

In the included solutions Dudeney claims, "The 8 cheeses can be removed in 33 moves, 10 cheeses in 49 moves, and 21 cheeses in 321 moves." Dudeney provides a method by which the reader may find these results, but does not prove its optimality [7].

While an optimal algorithm is easy to establish for the classical Tower of Hanoi, it is not the case for the general Reve's Puzzle. In fact, this is still an open problem. However, we may still check the particular solutions given by Dudeney.

The idea of Hanoi graphs may be naturally extended to this puzzle. They will be denoted H_4^n. In particular the Reve has given H_4^8, H_4^{10}, and H_4^{21}. So, a geodesic from the vertex representing the initial state, v_i, to the vertex representing the final state, v_f, is the minimum number of moves.

One can employ a breadth-first search to find $d(v_i, v_f)$. Due to symmetries in the puzzle, the search need only consider three levels at a time, and it may stop when any other perfect state has been met [13]. Using such a method Korf found the minimum number of moves for the Reve's Puzzle up to 31 discs [15]. In doing so Dudeney's claims were proved. These are given in Table 1.

Table 1: The Minimum Number of Moves for $n \in [31]$ for the Reve's Puzzle [15]

n	M_n	n	M_n
1	1	17	193
2	3	18	225
3	5	19	257
4	9	20	289
5	13	**21**	**321**
6	17	22	385
7	25	23	449
8	**33**	24	513
9	41	25	577
10	**49**	26	641
11	65	27	705
12	81	28	769
13	97	29	897
14	113	30	1025
15	129	31	1153
16	161		

2.2 Frame-Stewart

A natural extension to the Tower of Hanoi is to consider any number of discs on any number of pegs. The problem was first formally proposed by B. M. Stewart in 1939, as Problem 3918 in the *American Mathematical Monthly* as follows:

> Given a block in which k pegs and a set of n washers, no two alike in size, and arranged on one peg so that no washer is above a smaller washer. What is the minimum number of moves in which the n washers can be placed on another peg, if the washers must be moved one at a time, subject always to the condition that no washer be placed on a smaller washer [21]?

This is clearly a variant as we follow the exact same rules as the Tower of Hanoi and the Reve's Puzzle, but have k distinct pegs rather than 3 or 4, respectively. Since

finding an optimal algorithm for the Reve's Puzzle is an open problem, certainly the n discs on k pegs variant is an open problem. There is, however, a "presumed optimal algorithm," which was introduced by J. S. Frame and edited by Stewart two years after the problem was proposed [22].

Conjecture 2.2. *(Frame-Stewart Conjecture [22]) $F_p^n = T(n,p) = 2T(k,p) + T(n-k, p-1)$ choosing k to give the minimum value, where F_p^n is a Frame-Stewart number and $T(n,p)$ is the minimum number of moves required to complete the puzzle. In other words, the number of moves required to transfer all n discs to one of the $p-1$ other towers is the number required to transfer the top k discs to a tower that is not the destination peg. Then, without disturbing the peg that now contains the top k discs, transfer the remaining $n-k$ discs to the destination peg, using only the remaining $p-1$ pegs, taking $T(n-k, p-1)$ moves. Finally transfer the top k discs to the destination peg, taking $T(k,p)$ moves.*

This conjecture has been attempted and believed to have been proven by several mathematicians. Yet there are logical gaps in all known such attempts to prove the conjecture. Most such efforts are similar to the original Frame-Stewart paper [13].

2.3   Bottle-Neck Tower

In 1981, D. Wood created a variant of the Tower of Hanoi in which the Divine Rule is relaxed. Precisely, for $k \geq 1$: If disc j is placed higher than disc i on the same peg, then $j - i \leq k$. This is played on 3 pegs [23].

In 2006, Dinitz and Solomon proved an optimal algorithm in which the top $n-1$ discs are moved to the auxiliary peg, then disc n is moved to the destination peg, and

finally the $n - 1$ discs are moved to the destination peg [6].

2.4   Sinner's Tower and Santa Claus' Tower

The Sinner's Tower and Santa Claus' Tower were introduced in a 2007 paper by Chen, Tian, and Wang [4]. In the Sinner's variant the objective is the same as the classical Tower of Hanoi, but the divine rule may be violated k times. In other words at most k times a disc may be placed directly on top of smaller disc [4]. Denote the minimum number of moves needed to complete the Sinner's variant as $S(n, k)$. Let

$$g(n, k) = \begin{cases} 2n - 1 & : n \leq k + 2 \\ 4n - 2k - 5 & : k + 2 \leq n \leq 2k + 2 \\ 2^{n-2k} + 6k - 1 & : 2k + 2 \leq n \end{cases}$$

for all n and k, $S(n, k) = g(n, k)$ [4].

In the Santa Claus' variant for a number d when a disc x is put on a pile of other discs, before the move, discs smaller than x can only occur at the top up to $d - 1$ positions in that pile, but none of them can be of size less than or equal to $x - d$. Let $C(n, d)$ denote the minimum number of moves to complete Santa Claus' Tower. Let $n - 1 = dq + r$, where $1 \leq r \leq d$. Then, $C(n, d) = 1 + d(2^{q+1} - 2) + r2^{q+1}$ [4].

3 THE APPRENTICES' TOWER OF HANOI

3.1 The Legend

As with Lucas, we will construct a legend to describe our variant. The priests who work at the Tower of Hanoi have been doing so since the dawn of time. As such they have become quite old and tired. It is with this in mind that they have decided to train a new generation of priests to take over the transfer of the golden discs.

As we are all well aware, the youth of today is quite lazy and does not understand the need for all these so-called rules. So, if left to their own accord, the young apprentices would tend to ignore the rules for the transfer process.

However, the apprentices are under the watchful eyes of the vigilant priests whose sworn duty is to ensure the established and sacrosanct laws of Brahma are obeyed. The wise old priests can keep track of how many golden discs are moved at once, and ensure only the top golden disc may be moved.

Unfortunately, with age comes poor eye-sight. Thus, so long as the stacks only have one misplaced golden disc per diamond needle, the vigilant priests will not notice. Does this spell doom for us all?

3.2 The Rules

The Apprentices' Tower of Hanoi is a Tower of Hanoi variant allowing the player to "sin," meaning break the divine rule exactly once on each tower at any given time. Thus, on each peg there is at most one place in which a larger disc is on top of a smaller disc. After which discs must decrease in size. A configuration is *regular* if

only smaller discs are on top of larger discs. A configuration is *perfect* if all discs are on a single peg.

The objective of The Apprentices' Tower of Hanoi is the same as the classical Tower of Hanoi, i.e., to start with a perfect configuration with an n regular stack on peg-1 and conclude with a perfect configuration with an n regular stack on peg-3.

3.3   Labeling

As before, suppose the discs are distinctly labeled $1, 2, \ldots, n$ with larger discs being assigned greater numbers. When listing an arrangement, we shall list discs left to right to represent top to bottom.

On each of the three pegs, the discs may be in one of the following states:

i) Empty(E)- There are no discs on the peg.

ii) Regular(R)- There is at least one disc on the peg. The discs conform to the Divine Rule (i.e., no larger disc is atop a smaller disc and thus, the permutation is strictly increasing). In particular, k regular, if top disc (left disc in notation) has value k.

iii) Sinful(S)- There are at least two discs on the peg, the discs violate the divine rule in exactly one place. In other words, there is exactly one fall in the associated permutation. In particular, k sinful if top disc has value k.

Hence, any "allowed" configuration can be put into one of the following categories in Table 2:

25

Table 2: Allowed Configurations

Configuration	Configuration
E, E, E (possible only when n=0)	S, E, E
R, E, E (Perfect Regular Configuration)	S, S, E
S, R, E	R, R, E
S, S, S	S, S, R
S, R, R	R, R, R

Example 3.1. *Consider* 16|34|52 *seen in Figure 4. Here we have an R, R, S configuration, which due to symmetry is an S, R, R from above.*

The Example 3.1 is shown in Figure 4.

Figure 4: 16|34|52

4 QUEST FOR AN OPTIMAL ALGORITHM

As with all variants of the Tower of Hanoi we seek an optimal algorithm. In general the number of moves required to transfer n discs in regular order from peg-1 to peg-3 in regular order allowing s_i sins on peg-i will be denoted $S_n(s_1, s_2, s_3)$. So the traditional Tower of Hanoi can be solved in $S_n(0, 0, 0)$ moves, and the Apprentices' Tower of Hanoi can be solved in $S_n(1, 1, 1)$ moves. Unless otherwise noted we will assume $S_n = S_n(1, 1, 1)$.

Observation 4.1. $S_1 = 1$, $S_2 = 3$, and $S_3 = 5$.

For $S_1 = 1$ simply move the largest disc to the destination. For $S_2 = 3$ move the top disc to the auxiliary peg. Then, move the largest disc to the target peg. Finally, move the disc from the auxiliary peg to the target peg. For $S_3 = 5$ move the top disc to the auxiliary peg. Then, move the next disc to the auxiliary peg. Next, move the top disc to the target peg. Then, move the top disc of the auxiliary peg to the target peg. Lastly, move the disc from the auxiliary peg to the target peg.

Theorem 4.2. $S_n \geq 2n - 1$ for all n.

Proof. Solving the puzzle requires the largest disc be moved to the destination peg. In order to move the largest disc, the origin peg must contain only the largest disc. So the top $n - 1$ discs must be moved from the origin peg. This requires at least $n - 1$ moves. The largest disc must eventually be placed on the destination peg. This requires at least one move. Lastly, the $n - 1$ discs must be placed on the destination peg. Hence, $S_n \geq n - 1 + 1 + n - 1 = 2n - 1$. Q.E.D.

Theorem 4.3. *The number of moves required to solve an n disc Apprentices' Tower of Hanoi with an optimal algorithm is less than or equal to $2^n - 1$.*

Proof. The number of moves to solve the classical Tower of Hanoi is $2^n - 1$. If we ignore the ability to sin, then we are reduced to the classical Tower of Hanoi. Thus $S_n \leq 2^n - 1$.   Q.E.D.

This bound is sharp for $n \leq 2$. Since we can solve The Apprentices' Tower with three discs in five moves, as given above, this is not an optimal algorithm.

Theorem 4.4. *The largest disc in a stack of regular order may be moved only if all other discs are first moved.*

Proof. A disc cannot be moved while other discs are on top of it, and the largest disc is on the bottom. Thus, the largest disc in a regular stack must be moved last.   Q.E.D.

Theorem 4.5. *The minimum number of moves required to move n discs to another peg in regular order is equal to the minimum number of moves required to return them in regular order.*

Proof. By definition S_n is the minimum number of moves required to move a regular stack of n discs to another peg in regular order. So, reversing this sequence of moves will be the minimum number required to return the discs.   Q.E.D.

Theorem 4.6. *The number of moves required to move a stack of n discs in any order to another peg in another order is the same as the minimum number of moves required to return them.*

Proof. Reversing the sequence of moves will be the minimum number required to return the discs. Otherwise we have a contradiction with the assumption that we have used the minimum number of moves to begin with. Q.E.D.

Theorem 4.7. *The number of moves required to solve an n disc Apprentices' Tower of Hanoi with an optimal algorithm is less than or equal to* $2S_{n-2} + 3$.

Proof. First, move the top $n - 2$ discs to the auxiliary peg, taking S_{n-2} moves. Next, move disc $n - 1$ to the auxiliary peg, taking one move. Now, move disc n to the destination peg, taking one move. Then, move disc $n - 1$ to the destination peg, taking one move. Lastly, reverse and mirror the S_{n-2} initial moves in order to place the top $n - 2$ discs on the destination peg. This gives a total of

$$S_{n-2} + 1 + 1 + 1 + S_{n-2} = 2S_{n-2} + 3$$

moves. Thus, $S_n \leq 2S_{n-2} + 3$. Q.E.D.

This bound is sharp for $S_3 = 2S_{3-2} + 3 = 2 + 3 = 5$. As n gets larger, it is presumed the bound is no longer sharp and will exponentially diverge. Table 3 provides a list for small n.

Table 3: $2S_{n-2} + 3$

n	$2S_{n-2} + 3$
1	1
2	3
3	5
4	9
5	13
6	21
7	29
8	45
9	61
10	93

Theorem 4.8. *The number of moves required to solve an n disc Apprentices' Tower of Hanoi with an optimal algorithm obeys*

$$S_n \leq 2S_{n-k}(1,1,1) + 2S_{k-1}(1,1,0) + 1,$$

where we choose k to minimize the right side of the inequality.

Proof. Consider the following algorithm:

1. Move $n-k$ discs to the auxiliary peg in a regular order. This takes $S_{n-k}(1,1,1)$ moves.

2. Move $k-1$ discs to the auxiliary peg in a regular order. This takes $S_{k-1}(1,1,0)$ moves.

3. Move disc n to the destination peg. This takes one move.

4. Move $k-1$ discs from the auxiliary peg to the destination peg by mirroring and reversing the $S_{k-1}(1,1,0)$ moves.

5. Move $n-k$ discs from the auxiliary peg to the destination peg by mirroring and reversing the $S_{n-k}(1,1,1)$ moves. This takes S_{n-k} moves.

So, we have a total of

$$S_{n-k}(1,1,1) + S_{k-1}(1,1,0) + 1 + S_{k-1}(1,1,0) + S_{n-k}(1,1,1)$$

$$= 2S_{n-k}(1,1,1) + 2S_{k-1}(1,1,0) + 1$$

moves. $\hfill$ Q.E.D.

The floor function rounds the input down to the nearest integer. It is denoted $\lfloor x \rfloor$. Table 4 provides a list for $2S_{n-k}(1,1,1) + 2S_{k-1}(1,1,0) + 1$ where $k = \lfloor \frac{n}{2} \rfloor$.

Table 4: $2S_{n-k}(1,1,1) + 2S_{k-1}(1,1,0) + 1$

n	$2S_{n-k}(1,1,1) + 2S_{k-1}(1,1,0) + 1$	$k = \lfloor \frac{n}{2} \rfloor$
1	1	0
2	3	1
3	5	1
4	9	2
5	13	2
6	17	3
7	25	3
8	29	4
9	37	4
10	45	5

The following notation is used heavily in the following theorems. Let S'_n denote the number of moves required to move an n regular stack allowing for sinful order. Let S''_n denote the minimum number of moves required to move an n sinful stack allowing for sinful order. Let S'''_n denote the minimum number of moves required to move an n sinful stack to regular order given that it was created using S''_n moves.

Theorem 4.9. *The number of moves required to solve an n disc Apprentices' Tower of Hanoi with an optimal algorithm obeys* $S_n \le 2S'_{n-1} + 1$ *for all* $n \in \mathbb{N}$.

Proof. Let us consider the following sequence of moves.

1. Move $n - 1$ discs to the auxiliary peg in a sinful order. This takes S'_{n-1} moves.

2. Move disc n to the destination peg.

3. Move $n - 1$ discs to the destination peg by reversing and mirroring the original S'_{n-1} moves. This takes S_{n-1} moves.

31

The total of this sequence is $S'_{n-1} + 1 + S'_{n-1} = 2S'_{n-1} + 1$.   Q.E.D.

Theorem 4.10. *The number of moves required to solve an n disc Apprentices' Tower of Hanoi with an optimal algorithm obeys*

$$S_n \leq 2S_{k-1}(1,0,1) + 2S'_{n-k}(1,1,1) + 2S'''_{n-k}(1,1,0) + 1.$$

Proof. Consider the following sequence of moves:

1. Move the top $n - k$ discs to the destination peg in a sinful order. This takes S'_{n-k} moves.

2. Move the next $k-1$ to the auxiliary peg in regular order. This takes $S_{k-1}(1,0,1)$ moves.

3. Move the $n - k$ discs from the destination peg to the auxiliary peg in sinful order. This takes $S'''_{n-k}(1,1,0)$ moves.

4. Move disc n to the the destination peg. This takes one move.

5. Move the top $n-k$ discs from the auxiliary peg to the origin peg into the sinful order from step 1. This takes $S'''_{n-k}(1,1,0)$ moves by symmetry.

6. Move the $k - 1$ discs from the auxiliary peg to the destination peg in regular order. This takes $S_{k-1}(1,0,1)$ moves.

7. Move the $n-k$ discs from the origin peg to the destination peg in regular order. This takes S'_{n-k}.

So, in total there are

$$S'_{n-k} + S_{k-1}(1,0,1) + S'''_{n-k}(1,1,0) + 1 + S'''_{n-k}(1,1,0) + S_{k-1}(1,0,1) + S'_{n-k}$$

moves.

Whence,

$$S_n \leq 2S_{k-1}(1,0,1) + 2S'_{n-k}(1,1,1) + 2S'''_{n-k}(1,1,0) + 1.$$

<div align="right">Q.E.D.</div>

Theorem 4.11. *The number of moves required to move a stack of n discs from a peg to another, $S'_n \leq 2S'_{n-2} + 2$.*

Proof. Consider the following sequence of moves:

1. Move the top $n-2$ discs to the auxiliary peg in a sinful order. This takes S'_{n-2} moves.

2. Move disc $n-1$ to the destination peg. This takes one move.

3. Move the top $n-2$ discs from the auxiliary peg to the destination peg. Place these in regular order by symmetry. This takes S'_{n-2} moves.

4. Move disc n to the the destination peg. This takes one move.

Therefore, $S'_n \leq 2S'_{n-2} + 2$. Q.E.D.

Thus far we have assumed the largest disc only moves once. Now we will look at what happens if the largest disc moves more than once.

Theorem 4.12. *The largest disc can be moved only if among the other pegs there is an empty or regular stack.*

<div align="center">33</div>

Proof. The largest disc can be moved atop an empty or regular stack. If the stack is sinful, then the largest disc cannot be placed atop the stack else we break the divine rule twice. Q.E.D.

Theorem 4.13. *If the largest disc is moved twice, then for optimality we require the destination peg or the auxiliary peg to have a stack of regular order for the first.*

Proof. By Theorem 4.12, we know the peg the largest disc is moved to is empty or has a regular stack. If the peg is empty and we plan to move the largest disc twice, then this is not an optimal move as we would then just have to move the other $n-1$ discs atop it. Thus, the stack to which the largest disc is to be moved must be regular. Q.E.D.

Theorem 4.14. *If the largest disc is moved twice, then*

$$S_n \leq 2S_{n-k}(1,1,1) + 3S_{k-1}(1,0,1) + 2.$$

Proof. Consider the following sequence of moves:

1. Move the top $n-k$ discs to the auxiliary peg in a regular order. This takes S_{n-k} moves.

2. Move the next $k-1$ discs to the destination peg in regular order. This takes $S_{k-1}(1,0,1)$ moves.

3. Move the largest disc to the auxiliary peg. This takes one move.

4. Move the $k-1$ discs from the destination peg to the origin peg. Place these in regular order by symmetry. This takes $S_{k-1}(1,0,1)$ moves.

34

5. Move disc n to the the destination peg. This takes one move.

6. Move the $k-1$ discs from the origin peg to the destination peg in regular order. This takes $S_{k-1}(1,0,1)$ moves.

7. Move the $n-k$ discs from the auxiliary peg to the destination peg in a regular order. This takes S_{n-k} moves.

In total this is

$$S_{n-k} + S_{k-1}(1,0,1) + 1 + S_{k-1}(1,0,1) + 1 + S_{k-1}(1,0,1) + S_{n-k}$$

$$= 2S_{n-k}(1,1,1) + 3S_{k-1}(1,0,1) + 2$$

moves. $\hfill$ Q.E.D.

Theorem 4.15. *If the largest disc is moved twice, then*

$$S_n \leq 2S_{n-k}(1,1,1) + 2S'_{k-1}(1,0,1) + S_{k-1}(1,0,1) + 2.$$

Proof. Consider the following sequence of moves:

1. Move the top $n-k$ discs to the auxiliary peg in a regular order. This takes S_{n-k} moves.

2. Move the next $k-1$ discs to the destination peg in sinful order. This takes $S'_{k-1}(1,0,1)$ moves.

3. Move the largest disc to the auxiliary peg. This takes one move.

4. Move the $k-1$ discs from the destination peg to the origin peg. Place these in regular order by symmetry. This takes $S'_{k-1}(1,0,1)$ moves.

5. Move disc n to the the destination peg. This takes one move.

6. Move the $k-1$ discs from the origin peg to the destination peg in regular order. This takes $S_{k-1}(1, 0, 1)$ moves.

7. Move the $n-k$ discs from the auxiliary peg to the destination peg in a regular order. This takes S_{n-k} moves.

In total this is

$$S_{n-k} + S'_{k-1}(1, 0, 1) + 1 + S'_{k-1}(1, 0, 1) + 1 + S_{k-1}(1, 0, 1) + S_{n-k}$$

$$= 2S_{n-k}(1, 1, 1) + 2S'_{k-1}(1, 0, 1) + S_{k-1}(1, 0, 1) + 2$$

moves.   Q.E.D.

Theorem 4.16. *If the largest disc is moved twice, then*

$$S_n \leq 2S_{n-k}(1, 1, 1) + S'_{k-1}(1, 0, 1) + S''_{k-1}(1, 0, 1) + S'''_{k-1}(1, 0, 1) + 2.$$

Proof. Consider the following sequence of moves:

1. Move the top $n-k$ discs to the auxiliary peg in a regular order. This takes S_{n-k} moves.

2. Move the next $k-1$ discs to the destination peg in sinful order. This takes $S'_{k-1}(1, 0, 1)$ moves.

3. Move the largest disc to the auxiliary peg. This takes one move.

4. Move the $k-1$ discs from the destination peg to the origin peg. Place these in sinful order. This takes $S'''_{k-1}(1, 0, 1)$ moves.

36

5. Move disc n to the the destination peg. This takes one move.

6. Move the $k-1$ discs from the origin peg to the destination peg in regular order. This takes $S_{k-1}'''(1, 0, 1)$ moves.

7. Move the $n-k$ discs from the auxiliary peg to the destination peg in a regular order. This takes S_{n-k} moves.

In total this is

$$S_{n-k} + S_{k-1}'(1, 0, 1) + 1 + S_{k-1}''(1, 0, 1) + 1 + S_{k-1}'''(1, 0, 1) + S_{n-k}$$

$$= 2S_{n-k}(1, 1, 1) + S_{k-1}'(1, 0, 1) + S_{k-1}''(1, 0, 1) + S_{k-1}'''(1, 0, 1) + 2$$

moves.   Q.E.D.

5 AH_n GRAPH PROPERTIES

As with all variants of the Tower of Hanoi, we can generate a family graphs where the states of the puzzle form the vertex set and the moves between them form the edge set. Let AH_n be the graph associated with the Apprentices' Tower of Hanoi with n discs, where the vertices represent the states of the puzzle, and there is an edge between two vertices if and only if the states are separated by a single move.

Theorem 5.1. *The associated graph AH_n is a subgraph of AH_N for all $N \geq n$.*

Proof. Clearly, if $n = N$, then the claim follows immediately. So, let $N > n$ and consider the puzzle with N discs. Ignore the ability to move disc $n + 1$. This creates the puzzle with n discs.   Q.E.D.

5.1 The Number of Vertices

Let us now determine the number of states in the Apprentices' Tower of Hanoi. First let $[k]$ denote the set of all natural numbers less than or equal to k. So on any peg, with k discs, we have a a permutation on $[k]$ in which there is at most one fall. In other words there may be only one greater number to the left of a lesser number in the vertex notation on a particular peg for a state.

Lemma 5.2. [18] *The number of permutations on $[k]$ in which there is at most one fall is given by $2^k - k$.*

In this section, we make use of the Multinomial Theorem.

Theorem 5.3. *(Multinomial Theorem [12])*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{1 \le t \le m} x_t^{k_t}.$$

Theorem 5.4. *The number of states of the Apprentices' Tower of Hanoi, and thus, the number of vertices in the associated graph, is given by*

$$n(AH_n) = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \ge 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} (2^{\lambda_1} - \lambda_1)(2^{\lambda_2} - \lambda_2)(2^{\lambda_3} - \lambda_3). \tag{2}$$

Proof. We count this in $\frac{(n+1)(n+2)}{2}$ disjoint exhaustive classes.

The $(\lambda_1, \lambda_2, \lambda_3)th$ set is the set of all configurations in which λ_1 discs are on peg 1, λ_2 discs are on peg 2, and λ_3 discs are on peg 3.

The number of configurations in this set can be counted by:

1. Choosing λ_1 discs, λ_2 discs, and λ_3 discs to place on pegs 1, 2, and 3, respectively.

   There are $\binom{n}{\lambda_1, \lambda_2, \lambda_3}$ ways to do this.

2. Arranging the λ_1 discs on peg 1 such that there is at most one fall.

   There are $2^{\lambda_1} - \lambda_1$ ways to do this by Lemma 5.2.

3. Arranging the λ_2 discs on peg 2 such that there is at most one fall.

   There are $2^{\lambda_2} - \lambda_2$ ways to do this by Lemma 5.2.

4. Arranging the λ_3 discs on peg 1 such that there is at most one fall.

   There are $2^{\lambda_3} - \lambda_3$ ways to do this by Lemma 5.2.

So, there are $\binom{n}{\lambda_1, \lambda_2, \lambda_3}(2^{\lambda_1} - \lambda_1)(2^{\lambda_2} - \lambda_2)(2^{\lambda_3} - \lambda_3)$ configurations in set $(\lambda_1, \lambda_2, \lambda_3)$ by the Multiplication Principle. Ergo, by the Addition Principle,

$$n(AH_n) = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \ge 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3}(2^{\lambda_1} - \lambda_1)(2^{\lambda_2} - \lambda_2)(2^{\lambda_3} - \lambda_3). \qquad \text{Q.E.D.}$$

Corollary 5.5. *The number of states of The Apprentices' Tower of Hanoi is given*
by

$$n(AH_n) = 6^n - 3n \cdot 5^{n-1} + 3n(n-1)4^{n-2} - n(n-1)(n-2)3^{n-3}. \qquad (3)$$

Proof. First expansion yields

$$\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} (2^{\lambda_1} - \lambda_1)(2^{\lambda_2} - \lambda_2)(2^{\lambda_3} - \lambda_3)$$

$$= \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} (2^{\lambda_1+\lambda_2+\lambda_3} - 2^{\lambda_1+\lambda_2}\lambda_3 - 2^{\lambda_1+\lambda_3}\lambda_2$$

$$- 2^{\lambda_2+\lambda_3}\lambda_1 + \lambda_1\lambda_2 2^{\lambda_3} + \lambda_1\lambda_3 2^{\lambda_2} + \lambda_2\lambda_3 2^{\lambda_1} - \lambda_1\lambda_2\lambda_3).$$

The Multinomial Theorem gives

$$(x+y+z)^n = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} x^{\lambda_1} y^{\lambda_2} z^{\lambda_3}.$$

Consider

$$\frac{\partial}{\partial x}(x+y+z)^n = n(x+y+z)^{n-1} = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1 x^{\lambda_1-1} y^{\lambda_2} z^{\lambda_3}.$$

Choose $x = 1$, $y = 2$, and $z = 2$ to obtain

$$n5^{n-1} = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1 2^{\lambda_2+\lambda_3}.$$

Now consider,

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}(x+y+z)^n = n(n-1)(x+y+z)^{n-2}$$

$$= \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1\lambda_2 x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3}.$$

Choose $x = 1$, $y = 1$, and $z = 2$ to obtain

$$n(n-1)4^{n-1} = \sum_{\substack{\lambda_1 + \lambda_2 + \lambda_3 = n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1 \lambda_2 2^{\lambda_3}.$$

Finally, consider

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} (x + y + z)^n = n(n-1)(n-2)(x+y+z)^{n-3}$$

$$= \sum_{\substack{\lambda_1 + \lambda_2 + \lambda_3 = n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1 \lambda_2 \lambda_3 x^{\lambda_1 - 1} y^{\lambda_2 - 1} z^{\lambda_3 - 1}.$$

Choose $x = y = z = 1$ to obtain

$$n(n-1)(n-2)3^{n-3} = \sum_{\substack{\lambda_1 + \lambda_2 + \lambda_3 = n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1 \lambda_2 \lambda_3.$$

Hence, by substituting into Equation 3 we have

$$n(AH_n) = \sum_{\substack{\lambda_1 + \lambda_2 + \lambda_3 = n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} (2^{\lambda_1} - \lambda_1)(2^{\lambda_2} - \lambda_2)(2^{\lambda_3} - \lambda_3)$$

$$= 6^n - 3n5^{n-1} + 3n(n-1)4^{n-2} - n(n-1)(n-2)3^{n-3}.$$

Q.E.D.

Since this is an exponential of order 6^n we should expect the number of vertices to grow rapidly. Table 5 gives a list of small values of $n(AH_n)$.

Table 5: $n(AH_n)$ for small n.

n	$n(AH_n)$
0	1
1	3
2	12
3	57
4	300
5	1701
6	10206
7	63825
8	411096
9	2702349
10	17992506
11	120543561
12	808224372
13	5400815829
14	35868103734
15	236354531841
16	1544182760496
17	10001335837725
18	64233753928722
19	409298268016761
20	2589206145139596

We now classify the degree of all vertices based upon the states they represent. The *degree* of a vertex, $deg(v)$, is the number of edges incident on the vertex. Thus, here $deg(v)$ is the number of possible moves from a given state v.

Theorem 5.6. *A vertex v in AH_n has degree 0 if and only if it represents an E,E,E configuration.*

Proof. Assume $deg(v) = 0$. Then we may not move any discs. If any of the pegs are not empty, then we may move at least of disc to two locations. Hence, a contradiction. Therefore, all pegs are empty.

Now let us assume the converse that all pegs are empty. Clearly, nothing can be moved as there is nothing to be moved. Q.E.D.

Theorem 5.7. *A vertex v in AH_n has degree 2 if and only if it represents a perfect configuration or an S,E,E configuration.*

Proof. Let v represent a state where all discs are on the same peg. Without loss of generality, assume they are on the first peg. The only choices available are to move the top disc to the auxiliary peg or to move the destination peg. Hence, $deg(v) = 2$.

Now, let us assume $deg(v) = 2$. Assume to the contrary that all discs are not on the same peg. If one of the pegs were empty, then we could move either of the two top discs to the empty peg or move the smaller of the two atop the larger. This stipulation gives three options. So, assume none of the pegs are empty. Then, we have at least three options, moving the smallest atop either of the larger two discs or moving the medium disc atop the largest. Hence, a contradiction. Q.E.D.

Corollary 5.8. *There are $3(2^n - n)$ vertices of degree 2 in AH_n.*

Proof. A vertex v in AH_n has degree 2 if and only if it represents a state where all discs are on the same peg, by Theorem 5.7. Furthermore, by Lemma 5.2 there are $2^n - n$ such patterns on a peg. Since there are three pegs it follows from the Addition Principle that there are $3(2^n - n)$ degree 2 vertices in AH_n. Q.E.D.

Theorem 5.9. *If a vertex v in AH_n represents an R,R,E configuration, then $deg(v) = 4$.*

Proof. Suppose v in AH_n represents an R,R,E configuration. Accordingly, we may move either of the two top discs to either of the two other pegs. Therefore, $deg(v) = 4$. Q.E.D.

Theorem 5.10. *If a vertex v in AH_n represents an S,S,E configuration, then $deg(v) = 3$.*

Proof. Suppose v in AH_n represents an S,S,E configuration. Then, we may move either of the two top discs to the empty peg. Furthermore, we may move the smaller of the two top discs atop the larger. Hence, $deg(v) = 3$. Q.E.D.

Theorem 5.11. *If a vertex v in AH_n represents an S,R,E configuration, then $deg(v) = 4$ or $deg(v) = 3$.*

Proof. Assume v represents an S,R,E configuration. If the top disc of the sinful stack is larger than the top disc of the regular stack, then we may move either of two discs to either of the other two pegs. So, $deg(v) = 4$. If the top disc of the sinful stack is smaller than the top disc of the regular stack, then we may not move the top disc of the regular stack to the top of the sinful stack. So, $deg(v) = 3$. Q.E.D.

Theorem 5.12. *If a vertex v in AH_n represents an S,S,S configuration, then $deg(v) = 3$.*

Proof. Let v represent an S,S,S configuration. Then we may move the smallest of the top discs to either of the two other pegs. We may move the middle of the discs atop the largest. The largest disc may not move. Ergo, $deg(v) = 3$. Q.E.D.

Theorem 5.13. *If a vertex v in AH_n represents an S,S,R configuration, then $deg(v) = 3$, $deg(v) = 4$, or $deg(v) = 5$.*

44

Proof. Let v represent an S,S,R configuration. First, consider the case where the largest of the top discs is on the regular stack. In this case, we may move either of the other two top discs atop the largest disc. Furthermore, we may move the smallest disc atop the medium disc. Thus, $deg(v) = 3$.

Now, consider the case where the medium top disc is on the regular stack. In this case, we may move either of the other two top discs atop the medium disc. Moreover, we may move the smallest disc atop the largest disc, or we may move the medium disc atop the largest disc. Hence, $deg(v) = 4$.

Finally, consider the case where the smallest top disc is on the regular stack. In this case, we may move either of the other two top discs atop the smallest disc. Furthermore, we may move the smallest disc atop the largest disc, we may move the medium disc atop the largest, or we may move the smallest disc atop the medium disc. Therefore, $deg(v) = 5$.   Q.E.D.

Theorem 5.14. *If a vertex v in AH_n represents an S,R,R configuration, then $deg(v) = 4$, $deg(v) = 5$, or $deg(v) = 6$.*

Proof. Let v represent an S,R,R configuration. First, consider the case where the largest of the top discs is on the sinful stack. In this case, any of the top discs may be moved to the top of the other pegs. So, $deg(v) = 6$.

Next, consider the case where the smallest of the top discs is on the sinful stack. In this case, neither of the other two top discs may be placed on the sinful stack. However, any of the other moves are legal. Hence, $deg(v) = 4$.

Finally, consider the case where the medium top disc is on the sinful stack. In this case, the only move that is not allowed is to move the largest disc to the top of

the sinful stack. Thus, $deg(v) = 5$. Q.E.D.

Theorem 5.15. *If a vertex v in AH_n represents an R,R,R configuration, then $deg(v) = 6$.*

Proof. Assume a vertex v in AH_n represents an R,R,R configuration. Then, we may move any of the three top discs to the top of the other stacks. There are 6 ways to do this. Hence, $deg(v) = 6$. Q.E.D.

5.2   Figures of AH_n for $n \leq 3$

The following section contains the AH_n graphs for $n \leq 3$. They are shown in order such that n is increasing. The graphs are shown in Figures $5, 6, 7$, and 8.
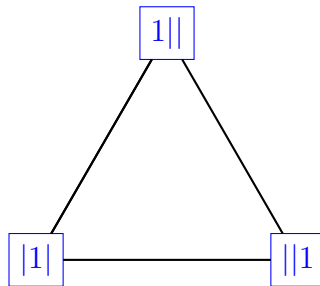

Figure 5: $AH_0 \cong$ Trivial Graph
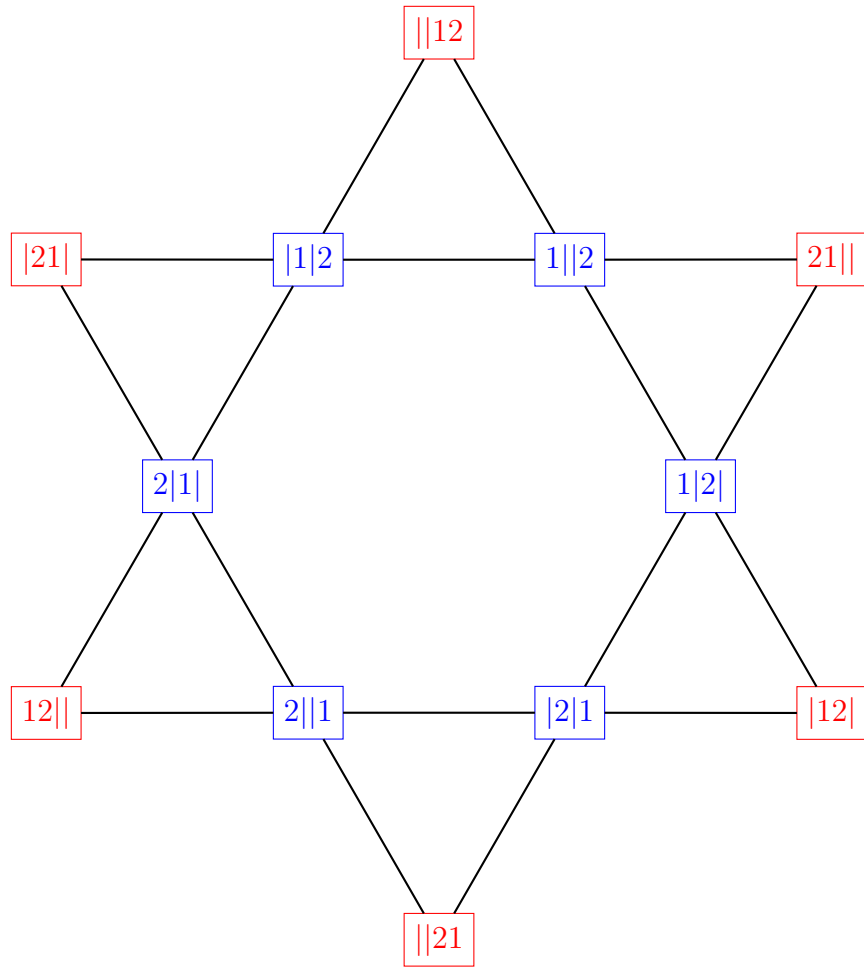

Figure 6: $AH_1 \cong K_3$

Figure 7: $AH_2 \cong SD(1) =$ Star of David Graph [20]
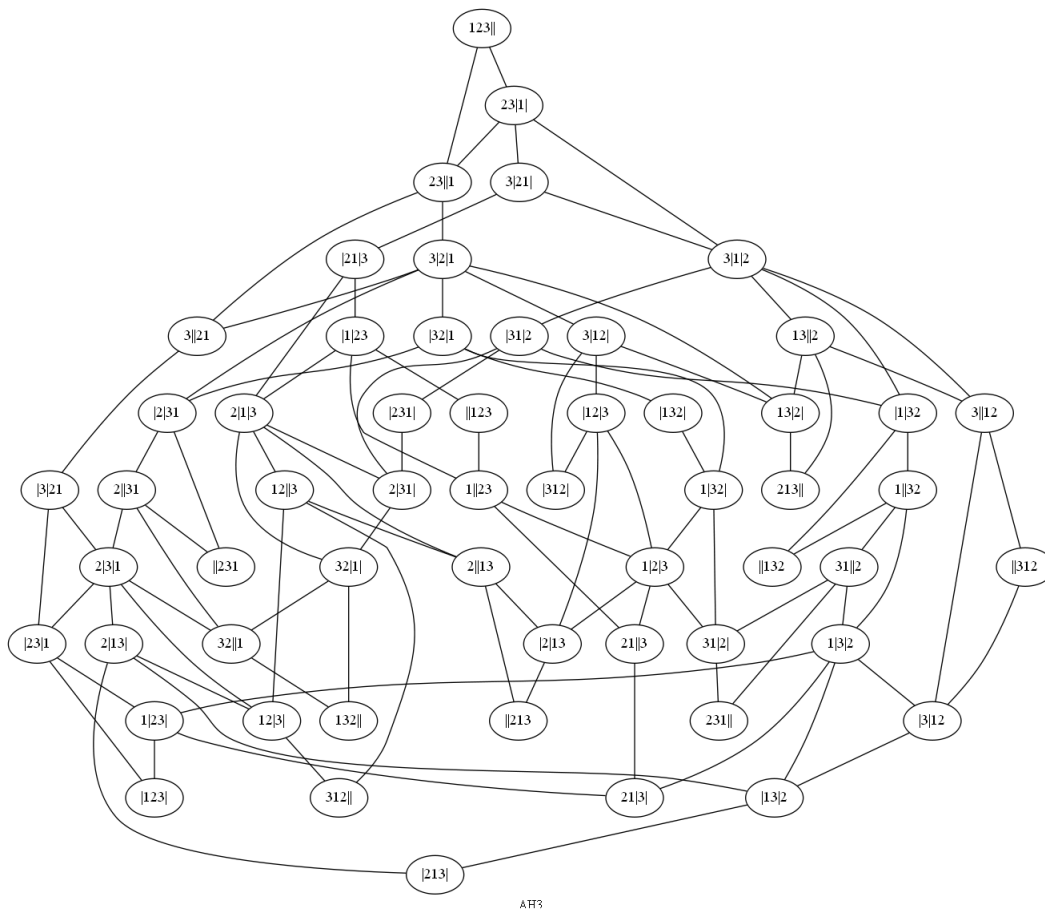
Figure 8: The graph AH_3.

5.3 Graph Invariants

A graph is *planar* if it can be drawn in the plane in such a way that no edge crosses another. Recall a *complete graph*, denoted K_n, has n vertices and all possible adjacencies between these vertices. A *complete bipartite graph*, denoted $K_{n,m}$, is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both incident vertices in the same subset, and every possible edge that could make adjacent vertices in different subsets is part of the graph. A *subdivision of an*

edge uv with vertices u, v gives a graph containing one new vertex w, and with an edge set replacing uv by two new edges, uw and wv. A *subdivision* of a graph G is a graph resulting from the subdivision of edges in G. An *induced subgraph* of a graph G is a subset of the vertex set of G and the set of edges such that both vertices incident to the edges are in the aforementioned vertex subset.

Let us now consider the planarity of AH_n. Clearly from above, AH_1 and AH_2 are planar. However, we must further examine the planarity of AH_3.

Theorem 5.16. *(Kuratowski's Theorem [2]) A graph G is planar if and only if G contains no subgraph that is a subdivision of a K_5 or $K_{3,3}$.*

Theorem 5.17. *The graph AH_n for $n \geq 3$ is non-planar.*

Proof. Let $n \geq 3$. AH_3 contains a subgraph that is a subdivision of $K_{3,3}$. To obtain this form the induced subgraph with the shaded vertices in Figure 9 below, i.e., $23|1|, 23||1, 3|21|, |21|3, 3|2|1, 3|1|2, 3||21, 3|12|, 2|1|3, |12|3, 3||12, |3|21, 12||3, 2|3|1, 2||13, 1|2|3, |23|1, |2|13, 21||3, 1|3|2, 1|23|, 12|3|, |3|12, 21|3|$. Furthermore, AH_3 is a subgraph of AH_n, by Theorem 5.1. So, AH_n contains a subgraph that is a subdivision of $K_{3,3}$. Therefore, by Kuratowski's Theorem AH_n is non-planar.   Q.E.D.
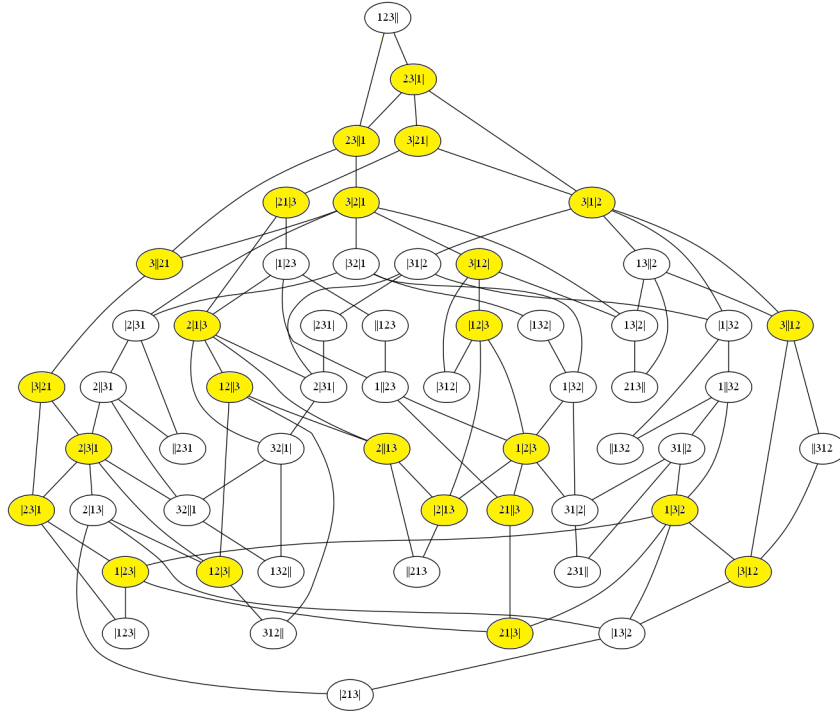
Figure 9: The graph AH_3 with selected vertices.

Theorem 5.18. [2] *A connected graph is Eulerian if and only if every vertex has even degree.*

Theorem 5.19. *The graph AH_n is not Eulerian for $n \geq 3$.*

Proof. Consider AH_n for $n \geq 3$. There exists a vertex of odd degree by Theorem 5.11 Hence, AH_n is not Eulerian.   Q.E.D.

Recall a *proper coloring* (or simply, a *coloring*) of a graph is an assignment of colors to the vertices, such that adjacent vertices have different colors [3]. The smallest number of colors in any coloring of a graph G is called the *chromatic number* of G, denote this $\chi(G)$ [3].

Theorem 5.20. *The chromatic number for the graph AH_n is 3 for all $n \geq 1$.*

Proof. Clearly, H_n is a subgraph of all AH_n. Hence, $\chi(AH_n) \geq \chi(H_n) = 3$. Now, color the H_n subgraph using a proper coloring. Suppose pegs 1, 2, and 3 have $m, \ell,$ and q discs, respectively. Then any other configuration with m, ℓ, and q discs on pegs 1, 2, and 3, respectively, may be put in the same color class, as no two of these states are adjacent. Since one of these configurations has all regular stacks, it follows AH_n has the same number of color classes as H_n.   Q.E.D.

The maximum degree of a vertex in a graph G is denoted $\Delta(G)$.

Theorem. *(Vizing's Theorem [2]) For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)$.*

Theorem 5.21. *For the graph AH_n, $6 \leq \chi'(AH_n) \leq 7$.*

Proof. The maximum degree of any vertex in AH_n is 6. Therefore, from Vizing's Theorem $6 \leq \chi'(AH_n) \leq 7$.   Q.E.D.

6 GENERALIZING THE APPRENTICES' TOWER OF HANOI

Suppose the eye-sight of the priests was a variable and there are p pegs. So, instead of allowing one sin per peg they allowed s_i for $i \in [p]$ sins on a given peg, and suppose each of the pegs were watched by different priests. This of course creates a more general puzzle than before.

For example, we shall denote The Apprentices' Tower of Hanoi with the usual ability to sin once per peg at any given time, as $(s_1, s_2, s_3) = (1, 1, 1)$. The ability to sin once only on the origin peg shall be denoted $(s_1, s_2, s_3) = (1, 0, 0)$. Similarly, the ability to sin once only on the auxiliary peg $(s_1, s_2, s_3) = (0, 1, 0)$.

Moreover, were we to consider the ability to sin three times on the origin peg, twice on the auxiliary, and once on the destination peg, then it would be denoted $(s_1, s_2, s_3) = (3, 2, 1)$. In general one could consider p pegs with n discs and s_i for $i \in [p]$ sins on a given peg as $(s_1, s_2, \ldots, s_i, \ldots, s_p)$.

6.1 s_i sins

First we shall consider the puzzle with three pegs and an arbitrary number of sins.

Theorem 6.1. *The number of moves required to solve the puzzle is atleast $2n - 1$, i.e., $S_n(s_1, s_2, s_3) \geq 2n - 1$ for all n.*

Proof. Solving the puzzle requires the largest disc be moved to the destination peg. In order to move the largest disc, the origin peg must contain only the largest disc. So the top $n - 1$ discs must be moved from the origin peg. This requires at least $n - 1$ moves. The largest disc must eventually be placed on the destination peg. This

requires at least one move. Lastly, the $n-1$ discs must be placed on the destination peg. Hence, $S_n \geq n-1+1+n-1 = 2n-1$. Q.E.D.

Theorem 6.2. $S_n(s_1, s_2, s_3) = S_n(s_3, s_2, s_1)$

Proof. By definition $S_n(s_1, s_2, s_3)$ is the minimum number of moves required to move a regular stack of n discs to another peg in regular order. So, reversing this sequence of moves will be the minimum number required to return the discs. Q.E.D.

Theorem 6.3. *The number of moves required to solve an n disc Apprentices' Tower of Hanoi with an optimal algorithm obeys $S_n \leq 2S_{n-k}(s_1, s_3, s_2)+2S_{k-1}(s_1, s_3, s_2-1)+1$, where we choose k to minimize the right side of the inequality.*

Proof. Consider the following algorithm:

1. Move $n-k$ discs to the auxiliary peg in a regular order. This takes $S_{n-k}(s_1, s_3, s_2)$ moves.

2. Move $k-1$ discs to the auxiliary peg in a regular order. This takes $S_{k-1}(s_1, s_3, s_2-1)+1$ moves.

3. Move disc n to the destination peg. This takes one move.

4. Move $k-1$ discs from the auxiliary peg to the destination peg by mirroring and reversing the $S_{k-1}(s_1, s_3, s_2-1)+1$ moves. This takes S_{k-1} moves.

5. Move $n-k$ discs from the auxiliary peg to the destination peg by mirroring and reversing the $S_{n-k}(s_1, s_3, s_2)$ moves. This takes S_{n-k} moves.

So, we have a total of $S_{n-k}(s_1, s_3, s_2) + S_{k-1}(s_1, s_3, s_2 - 1) + 1 + S_{k-1}(s_1, s_3, s_2 - 1) +$
$S_{n-k}(s_1, s_3, s_2) = 2S_{n-k}(s_1, s_3, s_2) + 2S_{k-1}(s_1, s_3, s_2 - 1) + 1$ moves.   Q.E.D.

Theorem 6.4. *If $n \leq 2$, then $S_n(s_1, s_2, s_3) = 2n - 1$.*

Proof. Assume $n = 2$. Move the top disc to the auxiliary peg. Then move the remaining disc to the destination peg. Lastly, move the disc from the auxiliary peg to the destination peg.   Q.E.D.

Theorem 6.5. *If $s_2 \geq n - 2$, then $S_n(s_1, s_2, s_3) = 2n - 1$.*

Proof. Move the top $n - 1$ discs from the origin peg to the auxiliary peg. Then move the largest disc to the destination peg. Lastly, mirror and reverse the first $n-1$ moves so that we have an n regular stack on the destination peg.   Q.E.D.

6.2 p pegs

We now consider p pegs with n discs and s_i for $i \in [p]$ sins on a given peg. Denote the minimum number of moves required to solve this variant, i.e., move a regular stack on peg 1 to a regular stack on peg 2, as $S_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$.

Theorem 6.6. *If $p \geq n + 1$, then $S_n(s_1, s_2, \ldots, s_i, \ldots, s_p) = 2n - 1$.*

Proof. Let $p \geq n + 1$. Move the top disc on the origin peg to the first empty peg. Repeat this $n-1$ times so that the first n pegs have precisely one disc. Since $p \geq n+1$ it follows that the destination peg is empty. Move the largest disc to the last peg. Mirror and reverse the $n - 1$ initial moves so that we have a regular of n discs on peg p.

Therefore, $S_n(s_1, s_2, \ldots, s_i, \ldots, s_p) = n - 1 + 1 + n - 1 = 2n - 1$.   Q.E.D.

Theorem 6.7. *If $\sum_{i=2}^{p-1}(s_i + 1) \geq n - 1$, then $S_n(s_1, s_2, \ldots, s_i, \ldots, s_p) = 2n - 1$.*

Proof. Let $\sum_{i=2}^{p-1}(s_i + 1) \geq n - 1$. Move the top $s_2 + 1$ discs to peg 2. Then move the top $s_3 + 1$ discs to peg 3. Repeat this for all $i \in \{2, 3, \ldots, p-1\}$ until the the largest disc is the only remaining disc on the first peg. Move the largest disc to the destination peg. Then mirror and reverse the initial $n - 1$ moves. Q.E.D.

Corollary 6.8. *If $\sum_{i=2}^{p-1} s_i + p \geq n + 1$, then $S_n(s_1, s_2, \ldots, s_i, \ldots, s_p) = 2n - 1$.*

Proof. Examine the sum.

$$\sum_{i=2}^{p-1}(s_i + 1) = \sum_{i=2}^{p-1} s_i + \sum_{i=2}^{p-1} 1$$
$$= \sum_{i=2}^{p-1} s_i + p - 2.$$

Thus, $\sum_{i=2}^{p-1} s_i + p - 2 \geq n - 1$ if and only if $\sum_{i=2}^{p-1} s_i + p \geq n + 1$. Whence, $\sum_{i=2}^{p-1} s_i + p \geq n + 1$, implies $S_n(s_1, s_2, \ldots, s_i, \ldots, s_p) = 2n - 1$. Q.E.D.

6.3   General Graphs

We will now examine graphs representing the generalized Apprentices' Tower of Hanoi. Let $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ denote the generalized Apprentices' Tower of Hanoi graphs corresponding to n discs, p pegs, and s_i sins per peg. We shall assume $n \geq 1$ otherwise all graphs are the null graph, i.e., the graph with zero vertices.

Theorem 6.9. *The graph $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ contains a K_p subraph.*

Proof. The top disc may be moved to any of the p pegs. Q.E.D.

Theorem 6.10. *If $p \geq 5$, then $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ is non-planar.*

Proof. The graph will contain a K_5 subgraph. Q.E.D.

Theorem 6.11. *If $p \geq 3$, $n \geq 3$, and $|\{s_i|s_i \geq 1\}| \geq 3$, then $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ is non-planar.*

Proof. Let $p \geq 3$, $n \geq 3$, and $|\{s_i|s_i \geq 1\}| \geq 3$. Thus, $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ contains an AH_3 subgraph which is non-planar. Q.E.D.

Theorem 6.12. *For the graph AH_n, $\chi(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) \geq p$.*

Proof. $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ contains a complete subgraph on p vertices. Q.E.D.

Theorem 6.13. *For the graph AH_n, $\Delta(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) \geq p - 1$.*

Proof. The top disc may be moved from any peg to the $p - 1$ other pegs. Q.E.D.

Theorem 6.14. *For the graph AH_n, $\Delta(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) \leq p(p - 1)$.*

Proof. At most all p pegs will be non-empty. The top disc on each of these pegs may be moved to at most $p - 1$ other discs. Giving a total of $p(p - 1)$ possible moves. Q.E.D.

Corollary 6.15. *For the graph AH_n, $p - 1 \leq \chi'(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) \leq p(p - 1) + 1$*

Proof. Follows immediately from Vizing's Theorem and Theorems 6.13 and 6.14. Q.E.D.

The minimum degree of a vertex in a graph G is denoted $\delta(G)$.

Theorem 6.16. *For the graph AH_n, $\delta(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) \geq p - 1$.*

Proof. At any point in the puzzle each peg has a top disc or is empty. Thus, there is a smallest top disc. This smallest top disc may be moved to any of the other $p-1$ pegs. Other discs may be moved as well. Ergo, $\delta(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) \geq p-1$.   Q.E.D.

Theorem 6.17. *For the graph* AH_n, $\delta(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) = p - 1$.

Proof. From Theorem 6.16, $\delta(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) \geq p - 1$. Furthermore, in each puzzle the initial state offers $p - 1$ available moves. Therefore,

$$\delta(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) = p - 1$$

.   Q.E.D.

Theorem 6.18. *When* p *is even,* $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ *is not Eulerian.*

Proof. Let p be even. Then, $p - 1$ is odd. From Theorem 6.17,

$$\delta(AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)) = p - 1$$

. Thus, $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p))$ contains a vertex of odd degree.
Hence, $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ is not Eulerian.   Q.E.D.

Theorem 6.19. *If* $n > s_i$ *for some* $i \in [p]$, *then* $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ *is not Eulerian.*

Proof. Let $n > s_i$. Then, there is an arrangement of discs such that ith peg has the maximum number of sins, and on another peg there is a regular stack with the remaining discs. Clearly nothing may be moved to the ith peg. So, the top disc of the regular stack may be moved to $p - 2$ pegs. While the top disc of the ith peg may

be moved to $p-1$ pegs. Thus, in total there are $2p-3$ possible moves. So, there is a vertex of odd degree in $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$. Ergo, $AH_n(s_1, s_2, \ldots, s_i, \ldots, s_p)$ is not Eulerian. Q.E.D.

7   FURTHER QUESTIONS

This section will contain a list of questions related to the Apprentices' Tower of Hanoi. It will also contain some questions related the Tower of Hanoi in general that I would like to see answered. Where I have some insight, a conjecture will accompany the question.

Question 1. *What is an optimal algorithm for solving the Apprentices' Tower of Hanoi? What are its properties?*

Conjecture 7.1. *An optimal algorithm for solving the Apprentices' Tower of Hanoi requires the largest disc to move only once.*

Conjecture 7.2. *An optimal algorithm for solving the Apprentices' Tower of Hanoi requires $2S'_{n-1} + 1$ moves. (Note that this follows immediately from Conjecture 7.1.)*

Question 2. *What is an optimal algorithm for solving the generalized Apprentices' Tower of Hanoi? What are its properties?*

Question 3. *What is the domination number for AH_n, i.e., $\gamma(AH_n)$?*

Question 4. *Is AH_n Hamiltonian?*

Conjecture 7.3. *The graph AH_n is not Hamiltonian.*

Question 5. *What are the topological properties of AH_n?*

Question 6. *What is the automorphism group of AH_n?*

Question 7. *Given a graph G, is it possible to create a "non-trivial" Tower of Hanoi variant corresponding to G? That is, a version dissimilar to having a peg corresponding to each vertex and allowing moves only between pegs representing adjacent vertices.*

Question 8. *Is there a way to have a computer generate the graphs based on the rule set of a Tower of Hanoi variant?*

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