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Distance-2 Domatic Numbers of Graphs

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Distance-2 Domatic Numbers of Graphs

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by

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ABSTRACT

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The *distance* $d(u, v)$ between two vertices u and v in a graph G equals the length of a shortest path from u to v. A set S of vertices is called a *distance*-2 *dominating set* if every vertex in $V \setminus S$ is within distance-2 of at least one vertex in S. The *distance*-2 *domatic number* is the maximum number of sets in a partition of the vertices of G into *distance*-2 *dominating sets.* We give bounds on the *distance*-2 *domatic number* of a graph and determine the *distance*-2 *domatic number* of selected classes of graphs.

DEDICATION

I would like to dedicate this thesis to my late grandfather, Nelson Kiser. Through the observation of his life I learned to love well and to appreciate the uniqueness of all those around us.

ACKNOWLEDGMENTS

I want to thank Dr. Robert Beeler and Dr. Robert Gardner for taking the time to read and make corrections to the original draft of my theis as well as Dr. Hedetniemi for proposing the topic.

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1 INTRODUCTION

1.1 Introduction to Graph Theory

A *graph* $G = (V, E)$ is made up of a finite nonempty set V and a possibly empty set E. The elements of V are referred to as *vertices* and the elements of E, two element subsets of V, are referred to as *edges*. The number of vertices, $|V|$, in a graph is known as the *order* of G, and the number of edges, $|E|$, is known as the *size* of G. The letters n and m are typically used to denote the order and size respectively of a graph. An edge denoted by uv, represents *adjacent* vertices u and v.

The graphs we consider are finite, undirected, have edges that must join two vertices, and the maximum number of edges allowed to join two vertices is one.

The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \,|\, uv \in V\}$ of vertices adjacent to v. Each vertex in $u \in N(v)$ is called a *neighbor* of v. The *degree* of a vertex v is $deg(v) = |N(v)|$. The minimum and maximum degrees of any vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. A vertex $v \in V$ is called an *isolated vertex* if it has no neighbors, that is, $deg(v) = 0$. A vertex with exactly one neighbor is called a *leaf*, and its neighbor is called a *support vertex*. The *closed neighborhood* of a vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ of vertices is $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of a set* S is the set $N[S] = \bigcup_{v \in S} N[v]$. The S-*private neighborhood* of a vertex $v \in S$ is the set $pn[v, S] = N[v] \setminus N[S \setminus \{v\}]$; vertices in the set $pn[v, S]$ are called *private neighbors* of v (with respect to S).

A *walk* in a graph G is a sequence of vertices starting with a vertex u and ending at a vertex v, where the consecutive vertices in the walk are adjacent in G. A walk in a graph G such that no vertex is repeated is referred to as a *path*. The *distance* $d(u, v)$ between two vertices u and v in a graph G equals the length of a shortest path from u to v. Two vertices u and v in a graph are *connected* if the graph contains a path between u and v. A graph G itself is said to be *connected* if every two vertices of G are connected.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of a graph G and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a *proper subgraph* of G. For a nonempty subset S of $V(G)$, the *subgraph* $G[S]$ *of* G *induced by* S has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. A subgraph H of a graph G is called an *induced subgraph* if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. A connected subgraph H of a graph G is a *component* of G if H is not a proper subgraph of any connected subgraph of G.

The *complement* $\overline{G}$ of a graph G is a graph with the vertex set $V(G)$ where two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

The *eccentricity* of a vertex v of a connected graph G is the distance between v and the vertex farthest from v in G. The *diameter* of G denoted $diam(G)$ is the largest eccentricity of the vertices of a graph G, while the *radius* of G is the smallest eccentricity of the vertices.

A *path* P_n is a graph of order n and size $n - 1$ with vertices denoted $v_1, v_2, ..., v_n$ and edges $v_iv_i + 1$ for $i = 1, 2, ..., n - 1$. A *cycle* C_n is a graph of order and size n with vertices denoted $v_1, v_2, ..., v_n$ and edges v_iv_{i+1} for $i = 1, 2, ..., n - 1$ and v_1v_n. A

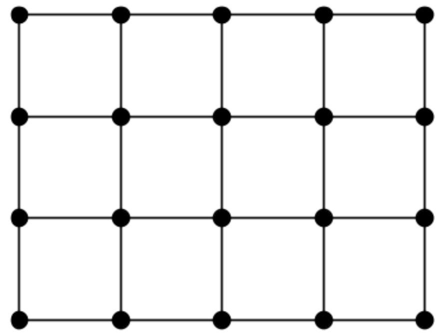

Figure 1: $G_{4,5} = P_4 \times P_5$

grid $G_{r,c}$ is a cartesian product of two paths $P_r \times P_c$ (where the *cartesian product* $K = G_1 \times G_2$ has vertex set $V(K) = V(G_1) \times V(G_2)$ and vertices (u_1, v_1) and (u_2, v_2) in $V(K)$ are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$. An example of a $G_{3,4}$ is given in Figure 1.

A *complete graph* K_n is a graph in which every two distinct vertices are adjacent. A graph G is a *complete bipartite graph* $K_{m,n}$ if $V(G)$ can be partitioned into two sets U and W so that uw is an edge of G if and only if $u \in U$ and $w \in W$.

A *vertex coloring* of a graph G is an assignment of colors to the vertices of G where one color is assigned to each vertex. Vertex colorings in which adjacent vertices are colored differently are known as a *proper vertex coloring.* A *k-coloring* is a coloring in which each color used is one of k colors. A graph G is *k-colorable* if there exists a coloring of G from a set of k colors. The minimum positive integer k for which G is k-colorable is the *chromatic number* of G, denoted $\chi(G)$.

2 DOMINATION

2.1 Introduction to Domination

A set S is a *dominating set* of a graph G if $N[S] = V$, that is, for every $v \in V$, either $v \in S$ or $v \in N(u)$ for some vertex $u \in S$. The minimum cardinality of a dominating set in a graph G is called the *domination number* and is denoted $\gamma(G)$. A dominating set of minimum cardinality is called a *γ-set*. A dominating set that contains no dominating set as a proper subset is called a *minimal dominating set*.

A set S of vertices in a graph G is *independent* if no two vertices in S are adjacent. A set S of vertices in a graph G is said to be an *independent dominating set* of G if S is both a dominating and independent set of G.

A set S is a *total dominating set* of a graph G if $N(S) = V$, that is, every $v \in V$ is adjacent to at least one vertex in S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G, and a total dominating set of minimum cardinality is called a *γ_t-set*.

The *domatic number* $d(G)$ is the maximum number of sets in a partition of $V(G)$ into dominating sets, and the *total domatic number* $d_t(G)$ is the maximum number of sets in a partition of $V(G)$ into total dominating sets.

Figure 4 provides an example of a dominating set of a graph G, a minimum dominating set of the same graph, and a partition of $V(G)$ into the maximum number dominating sets, where the different colored vertices represent different dominating sets in the partition.

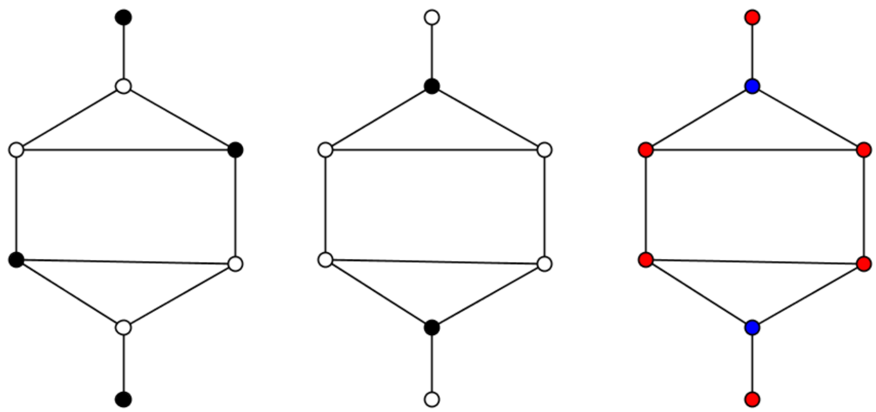

Figure 2: Modified Houses

2.2 Domination and Domatic Number Background

The origin of a dominating set or more precisely a minimum dominating set is credited to one Carl Friedrich de Jaenisch. In 1862 he proposed the question of determining the minimum number of queens needed to dominate every square on a standard chess board [2]. In the late 1800's chess players were also studying the following three questions

1. What is the minimum number of other chess pieces which are necessary to dominate every square of an $n \times n$ board? Other examples of a minimum dominating set.

2. What is the minimum number of mutually nonattacking chess pieces needed to dominate every square of an $n \times n$ board? An example of an independent set of minimum cardinality.

3. What is the maximum number of chess pieces which can be placed on an $n \times n$ chessboard in such a way that no two of them dominate each other? An example of

an independent set of maximum cardinality [6].

These three questions received more attention in 1964 when twin brothers, Isaak and Akiva Yaglom, provided detailed solutions for the rooks, knights, bishops and kings [6].

The first formal graph theoretical definition of a domination number appeared in Claude Berge's 1958 work "The Theory of Graphs" [1], in which he referred to the number as the coefficient of external stability. In 1962, Oystein Ore [8] used the current names of a dominating set and the domination number when he defined the two terms in his book "Theory of Graphs."

In 1977, Ernest Cockayne and Stephen Hedetniemi [3] published the paper, "Towards a Theory of Domination in Graphs," in which they provided a survey of the few results known about dominating sets at that time and introduced the domatic number of a graph. Two of the results included the following two given by Ore in [8] pertaining to minimal dominating sets.

Theorem 2.1 (Ore's Theorem) *In any graph $G = (V, E)$ having no isolated vertices, the complement $V \setminus S$ of any minimal dominating set S is a dominating set.*

Theorem 2.2 [8] *A dominating set S is a minimal dominating set if and only if for each $v \in S$ one of the following two conditions hold:*

1. *v is not adjacent to any vertex in S, or*
2. *there is a vertex $u \notin S$ such that $N(u) \cap S = \{v\}$.*

Hedetniemi and Cockayne defined [3] a *D-partition* of a graph G as a partition of the vertices of G into dominating sets. They noted that one way of obtaining a

D-partition was by assigning colors to the vertices of a graph such that each vertex is adjacent to a vertex of every color different from itself. If a graph G then has a domatic number k, it follows that every vertex must be adjacent to at least $k-1$ vertices, one in each dominating subset of a D-partition of order k. Thus the upper bound for the domatic number was given.

Proposition 2.3 [3] *For any graph G, $d(G) \leq \delta(G) + 1$.*

We next note the proposition presented that states $d(G)$ for special classes of graphs.

Proposition 2.4 [3]

i. $d(K_n + G) = n + d(G)$.

ii. $d(K_n) = n$; $d(\overline{K}_n) = 1$.

iii. (Ore) $d(G) \geq 2$ if and only if G has no isolated vertices.

iv. For any tree with $p \geq 2$ vertices, $d(T) = 2$.

v. For any $n \geq 1$, $d(C_{3n}) = 3$, and $d(C_{3n+1}) = d(C_{3n+2}) = 2$.

vi. For any $2 \leq m \leq n$, $d(K_{m,n}) = m$.

Hedetniemi and Cockayne [3] then established a bound by considering the product of the invariant (domatic number) and its complement in respect to the order of a graph. A method introduced by Nordhaus and Gaddum [9] in their study of the *chromatic number* of a graph.

Proposition 2.5 [3] *For any graph G, having n vertices, $d(G) + d(\overline{G}) \leq n + 1$.*

During their study of the domatic number of a graph, Hedetniemi and Cockayne [3] noticed the fact that the minimum order of a graph's clique (maximal complete subgraph), denoted $c(G)$, seemed to bound the graph's domatic number. They began checking graphs with order less than or equal to five and found the conjecture held. They continued to check the conjecture for graphs of higher order and in doing so discovered the counterexample provided in Figure 3, where $c(G) = 4$ and $d(G) = 3$. Attempts to settle the $c \leq d$ conjecture did however lead to the following results.

Let A be a set and let S be a collection of nonempty subsets of A. The *intersection graph* of S is the graph whose vertices are the elements of S and where two vertices are adjacent if the subsets have a nonempty intersection.

We note that a graph is said to be *indominable* if it has a D-partition in which every subset is independent and a *clique graph* $K(G)$ of a graph G is the intersection graph of the cliques of G.

Proposition 2.6 [3] *If G is indominable, then $c(G) \leq d(G)$.*

Theorem 2.7 [3] *If $K(G) = C_n$ where n is even, then $c(G) \leq d(G)$.*

Theorem 2.8 [3] *If $K(G)$ is a tree, then $c(G) \leq d(G)$.*

2.3 Total Domatic Number Background

In 1980 Cockayne, Hedetniemi, and Dawes [4] defined a total dominating set of a graph. The authors were concerned with sets of vertices in graphs which not only cover vertices outside the set, but sets that cover all vertices.

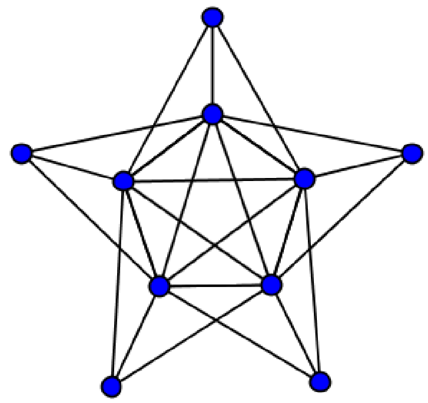

Figure 3: Counterexample to the $c \leq d$ conjecture

After presenting bounds for the total dominating number, the total domatic number of a graph was introduced and the following bounds were established.

Theorem 2.9 [4] *For the complete graph K_n, complete bipartite graph $K_{r,s}$, and cycle C_{4k}, we have $d_t(K_n) = \lfloor \frac{n}{2} \rfloor$, $d_t(K_{m,n}) = min(m, n)$, and $d_t(C_{4n}) = 2$.*

Theorem 2.10 [4] *For any graph G with no isolated vertices,*

$$d_t(G) \leq min\ (\delta(G), \tfrac{n}{\gamma_t(G)}).$$

Theorem 2.11 [4] *If G has n vertices and no isolates, then $\gamma_t(G) + d_t(G) \leq n + 1$, with equality if and only if $G = mK_2$.*

Theorem 2.12 [4] *If G is connected and has $n \geq 3$ vertices, then $\gamma_t(G) + d_t(G) \leq n$, with equality if and only if G is $K_{1,2}, K_3, C_4, K_4-$edge or K_4.*

Theorem 2.13 [4] *If G has n vertices, no isolates, and $\Delta(G) < n-1$, then $d_t(G) + d_t(\overline{G}) \leq n - 1$, with equality if and only if G or $\overline{G} = C_4$.*

This section has acted as a brief introduction to the concept of the domatic and total domatic number of a graph and some methods of determing bounds. Next we introduce *distance-2 domination* and then present our results on the *distance-2 domatic number* of a graph. We determine bounds for the distance-2 domatic number and determine the value of $d_{\leq 2}(G)$ for special classes of graphs.

3 DISTANCE-2 DOMINATION

3.1 Distance-2 Domination

In the following sections we present the results of my research.

A set S is called a *distance-k dominating set* if every vertex in $V \setminus S$ is within distance-k of at least one vertex in S, that is, for every vertex $v \in V \setminus S$, there exists a vertex $u \in S$ such that $d(u, v) \leq k$. It follows from this definition that dominating sets and distance-1 dominating sets are equivalent concepts.

Thus a set S is called a *distance-*2 *dominating set* if every vertex in $V \setminus S$ is within distance-2 of at least one vertex in S. The minimum cardinality of a distance-2 dominating set in G is called the *distance-*2 *domination number* and is denoted $\gamma_{\leq 2}(G)$. A distance-2 dominating set of cardinality $\gamma_{\leq 2}(G)$ is called a $\gamma_{\leq 2}$*-set*. The *distance-2 domatic number* $d_{\leq 2}(G)$ is the maximum number of sets in a partition of $V(G)$ into *distance-2 dominating sets*. Figure 4 provides an example of a distance-2 dominating set of a graph G, a minimum distance-2 dominating set of the same graph, and a partition of $V(G)$ into the maximum number of distance-2 dominating sets, where the different colored vertices represent different distance-2 dominating sets in the partition.

The *distance-2 open neighborhood of a vertex* $v \in V$ is the set, $N_2(v)$, of vertices within a distance of two of v.

A *minimal distance-2 dominating set* in a graph G is a distance-2 dominating set that contains no distance-2 dominating set as a proper subset.

A set S of vertices in a graph G is said to be an *independent distance-2 dominating*

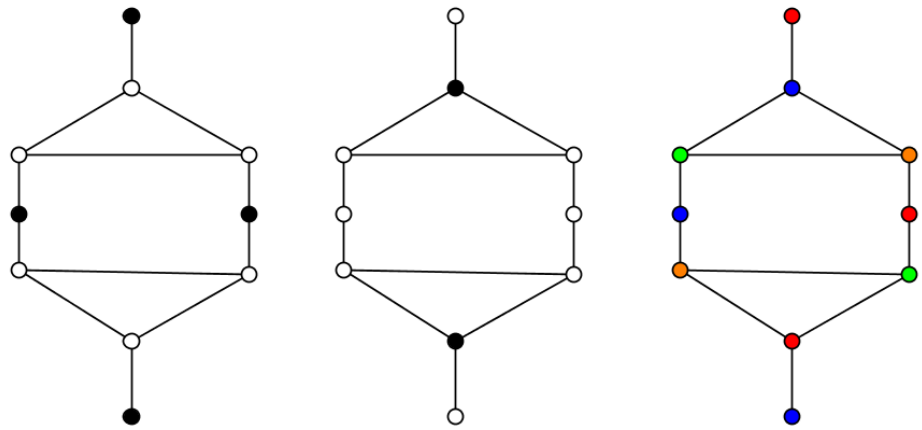

Figure 4: Modified Houses 2

set of G if S is both a distance-2 dominating and independent set of G. A graph G is *distance-2 indominable* if it has a distance-2 partition in which every subset is independent.

Proposition 3.1 *The distance-2 dominating set S is a minimal distance-2 dominating set if and only if for each $v \in S$ at least one of the following two conditions hold:*

1. *v is not within a distance of two of any vertex in S;*
2. *there is a vertex $w \notin S$ such that $N_2(w) \cap S = \{v\}$.*

Proof. Assume S is a minimal distance-2 dominating set. It follows that for any $v \in S$, $S \setminus \{v\}$ is not a distance-2 dominating set. Thus, there exists a vertex w that is not within a distance of two of any vertex in $S \setminus \{v\}$. If $w = v$, then v is not within a distance of two of any vertex in S. If $w \neq v$, we note that since S is a distance-2 dominating set, then w must be within a distance of two of some vertex in S. Therefore, since w is not within a distance of two of any vertex in $S \setminus \{v\}$,

$N_2(w) \cap S = \{v\}$.

Next, assume that at least one of the two conditions are true for each $v \in S$. Then it follows that $S \setminus v$ is not a distance-2 dominating set, and S is a minimal distance-2 dominating set. $\square$

4 DISTANCE-2 DOMATIC NUMBER RESULTS

4.1 Bounds on the Distance-2 Domatic Number

In this section, bounds on the distance-2 domatic number of a graph are established. We note that our results on paths, cycles, and graphs with diameter two were obtained in a more general form by Zelinka [10] for distance-k domatic numbers.

Observation 4.1 *If G is a graph on order n with $diam(G) \leq 2$, then $d_{\leq 2}(G) = n$.*

Observation 4.2 *For any graph G, $d_{\leq 2}(G) \leq \left\lfloor \frac{n}{\gamma_{\leq 2}(G)} \right\rfloor$.*

Since every dominating set is a distance-2 dominating set, we have the following bounds.

Observation 4.3 *For any graph G, $d(G) = d_{\leq 1}(G) \leq d_{\leq 2}(G)$.*

Thus by Ore's Theorem it follows, for isolate free graphs G, $2 \leq d(G) \leq d_{\leq 2}(G)$.

Our next result shows that every total dominating set of G can be partitioned into two distance-2 dominating sets.

Lemma 4.4 *If G is a graph with no isolated vertices and S is a total dominating set of G, then S can be partitioned into two distance-2 dominating sets.*

Proof. Let S be a total dominating set of G. Then there are no isolated vertices in the induced subgraph $G[S]$. Hence, by Ore's Theorem, we can partition S into two dominating sets of the induced subgraph $G[S]$, say S_1 and S_2.

Since S_1 and S_2 are dominating sets of $G[S]$, in order to show that S_1 and S_2 are distance-2 dominating sets of G, it suffices to show that each vertex in $V \setminus S$ is at most distance-2 from at least one vertex in S_1 and at least one vertex in S_2.

Consider any vertex $v \in V \setminus S$. Since S is a total dominating set of G, v has at least one neighbor, say x, in S. Now $x \in S_1$ or $x \in S_2$. Without loss of generality, assume that $x \in S_1$. Since S_i for $i = 1, 2$ are dominating sets of $G[S]$, x has a neighbor in S_2. Hence, v is distance-1 dominated by S_1 and at most distance-2 dominated by S_2. Similarly, if $x \in S_2$, v is distance-1 dominated by S_2 and distance-2 dominated by S_1. In any case, each of S_1 and S_2 is a distance-2 dominating set of G. $\square$

Since any total dominating set of G is a distance-2 dominating set, we have the following corollary.

Corollary 4.5 *If G is a graph with no isolated vertices, then $d_{\leq 2}(G) \geq 2d_t(G)$.*

In addition, one can also show that the vertices of a connected graph with minimum degree two can be partitioned into one dominating set and two distance-2 dominating sets. In order to show this we will use the following theorem of Henning and Southey [7].

Theorem 4.6 [7] *If G is a graph with $\delta(G) \geq 2$ and G has no C_5 component, then the vertices of G can be partitioned into a dominating set and a total dominating set.*

Theorem 4.7 *If G is a connected graph of order $n \geq 3$ and minimum degree $\delta(G) \geq 2$, then $d_{\leq 2}(G) \geq 3$.*

Proof. Since $diam(C_5) = 2$, by Observation 4.1, $d_{\leq 2}(C_5) = 5 \geq 3$.

Hence, assume $G \neq C_5$ is a connected graph with minimum degree $\delta(G) \geq 2$. By Theorem 4.6, we can partition the vertices of G into a dominating set S_0 and a total dominating set S. By Lemma 4.4, S can be partitioned into two distance-2 dominating sets, S_1 and S_2. Then $\{S_0, S_1, S_2\}$ is a partition of $V(G)$, where each S_i is a distance-2 dominating set, and so $d_{\leq 2}(G) \geq 3$. $\square$

Theorem 4.8 *If G is a graph with no isolated vertices and no K_2 component, then $d_{\leq 2}(G) \geq 3$.*

Proof. We first show that any graph with no isolated vertices and no K_2 component can be partitioned into a total dominating set and a distance-2 dominating set. Let S be a $\gamma_t(G)$-set. If $V \setminus S$ is a distance-2 dominating set, then we are finished. Hence, suppose that $V \setminus S$ is not a distance-2 dominating set. Then there exists a vertex $u \in S$ that is not distance-2 dominated by $V \setminus S$. Since G has no isolated vertices, u has at least one neighbor, say w, in S. Moreover, since G has no K_2 component, at least one of u and w has another neighbor, say v, in S. Since u is not distance-2 dominated by $V \setminus S$, no neighbor of u has a neighbor in $V \setminus S$. If $v \in N(u)$, then $S \setminus \{w\}$ is a total dominating set with cardinality less than $\gamma_t(G)$, a contradiction. If $v \in N(w)$, then $S \setminus \{u\}$ is a total dominating set, again a contradiction.

Hence, we can partition the vertices of G into a distance-2 dominating set $V \setminus S$ and a total-dominating set S. It follows from Lemma 4.4 that $d_{\leq 2}(G) \geq 3$. $\square$

Let S be the set of support vertices of G, and let $\delta_S(G)$ denote the minimum degree in G of any vertex in S. Since any leaf adjacent to a support vertex v is distance-2 dominated only the vertices of $N[v]$, we have the following observation.

Observation 4.9 *If G is a graph with set of support vertices S, then $d_{\leq 2}(G) \leq 1 + \delta_S(G)$.*

Note that $deg_2(v) = |N(v)| \cup |N_2(v)|$ and $\delta_2(G) = \min(deg_2(v))$.

Proposition 4.10 *For any graph G, $d_{\leq 2}(G) \leq \delta_2(G) + 1$.*

We will say that a graph G is *distance-2 domatically full* if $d_{\leq 2}(G) = \delta_2(G) + 1$.

4.2 Specific Families

In this section we determine the distance-2 domatic number of paths P_n, cycles C_n, and grids $G_{r,c}$, as well as classifying which graphs in each class are distance-2 domatically full.

Proposition 4.11 *If P_n is a path with order $n \geq 3$, then $d_{\leq 2}(P_n) = 3$.*

Proof. By Observation 4.9, $d_{\leq 2}(P_n) \leq 3$.

Let $P_n = v_1, v_2, ..., v_n$, for $n \geq 3$. Let $S_i = \{v_j \,|\, j \equiv i \pmod 3\}$ for $i \in \{0, 1, 2\}$. It is simple to see that each S_i, $0 \leq i \leq 2$, is a distance-2 dominating set, implying that $d_{\leq 2}(P_n) \geq 3$. Hence, $d_{\leq 2}(P_n) = 3$. $\square$

Proposition 4.12 *If C_n is a cycle, then*

$$d_{\leq 2}(C_n) = \begin{cases} 3 & \text{if } n = 3, 6, 7, 11 \\ 5 & \text{if } 5|n \\ 4 & \text{otherwise.} \end{cases}$$

Proof. First we consider small values of n. If $n = 3$, then C_3 has diameter two and by Observation 4.1, $d_{\leq 2}(C_3) = 3$.

Let $n = 6$. Note that $\gamma_{\leq 2}(C_6) = 2$. Thus, $d_{\leq 2}(C_6) \leq \frac{6}{2} = 3$. Figure 5 gives a partition of the vertices of C_6 into three distance-2 dominating sets (each color class represents a set in the partition.) Thus, $d_{\leq 2}(C_6) \geq 3$ and so, $d_{\leq 2}(C_6) = 3$.

Let $n = 7$. Note $\gamma_{\leq 2}(C_7) = 2$. Thus, $d_{\leq 2}(C_7) \leq \lfloor \frac{6}{2} \rfloor = 3$. Figure 5 gives a partition of the vertices of C_7 into three distance-2 dominating sets. Thus, $d_{\leq 2}(C_7) \geq 3$ and so, $d_{\leq 2}(C_7) = 3$.

Let $n = 11$. Note $\gamma_{\leq 2}(C_{11}) = 3$. Thus, $d_{\leq 2}(C_{11}) \leq \lfloor \frac{11}{3} \rfloor = 3$. Again we illustrate that $d_{\leq 2}(C_{11}) \geq 3$, and hence, obtain equality by showing a partition of the vertices of C_{11} into three distance-2 dominating sets in Figure 5.

Since a vertex in a cycle distance-2 dominates exactly five consecutive vertices on the cycle, it follows that $d_{\leq 2}(C_n) \leq 5$. Let $\{v_1, v_2, ..., v_n\}$ be the vertex set of C_n. Assume first that $5|n$. Then $S_i = \{v_j \mid j \equiv i \pmod 5\ \}$ for $0 \leq i \leq 4$ is a distance-2 dominating set of C_n. Hence, $d_{\leq 2}(C_n) \geq 5$ and so, $d_{\leq 2}(C_n) = 5$.

For each vertex v_i on the cycle to be distance-2 dominated five times, each vertex in $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$ must be in a different set in the $d_{\leq 2}$-partition. In other words, each set of five consecutive vertices on the cycle must have a nonempty intersection with each of the five different distance-2 dominating sets of C_n. Hence, if $5 \nmid n$, it follows that $d_{\leq 2}(C_n) \leq 4$.

Assume that $5 \nmid n$ and $n \notin \{3, 6, 7, 11\}$. To show that $d_{\leq 2} \geq 4$, we consider the four possibilities.

Case 1. $n \equiv 0, 1$ (mod 4).

Let $S_j = \{v_j \,|\, j \equiv i \ (mod\ 4) \text{ for } 0 \leq i \leq 3\}$.

Case 2. $n \equiv 2\ (mod\ 4)$. Note that $n \geq 14$.

Let $S_0 = \{v_4, v_9\} \cup \{v_i \,|\, i \geq 11 \text{ and } (i-10) \equiv 0\ (mod\ 4)$,

$S_1 = \{v_1, v_5, v_6, v_{10}\} \cup \{v_i \,|\, i \geq 11 \text{ and } (i-10) \equiv 1\ (mod\ 4)$,

$S_2 = \{v_2, v_7\} \cup \{v_i \,|\, i \geq 11 \text{ and } (i-10) \equiv 2\ (mod\ 4)$,

and $S_3 = \{v_3, v_8\} \cup \{v_i \,|\, i \geq 11 \text{ and } (i-10) \equiv 3\ (mod\ 4)$.

Case 3. $n \equiv 3\ (mod\ 4)$. Note that $n \geq 19$.

Let $S_0 = \{v_4, v_9, v_{14}\} \cup \{v_i \,|\, i \geq 16 \text{ and } (i-15) \equiv 0\ (mod\ 4)$.

$S_1 = \{v_1, v_5, v_6, v_{10}, v_{11}, v_{15}\} \cup \{v_i \,|\, i \geq 16 \text{ and } (i-15) \equiv 1\ (mod\ 4)$,

$S_2 = \{v_2, v_7, v_{12}\} \cup \{v_i \,|\, i \geq 16 \text{ and } (i-15) \equiv 2\ (mod\ 4)$,

and $S_3 = \{v_3, v_8, v_{13}\} \cup \{v_i \,|\, i \geq 16 \text{ and } (i-15) \equiv 3\ (mod\ 4)$.

In all cases S_i for $0 \leq i \leq 3$ is a distance-2 dominating set of C_n, and so, $\{S_0, S_1, S_2, S_3\}$ is a distance-2 partition of C_n. Hence, $d_{\leq 2}(C_n) \geq 4$, and so, $d_{\leq 2}(C_n) = 4$. □

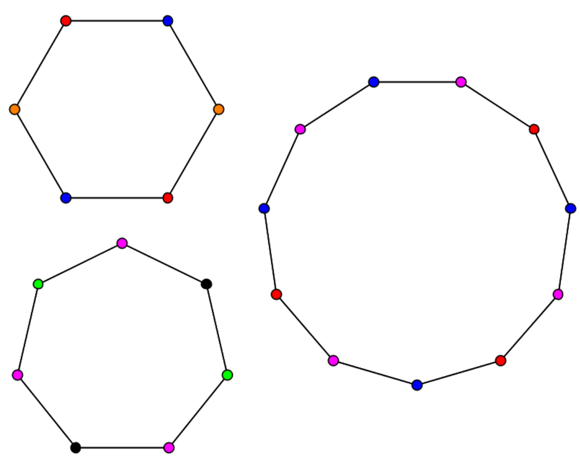

Figure 5: Cycles

We next determine the distance-2 domatic number of grids, $G_{r,c} = P_r \times P_c$, where $2 \le r \le c$. We consider the grid as an"array" where $v_{i,j}$ represents the vertex in the i^{th} row and j^{th} column. To aid in our discussion of grids, we sometimes list a pattern of numbers in array format and say that each vertex is labeled by its corresponding (positional) entry in the pattern.

Proposition 4.13 *If $G_{r,c}$ is a grid where $r = 2 \le c$, then*

$$d_{\le 2}(G_{2,c}) = \begin{cases} 4 & \textit{for } c \in \{2,3,4\} \\ 5 & \textit{otherwise.} \end{cases}$$

Proof. First we consider small values of c. If $c = 2$, then $G_{2,2} = C_4$ has diameter two and by Observation 4.1, $d_{\le 2}(G_{2,2}) = 4$.

For $c = 3$, we note that any distance-2 dominating set containing a corner vertex has cardinality at least two. Since $n = 6$, it follows that $d_{\le 2}(G_{2,3}) \le 4$.

From the following labeling, note that $d_{\le 2}(G_{2,3}) \ge 4$, and hence, $d_{\le 2}(G_{2,3}) = 4$.

$$\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 4 & 1 \end{array}$$

If $c = 4$, then no single vertex distance-2 dominates $G_{2,4}$. Thus, $\gamma_{\le 2}(G_{2,4}) \ge 2$, and so, $G_{2,4} \le \frac{n}{2} = 4$.

From the following labeling, note that $d_{\le 2}(G_{2,4}) \ge 4$, and hence, $d_{\le 2}(G_{2,4}) = 4$.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{array}$$

Assume now that $c \ge 5$. Note that the corner vertex $v_{1,1}$ is distance-2 dominated only by $v_{1,1}$, $v_{1,2}$, $v_{1,3}$, $v_{2,1}$, and $v_{2,2}$, and hence, $d_{\le 2}(G_{2,c}) \le 5$.

We define patterns as follows. Let

$\begin{matrix} 1 & 3 & 5 & 2 & 4 \\ 2 & 4 & 5 & 1 & 3 \end{matrix}$ be called a 0-block,

$\begin{matrix} 1 & 3 & 5 & 5 & 2 & 4 \\ 2 & 4 & 5 & 5 & 1 & 3 \end{matrix}$ be called a 1-block, and

$\begin{matrix} 1 & 3 & 5 & 2 & 5 & 4 & 1 \\ 2 & 4 & 5 & 1 & 5 & 3 & 2 \end{matrix}$ be called a 2-block.

Let $c = 5k + j$ for integers i and k where $0 \leq j \leq 2$. Now label $G_{2,c}$ by repeating the 0-block $k - 1$ times and adding a j-block.

If $j = 3$, then label $G_{2,c}$ by repeating the 0-block k times and ending with the following pattern $\begin{matrix} 5 & 1 & 3 \\ 5 & 2 & 4 \end{matrix}$ for the last three columns.

If $j = 4$, then label $G_{2,c}$ by repeating the 0-block pattern k times and ending with the following pattern $\begin{matrix} 5 & 5 & 1 & 3 \\ 5 & 5 & 2 & 4 \end{matrix}$ for the last four columns.

For $1 \leq i \leq 5$, let S_i be the set of vertices of $G_{2,c}$ labeled i by the patterns in each case. The collection of sets S_i where $1 \leq i \leq 5$ is a distance-2 domatic partition of $G_{2,c}$ in all five cases. Hence, $d_{\leq 2}(G_{2,c}) \geq 5$, and thus, $d_{\leq 2}(G_{2,c}) = 5$. $\square$

Proposition 4.14 *If $G_{r,c}$ is a grid where $r = 3 \leq c$, then*

$$d_{\leq 2}(G_{3,c}) = \begin{cases} 5 & \textit{for } c \in \{3, 5\} \\ 6 & \textit{otherwise.} \end{cases}$$

Proof. First note that every corner vertex is distance-2 dominated by exactly six vertices of $G_{3,c}$. Hence, $d_{\leq 2}(G_{3,c}) \leq 6$.

Let $c = 3$. Note $\gamma_{\leq 2}(G_{3,3}) = 1$. However, the only vertex that distance-2 dominates $G_{3,3}$ is the vertex $v_{2,2}$. Hence, all other distance-2 dominating sets of $G_{3,3}$ must contain at least two vertices. Thus, $d_{\leq 2}(G_{3,3}) \leq 1 + \frac{8}{2} = 5$.

From the following labeling, note that $d_{\leq 2}(G_{3,3}) \geq 5$, and hence, $d_{\leq 2}(G_{3,3}) = 5$.

1	2	3
4	5	4
3	2	1

Let $c = 4$. Then $d_{\leq 2}(G_{3,4}) \leq 6$, the following labeling insist $d_{\leq 2}(G_{3,4}) \geq 6$, and hence, $d_{\leq 2}(G_{3,4}) = 6$.

1	4	6	3
2	5	1	2
3	6	4	5

Assume $c = 5$. Note $\gamma_{\leq 2}(G_{3,5}) = 2$. However, the only way to form a distance-2 dominating set of $G_{3,5}$ consisting of only two vertices is to consider the vertices $v_{2,2}$ and $v_{2,4}$. Hence, all other distance-2 dominating sets of $G_{3,5}$ must contain at least three vertices. Therefore $d_{\leq 2}(G_{3,5}) \leq 1 + \lfloor \frac{13}{3} \rfloor = 5$.

From the following labeling note that $d_{\leq 2}(G_{3,5}) \geq 5$, and hence, $d_{\leq 2}(G_{3,5}) = 5$.

1	4	2	3	4
2	5	3	5	1
3	1	4	3	2

Let $G_{3,c}$ be a grid where $c \geq 6$.

Let

1	4	6	3	
2	5	5	2	be a 4-block pattern,
3	6	4	1	

1	4	6	3	5	
2	5	5	2	6	be a 5-block pattern, and
3	6	4	1	4	

1	4	5	3	4	5	1	
2	6	6	2	6	6	2	be a 7-block pattern.
3	5	4	1	5	4	3	

We consider four cases.

Case 1. $c = 4k$ for some integer k.

Label $G_{3,c}$ by repeating the 4-block pattern k times.

Case 2. $c = 4k + 1$ for some integer k.

Since $c \geq 6$, we have that $c \geq 9$. Label $G_{3,c}$ by repeating a 4-block $k - 1$ times followed by a 5-block.

Case 3.$c = 4k + 2$ for some integer k.

Label $G_{3,c}$ by repeating a 4-block $k-1$ times and ending in the following pattern for the last two columns $\begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{matrix}$.

Case 4. $c = 4k + 3$ for some integer k. Label $G_{3,c}$ by repeating a 4-block $k - 1$ times followed by one 7-block.

Let S_i, for $1 \leq i \leq 6$, be the set of vertices labeled i in each of the cases. The collection of sets S_i where $1 \leq i \leq 6$ is a distance-2 domatic partition of $G_{3,c}$ in all four cases. Hence, $d_{\leq 2}(G_{3,6}) \geq 6$, and thus, $d_{\leq 2}(G_{3,6}) = 6$. $\square$

Proposition 4.15 *If $G_{r,c}$ is a grid where $r = 4 \leq c$, then*

$$d_{\leq 2}(G_{4,c}) = \begin{cases} 5 & \text{if } c = 4 \\ 6 & \text{otherwise.} \end{cases}$$

Proof. Assume that $c = 4$. Then $n = 16$. Since two vertices can distance-2 dominate at most fifteen vertices, it follows that $\gamma_{\leq 2}(G_{4,c}) \geq 3$. Hence, $d_{\leq 2}(G_{4,4}) \leq \lfloor \frac{16}{3} \rfloor = 5$.

From the following illustration note $d_{\leq 2}(G_{4,4}) \geq 5$, and hence, $d_{\leq 2}(G_{4,4}) = 5$.

$$\begin{matrix} 1 & 4 & 2 & 3 \\ 2 & 5 & 5 & 4 \\ 3 & 5 & 5 & 1 \\ 4 & 1 & 3 & 2 \end{matrix}$$

Let $G_{4,c}$ be a grid where $c \geq 5$. Then the vertex $v_{1,1}$ is distance-2 dominated only by $v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}$ and $v_{3,1}$, and hence, $d_{\leq 2}(G_{2,c}) \leq 6$.

Let

$$\begin{array}{ccc} 1 & 3 & 6 \\ 2 & 5 & 4 \\ 4 & 1 & 2 \\ 6 & 3 & 5 \end{array}$$ be a 3-block pattern and

$$\begin{array}{cccc} 5 & 1 & 3 & 6 \\ 3 & 2 & 5 & 4 \\ 5 & 4 & 1 & 2 \\ 1 & 6 & 3 & 5 \end{array}$$ be a 4-block pattern.

We consider three cases.

Case 1. $c = 3k$ for some integer k.

Label $G_{4,c}$ by repeating the 3-block pattern k times.

Case 2. $c = 3k + 1$ for some integer k.

Since $c \geq 5$, we have $c \geq 7$. Label $G_{4,c}$ by repeating the 3-block pattern $k - 1$ times followed by a 4-block pattern.

Case 3. $c = 3k + 2$ for some integer k.

Label $G_{4,c}$ by repeating the 3-block pattern k times and ending with the pattern $\begin{array}{cc} 4 & 5 \\ 1 & 2 \\ 4 & 3 \\ 1 & 6 \end{array}$ for the last two columns.

Let S_i, for $1 \leq i \leq 6$, be the set of vertices labeled i by the pattern in each case. The collection of sets S_i where $1 \leq i \leq 6$ is a distance-2 partition of $G_{4,c}$ in all three cases. Hence, $d_{\leq 2}(G_{4,c}) \geq 6$, and thus, $d_{\leq 2}(G_{4,c}) = 6$. $\square$

Proposition 4.16 *If $G_{r,c}$ is a grid for $r = 5 \leq c$, then $d_{\leq 2}(G_{5,c}) = 6$.*

Proof. Let $G_{5,c}$ be a grid where $c \geq 5$. Then the vertex $v_{1,1}$ is distance-2 dominated only by $v_{1,1}$,$v_{1,2}$, $v_{1,3}$, $v_{2,1}$, $v_{2,2}$, and $v_{3,1}$, and hence, $d_{\leq 2}(G_{2,c}) \leq 6$.

Let

$$\begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & 5 & 4 & 5 \\ 6 & 1 & 1 & 1 \\ 3 & 2 & 3 & 2 \\ 1 & 5 & 4 & 6 \end{array} \quad \text{be a 0-block pattern,}$$

$$\begin{array}{ccccc} 1 & 5 & 4 & 3 & 6 \\ 6 & 2 & 2 & 5 & 1 \\ 3 & 4 & 1 & 6 & 2 \\ 5 & 1 & 3 & 1 & 4 \\ 4 & 2 & 6 & 5 & 3 \end{array} \quad \text{be a 1-block pattern,}$$

$$\begin{array}{cccccc} 2 & 6 & 4 & 3 & 5 & 1 \\ 5 & 1 & 1 & 1 & 2 & 6 \\ 3 & 4 & 6 & 5 & 1 & 4 \\ 6 & 2 & 3 & 1 & 1 & 3 \\ 5 & 1 & 4 & 6 & 2 & 5 \end{array} \quad \text{be a 2-block pattern, and}$$

$$\begin{array}{ccccccc} 2 & 6 & 4 & 4 & 3 & 5 & 1 \\ 5 & 1 & 3 & 5 & 1 & 2 & 6 \\ 3 & 4 & 1 & 2 & 6 & 1 & 4 \\ 6 & 2 & 1 & 5 & 4 & 1 & 3 \\ 5 & 1 & 4 & 3 & 6 & 2 & 5 \end{array} \quad \text{be a 3-block pattern.}$$

Let $c = 4k + j$ for integers j and k, where $0 \leq j \leq 3$. Label $G_{5,c}$ by repeating a 4-block $k - 1$ times followed by a j-block.

The collection of sets S_i where $1 \leq i \leq 6$ is a distance-2 partition of $G_{5,c}$ for all $c \geq 5$. Hence, $d_{\leq 2}(G_{5,c}) \geq 6$, and thus, $d_{\leq 2}(G_{5,c}) = 6$. $\square$

Proposition 4.17 *If $G_{r,c}$ is a grid where $r \geq 3$ and $c \geq 6$, then $d_{\leq 2}(G_{r,c}) = 6$.*

Proof. Assume that $r \geq 3$ and $c \geq 6$. Note that each corner of a $G_{r,c}$ is distance-2 dominated by exactly six vertices. Hence, $d_{\leq 2}(G_{r,c}) \leq 6$.

Note by the Problem of Frobenius [5], any integer $x \geq 3$ can be expressed as the sum of multiples of the three integers 3, 4, and 5. Since $r \geq 3$, we can express r as $3x + 4y + 5z$, where x, y, and z are non-negative integers.

Thus we can consider $G_{r,c}$ as x copies of $G_{3,c}$, y copies of $G_{4,c}$, and z copies of $G_{5,c}$. Using the labeling given in Proposition 4.14, 4.15, and 4.16 for these subgrids, we can obtain a distance-2 partition of $G_{r,c}$ of cardinality six. Hence, $d_{\leq 2}(G_{r,c}) \geq 6$, and so, $d_{\leq 2}(G_{r,c}) = 6$. $\square$

As an example, using the method described in the proof, we can partition the grid in Figure 6 into six different distance-2 dominating sets, where each is being represented with a different color. We begin by noting that $r = 11 = 4 \cdot 2 + 3 \cdot 1$. Thus, we can consider $G_{11,13}$ to be two copies of $G_{4,13}$ and one copy of $G_{3,13}$. Using the labellings given in Proposition 4.14 and 4.15, we obtain the distance-2 domatic partition.

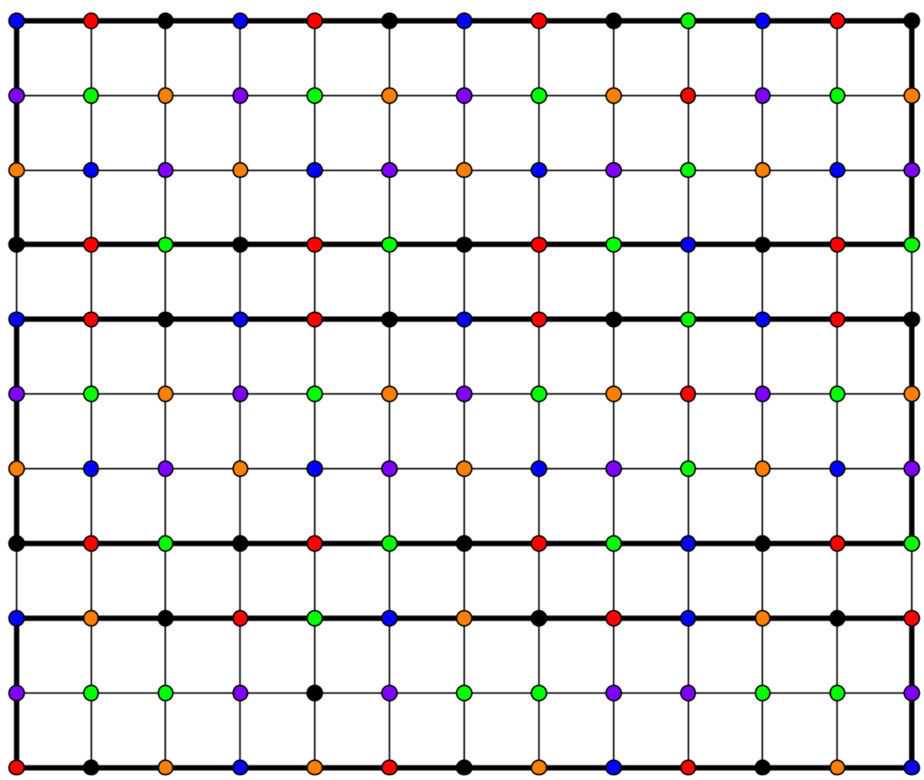

Figure 6: $d_{\leq 2}(G_{11,13}) \geq 6$

Proposition 4.18 *Let $G_{\infty,\infty}$ be the infinite grid. Then $d_{\leq 2}(G_{r,c}) = 13$.*

Proof. Since a vertex in an infinite grid distance-2 dominates exactly thirteen vertices, note that $d_{\leq 2}(G_{r,c}) \leq 13$.

Consider the following labeling:

$$
\begin{array}{ccccccccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 6 & 11 & 3 & 8 & 13 & 5 & 10 & 2 & 7 & 12 & 4 & 9 & \cdots \\
\cdots & 2 & 7 & 12 & 4 & 9 & 1 & 6 & 11 & 3 & 8 & 13 & 5 & 10 & \cdots \\
\cdots & 3 & 8 & 13 & 5 & 10 & 2 & 7 & 12 & 4 & 9 & 1 & 6 & 11 & \cdots \\
\cdots & 4 & 9 & 1 & 6 & 11 & 3 & 8 & 13 & 5 & 10 & 2 & 7 & 12 & \cdots \\
\cdots & 5 & 10 & 2 & 7 & 12 & 4 & 9 & 1 & 6 & 11 & 3 & 8 & 13 & \cdots \\
\cdots & 6 & 11 & 3 & 8 & 13 & 5 & 10 & 2 & 7 & 12 & 4 & 9 & 1 & \cdots \\
\cdots & 7 & 12 & 4 & 9 & 1 & 6 & 11 & 3 & 8 & 13 & 5 & 10 & 2 & \cdots \\
\cdots & 8 & 13 & 5 & 10 & 2 & 7 & 12 & 4 & 9 & 1 & 6 & 11 & 3 & \cdots \\
\cdots & 9 & 1 & 6 & 11 & 3 & 8 & 13 & 5 & 10 & 2 & 7 & 12 & 4 & \cdots \\
\cdots & 10 & 2 & 7 & 12 & 4 & 9 & 1 & 6 & 11 & 3 & 8 & 13 & 5 & \cdots \\
\cdots & 11 & 3 & 8 & 13 & 5 & 10 & 2 & 7 & 12 & 4 & 9 & 1 & 6 & \cdots \\
\cdots & 12 & 4 & 9 & 1 & 6 & 11 & 3 & 8 & 13 & 5 & 10 & 2 & 7 & \cdots \\
\cdots & 13 & 5 & 10 & 2 & 7 & 12 & 4 & 9 & 1 & 6 & 11 & 3 & 8 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

The labeling is achieved by repeating the integers 1 through 13 on each column such that the row entries differ by 5 $(mod\ 13)$, that is $a_{i,j} = (a_{i,j-1} + 5)\ (mod\ 13)$.

For $1 \leq i \leq 13$, let S_i be the infinite set of vertices of $G_{\infty,\infty}$ labeled i by the pattern. The collection of sets S_i for $1 \leq i \leq 13$ is a distance-2 domatic partition of the grid. Hence, $d_{\leq 2}(G_{r,c}) \geq 13$, and thus, $d_{\leq 2}(G_{r,c}) = 13$. $\square$

We conclude this section by noting that the following graphs are distance-2 domatically full

- Paths – P_n for $n \geq 3$.

- Cycles – C_n for $n = 3, 4, 5$ and $5|n$.

- Grids – $G_{r,c}$
  - $G_{2,2}$ and $G_{2,c}$ where $c \geq 5$.
  - $G_{3,4}$ and $G_{3,c}$ where $c \geq 6$.
  - $G_{4,c}$ where $c \geq 5$.
  - $G_{5,c}$ where $c \geq 5$.
  - $G_{r,c}$ where $r \geq 6$ and $n \geq 6$.
  - $G_{\infty,\infty}$.

5 CONCLUDING REMARKS

Cockayne and Hedetniemi [3] stated that the theory of domatic numbers resembles the well known theory of graph colorings. With this in mind, we conclude this thesis by exploring the relationship between the distance-2 domatic number of a graph and the chromatic number of a graph. In particular the observation that the distance-2 domatic number seems to be bounded below by the chromatic number in all graphs.

For distance-2 indominable graphs, we first note the following proposition.

Proposition 5.1 *If G is distance-2 indominable, then $d_{\leq 2}(G) \geq \chi(G)$.*

Proof

Let G be distance-2 indominable. Then we can find an independent distance-2 domatic partition of G by labeling each subset of vertices in the partition with some integer $1, 2, ..., k$. Since this is a proper coloring of G, it follows $d_{\leq 2}(G) \geq \chi(G)$. □

The bound of Proposition 5.1 is not true in general, however. For example see the graph G in Figure 7, where $d_{\leq 2}(G) = 3$ and $\chi(G) = 4$.

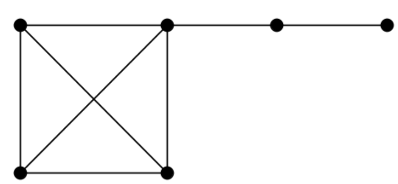

Figure 7: Graph G

Since we have yet to find a counterexample for graphs with no induced K_4, we make the following conjecture. Let $w(G)$ denote the number of vertices in a maximum clique of G.

Conjecture 5.2 *If $w(G) \geq 4$, then $d_{\leq 2}(G) \geq \chi(G)$.*

In this thesis we defined the distance-2 domatic number of graph G as being the maximum number of sets in a partition of the vertices of G into distance-2 dominating sets. We calculated the distance-2 domatic number for paths, cycles, and grids, and determined upper and lower bounds for the distance-2 domatic number. Some topics for further investigation include the following:

i. Determine the $d_{\leq 2}$- domatic number of a cylinder $P_m \times C_n$.

ii. Determine the $d_{\leq 2}$- domatic number of a torus $C_m \times C_n$.

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