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The Number of Zeros of a Polynomial in a Disk as a Consequence of Restrictions on the Coefficients

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by

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ABSTRACT

The Number of Zeros of a Polynomial in a Disk as a Consequence of Restrictions on the Coefficients

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In this thesis, we put restrictions on the coefficients of polynomials and give bounds concerning the number of zeros in a specific region. Our results generalize a number of previously known theorems, as well as implying many new corollaries with hypotheses concerning monotonicity of the modulus, real, as well as real and imaginary parts of the coefficients separately. We worked with Eneström-Kakeya type hypotheses, yet we were only concerned with the number of zeros of the polynomial. We considered putting the same type of restrictions on the coefficients of three different types of polynomials: polynomials with a monotonicity "flip" at some index k, polynomials split into a monotonicity condition on the even and odd coefficients independently, and $\mathcal{P}_{n,\mu}$ polynomials that have a gap in between the leading coefficient and the proceeding coefficient, namely the μ^{th} coefficient.

DEDICATION

I would like to dedicate this thesis to my son, Braeden Kye McCarter-Shields. You are my heart and soul and you do not realize this yet, but you kept me trekking on all of these years. I love you little man.

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1 INTRODUCTION

Research on the zeros of polynomials has a plethora of applications and is studied vastly by both the theoretical and applied mathematical communities. In general it can be quite difficult to find the zeros of a polynomial; therefore it is desirable to apply restrictions to the coefficients in order to restrict the locations of zeros. Historically, the study of finding zeros began with Gauss, who proved the classical complex analysis result, The Fundamental Theorem of Algebra, and Cauchy, who is thought of as the father of complex analysis. This was around the same time the geometric representation of the complex numbers was introduced into mathematics in the 1800s [16].

In the early 1900s Gustaf Hjalmar Eneström, a Swedish mathematician and Soichi Kakeya, a Japanese mathematician simultaneously worked on a result that would give the bound of the location of the zeros of a polynomial with nonnegative monotonically increasing coefficients. Eneström was best known for creating the Eneström Index, which is used to identify Eulers writings [13]. Kakeya is most noted for solving the transportation problem, a very famous and important problem sought out during World War II. Although, both Eneström and Kakeya proved the same result, they did so independently, and both are given credit for the proof [13],[28]. The Eneström-Kakeya Theorem concerns the location of zeros of a polynomial with monotonically increasing real, nonnegative coefficients. This result was quite extraordinary in that it restricts the location of zeros, of this type of polynomial, to the closed unit disk. This bound on the zero's locations makes the zeros easy to find, which leads to an easier way of finding the critical points of the polynomial by the Gauss-Lucas Theorem

[1], which lead us into applications. There are numerous applications including, but not limited to, "Cryptograph, Control Theory, Signal Processing, Communication Theory, Coding Theory, Combinatorics, and Bio-Mathematics" [7]. There is always interest in getting better results and faster ways of locating and counting the zeros of polynomials, whether it is in general or with specific hypotheses.

Here we introduce the Eneström-Kakeya Theorem and begin to give the background of our work, leading up to the research we have accomplished.

Theorem 1.1 (Eneström-Kakeya) *For polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$, if the coefficients satisfy $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| \leq 1$ (see section 8.3 of [26]).*

In connection with the location of zeros of an analytic function $f(z) = \sum_{j=0}^{\infty} a_j z^j$, where $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$, Aziz and Mohammad imposed the condition $0 < \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots$ (and a similar condition on the β_j's) [3]. We denote this restriction of the coefficients as a "flip at k", where the monotonicity of the coefficients changes from increasing to decreasing. These types of conditions have also been put on the coefficients of polynomials in order to get a restriction on the location of zeros [15]. In Chapter 2, we impose these types of restrictions on the coefficients of polynomials in order to count the number of zeros in a certain region.

We introduce the idea of counting zeros of a polynomial with Jensen's Formula, which Titchmarsh used to get a bound on the number of zeros in a specific region.

Theorem 1.2 (Jensen's Formula) *(From Conway's Function's of One Complex Variable I, page 280.)*

Let f be an analytic function on a region containing $\overline{B}(0;R)$ and suppose that $a_1, a_2, \ldots, a_n$ are the zeros of f in $B(0,R)$, repeated according to multiplicity. If $f(0) \neq 0$ then

$$\log|f(0)| = -\sum_{k=1}^{n} \log\frac{R}{|a_k|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta.$$

Theorem 1.3 (Titchmarsh's Number of Zeros Theorem) *[29]*

Let f be analytic in $|z| < R$. Let $|f(z)| \leq M$ in the disk $|z| \leq R$ and suppose $f(0) \neq 0$. Then for $0 < \delta < 1$ the number of zeros of $f(z)$ in the disk $|z| \leq \delta R$ is less than

$$\frac{1}{\log 1/\delta} \log\frac{M}{|f(0)|}.$$

Here we show the proof of Titchmarsh's Number of Zeros Theorem. This proof uses the idea that we can get a bound on the number of zeros by applying Jensen's Formula for analytic functions on a closed disk. This result is the foundation of our research in that we always relate back to Titchmarsh's result to get the number of zeros. We seek out a specific value of M such that we have a new, or better, bound on the number of zeros of specific polynomials.

Proof of Theorem 1.3 Let f have n zeros in the disk $|z| \leq \delta R$, say $a_1, a_2, \ldots, a_n$. Then for $1 \leq k \leq n$ we have $|a_k| \leq \delta R$, or $\dfrac{R}{|a_k|} \geq \dfrac{1}{\delta}$. So

$$\sum_{k=1}^{n} \log\frac{R}{|a_k|} = \log\frac{R}{|a_1|} + \log\frac{R}{|a_2|} + \cdots + \log\frac{R}{|a_n|} \geq n\log\frac{1}{\delta}. \tag{1}$$

By Jensen's Formula, we have

$$\begin{aligned}\sum_{k=1}^{n} \log \frac{R}{|a_k|} &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \log M\, d\theta - \log|f(0)| \\ &= \log M - \log|f(0)| \\ &= \log \frac{M}{|f(0)|}. \end{aligned} \tag{2}$$

Combining (1) and (2) gives

$$n \log \frac{1}{\delta} \leq \sum_{k=1}^{n} \log \frac{R}{|a_k|} \leq \log \frac{M}{|f(0)|},$$

or

$$n \leq \frac{1}{\log 1/\delta} \log M|f(0)|.$$

Since n is the number of zeros of f in $|z| \leq \delta R$, the result follows. $\square$

We will now discuss what others have researched concerning the number of zeros of a polynomial using Titchmarsh type results. This is the background needed to understand the type of results we have derived. There is much research still active in this field of mathematics on counting the number of zeros in a specific region. Even though some of this work was accomplished in the early/mid 1900's, much of this is recently discovered and many papers have been recently published concerning the number of zeros result.

By putting a restriction on the coefficients of a polynomial similar to that of the Eneström-Kakeya Theorem, Mohammad used a special case of Theorem 1.3 to prove the following [24].

Theorem 1.4 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be such that* $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_{n-1} \le a_n$. *Then the number of zeros in* $|z| \le \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log\left(\frac{a_n}{a_0}\right).$$

In her dissertation work, Dewan weakens the hypotheses of Theorem 1.4 and proves the following two results for polynomials with complex coefficients [8, 23].

Theorem 1.5 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be such that* $|arg(a_j) - \beta| \le \alpha \le \pi/2$ *for all* $1 \le j \le n$ *and some real* α *and* β, *and* $0 < |a_0| \le |a_1| \le |a_2| \le \cdots \le |a_{n-1}| \le |a_n|$. *Then the number of zeros of* p *in* $|z| \le 1/2$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Theorem 1.6 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for all* j *and* $0 < \alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{n-1} \le \alpha_n$, *then the number of zeros of* p *in* $|z| \le 1/2$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|\alpha_0|}.$$

Pukhta generalized Theorems 1.5 and 1.6 by finding the number of zeros in $|z| \le \delta$ for $0 < \delta < 1$ [25]. The next theorem, due to Pukhta, deals with a monotonicity condition on the moduli of the coefficients.

Theorem 1.7 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be such that* $|arg(a_j) - \beta| \le \alpha \le \pi/2$ *for all* $1 \le j \le n$ *and some real* α *and* β*, and* $0 < |a_0| \le |a_1| \le |a_2| \le \cdots \le |a_{n-1}| \le |a_n|$*. Then the number of zeros of* p *in* $|z| \le \delta$*,* $0 < \delta < 1$ *, does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Pukhta also gave a result which involved a monotonicity condition on the real part of the coefficients [25]. Though the proof presented by Pukhta is correct, there was a slight typographical error in the statement of the result as it appeared in print. The correct statement of the theorem is as follows.

Theorem 1.8 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be such that* $|arg(a_j) - \beta| \le \alpha \le \pi/2$ *for all* $1 \le j \le n$ *and some real* α *and* β*, and* $0 < \alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{n-1} \le \alpha_n$*. Then the number of zeros of* p *in* $|z| \le \delta$*,* $0 < \delta < 1$*, does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{2\left(\alpha_n + \sum_{j=0}^{n} |\beta_j|\right)}{|a_0|}.$$

Aziz and Zargar [4] introduced the idea of imposing an inequality on the even index and odd index for the coefficients of a polynomial separately. Cao and Gardner [5] generalized this idea to impose the conditions

$$0 \ne \alpha_0 \le \alpha_2 t^2 \le \alpha_4 t^4 \le \cdots \le \alpha_{2k} t^{2k} \ge \alpha_{2k+2} t^{2k+2} \ge \cdots \ge \alpha_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \le \alpha_3 t^2 \le \alpha_5 t^4 \le \cdots \le \alpha_{2\ell-1} t^{2\ell-2} \ge \alpha_{2\ell+1} t^{2\ell} \ge \cdots \ge \alpha_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor}$$

on the real parts of the coefficients and gave a result restricting the location of the zeros of a polynomial. The hypotheses with restriction on the real and imaginary parts of the coefficients, split into even and odd indices imposed by Cao and Gardner can be seen here:

$$\alpha_0 \le \alpha_2 t^2 \le \alpha_4 t^4 \le \cdots \le \alpha_{2k} t^{2k} \ge \alpha_{2k+2} t^{2k+2} \ge \cdots \ge \alpha_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \le \alpha_3 t^2 \le \alpha_5 t^4 \le \cdots \le \alpha_{2\ell-1} t^{2\ell-2} \ge \alpha_{2\ell+1} t^{2\ell} \ge \cdots \ge \alpha_{2\lfloor (n+1)/2 \rfloor -1} t^{2\lfloor n/2 \rfloor}$$

$$\beta_0 \le \beta_2 t^2 \le \beta_4 t^4 \le \cdots \le \beta_{2s} t^{2s} \ge \beta_{2s+2} t^{2s+2} \ge \cdots \ge \beta_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor}$$

$$\beta_1 \le \beta_3 t^2 \le \beta_5 t^4 \le \cdots \le \beta_{2q-1} t^{2q-2} \ge \beta_{2q+1} t^{2q} \ge \cdots \ge \beta_{2\lfloor (n+1)/2 \rfloor -1} t^{2\lfloor n/2 \rfloor}.$$

In Chapter 3, we use the same hypotheses to count the number of zeros of the polynomial by considering the moduli, real, as well as real and imaginary restrictions of the even and odd indices.

In Chapter 4 we consider a type of polynomial with a gap between the leading coefficient and the following coefficient, we denote the class of all of such polynomials as $\mathcal{P}_{n,\mu}$, where the polynomial is of the form $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$. While studying Bernstein type inequalities, Chan and Malik [6] introduced this particular class of polynomials. Notice that when $\mu = 1$, we simply have the class of all polynomials of degree n. This class has been extensively studied in connection with Bernstein type inequalities (see, for example, [2, 11, 10, 31, 27]). $\mathcal{P}_{n,\mu}$ polynomials are the last class of polynomials we considered with restricting the coefficients.

2 A MONOTONICITY CONDITION ON ALL OF THE COEFFICIENTS

In this chapter, we consider a monotonicity condition on all of the coefficients. First, we imposed the condition on the moduli of the coefficients, similar to what Dewan did for locations of zeros. In section 2.2 we split the coefficients into the real and imaginary parts and put a monotonicity restriction on only the real part, much like Pukta's generalization of Theorem 1.6. Finally, we consider the monotonicity restriction on the real and imaginary parts of the coefficients in 2.3. This is done with the number of zeros in mind, where $|z| \leq \delta$ where $0 < \delta < 1$, is the specific region in the complex plane we are considering. Our results of this chapter appear in the *Journal of Classical Analysis* [14].

2.1 Restrictions on the moduli of the Coefficients

In this section, we first consider the number of zeros in an annulus with restrictions on the moduli of the coefficients where the monotonicity flips at some position k. These are related to Puhkta type results with the monotonicity flip at k.

Theorem 2.1 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $t > 0$ *and some* $0 \leq k \leq n$,

$$0 < |a_0| \leq t|a_1| \leq t^2|a_2| \leq \cdots \leq t^{k-1}|a_{k-1}| \leq$$

$$t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^{n-1}|a_{n-1}| \geq t^n|a_n|$$

and $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for* $1 \leq j \leq n$ *and for some real* α *and* β. *Then for*

$0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $M = |a_0|t(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha + |a_n|t^{n+1}(1 + \sin\alpha - \cos\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|t^{j+1}$.

As is traditional in classical complex analysis, we first discuss the results and then offer a proof at the end of the section. Notice that when $t = 1$ in Theorem 2.1, we get the following.

Corollary 2.2 *Let* $P(z) = \displaystyle\sum_{j=0}^{n} a_j z^j$ *where for some* $t > 0$ *and some* $0 \leq k \leq n$,

$$0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{k-1}| \leq |a_k| \geq |a_{k+1}| \geq \cdots \geq |a_{n-1}| \geq |a_n|$$

and $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for* $1 \leq j \leq n$ *and for some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $M = |a_0|(1-\cos\alpha-\sin\alpha)+2|a_k|\cos\alpha+|a_n|(1+\sin\alpha-\cos\alpha)+2\sin\alpha \sum_{j=0}^{n-1} |a_j|$.

With $k = n$ in Corollary 2.2, the hypothesis becomes $0 < |a_0| \leq |a_1| \leq \cdots \leq |a_n|$, and the value of M becomes $|a_0|(1 - \cos\alpha - \sin\alpha) + |a_n|(1 + \sin\alpha + \cos\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|$. Since $0 \leq \alpha \leq \pi/2$, we have $1 - \cos\alpha - \sin\alpha \leq 0$. So the value of M given by Theorem 1 is less than or equal to $|a_n|(1 + \sin\alpha + \cos\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|$, and Theorem 2.1 implies Theorem 1.7.

In order to prove Theorem 2.1 we need the following, which is due to Govil and Rahman and appears in [17].

Lemma 2.3 *Let $z, z' \in \mathbb{C}$ with $|z| \geq |z'|$. Suppose $|\arg z^* - \beta| \leq \alpha \leq \pi/2$ for $z^* \in \{z, z'\}$ and for some real α and β. Then*

$$|z - z'| \leq (|z| - |z'|)\cos\alpha + (|z| + |z'|)\sin\alpha.$$

Proof of Theorem 2.1. Consider

$$\begin{aligned} F(z) &= (t-z)P(z) = (t-z)\sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n}(a_j t z^j - a_j z^{j+1}) \\ &= a_0 t + \sum_{j=1}^{n} a_j t z^j - \sum_{j=1}^{n} a_{j-1} z^j - a_n z^{n+1} \\ &= a_0 t + \sum_{j=1}^{n}(a_j t - a_{j-1}) z^j - a_n z^{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$\begin{aligned} |F(z)| &\leq |a_0| t + \sum_{j=1}^{n} |a_j t - a_{j-1}| t^j + |a_n| t^{n+1} \\ &= |a_0| t + \sum_{j=1}^{k} |a_j t - a_{j-1}| t^j + \sum_{j=k+1}^{n} |a_{j-1} - a_j t| t^j + |a_n| t^{n+1} \end{aligned}$$

$$\begin{aligned} \leq\ & |a_0| t + \sum_{j=1}^{k} \{(|a_j| t - |a_{j-1}|)\cos\alpha + (|a_{j-1}| + |a_j| t)\sin\alpha\}\, t^j \\ & + \sum_{j=k+1}^{n} \{(|a_{j-1}| - |a_j| t)\cos\alpha + (|a_j| t + |a_{j-1}|)\sin\alpha\}\, t^j + |a_n| t^{n+1} \end{aligned}$$

by Lemma 2.3 with $z = a_j t$ and $z' = a_{j-1}$ when $1 \le j \le k$,

and with $z = a_{j-1}$ and $z' = a_j t$ when $k+1 \le j \le n$

$$
\begin{aligned}
&= |a_0|t + \sum_{j=1}^{k} |a_j| t^{j+1} \cos\alpha - \sum_{j=1}^{k} |a_{j-1}| t^j \cos\alpha + \sum_{j=1}^{k} |a_{j-1}| t^j \sin\alpha \\
&\quad + \sum_{j=1}^{k} |a_j| t^{j+1} \sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^j \cos\alpha - \sum_{j=k+1}^{n} |a_j| t^{j+1} \cos\alpha \\
&\quad + \sum_{j=k+1}^{n} |a_j| t^{j+1} \sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^j \sin\alpha + |a_n| t^{n+1} \\
&= |a_0|t + |a_k| t^{k+1} \cos\alpha + \sum_{j=1}^{k-1} |a_j| t^{j+1} \cos\alpha - |a_0| t \cos\alpha - \sum_{j=1}^{k-1} |a_j| t^{j+1} \cos\alpha \\
&\quad + |a_0| t \sin\alpha + \sum_{j=1}^{k-1} |a_j| t^{j+1} \sin\alpha + |a_k| t^{k+1} \sin\alpha + \sum_{j=1}^{k-1} |a_j| t^{j+1} \sin\alpha \\
&\quad + |a_k| t^{k+1} \cos\alpha + \sum_{j=k+1}^{n-1} |a_j| t^{j+1} \cos\alpha - |a_n| t^{n+1} \cos\alpha - \sum_{j=k+1}^{n-1} |a_j| t^{j+1} \cos\alpha \\
&\quad + |a_n| t^{n+1} \sin\alpha + \sum_{j=k+1}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_k| t^{k+1} \sin\alpha + \sum_{j=k+1}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1} \\
&= |a_0|t + |a_k| t^{k+1} \cos\alpha - |a_0| t \cos\alpha + |a_0| t \sin\alpha + |a_k| t^{k+1} \sin\alpha + 2\sum_{j=1}^{k-1} |a_j| t^{j+1} \sin\alpha \\
&\quad + |a_k| t^{k+1} \cos\alpha - |a_n| t^{n+1} \cos\alpha + |a_n| t^{n+1} \sin\alpha + |a_k| t^{k+1} \sin\alpha \\
&\quad + 2\sum_{j=k+1}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1} \\
&= |a_0| t (1 - \cos\alpha - \sin\alpha) + |a_k| (2t^{k+1} \cos\alpha + 2t^{k+1} \sin\alpha) + |a_n| t^{n+1} (1 + \sin\alpha - \cos\alpha) \\
&\quad + 2\sum_{j=0}^{k-1} |a_j| t^{j+1} \sin\alpha + 2\sum_{j=k+1}^{n-1} |a_j| t^{j+1} \sin\alpha \\
&= |a_0| t (1 - \cos\alpha - \sin\alpha) + 2|a_k| t^{k+1} \cos\alpha + |a_n| t^{n+1} (1 + \sin\alpha - \cos\alpha) \\
&\quad + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| t^{j+1} \\
&= M.
\end{aligned}
$$

Now $F(z)$ is analytic in $|z| \leq t$, and $|F(z)| \leq M$ for $|z| = t$. So by Theorem 1.3 and the Maximum Modulus Theorem, the number of zeros of F (and hence of P) in $|z| \leq \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

The theorem follows. $\square$

2.2 Restrictions on the Real Part of the Coefficients

In this section, we impose the condition of having only a restriction on the real part of the coefficients, along with the t condition and a flip of the monotonicity at some position k. Again we have the number of zeros result in mind and we seek out a different M value. We show Theorem 2.4:

Theorem 2.4 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that for some* $t > 0$ *and some* $0 \leq k \leq n$ *we have*

$$0 \neq \alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^n\alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = (|\alpha_0| - \alpha_0)t + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=0}^{n} |\beta_j| t^{j+1}$.

Notice that with $t = 1$ in Theorem 2.4, we get the following.

Corollary 2.5 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leq j \leq n$. *Suppose we have*

$$0 \neq \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_{n-1} \geq \alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| - \alpha_0) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

With $k = n$ and $0 < \alpha_0$ in Theorem 2.4, the hypothesis becomes $0 < \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ and the value of M becomes $2(\alpha_n + \sum_{j=0}^{n} \beta_j)$; therefore Theorem 1.8 follows from Theorem 2.4. With $\beta_j = 0$ for $1 \leq j \leq n$ and $\delta = 1/2$, Corollary 2.5 reduces to a result of Dewan and Bidkham [9].

Example: Consider the polynomial $p(z) = (z+0.1)^2(z+10)^2 = 1+20.2z+104.01z^2+20.2z^3+z^4$. With $\alpha_0 = \alpha_4 = 1$, $\alpha_1 = \alpha_3 = 20.2$, $\alpha_2 = 104.01$, and each $\beta_j = 0$, we see that Corollary 2.5 applies to p with $k = 2$, however none of Theorems 1.5 through 1.8 apply to p. With $\delta = 0.1$, Corollary 2.5 implies that the number of zeros in $|z| \leq \delta = 0.1$ is less than $\frac{1}{\log(1/0.1)} \log \frac{2(104.01)}{1} \approx 2.318$, which implies that p has at most two zeros in $|z| \leq 0.1$, and of course p has exactly two zeros in this region. We also observe that Theorem 1.3 applies to p, but requires that we find a bound for $|p(z)|$ for $|z| = R = 1$; this fact makes it harder to determine the bound given by the conclusion of Theorem 1.3, as opposed to the other results mentioned above which give bounds in terms of the coefficients of p. Since all the coefficients of p in this are positive, it is quite easy to find this maximum, and Theorem 1.3 also implies that p

has at most two zeros in $|z| \le \delta = 0.1$.

Proof of Theorem 2.4 As in the proof of Theorem 2.1,

$$F(z) = (t-z)P(z) = a_0 t + \sum_{j=1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1},$$

and so

$$\begin{aligned} F(z) &= (\alpha_0 + i\beta_0)t + \sum_{j=1}^{n}((\alpha_j + i\beta_j)t - (\alpha_{j-1} + i\beta_{j-1}))z^j - (\alpha_n + i\beta_n)z^{n+1} \\ &= (\alpha_0 + i\beta_0)t + \sum_{j=1}^{n}(\alpha_j t - \alpha_{j-1})z^j + i\sum_{j=1}^{n}(\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1} \end{aligned}$$

For $|z| = t$ we have

$$\begin{aligned} |F(z)| &\le (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{n} |\alpha_j t - \alpha_{j-1}|t^j + \sum_{j=1}^{n}(|\beta_j|t + |\beta_{j-1}|)t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\ &= (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{k}(\alpha_j t - \alpha_{j-1})t^j + \sum_{j=k+1}^{n}(\alpha_{j-1} - \alpha_j t)t^j + \sum_{j=1}^{n-1}|\beta_j|t^{j+1} \\ &\quad +|\beta_n|t^{n+1} + |\beta_0|t + \sum_{j=1}^{n-1}|\beta_j|t^{j+1} + (|\alpha_n| + |\beta_n|)t^{n+1} \end{aligned}$$

$$\begin{aligned} &= |\alpha_0|t + \sum_{j=1}^{k-1}\alpha_j t^{j+1} + \alpha_k t^{k+1} - \alpha_0 t - \sum_{j=1}^{k-1}\alpha_j t^{j+1} + \alpha_k t^{k+1} \\ &\quad + \sum_{j=k+1}^{n-1}\alpha_j t^{j+1} - \alpha_n t^{n+1} - \sum_{j=k+1}^{n-1}\alpha_j t^{j+1} + 2\sum_{j=0}^{n}|\beta_j|t^{j+1} + |\alpha_n|t^{n+1} \\ &= (|\alpha_0| - \alpha_0)t + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=0}^{n}|\beta_j|t^{j+1} \\ &= M. \end{aligned}$$

The result now follows as in the proof of Theorem 2.1. □

2.3 Restrictions on the Real and Imaginary Parts of the Coefficients

In this section, we now consider the same type of result with a restriction on the real and imaginary parts of the coefficients. Both the real and imaginary parts have a monotonicity flip at k and ℓ, respectively. This gives a more specific bound on the number of zeros for polynomials which satisfy the hypotheses stated here.

Theorem 2.6 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that for some* $t > 0$, *for some* $0 \leq k \leq n$ *we have*

$$0 \neq \alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^n\alpha_n,$$

and for some $0 \leq \ell \leq n$ *we have*

$$\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq \cdots \leq t^{\ell-1}\beta_{\ell-1} \leq t^\ell\beta_\ell \geq t^{\ell+1}\beta_{\ell+1} \geq \cdots \geq t^{n-1}\beta_{n-1} \geq t^n\beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|},$$

where $M = (|\alpha_0| - \alpha_0)t + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + (|\beta_0| - \beta_0)t + 2\beta_\ell t^{\ell+1} + (|\beta_n| - \beta_n)t^{n+1}$.

Theorem 2.6 gives several corollaries with hypotheses concerning monotonicity of the real and imaginary parts. For example, with $t = 1$ and $k = \ell = n$ we have the hypotheses that $0 \neq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$, resulting in the following.

Corollary 2.7 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that we have*

$$0 \neq \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| - \alpha_0) + (|\alpha_n| + \alpha_n) + (|\beta_0| - \beta_0) + (|\beta_n| + \beta_n)}{|a_0|}.$$

With $t = 1$ and $k = \ell = 0$, Theorem 2.6 gives the following.

Corollary 2.8 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that we have*

$$0 \neq \alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} \geq \alpha_n \text{ and } \beta_0 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1} \geq \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| + \alpha_0) + (|\alpha_n| - \alpha_n) + (|\beta_0| + \beta_0) + (|\beta_n| - \beta_n)}{|a_0|}.$$

With $t = 1$, we can let $k = n$ and $\ell = 0$ (or $k = 0$ and $\ell = n$), Theorem 2.6 gives the next two results.

Corollary 2.9 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that we have*

$$0 \neq \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n \text{ and } \beta_0 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1} \geq \beta_n.$$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| - \alpha_0) + (|\alpha_n| + \alpha_n) + (|\beta_0| + \beta_0) + (|\beta_n| - \beta_n)}{|a_0|}.$$

Corollary 2.10 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $0 \leq j \leq n$. Suppose that we have*

$$0 \neq \alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} \geq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \beta_n.$$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| + \alpha_0) + (|\alpha_n| - \alpha_n) + (|\beta_0| - \beta_0) + (|\beta_n| + \beta_n)}{|a_0|}.$$

Proof of Theorem 2.6 As in the proof of Theorem 2.4,

$$F(z) = (\alpha_0 + i\beta_0)t + \sum_{j=1}^{n}(\alpha_j t - \alpha_{j-1})z^j + i\sum_{j=1}^{n}(\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}.$$

For $|z| = t$ we have

$$\begin{aligned}
|F(z)| &\leq (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{n} |\alpha_j t - \alpha_{j-1}|t^j + \sum_{j=1}^{n} |\beta_j t - \beta_{j-1}|t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{k} |\alpha_j t - \alpha_{j-1}|t^j + \sum_{j=k+1}^{n} |\alpha_j t - \alpha_{j-1}|t^j \\
&\quad + \sum_{j=1}^{\ell} |\beta_j t - \beta_{j-1}|t^j + \sum_{j=\ell+1}^{n} |\beta_j t - \beta_{j-1}|t^j + (|\alpha_n| + |\beta_n|)t^{n+1}
\end{aligned}$$

$$
\begin{aligned}
&= (|\alpha_0| + |\beta_0|)t + \sum_{j=1}^{k}(\alpha_j t - \alpha_{j-1})t^j + \sum_{j=k+1}^{n}(\alpha_{j-1} - \alpha_j t)t^j \\
&\quad + \sum_{j=1}^{\ell}(\beta_j t - \beta_{j-1})t^j + \sum_{j=\ell+1}^{n}(\beta_{j-1} - \beta_j t)t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= |\alpha_0|t + \sum_{j=1}^{k-1}\alpha_j t^{j+1} + \alpha_k t^{k+1} - \alpha_0 t - \sum_{j=1}^{k-1}\alpha_j t^{j+1} + \alpha_k t^{k+1} + \sum_{j=k+1}^{n-1}\alpha_j t^{j+1} \\
&\quad -\alpha_n t^{n+1} - \sum_{j=k+1}^{n-1}\alpha_j t^{j+1} + |\alpha_n|t^{n+1} + |\beta_0|t + \sum_{j=1}^{\ell-1}\beta_j t^{j+1} + \beta_\ell t^{\ell+1} \\
&\quad -\beta_0 t - \sum_{j=1}^{\ell-1}\beta_j t^{j-1} + \beta_\ell t^{\ell+1} + \sum_{j=\ell+1}^{n-1}\beta_j t^{j+1} - \beta_n t^{n+1} - \sum_{j=\ell+1}^{n-1}\beta_j t^{j+1} + |\beta_n|t^{n+1} \\
&= (|\alpha_0| - \alpha_0)t + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + (|\beta_0| - \beta_0)t \\
&\quad +2\beta_\ell t^{\ell+1} + (|\beta_n| - \beta_n)t^{n+1} \\
&= M.
\end{aligned}
$$

The result now follows as in the proof of Theorem 2.1. $\square$

3 A MONOTONICITY CONDITION ON THE COEFFICIENTS OF EVEN POWERS AND COEFFICIENTS OF ODD POWERS OF THE VARIABLE

In this chapter, we explore the same type restrictions of the coefficients of the polynomial, yet we impose the monotonicity condition on the even and odd indexed coefficients separately, as did Cao and Gardner for the locations of zeros [5]. The number of zeros result has not been previously worked on concerning the even and odd indexed restriction on the coefficients of the polynomials and this research is novel in that aspect. Due to the type of restrictions we impose we obtain a large number of corollaries and results that trump previous research for specific polynomials.

3.1 Restrictions on the Moduli of the Coefficients

In this section, we first consider the moduli of the coefficients with a flip at $2k$ for the even indices and a flip at $2\ell - 1$ for the odd indices and seek out a new bound on the M value.

Theorem 3.1 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$. Suppose that for some $t > 0$, some nonnegative integers k and s:*

$$0 \neq |a_0| \leq |a_2 t^2| \leq |a_4 t^4| \leq \cdots \leq |a_{2k} t^{2k}| \geq |a_{2k+2} t^{2k+2}| \geq \cdots \geq |a_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor}|,$$

$$|a_1| \leq |a_3 t^2| \leq |a_5 t^4| \leq \cdots \leq |a_{2\ell-1} t^{2\ell-2}| \geq |a_{2\ell+1} t^{2\ell}| \geq \cdots \geq |a_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor}|$$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2})(1 - \cos\alpha - \sin\alpha)$$
$$+2\cos\alpha(|a_{2k}|t^{2k+2} + |a_{2\ell-1}|t^{2\ell+1}) + 2\sin\alpha\sum_{j=0}^{n}|a_j|t^{j+2}.$$

Notice when $t = 1$ in Theorem 3.1 we get the following,

Corollary 3.2 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$. Suppose that for some $t > 0$, some nonnegative integers k and s:*

$$0 \neq |a_0| \leq |a_2| \leq |a_4| \leq \cdots \leq |a_{2k}| \geq |a_{2k+2}| \geq \cdots \geq |a_{2\lfloor n/2\rfloor}|,$$

$$|a_1| \leq |a_3| \leq |a_5| \leq \cdots \leq |a_{2\ell-1}| \geq |a_{2\ell+1}| \geq \cdots \geq |a_{2\lfloor (n+1)/2\rfloor-1}|$$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta}\log\frac{M}{|a_0|}$$

where

$$M = (|a_0| + |a_1| + |a_{n-1}| + |a_n|)(1 - \cos\alpha - \sin\alpha)$$
$$+2\cos\alpha(|a_{2k}| + |a_{2\ell-1}|) + 2\sin\alpha\sum_{j=0}^{n}|a_j|.$$

From Corollary 3.2 with $2k = 2\lfloor n/2\rfloor$ and $2\ell - 1 = 2\lfloor (n+1)/2\rfloor - 1$ we get the following,

Corollary 3.3 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$*. Suppose that for some* $t > 0$*, some nonnegative integers* k *and* s*:*

$$0 \neq |a_0| \leq |a_2| \leq |a_4| \leq \cdots \leq |a_{2\lfloor n/2 \rfloor}|,$$

$$|a_1| \leq |a_3| \leq |a_5| \leq \cdots \leq |a_{2\lfloor (n+1)/2 \rfloor - 1}|$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|a_0| + |a_1|)(1 - \cos\alpha - \sin\alpha) + (|a_{n-1}| + |a_n|)(1 + \cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n} |a_j|.$$

From Corollary 3.2, when $k = 0$ and $\ell = 1$ we get,

Corollary 3.4 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$*. Suppose that for some* $t > 0$*, some nonnegative integers* k *and* s*:*

$$0 \neq |a_0| \geq |a_2| \geq \cdots \geq |a_{2\lfloor n/2 \rfloor}|,$$

$$|a_1| \geq |a_3| \geq \cdots \geq |a_{2\lfloor (n+1)/2 \rfloor - 1}|$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|a_0| + |a_1|)(1 + \cos\alpha - \sin\alpha) + (|a_{n-1}| + |a_n|)(1 - \cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n} |a_j|.$$

From Corollary 3.2 when $2k = 2\lfloor n/2 \rfloor$ and $\ell = 1$ we get,

Corollary 3.5 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$*. Suppose that for some* $t > 0$*, some nonnegative integers* k *and* s*:*

$$0 \neq |a_0| \leq |a_2| \leq |a_4| \leq \cdots \leq |a_{2\lfloor n/2 \rfloor}|,$$

$$|a_1| \geq |a_3| \geq \cdots \geq |a_{2\lfloor (n+1)/2 \rfloor - 1}|$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|a_0|+|a_1|+|a_{n-1}|+|a_n|)(1-\cos\alpha-\sin\alpha)+2\cos\alpha(|a_{2\lfloor n/2 \rfloor}|+|a_1|)+2\sin\alpha \sum_{j=0}^{n} |a_j|.$$

From Corollary 3.2 when $k = 0$ and $2\ell - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ we get,

Corollary 3.6 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$*. Suppose that for some* $t > 0$*, some nonnegative integers* k *and* s*:*

$$0 \neq |a_0| \geq |a_{2k+2}| \geq \cdots \geq |a_{2\lfloor n/2 \rfloor}|,$$

$$|a_1| \leq |a_3| \leq |a_5| \leq \cdots \leq |a_{2\lfloor (n+1)/2 \rfloor - 1}|$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$\begin{aligned} M \quad &= (|a_0|t^2 + |a_1| + |a_{n-1}| + |a_n|)(1 - \cos\alpha - \sin\alpha) \\ &+2\cos\alpha(|a_0| + |a_{2\lfloor (n+1)/2\rfloor -1}|) + 2\sin\alpha \sum_{j=0}^{n} |a_j| t^{j+2}. \end{aligned}$$

We now present the proof of Theorem 3.1.

Proof of Theorem 3.1 Define

$$G(z) = (t^2 - z^2)P(z) = t^2 a_0 + a_1 t^2 z + \sum_{j=2}^{n}(a_j t^2 - a_{j-2})z^j - a_{n-1}z^{n+1} - a_n z^{n+2}.$$

For $|z| = t$ we have

$$G(z) \quad \leq \quad |a_0|t^2 + |a_1|t^3 + \sum_{j=2}^{n} |a_j t^2 - a_{j-2}|t^j + |a_{n-1}|t^{n+1} + |a_n|t^{n+2}$$

$$
\begin{aligned}
&= |a_0|t^2 + |a_1|t^3 + \sum_{\substack{j=2\\ j \text{ even}}}^{2k} |\alpha_j t^2 - \alpha_{j-2}|t^j + \sum_{\substack{j=2k+2\\ j \text{ even}}}^{2\lfloor n/2\rfloor} |a_j t^2 - a_{j-2}|t^j \\
&\quad + \sum_{\substack{j=3\\ j \text{ odd}}}^{2s-1} |a_j t^2 - a_{j-2}|t^j + \sum_{\substack{j=2s+1\\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor -1} |a_j t^2 - a_{j-2}|t^j + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
&\leq |a_0|t^2 + |a_1|t^3 + \sum_{\substack{j=2\\ j \text{ even}}}^{2k} \{(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\}t^j \\
&\quad + \sum_{\substack{j=2k+2\\ j \text{ even}}}^{2\lfloor n/2\rfloor} \{(|a_{j-2}| - |a_j|t^2)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\}t^j \\
&\quad + \sum_{\substack{j=3\\ j \text{ odd}}}^{2s-1} \{(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\}t^j \\
&\quad + \sum_{\substack{j=2s+1\\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor -1} \{(|a_{j-2}| - |a_j|t^2)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\}t^j \\
&\quad + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \text{ by Lemma 2.1}
\end{aligned}
$$

$$
\begin{aligned}
&= |a_0|t^2 + |a_1|t^3 - |a_0|t^2\cos\alpha + |a_{2k}|t^{2k+2}\cos\alpha + \cos\alpha \sum_{\substack{j=2\\ j \text{ even}}}^{2k-2} |a_j|t^{j+2} \\
&\quad - \cos\alpha \sum_{\substack{j=2\\ j \text{ even}}}^{2k-2} |a_j|t^{j+2} + |a_0|t^2\sin\alpha + |a_{2k}|t^{2k+2}\sin\alpha + \sin\alpha \sum_{\substack{j=2\\ j \text{ even}}}^{2k-2} |a_j|t^{j+2} \\
&\quad + \sin\alpha \sum_{\substack{j=2\\ j \text{ even}}}^{2k-2} |a_j|t^{j+2} + |a_{2k}|t^{2k+2}\cos\alpha - |a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}\cos\alpha \\
&\quad + \cos\alpha \sum_{\substack{j=2k+2\\ j \text{ even}}}^{2\lfloor n/2\rfloor -2} |a_j|t^{j+2} - \cos\alpha \sum_{\substack{j=2k+2\\ j \text{ even}}}^{2\lfloor n/2\rfloor -2} |a_j|t^{j+2} \\
&\quad + |a_{2k}|t^{2k+2}\sin\alpha + |a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}\sin\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+\sin\alpha \sum_{\substack{j=2k+2\\ j \text{ even}}}^{2\lfloor n/2\rfloor-2} |a_j|t^{j+2} + \sin\alpha \sum_{\substack{j=2k+2\\ j \text{ even}}}^{2\lfloor n/2\rfloor-2} |a_j|t^{j+2} \\
&-|a_1|t^3\cos\alpha + |a_{2s-1}|t^{2s+1}\cos\alpha + \cos\alpha \sum_{\substack{j=3\\ j \text{ odd}}}^{2s-3} |a_j|t^{j+2} - \cos\alpha \sum_{\substack{j=3\\ j \text{ odd}}}^{2s-3} |a_j|t^{j+2} \\
&+|a_1|t^3\sin\alpha + |a_{2s-1}|t^{2s+1}\sin\alpha + \sin\alpha \sum_{\substack{j=3\\ j \text{ odd}}}^{2s-3} |a_j|t^{j+2} + \sin\alpha \sum_{\substack{j=3\\ j \text{ odd}}}^{2s-3} |a_j|t^{j+2} \\
&+|a_{2s-1}|t^{2s+1}\cos\alpha - |a_{2\lfloor (n_1)/2\rfloor-1}|t^{2\lfloor (n_1)/2\rfloor+1}\cos\alpha \\
&+\cos\alpha \sum_{\substack{j=2s+1\\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-3} |a_j|t^{j+2} - \cos\alpha \sum_{\substack{j=2s+1\\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-3} |a_j|t^{j+2} \\
&+|a_{2s-1}|t^{2s+1}\sin\alpha + |a_{2\lfloor (n+1)/2\rfloor-1}|t^{2\lfloor (n+1)/2\rfloor+1}\sin\alpha \\
&+\sin\alpha \sum_{\substack{j=2s+1\\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-3} |a_j|t^{j+2} + \sin\alpha \sum_{\substack{j=2s+1\\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-3} |a_j|t^{j+2} \\
&+|a_{n-1}|t^{n+1} + |a_n|t^{n+2}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & |a_0|t^2 + |a_1|t^3 - |a_0|t^2\cos\alpha + |a_{2k}|t^{2k+2}\cos\alpha + |a_0|t^2\sin\alpha + |a_{2k}|t^{2k+2}\sin\alpha \\
&+2\sin\alpha \sum_{\substack{j=2\\ j \text{ even}}}^{2k-2} |a_j|t^{j+2} + |a_{2k}|t^{2k+2}\cos\alpha - |a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}\cos\alpha \\
&+|a_{2k}|t^{2k+2}\sin\alpha + |a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}\sin\alpha + 2\sin\alpha \sum_{\substack{j=2k+2\\ j \text{ even}}}^{2\lfloor n/2\rfloor-2} |a_j|t^{j+2} \\
&-|a_1|t^3\cos\alpha + |a_{2s-1}|t^{2s+1}\cos\alpha + |a_1|t^3\sin\alpha + |a_{2s-1}|t^{2s+1}\sin\alpha \\
&+2\sin\alpha \sum_{\substack{j=3\\ j \text{ odd}}}^{2s-3} |a_j|t^{j+2}
\end{aligned}
$$

$$
\begin{aligned}
& +|a_{2s-1}|t^{2s+1}\cos\alpha - |a_{2\lfloor (n+1)/2\rfloor -1}|t^{2\lfloor (n+1)/2\rfloor +1}\cos\alpha + |a_{2s-1}|t^{2s+1}\sin\alpha \\
& +|a_{2\lfloor (n+1)/2\rfloor -1}|t^{2\lfloor (n+1)/2\rfloor +1}\sin\alpha + 2\sin\alpha \sum_{\substack{j=2s+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor -3} |a_j|t^{j+2} \\
& +|a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
= \; & |a_0|t^2(1-\cos\alpha+\sin\alpha) + |a_1|t^3(1-\cos\alpha+\sin\alpha) + 2|a_{2k}|t^{2k+2}(\cos\alpha+\sin\alpha) \\
& +2|a_{2s-1}|t^{2s+1}(\cos\alpha+\sin\alpha) + 2\sin\alpha \sum_{\substack{j=2 \\ j \text{ even}}}^{2k-2} |a_j|t^{j+2} + 2\sin\alpha \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor -2} |a_j|t^{j+2} \\
& +2\sin\alpha \sum_{\substack{j=3 \\ j \text{ odd}}}^{2s-2} |a_j|t^{j+2} + 2\sin\alpha \sum_{\substack{j=2s+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor -3} |a_j|t^{j+2} \\
& +|a_{n-1}|t^{n+1}(1-\cos\alpha+\sin\alpha) + |a_n|t^{n+2}(1-\cos\alpha+\sin\alpha) \\
= \; & (|a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2})(1-\cos\alpha+\sin\alpha) \\
& +2\cos\alpha(|a_{2k}|t^{2k+2} + |a_{2s-1}|t^{2s-1}) + 2\sin\alpha \sum_{j=2}^{n-2} |a_j|t^{j+2} \\
= \; & (|a_0|t^2 + |a_1|t^3 + |a_{n-1}|t^{n+1} + |a_n|t^{n+2})(1-\cos\alpha-\sin\alpha) \\
& +2\cos\alpha(|a_{2k}|t^{2k+2} + |a_{2s-1}|t^{2s-1}) + 2\sin\alpha \sum_{j=0}^{n} |a_j|t^{j+2} \\
= \; & M.
\end{aligned}
$$

Now $G(z)$ is analytic in $|z| \leq t$, and $|G(z)| \leq M$ for $|z| = t$. So by Titchmarsh's theorem and the Maximum Modulus Theorem, the number of zeros of G (and hence of P) in $|z| \leq \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

The theorem follows. $\square$

3.2 Restrictions on the Real Part of the Coefficients

In this section, we now put the restriction on the real part only as before in Chapter 2, yet with the even and odd restriction on the coefficients.

Theorem 3.7 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $0 \le j \le n$. *Suppose that for some* $t > 0$, *some nonnegative integers* k *and* s, *and positive integers* ℓ *and* q:

$$0 \ne \alpha_0 \le \alpha_2 t^2 \le \alpha_4 t^4 \le \cdots \le \alpha_{2k} t^{2k} \ge \alpha_{2k+2} t^{2k+2} \ge \cdots \ge \alpha_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \le \alpha_3 t^2 \le \alpha_5 t^4 \le \cdots \le \alpha_{2\ell-1} t^{2\ell-2} \ge \alpha_{2\ell+1} t^{2\ell} \ge \cdots \ge \alpha_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor}$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \le \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| - \alpha_0)t^2 + (|\alpha_1| - \alpha_1)t^3 + 2\alpha_{2k} t^{2k+2} + \alpha_{2\ell-1} t^{2\ell+1} + (|\alpha_{n-1}| - \alpha_{n-1})t^{n+1}$$
$$+ (|\alpha_n| - \alpha_n)t^{n+2} + 2\sum_{j=0}^{n} |\beta_j| t^{j+2}.$$

With $t = 1$ in Theorem 3.7 , we get the following,

Corollary 3.8 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $0 \le j \le n$. *Suppose that for some* $t > 0$, *some nonnegative integers* k *and* s, *and positive integers* ℓ *and* q:

$$0 \ne \alpha_0 \le \alpha_2 \le \alpha_4 \le \cdots \le \alpha_{2k} \ge \alpha_{2k+2} \ge \cdots \ge \alpha_{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \le \alpha_3 \le \alpha_5 \le \cdots \le \alpha_{2\ell-1} \ge \alpha_{2\ell+1} \ge \cdots \ge \alpha_{2\lfloor (n+1)/2 \rfloor - 1}$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \le \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| - \alpha_0) + (|\alpha_1| - \alpha_1) + 2\alpha_{2k} + \alpha_{2\ell-1} + (|\alpha_{n-1}| - \alpha_{n-1})$$
$$+ (|\alpha_n| - \alpha_n) + 2\sum_{j=0}^{n} |\beta_j|.$$

With $t = 1, 2k = 2\lfloor n/2 \rfloor$ and $2\ell - 1 = 2\lfloor n + 1/2 \rfloor - 1$ we get,

Corollary 3.9 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $0 \le j \le n$. *Suppose that for some* $t > 0$, *some nonnegative integers* k *and* s, *and positive integers* ℓ *and* q*:*

$$0 \ne \alpha_0 \le \alpha_2 \le \alpha_4 \le \cdots \le \alpha_{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \le \alpha_3 \le \alpha_5 \le \cdots \le \alpha_{2\lfloor (n+1)/2 \rfloor - 1}$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \le \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| - \alpha_0) + (|\alpha_1| - \alpha_1) + 2\alpha_{2\lfloor n/2 \rfloor} + \alpha_{2\lfloor n+1/2 \rfloor - 1}$$
$$+ (|\alpha_{n-1}| - \alpha_{n-1}) + (|\alpha_n| - \alpha_n) + 2\sum_{j=0}^{n} |\beta_j|.$$

When, $t = 1, k = 0$ and $\ell - 1 = 0$ in Theorem 3.7 we get,

Corollary 3.10 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $0 \le j \le n$. *Suppose that for some* $t > 0$, *some nonnegative integers* k *and* s, *and positive integers* ℓ *and* q:

$$0 \neq \alpha_0 \geq \alpha_2 \geq \cdots \geq \alpha_{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \geq \alpha_3 \geq \cdots \geq \alpha_{2\lfloor (n+1)/2 \rfloor - 1}$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \le \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| + 2\alpha_0) + (|\alpha_1| - \alpha_1) + (|\alpha_{n-1}| - \alpha_{n-1}) + (|\alpha_n| - \alpha_n) + 2\sum_{j=0}^{n} |\beta_j|.$$

From Corollaries 3.9 and 3.10 we can easily obtain two more corollaries with the monotonicity of each the even and odd differing (i.e., we can have the even coefficients monotonically increasing as the odd are monotonically decreasing, and visa versa). We now present the proof of Theorem 3.7.

Proof theorem 3.7 Define

$$G(z) = (t^2 - z^2)P(z) = t^2 a_0 + a_1 t^2 z + \sum_{j=2}^{n} (a_j t^2 - a_{j-2}) z^j - a_{n-1} z^{n+1} - a_n z^{n+2}.$$

For $|z| = t$ we have

$$
\begin{aligned}
|G(z)| \;&\le\; (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 \\
&\quad + \sum_{j=2}^{n} |\alpha_j t^2 - \alpha_{j-2}| t^j + \sum_{j=2}^{n} (|\beta_j| t^2 + |\beta_{j-2}|) t^j \\
&\quad + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} |\alpha_j t^2 - \alpha_{j-2}| t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor - 1} |\alpha_j t^2 - \alpha_{j-2}| t^j \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} (|\beta_j| t^2 + |\beta_{j-2}|) t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor - 1} (|\beta_j| t^2 + |\beta_{j-2}|) t^j \\
&\quad + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} (\alpha_j t^2 - \alpha_{j-2}) t^j + \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} (\alpha_{j-2} - \alpha_j t^2) t^j \\
&\quad + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell - 1} (\alpha_j t^2 - \alpha_{j-2}) t^j + \sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor - 1} (\alpha_{j-2} - \alpha_j t^2) t^j \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} |\beta_j| t^{j+2} + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor n+1/2\rfloor - 1} |\beta_j| t^{j+2} \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} |\beta_{j-2}| t^{j} + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor n+1/2\rfloor - 1} |\beta_{j-2}| t^{j} \\
&\quad + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_1| + |\beta_1|)t^3 \\
&\quad + \left(\sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \alpha_j t^{j+2} - \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \alpha_{j-2} t^j \right) + \left(\sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} \alpha_{j-2} t^j - \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor} \alpha_j t^{j+2} \right)
\end{aligned}
$$

$$
\begin{aligned}
& + \left(\sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} \alpha_j t^{j+2} - \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} \alpha_{j-2} t^j \right) + \left(\sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor -1} \alpha_{j-2} t^j - \sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor -1} \alpha_j t^{j+2} \right) \\
& + |\beta_0| t^4 + 2 \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |\beta_j| t^{j+2} + 2 \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor n+1/2 \rfloor -1} |\beta_j| t^{j+2} + |\beta_1| t^3 \\
& + (|\alpha_{n-1}| + |\beta_{n-1}|) t^{n+1} + (|\alpha_n| + |\beta_n|) t^{n+2} \\
= \; & (|\alpha_0| + |\beta_0|) t^2 + (|\alpha_2| + |\beta_1|) t^3 \\
& + \left(\alpha_{2k} t^{2k+2} - \alpha_0 t^2 \right) + \left(\alpha_{2k} t^{2k+2} - \alpha_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor + 2} \right) \\
& + \left(\alpha_{2\ell-1} t^{2\ell+1} - \alpha_1 t^3 \right) + \left(\alpha_{2\ell-1} t^{2\ell+1} - \alpha_{2\lfloor (n+1)/2 \rfloor -1} t^{2\lfloor (n+1)/2 \rfloor + 1} \right) \\
& + 2 \left(\sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |\beta_j| t^{j+2} + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor n+1/2 \rfloor -1} |\beta_j| t^{j+2} \right) + |\beta_0| t^4 + |\beta_1| t^3 \\
& + (|\alpha_{n-1}| + |\beta_{n-1}|) t^{n+1} + (|\alpha_n| + |\beta_n|) t^{n+2} \\
= \; & (|\alpha_0| - \alpha_0) t^2 + (|\alpha_1| - \alpha_1) t^3 + 2\alpha_{2k} t^{2k+2} + \alpha_{2\ell-1} t^{2\ell+1} + (|\alpha_{n-1}| - \alpha_{n-1}) t^{n+1} \\
& + (|\alpha_n| - \alpha_n) t^{n+2} + 2 \sum_{j=0}^{n} |\beta_j| t^{j+2}. \\
= \; & M.
\end{aligned}
$$

Now $G(z)$ is analytic in $|z| \leq t$, and $|G(z)| \leq M$ for $|z| = t$. So by Titchmarsh's theorem and the Maximum Modulus Theorem, the number of zeros of G (and hence of P) in $|z| \leq \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

The theorem follows. □

3.3 Restrictions on the Real and Imaginary Part of the Coefficients

In this section, we impose the same restriction as in section 2.3 on the real and imaginary part of the coefficients, yet we also have the coefficients restricted with the even and odd indices. This gives four restrictions in the hypotheses: even and real, even and imaginary, odd and real, as well as odd and imaginary. Because of the restrictions, this section gives gratuitous amounts of corollaries. Although we do not list them all, we state the ones of greatest deviation and note how we can easily obtain the remaining corollaries with standard algebra.

Theorem 3.11 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $0 \leq j \leq n$. Suppose that for some $t > 0$, some nonnegative integers k and s, and positive integers ℓ and q:*

$$0 \neq \alpha_0 \leq \alpha_2 t^2 \leq \alpha_4 t^4 \leq \cdots \leq \alpha_{2k} t^{2k} \geq \alpha_{2k+2} t^{2k+2} \geq \cdots \geq \alpha_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \leq \alpha_3 t^2 \leq \alpha_5 t^4 \leq \cdots \leq \alpha_{2\ell-1} t^{2\ell-2} \geq \alpha_{2\ell+1} t^{2\ell} \geq \cdots \geq \alpha_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor}$$

$$\beta_0 \leq \beta_2 t^2 \leq \beta_4 t^4 \leq \cdots \leq \beta_{2s} t^{2s} \geq \beta_{2s+2} t^{2s+2} \geq \cdots \geq \beta_{2\lfloor n/2 \rfloor} t^{2\lfloor n/2 \rfloor}$$

$$\beta_1 \leq \beta_3 t^2 \leq \beta_5 t^4 \leq \cdots \leq \beta_{2q-1} t^{2q-2} \geq \beta_{2q+1} t^{2q} \geq \cdots \geq \beta_{2\lfloor (n+1)/2 \rfloor - 1} t^{2\lfloor n/2 \rfloor}.$$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$\begin{aligned} M \quad &= (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0) t^2 + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1) t^3 \\ &+ 2(\alpha_{2k} t^{2k+2} + \alpha_{2\ell-1} t^{2\ell+1} + \beta_{2s} t^{2s+2} + \beta_{2q-1} t^{2q+1}) \\ &+ (|\alpha_{n-1}| - \alpha_{n-1} + |\beta_{n-1}| - \beta_{n-1}) t^{n+1} + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n) t^{n+2}. \end{aligned}$$

With $t = 1$ in Theorem 3.11, we have:

Corollary 3.12 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that for some nonnegative integers* k *and* s, *and positive integers* ℓ *and* q:

$$0 \neq \alpha_0 \leq \alpha_2 \leq \alpha_4 \leq \cdots \leq \alpha_{2k} \geq \alpha_{2k+2} \geq \cdots \geq \alpha_{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \leq \alpha_3 \leq \alpha_5 \leq \cdots \leq \alpha_{2\ell-1} \geq \alpha_{2\ell+1} \geq \cdots \geq \alpha_{2\lfloor (n+1)/2 \rfloor - 1}$$

$$\beta_0 \leq \beta_2 \leq \beta_4 \leq \cdots \leq \beta_{2s} \geq \beta_{2s+2} \geq \cdots \geq \beta_{2\lfloor n/2 \rfloor}$$

$$\beta_1 \leq \beta_3 \leq \beta_5 \leq \cdots \leq \beta_{2q-1} \geq \beta_{2q+1} \geq \cdots \geq \beta_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0) + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1) + 2(\alpha_{2k} + \alpha_{2\ell-1} + \beta_{2s} + \beta_{2q-1})$$

$$+(|\alpha_{n-1}| - \alpha_{n-1} + |\beta_{n-1}| - \beta_{n-1}) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n).$$

By manipulating the parameters k, ℓ, s, and q we easily get over sixteen more corollaries from Corollary 3.12. For example, with $k = s = 2\lfloor n/2 \rfloor$ and $\ell = q = 2\lfloor (n+1)/2 \rfloor - 1$ we have:

Corollary 3.13 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that:*

$$0 \neq \alpha_0 \leq \alpha_2 \leq \alpha_4 \leq \cdots \leq \alpha_{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \leq \alpha_3 \leq \alpha_5 \leq \cdots \leq \alpha_{2\lfloor (n+1)/2 \rfloor - 1}$$

$$\beta_0 \leq \beta_2 \leq \beta_4 \leq \cdots \leq \beta_{2\lfloor n/2 \rfloor}$$

$$\beta_1 \leq \beta_3 \leq \beta_5 \leq \cdots \leq \beta_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0) + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1) + (|\alpha_{n-1}| + \alpha_{n-1} + |\beta_{n-1}| + \beta_{n-1})$$

$$+ (|\alpha_n| + \alpha_n + |\beta_n| + \beta_n).$$

With $k = s = 0$ and $\ell = q = 1$ in Corollary 3.12 we have:

Corollary 3.14 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $0 \leq j \leq n$. Suppose that:*

$$0 \neq \alpha_0 \geq \alpha_2 \geq \alpha_4 \geq \cdots \geq \alpha_{2\lfloor n/2 \rfloor}$$

$$\alpha_1 \geq \alpha_3 \geq \alpha_5 \geq \cdots \geq \alpha_{2\lfloor (n+1)/2 \rfloor - 1}$$

$$\beta_0 \geq \beta_2 \geq \beta_4 \geq \cdots \geq \beta_{2\lfloor n/2 \rfloor}$$

$$\beta_1 \geq \beta_3 \geq \beta_5 \geq \cdots \geq \beta_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = (|\alpha_0| + \alpha_0 + |\beta_0| + \beta_0) + (|\alpha_1| + \alpha_1 + |\beta_1| + \beta_1) + (|\alpha_{n-1}| - \alpha_{n-1} + |\beta_{n-1}| - \beta_{n-1})$$

$$+(|\alpha_n| - \alpha_n + |\beta_n| - \beta_n).$$

With $k = \lfloor n/r \rfloor$, $\ell = 1$, and each a_j real in Corollary 3.12 we have:

Corollary 3.15 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $a_j \in \mathbb{R}$ *for* $0 \leq j \leq n$. *Suppose that:*

$$0 \neq a_0 \leq a_2 \leq a_4 \leq \cdots \leq a_{2\lfloor n/2 \rfloor},$$

$$a_1 \geq a_3 \geq a_5 \geq \cdots \geq a_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where

$$M = |a_0| + a_0 + |a_{2\lfloor n/2 \rfloor}| + a_{2\lfloor n/2 \rfloor}.$$

Example: Consider the polynomial $P(z) = 1 + 10z + 2z^2 + 0z^3 + 3z^4 + 0z^5 + 4z^6 + 0z^7 + 8z^8$. The zeros of P are approximately -0.102119, -0.872831, $-0.629384 \pm 0.855444i$, $0.22895 \pm 1.05362i$, and $0.887908 \pm 0.530244i$. Applying Corollary 3.15 with $\delta = 0.15$ we see that it predicts no more than 1.888926 zeros in $|z| \leq 0.15$. In other words, Corollary 3.15 predicts at most one zero in $|z| \leq 0.15$ In fact, P does have exactly one zero in $|z| \leq 0.15$, and Corollary 3.15 is sharp for this example.

There are many more corollaries that can come from Theorem 3.11, yet we merely note the remaining can easily be obtained with standard algebra. Here we now present the proof of Theorem 3.11 which involves restrictions on the monotonicity of both the real and imaginary coefficients as well as the even and odd restriction as before in sections 3.1 and 3.2.

Proof Theorem 3.11 Define

$$G(z) = (t^2 - z^2)P(z) = t^2 a_0 + a_1 t^2 z + \sum_{j=2}^{n} (a_j t^2 - a_{j-2}) z^j - a_{n-1} z^{n+1} - a_n z^{n+2}.$$

For $|z| = t$ we have

$$\begin{aligned}
|G(z)| &\leq (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 \\
&\quad + \sum_{j=2}^{n} |\alpha_j t^2 - \alpha_{j-2}| t^j + \sum_{j=2}^{n} |\beta_j t^2 - \beta_{j-2}| t^j \\
&\quad + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |\alpha_j t^2 - \alpha_{j-2}| t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} |\alpha_j t^2 - \alpha_{j-2}| t^j \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} |\beta_j t^2 - \beta_{j-2}| t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor - 1} |\beta_j t^2 - \beta_{j-2}| t^j \\
&\quad + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2}
\end{aligned}$$

$$
\begin{aligned}
&= (|\alpha_0|+|\beta_0|)t^2 + (|\alpha_2|+|\beta_1|)t^3 \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} (\alpha_j t^2 - \alpha_{j-2})t^j + \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} (\alpha_{j-2} - \alpha_j t^2)t^j \\
&\quad + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} (\alpha_j t^2 - \alpha_{j-2})t^j + \sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor -1} (\alpha_{j-2} - \alpha_j t^2)t^j \\
&\quad + \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} (\beta_j t^2 - \beta_{j-2})t^j + \sum_{\substack{j=2s+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} (\beta_{j-2} - \beta_j t^2)t^j \\
&\quad + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2q-1} (\beta_j t^2 - \beta_{j-2})t^j + \sum_{\substack{j=2q+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor -1} (\beta_{j-2} - \beta_j t^2)t^j \\
&\quad + (|\alpha_{n-1}|+|\beta_{n-1}|)t^{n+1} + (|\alpha_n|+|\beta_n|)t^{n+2}
\end{aligned}
$$

$$
\begin{aligned}
&= (|\alpha_0|+|\beta_0|)t^2 + (|\alpha_2|+|\beta_1|)t^3 \\
&\quad + \left(\sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \alpha_j t^{j+2} - \sum_{\substack{j=2 \\ j \text{ even}}}^{2k} \alpha_{j-2} t^j \right) + \left(\sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \alpha_{j-2} t^j - \sum_{\substack{j=2k+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \alpha_j t^{j+2} \right) \\
&\quad + \left(\sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} \alpha_j t^{j+2} - \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\ell-1} \alpha_{j-2} t^j \right) + \left(\sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor -1} \alpha_{j-2} t^j - \sum_{\substack{j=2\ell+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor -1} \alpha_j t^{j+2} \right) \\
&\quad + \left(\sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \beta_j t^{j+2} - \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \beta_{j-2} t^j \right) + \left(\sum_{\substack{j=2s+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \beta_{j-2} t^j - \sum_{\substack{j=2s+2 \\ j \text{ even}}}^{2\lfloor n/2 \rfloor} \beta_j t^{j+2} \right) \\
&\quad + \left(\sum_{\substack{j=3 \\ j \text{ odd}}}^{2q-1} \beta_j t^{j+2} - \sum_{\substack{j=3 \\ j \text{ odd}}}^{2q-1} \beta_{j-2} t^j \right) + \left(\sum_{\substack{j=2q+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor -1} \beta_{j-2} t^j - \sum_{\substack{j=2q+1 \\ j \text{ odd}}}^{2\lfloor (n+1)/2 \rfloor -1} \beta_j t^{j+2} \right) \\
&\quad + (|\alpha_{n-1}|+|\beta_{n-1}|)t^{n+1} + (|\alpha_n|+|\beta_n|)t^{n+2}
\end{aligned}
$$

$$
\begin{aligned}
&= (|\alpha_0| + |\beta_0|)t^2 + (|\alpha_2| + |\beta_1|)t^3 \\
&\quad + \left(\alpha_{2k}t^{2k+2} - \alpha_0 t^2\right) + \left(\alpha_{2k}t^{2k+2} - \alpha_{2\lfloor n/2\rfloor}t^{2\lfloor n/2\rfloor+2}\right) \\
&\quad + \left(\alpha_{2\ell-1}t^{2\ell+1} - \alpha_1 t^3\right) + \left(\alpha_{2\ell-1}t^{2\ell+1} - \alpha_{2\lfloor (n+1)/2\rfloor-1}t^{2\lfloor (n+1)/2\rfloor+1}\right) \\
&\quad + \left(\beta_{2s}t^{2s+2} - \beta_0 t^2\right) + \left(\beta_{2s}t^{2s+2} - \beta_{2\lfloor n/2\rfloor}t^{2\lfloor n/2\rfloor+2}\right) \\
&\quad + \left(\beta_{2q-1}t^{2q+1} - \beta_1 t^3\right) + \left(\beta_{2q-1}t^{2q+1} - \beta_{2\lfloor (n+1)/2\rfloor-1}t^{2\lfloor (n+1)/2\rfloor+1}\right) \\
&\quad + (|\alpha_{n-1}| + |\beta_{n-1}|)t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+2} \\
&= (|\alpha_0| - \alpha_0 + |\beta_0| - \beta_0)t^2 + (|\alpha_1| - \alpha_1 + |\beta_1| - \beta_1)t^3 \\
&\quad + 2(\alpha_{2k}t^{2k+2} + \alpha_{2\ell-1}t^{2\ell+1} + \beta_{2s}t^{2s+2} + \beta_{2q-1}t^{2q+1}) \\
&\quad + (|\alpha_{n-1}| - \alpha_{n-1} + |\beta_{n-1}| - \beta_{n-1})t^{n+1} + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+2} \\
&= M.
\end{aligned}
$$

Now $G(z)$ is analytic in $|z| \leq t$, and $|G(z)| \leq M$ for $|z| = t$. So by Titchmarsh's theorem and the Maximum Modulus Theorem, the number of zeros of G (and hence of P) in $|z| \leq \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

The theorem follows. □

4 A MONOTONICITY CONDITION ON THE COEFFICIENTS OF POLYNOMIALS WITH A GAP

In this chapter, we consider the same three types of hypotheses concerning the monotonicity of the moduli, real, as well as real and imaginary parts of the coefficients. Yet, we put these restrictions on a class of polynomials we denote as $\mathcal{P}_{n,\mu}$. This polynomial has a gap between the leading coefficient and the preceding coefficient, which has an index of μ. These polynomials are studied greatly in connection with Bernstein type inequalities [6]. We obtain a number of new results and corollaries for $\mathcal{P}_{n,\mu}$ type polynomials concerning the number of zeros result.

4.1 Restrictions on the Moduli of the Coefficients

In this section, we consider the $\mathcal{P}_{n,\mu}$ class of polynomials with the same restriction as in section 2.1 on the real part of the coefficient only and seek a bound on M to count the number of zeros using Theorem 1.3.

Theorem 4.1 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$ *and for some* $t > 0$ *and some* k *with* $\mu \leq k \leq n$,

$$t^{\mu}|a_{\mu}| \leq \cdots \leq t^{k-1}|a_{k-1}| \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^{n-1}|a_{n-1}| \geq t^n|a_n|$$

and $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for* $\mu \leq j \leq n$ *and for some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $M = 2|a_0|t + |a_\mu|t^{\mu+1}(1-\cos\alpha-\sin\alpha) + 2|a_k|t^{k+1}\cos\alpha + |a_n|t^{n+1}(1-\cos\alpha - \sin\alpha) + 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha$.

With $t = 1$ in Theorem 4.1 we get the following.

Corollary 4.2 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$ *and for some* $t > 0$ *and some* k *with* $\mu \leq k \leq n$,

$$|a_\mu| \leq \cdots \leq |a_{k-1}| \leq |a_k| \geq |a_{k+1}| \geq \cdots \geq |a_{n-1}| \geq |a_n|$$

and $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for* $\mu \leq j \leq n$ *and for some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta}\log\frac{M}{|a_0|}$$

where $M = 2|a_0| + |a_\mu|(1-\cos\alpha-\sin\alpha) + 2|a_k|\cos\alpha + |a_n|(1-\cos\alpha-\sin\alpha) + 2\sum_{j=\mu}^{n}|a_j|\sin\alpha$.

With $k = n$ in Corollary 4.2 we get:

Corollary 4.3 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$,

$$|a_\mu| \leq \cdots \leq |a_{n-1}| \leq |a_n|$$

and $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for* $\mu \leq j \leq n$ *and for some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta}\log\frac{M}{|a_0|}$$

where $M = 2|a_0| + |a_\mu|(1-\cos\alpha-\sin\alpha) + |a_n|(1+\cos\alpha-\sin\alpha) + 2\sum_{j=\mu}^{n}|a_j|\sin\alpha$.

With $k = \mu$ in Corollary 4.2 we get:

Corollary 4.4 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$,

$$|a_\mu| \geq \cdots \geq |a_{n-1}| \geq |a_n|$$

and $|arg\, a_j - \beta| \leq \alpha \leq \pi/2$ *for* $\mu \leq j \leq n$ *and for some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $M = 2|a_0| + |a_\mu|(1 + \cos\alpha - \sin\alpha) + |a_n|(1 - \cos\alpha - \sin\alpha) + 2\sum_{j=\mu}^{n} |a_j| \sin\alpha$.

Proof of Theorem 4.1. Consider

$$\begin{aligned}
F(z) &= (t-z)P(z) = (t-z)(a_0 + \sum_{j=\mu}^{n} a_j z^j) = a_0 t + \sum_{j=\mu}^{n} a_j t z^j - a_0 z - \sum_{j=\mu}^{n} a_j z^{j+1} \\
&= a_0(t-z) + \sum_{j=\mu}^{n} a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j \\
&= a_0(t-z) + a_\mu t z^\mu + \sum_{j=\mu+1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1}.
\end{aligned}$$

For $|z| = t$ we have

$$\begin{aligned}
|F(z)| &\leq 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{n} |a_j t - a_{j-1}|t^j + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} |a_j t - a_{j-1}|t^j + \sum_{j=k+1}^{n} |a_{j-1} - a_j t|t^j + |a_n|t^{n+1}
\end{aligned}$$

$$\begin{aligned}
&\leq 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} \{(|a_j|t - |a_{j-1}|)\cos\alpha + (|a_{j-1}| + |a_j|t)\sin\alpha\} t^j \\
&\quad + \sum_{j=k+1}^{n} \{(|a_{j-1}| - |a_j|t)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\} t^j + |a_n|t^{n+1} \\
&\quad \text{by Lemma 2.3 with } z = a_j t \text{ and } z' = a_{j-1} \text{ when } 1 \leq j \leq k, \\
&\quad \text{and with } z = a_{j-1} \text{ and } z' = a_j t \text{ when } k+1 \leq j \leq n \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} |a_j|t^{j+1}\cos\alpha - \sum_{j=\mu+1}^{k} |a_{j-1}|t^j\cos\alpha + \sum_{j=\mu+1}^{k} |a_{j-1}|t^j\sin\alpha \\
&\quad + \sum_{j=\mu+1}^{k} |a_j|t^{j+1}\sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}|t^j\cos\alpha - \sum_{j=k+1}^{n} |a_j|t^{j+1}\cos\alpha \\
&\quad + \sum_{j=k+1}^{n} |a_j|t^{j+1}\sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}|t^j\sin\alpha + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} - |a_\mu|t^{\mu+1}\cos\alpha + |a_k|t^{k+1}\cos\alpha + |a_\mu|t^{\mu+1}\sin\alpha \\
&\quad + |a_k|t^{k+1}\sin\alpha + 2\sum_{j=\mu+1}^{k-1} |a_j|t^{j+1}\sin\alpha + |a_k|t^{k+1}\cos\alpha - |a_n|t^{n+1}\cos\alpha + |a_k|t^{k+1}\sin\alpha \\
&\quad + |a_n|t^{n+1}\sin\alpha + 2\sum_{j=k+1}^{n-1} |a_j|t^{j+1}\sin\alpha + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + |a_\mu|t^{\mu+1}(\sin\alpha - \cos\alpha) + 2\sum_{j=\mu+1}^{n-1} |a_j|t^{j+1}\sin\alpha \\
&\quad + 2|a_k|t^{k+1}\cos\alpha + (\sin\alpha - \cos\alpha + 1)|a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1}(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha \\
&\quad + |a_n|t^{n+1}(1 - \cos\alpha - \sin\alpha) + 2\sum_{j=\mu}^{n} |a_j|t^{j+1}\sin\alpha \\
&= M.
\end{aligned}$$

Now $F(z)$ is analytic in $|z| \leq t$, and $|F(z)| \leq M$ for $|z| = t$. So by Theorem 1.3 and the Maximum Modulus Theorem, the number of zeros of F (and hence of P) in

$|z| \leq \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

The theorem follows. $\square$

4.2 Restrictions on the Real Part of the Coefficients

In this section, we put a monotonicity restriction on the real part of the coefficients only for polynomials in the class $\mathcal{P}_{n,\mu}$.

Theorem 4.5 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, *Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $t > 0$ *and some* k *with* $\mu \leq k \leq n$ *we have*

$$t^\mu \alpha_\mu \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^n \alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n} |\beta_j| t^{j+1}$.

With $t = 1$ in Theorem 4.5, we get:

Corollary 4.6 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, *Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that we have*

$$\alpha_\mu \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_{n-1} \geq \alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \le \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|$.

Example. Consider the polynomial $P(z) = 0.1 + 0.001z^2 + 2z^3 + 0.002z^4 + 0.002z^5 + 0.001z^6$. The zeros of P are approximately $z_1 = -0.368602$, $z_2 = 0.184076 + 0.319010i$, $z_3 = 0.184076 - 0.319010i$, and $z_4 = 5.62344 + 10.92507i$, $z_5 = 5.62344 - 10.92507i$, and $z_6 = -13.2464$. Corollary 4.6 applies to P with $\mu = 2$ and $k = 3$. With $\delta = 0.37$ we see that it predicts no more than 3.75928 zeros in $|z| \le 0.37$. In other words, Corollary 4.6 predicts at most three zeros in $|z| \le 0.37$ In fact, P does have exactly three zeros in $|z| \le 0.37$, namely z_1, z_2, and z_3. So Corollary 4.6 is sharp for this example.

With $k = n$ in Corollary 4.6 we get:

Corollary 4.7 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that we have*

$$\alpha_\mu \le \cdots \le \alpha_{n-1} \le \alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \le \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + (|\alpha_n| + \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|$.

With $k = \mu$ in Corollary 4.6 we get:

Corollary 4.8 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$*,* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$*. Suppose that we have*

$$\alpha_\mu \geq \cdots \geq \alpha_{n-1} \geq \alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu) + (|\alpha_n| - \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|$.

It is easy to see from Corollary 4.6 that if $n = k$ or $k = 0$, M will not change drastically, and in fact the last three terms are the only ones affected.

Proof of Theorem 4.5. As in the proof of Theorem 4.1,

$$F(z) = (t-z)P(z) = a_0(t-z) + a_\mu t z^\mu + \sum_{j=\mu+1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1},$$

and so

$$\begin{aligned} F(z) &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^{n} ((\alpha_j + i\beta_j)t - (\alpha_{j-1} + i\beta_{j-1}))z^j \\ &\quad -(\alpha_n + i\beta_n)z^{n+1} \\ &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^{n} (\alpha_j t - \alpha_{j-1})z^j \\ &\quad +i \sum_{j=\mu+1}^{n} (\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$
\begin{aligned}
|F(z)| &\leq 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{n} |\alpha_j t - \alpha_{j-1}|t^j \\
&\quad + \sum_{j=\mu+1}^{n} (|\beta_j|t + |\beta_{j-1}|)t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_j t - \alpha_{j-1})t^j \\
&\quad + \sum_{j=k+1}^{n} (\alpha_{j-1} - \alpha_j t)t^j + |\beta_\mu|t^{\mu+1} + 2\sum_{j=\mu+1}^{n-1} |\beta_j|t^{j+1} + |\beta_n|t^{n+1} \\
&\quad +(|\alpha_n| + |\beta_n|)t^{n+1} \\
&= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} - \alpha_\mu t^{\mu+1} + 2\alpha_k t^{k+1} - \alpha_n t^{n+1} \\
&\quad +|\beta_\mu|t^{\mu+1} + 2\sum_{j=\mu+1}^{n} |\beta_j|t^{j+1} + |\alpha_n|t^{n+1} \\
&= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} \\
&\quad +2\sum_{j=\mu}^{n} |\beta_j|t^{j+1} \\
&= M.
\end{aligned}
$$

The result now follows as in the proof of Theorem 4.1. □

4.3 Restrictions on the Real and Imaginary Part of the Coefficients

In this section, we put the monotonicity restriction on both the real and imaginary parts of the coefficients for polynomials of the class $\mathcal{P}_{n,\mu}$.

Theorem 4.9 *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ where $a_0 \neq 0$, $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some $t > 0$ and some k with $\mu \leq k \leq n$ we have*

$$t^\mu \alpha_\mu \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^n \alpha_n$$

and for some $\mu \leq \ell \leq n$ *we have*

$$t^{\mu}\beta_{\mu} \leq \cdots \leq t^{\ell-1}\beta_{\ell-1} \leq t^{\ell}\beta_{\ell} \geq t^{\ell+1}\beta_{\ell+1} \geq \cdots \geq t^{n-1}\beta_{n-1} \geq t^{n}\beta_{n}.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2(\alpha_k t^{k+1} + \beta_\ell t^{\ell+1})$
$+ (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1}$.

In Theorem 4.9 if we let $t = 1$ we get the following.

Corollary 4.10 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* k *with* $\mu \leq k \leq n$ *we have*

$$\alpha_\mu \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_{n-1} \geq \alpha_n$$

and for some $\mu \leq \ell \leq n$ *we have*

$$\beta_\mu \leq \cdots \leq \beta_{\ell-1} \leq \beta_\ell \geq \beta_{\ell+1} \geq \cdots \geq \beta_{n-1} \geq \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + 2(\alpha_k + \beta_\ell) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)$.

In Corollary 4.10 if we let $k = \ell = n$ we get the following.

Corollary 4.11 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$*,* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$*. Suppose that for some* k *with* $\mu \leq k \leq n$ *we have*

$$\alpha_\mu \leq \cdots \leq \alpha_{n-1} \leq \alpha_n$$

and for some $\mu \leq \ell \leq n$ *we have*

$$\beta_\mu \leq \cdots \leq \beta_{n-1} \leq \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + (|\alpha_n| + \alpha_n + |\beta_n| + \beta_n)$*.*

In Corollary 4.10 if we let $k = \ell = \mu$ we get the following.

Corollary 4.12 *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$*,* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$*. Suppose that for some* k *with* $\mu \leq k \leq n$ *we have*

$$\alpha_\mu \geq \cdots \geq \alpha_{n-1} \geq \alpha_n$$

and for some $\mu \leq \ell \leq n$ *we have*

$$\beta_\mu \geq \cdots \geq \beta_{n-1} \geq \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

where $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu + |\beta_\mu| + \beta_\mu) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)$*.*

In Corollary 4.10, we get similar corollaries by letting $k = n$ and $\ell = \mu$, or $k = \mu$ and $\ell = n$.

Proof of Theorem 4.9. As in the proof of Theorem 4.1,

$$F(z) = (t-z)P(z) = a_0(t-z) + a_\mu^t z^\mu + \sum_{j=\mu}^{n}(a_j t - a_{j-1})z^j - a_n z^{n+1},$$

and so

$$\begin{aligned} F(z) &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^{n}((\alpha_j + i\beta_j)t - (\alpha_{j-1} + i\beta_{j-1}))z^j \\ &\quad -(\alpha_n + i\beta_n)z^{n+1} \\ &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^{n}(\alpha_j t - \alpha_{j-1})z^j \\ &\quad +i\sum_{j=\mu+1}^{n}(\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1} \end{aligned}$$

For $|z| = t$ we have

$$\begin{aligned} |F(z)| &\leq (|\alpha_0| + |\beta_0|)2t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{n}|\alpha_j t - \alpha_{j-1}|t^j \\ &\quad + \sum_{j=\mu+1}^{n}(|\beta_j t + \beta_{j-1}|)t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{k}(\alpha_j t - \alpha_{j-1})t^j \\ &\quad + \sum_{j=k+1}^{n}(\alpha_{j-1} - \alpha_j t)t^j + \sum_{j=\mu+1}^{\ell}(\beta_j t - \beta_{j-1})t^j \\ &\quad + \sum_{j=\ell+1}^{n}(\beta_{j-1} - \beta_j t)t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \end{aligned}$$

$$
\begin{aligned}
&= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} - \alpha_\mu t^{\mu+1} + 2\alpha_k t^{k+1} - \alpha_n t^{n+1} - \beta_\mu t^{\mu+1} \\
&\quad +2\beta_\ell t^{\ell+1} - \beta_n t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+1} \\
&= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2(\alpha_k t^{k+1} + \beta_\ell t^{\ell+1}) \\
&\quad +(|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1} \\
&= M.
\end{aligned}
$$

The result now follows as in the proof of Theorem 4.1. □

5 CONCLUSION

In this thesis, we considered three different classes of polynomials. First, a class of polynomials with monotonically increasing coefficients to a position k, where the monotonicity then decreased. Second, a class of polynomial with the same type of monotonicity flip, yet with the indices of the coefficients separated into an even and odd restriction. Last, we considered a class of polynomials with a gap between the leading and proceeding coefficient, where the proceeding coefficient had an index of μ. Each class of polynomial was considered with three hypotheses: restrictions on the moduli of the coefficients, restrictions on the real part only of the coefficients, and restrictions on the real and imaginary part of the coefficients. We put these restrictions on the coefficients of polynomials in order to count the number of zeros of each particular class of polynomial, in a specific region. We relied on Titchmarsh's result for counting the number of zeros, yet we sought out different M values for specific polynomials. This was done to give results which can be easily applied by simply plugging in coefficients of a polynomial, as opposed to the computation of the bound M, which could be quite difficult.

There has been much research done in this particular field of mathematics and the area is currently active. Our results from Chapter 2 have appeared in *The Journal of Classical Analysis* [14]; the results from Chapters 3 and 4 are also being submitted to journals. Furthermore, there is potential for further research in this area. From the work in this thesis, one could combine Chapters 3 and 4 and consider the number of zeros of the $\mathcal{P}_{n,\mu}$ class polynomials from Chapter 3 and impose the even and odd restriction from Chapter 4. This would be an original body of research.

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