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Very Cost Effective Domination in Graphs

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Very Cost Effective Domination in Graphs

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

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May 2014

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Keywords: very cost effective domination, very cost effective domination number

ABSTRACT

Very Cost Effective Domination in Graphs

by

Tony Rodriguez

A set S of vertices in a graph $G = (V, E)$ is a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in S, and the minimum cardinality of a dominating set of G is the domination number of G. A vertex v in a dominating set S is said to be very cost effective if it is adjacent to more vertices in $V \setminus S$ than to vertices in S. A dominating set S is very cost effective if every vertex in S is very cost effective. The minimum cardinality of a very cost effective dominating set of G is the very cost effective domination number of G. We first give necessary conditions for a graph to have equal domination and very cost effective domination numbers. Then we determine an upper bound on the very cost effective domination number for trees in terms of their domination number, and characterize the trees which attain this bound. Lastly, we show that no such bound exists for graphs in general, even when restricted to bipartite graphs.

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DEDICATION

This thesis is dedicated to those who came before me, whose shoulders have given me a place to stand that I may not simply see the beauties of mathematics and nature, but that I may comprehend and shape them.

ACKNOWLEDGMENTS

The construction and formation of this thesis was made possible through the efforts and dedication of several people who had nothing to gain except to see me succeed. This is in no way meant to be an exhaustive list of those individuals, but I feel it would simply be unacceptable to fail to mention these few.

Firstly, Dr. Teresa Haynes has been singularly instrumental in bringing about this thesis. Her love and consequent knowledge of Graph Theory inspired me to learn more about Graph Theory to begin with. I loved the classes I had under her, and collaborating with her in this particular work has been an honor. I am grateful to Dr. Stephen Hedetniemi, whose curiosity and excellent questions formed the basis for this research. I would also like to thank my committee members, Dr. Robert Beeler and Dr. Robert Gardner, not only for offering encouragement and support throughout this project and others, but for the patience and quality of instruction that you have both offered in the past. Also, I would like to thank Michael Marks for always being willing to help in any way possible, even when I don't yet realize I need the help. Thanks for keeping me in mind.

Lastly, I want to thank my mother, Evelyn Mullins, for all you have done and continue to do for me. You instilled a great love of knowledge and a deep desire to learn upon me at a young age. These are among the greatest gifts one person can give to another. Thanks.

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1 INTRODUCTION

1.1 Basic Definitions Within Graph Theory

A *graph* $G = (V, E)$ is a nonempty set $V(G)$, the elements of which are called *vertices*, together with a (possibly empty) set $E(G)$ of unordered pairs of elements of $V(G)$ called *edges*. For the sake of simplicity, we will denote edge $\{u, v\}$ as uv. The *order* of a graph, denoted $n(G)$, is equal to the number of vertices in the graph, and the *size* of a graph, denoted $m(G)$, is the number of edges in a graph. If $n(G) = 1, G$ is said to be *trivial*, otherwse G is *nontrivial*. Further, if $m(G) = 0, G$ is called *empty*, otherwise G is *nonempty*. When G is clear from context, we will denote the vertex and edge sets as V and E, respectively. A graph is said to be *simple* if it does not contain multiple edges between any pair of vertices and edges must have distinct endpoints. We will restrict our attention to simple graphs of finite order for this thesis.

For any two vertices u and v, if uv is an edge, then u and v are said to be *adjacent*, edge uv is *incident* to vertex u, and u is a *neighbor* of v. Further, if $uv \notin E$, u and v are said to be *nonadjacent*. Two edges are said to be *adjacent* if they share a common vertex. The *degree* of a vertex is the number of edges to which the vertex is incident. A vertex of degree 0 is an *isolated vertex*, while a vertex of degree 1 is a *leaf* or *pendant*. A vertex of odd degree is said to be *odd*, while one of even degree is said to be *even*. The *maximum degree* of a graph G, written $\Delta(G)$, is the maximum degree of any vertex in $V(G)$, while the *minimum degree* of G, denoted $\delta(G)$, is the minimum degree of any vertex in $V(G)$. A *u-v walk* W of G is a finite, alternating sequence $W : u = u_0, e_1, u_1, e_2, ..., u_{k-1}, e_k, u_k = v$ of vertices and edges, beginning with vertex u and ending with vertex v such that $e_i = u_{i-1}u_i, 1 \leq i \leq k$. A u-v walk is *closed* if $u = v$ and *open* if $u \neq v$. A u-v *trail* is a u-v walk in which no edges are repeated. A u-v *path* is a u-v walk in which no vertices are repeated. A nontrivial closed trail is called a *circuit*. A circuit on at least 3 vertices where no vertex appears more than once is called a *cycle*. A graph is *connected* if, given two distinct vertices u and v, then a u-v path exists. A connected graph that contains no cycles is called a *tree*. A path on n vertices is denoted P_n and cycles on n vertices are denoted C_n. A graph is *complete* if, given any two vertices $u, v \in V$, then $uv \in E$. The complete graph on n vertices is denoted K_n.

H K_4 C_4

Figure 1: The House Graph H, K_4, and C_4

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is said to be an *induced subgraph* if, for any pair of vertices $u, v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$. An induced subgraph with vertex set $S \subseteq V(G)$ is written $G[S]$ and is said to be *induced by* S. The *open neighborhood* of a vertex u is the set $N(u) = \{v | uv \in E\}$, while the *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$. The *open neighborhood* of a set $S \subseteq V$ is $N(S) = \bigcup_{u \in S} N(u)$. Similarly, the *closed neighborhood* of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is said to be *independent* if the vertices in S are pairwise nonadjacent. The *vertex independence number* $\beta_0(G)$ is the maximum cardinality of an independent set of G. A graph is said to be *k-partite* if its vertex set can be partitioned into k independent sets. In particular, a graph is *bipartite* if $k = 2$. $S \subseteq V$ is said to be a *dominating set* if $N[S] = V$. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. The *upper domination number*, written $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G. The *independent domination number* $i(G)$ is the minimum cardinality of a maximal independent set of G. A dominating set with cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*, and an independent dominating set of cardinality $i(G)$ is called an $i(G)$*-set*.

1.2 Very Cost Effective Domination

A vertex v in a set S is said to be *cost effective* if it is adjacent to at least as many vertices in $V \setminus S$ as in S, and is *very cost effective* if it is adjacent to more vertices in $V \setminus S$ than in S. A set S is said to be *cost effective* if every vertex in S is cost effective. Similarly, S is *very cost effective* if every vertex in S is very cost effective. Set S is a *cost effective dominating set* if it is both cost effective and dominating, and a *very cost effective dominating set* if it is both very cost effective and dominating. The *cost effective domination number* of a graph G, written $\gamma_{ce}(G)$, is the minimum cardinality of a cost effective dominating set. The *very cost effective domination number* $\gamma_{vce}(G)$ of a graph G is the minimum cardinality of a very cost effective dominating set. The *upper cost effective domination number* $\Gamma_{ce}(G)$ is the maximum cardinality of a minimal dominating set that is cost effective. The *upper very cost effective domination number* $\Gamma_{vce}(G)$ is the maximum cardinality of a minimal dominating set that is very cost effective. A cost effective dominating set of cardinality $\gamma_{ce}(G)$ is a $\gamma_{ce}(G)$*-set*, while a very cost effective dominating set of cardinality $\gamma_{vc\epsilon}(G)$ is a $\gamma_{vc\epsilon}(G)$-*set*. For example, consider the graph in Figures $2(a)$, $2(b)$, and $2(c)$. In Figure $2(a)$, the darkened vertices represent a $\gamma(G)$-set, in Figure $2(b)$, the darkened vertices represent a $\gamma_{c\epsilon}(G)$-set, and in Figure $2(c)$, the darkened vertices represent a $\gamma_{vc\epsilon}(G)$-set.

employee to have more clients outside than company than within, so S needs to be very cost effective also. Thus, S should be a very cost effective dominating set.

2 RELATED WORK

2.1 Unfriendly Partitions

Cost effective domination, and therefore very cost effective domination, is derived from the study of unfriendly partitions of graphs. Let C be a two-coloring of the vertices of a graph G such that $C : V \to \{Red, Blue\}$. For all $v \in V$, let $B(v) = \{u \in N(v)|C(u) = Blue\}$ and $R(v) = \{u \in N(v)|C(u) = Red\}$. For a set $S \subseteq V$, let $B(S) = \{v \in S|C(v) = Blue\}$ and $R(S) = \{v \in S|C(v) = Red\}$. Thus a two-coloring of V produces a bipartition $\pi = \{B(V), R(V)\}$. Given such a bipartition π, we say an edge $uv \in E$ is *bicolored* if $C(u) \neq C(v)$. A bipartition π is an *unfriendly partition* if every vertex $u \in B(V)$ has at least as many neighbors in $R(V)$ as in $B(V)$, and every vertex $u \in R(V)$ has at least as many neighbors in $B(V)$ as in $R(V)$. In other words, if $C(u) = Red$, then $|R(u)| \leq |B(u)|$, and if $C(u) = Blue$, then $|B(u)| \leq |R(u)|$. These partitions were defined and studied by Borodin and Koshtochka [3], Aharoni, Milner, and Prikry [1], and Shelah and Milner [12].

The notion of unfriendly partitions have influenced other ideas. In [13, 14], the concept of α-domination in graphs is defined and studied. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called an *α-dominating set* if for every vertex $v \in V \setminus S$, $|N(v) \cap S|/|N[v]| \geq \alpha$, where $0 \leq \alpha < 1$. In the case of $\alpha \geq 1/2$, every vertex in $V \setminus S$ meets the *unfriendly condition* in that it has at least as many neighbors in S as it has in $V \setminus S$. We note that no unfriendly condition is imposed on the vertices in S.

An idea closely related to unfriendly partitions is that of satisfactory partitions.

A bipartition $\pi = \{B(V), R(V)\}$ is called a *satisfactory partition* if, for every vertex $v \in B(V)$, $|B(v)| \geq |R(v)|$ and every vertex $u \in R(V)$, $|R(u)| \geq |B(u)|$. In other words, each vertex has at least as many neighbors in its own partition as the other partition. While it is known that every graph has an unfriendly partition, it is not true in general that every graph has a satisfactory partition. Indeed, determining whether or not an arbitrary graph has a satisfactory partition is NP-complete [2]. Satisfactory partitions have been studied in [4, 5, 6, 11].

2.2 Cost Effective Domination

Cost effective domination was introduced in [7]. The study of cost effective domination was motivated by the studies of unfriendly partitions and satisfactory partitions. As defined earlier, a set is a *cost effective dominating set* if it is both dominating and cost effective. The following inequality is a well-known result pertaining to domination.

Proposition 2.1 *For any graph G, $\gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G)$.*

It can be noted that every independent dominating set in an isolate-free graph is also a very cost effective dominating set. This observation allowed the previous proposition to be extended to include the cost effective and very cost effective domination parameters in the following two results.

Observation 2.2 [7] *Every independent dominating set S in an isolate-free graph G is a (very) cost effective dominating set.*

Corollary 2.3 [7] *For any isolate-free graph G,*

$$\gamma(G) \leq \gamma_{c\epsilon}(G) \leq \gamma_{vc\epsilon}(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_{vc\epsilon}(G) \leq \Gamma_{c\epsilon}(G) \leq \Gamma(G).$$

In [10], the cost effective domination number of several families of graphs was determined, in relation to the graphs' domination number. The authors were also able to bound $\gamma_{c\epsilon}(T)$ in terms of $\gamma(T)$ for all trees T and give a characterization for those trees which obtain the upper bound. We give some of these results below. First we give an additional definition. The *corona* of graphs G and H, denoted $G \circ H$, is the graph formed from one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex in $V(G)$ is adjacent to every vertex in the i^{th} copy of H.

Theorem 2.4 [10] *If G has maximum degree $\Delta(G) \leq 4$, then $\gamma(G) = \gamma_{c\epsilon}(G)$.*

Theorem 2.5 [10] *If $\gamma(G) \leq 3$, then $\gamma(G) = \gamma_{c\epsilon}(G)$.*

Theorem 2.6 [10] *If T is a tree with $\gamma(T) \geq 3$, then $\gamma(T) \leq \gamma_{c\epsilon}(T) \leq 2\gamma(T) - 3$. Further, these bounds are sharp.*

The authors of [10] noted that the upper bound of Theorem 2.6 does not hold for the very cost effective domination number of trees. For a counterexample, consider the tree $T = K_{1,t} \circ \overline{K}_t$ for which $\gamma(T) = \gamma_{c\epsilon}(T) = t + 1$ and $\gamma_{vc\epsilon}(T) = 2t > 2t - 1 = 2(t + 1) - 3 = 2\gamma(T) - 3$. This lead them to ask the following questions:

1. Is there a bound on $\gamma_{vc\epsilon}(T)$ in terms of $\gamma(T)$ for trees T?

2. Is there a bound on $\gamma_{c\epsilon}(G)$ in terms of $\gamma(G)$ for graphs G?

In this thesis, we address these and other questions motivated by this research. We answer the first question by giving a bound on $\gamma_{vc\epsilon}(T)$ in terms of $\gamma(T)$ for trees T and characterizing those trees which attain the bound. As for the second question, we answer it in the negative, showing that no such bound exists, even when restricted to bipartite graphs.

3 PRELIMINARY RESULTS

We first give some additional terminology. An *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted $epn(v, S)$. The neighbor of a leaf vertex is a *support vertex*. The *double star* $S_{r,s}$ is the tree with exactly two adjacent non-leaf vertices, one of which is adjacent to $r \geq 1$ leaves and the other to $s \geq 1$ leaves.

We note that the same conditions on a graph G given in Theorem 2.4 and Theorem 2.5 to ensure that $\gamma(G) = \gamma_{c\epsilon}(G)$ do not guarantee that $\gamma(G) = \gamma_{vc\epsilon}(G)$. To see this, consider the graph H shown in Figure 3. Note that $\Delta(H) = 4$ and $\gamma(G) = 3$, but $\gamma_{vc\epsilon}(H) = 4$.

H

Figure 3: The graph H does not have a very cost effective γ-set

However, if we tighten the restrictions on $\Delta(G)$ and $\gamma(G)$ in Theorem 2.4 and Theorem 2.5, respectively, similar results are attainable for the very cost effective domination number. Note that the graph in Figure 3 shows that the results of Theorems 3.1 and 3.2 are the best possible.

Theorem 3.1 *If a graph G has no isolated vertices and maximum degree $\Delta(G) \leq 3$,*

Proof. Among all $\gamma(G)$-sets, select S to be one with the minimum number of edges in $G[S]$. If S is very cost effective, we are finished. Thus, assume there is a vertex $x \in S$ that is not very cost effective, that is, $|N(x) \cap S| \geq |N(x) \cap (V \setminus S)|$. Since G has no isolated vertices and x is not very cost effective, we have $|N(x) \cap S| \geq 1$. By the minimality of S, we have $epn(x, S) \neq \emptyset$, and so $|N(x) \cap (V \setminus S)| \geq 1$. Since $\Delta(G) \leq 3$ and $|N(x) \cap S| \geq |N(x) \cap (V \setminus S)| \geq 1$, it follows that $|N(x) \cap (V \setminus S)| = 1$. Let $N(x) \cap (V \setminus S) = \{x_1\}$. Note that $S \setminus \{x\}$ dominates $G - x_1$. Thus, $S' = (S \setminus \{x\}) \cup \{x_1\}$ is a $\gamma(G)$-set with fewer edges in $G[S']$ than in $G[S]$, contradicting our choice of S. We conclude that S is very cost effective. $\square$

Theorem 3.2 *If a graph G has no isolated vertices and $\gamma(G) \leq 2$, then $\gamma(G) = \gamma_{vce}(G)$.*

Proof. If $\gamma(G) = 1$, then since G has no isolated vertices, $\gamma(G) = \gamma_{vce}(G)$. Let $\gamma(G) = 2$, and let $S = \{x, y\}$ be a $\gamma(G)$-set. If S is very cost effective, then we are finished. Thus, we may assume that S is not very cost effective. Since $|S| = 2$ and G has no isolated vertices, $xy \in E(G)$. Further, since $xy \in E(G)$, the minimality of S implies that $epn(x, S) \neq \emptyset$ and $epn(y, S) \neq \emptyset$. Since S is not very cost effective, at least one of x or y has exactly one S-external private neighbor. Without loss of generality, assume that $epn(x, S) = \{x_1\}$. Note that y dominates every vertex in $V(G) \setminus \{x_1\}$, and thus $\{x_1, y\}$ is an independent dominating set. By Observation 2.2, $\{x_1, y\}$ is a very cost effective dominating set of G. Therefore, $2 = \gamma(G) = \gamma_{vce}(G)$, as desired. $\square$

We conclude this section by noting that it was shown in [10] that if G is cubic, then

$\gamma(G) = \gamma_{vc\epsilon}(G)$. Theorem 3.1 is not a surprising result with this in mind. However, as previously mentioned, it is the best possible bound on $\Delta(G)$ to guarantee that $\gamma(G) = \gamma_{vc\epsilon}(G)$.

4 MAIN RESULTS

4.1 Trees

In this section, we address the first question posed in [10], namely: Is there a bound on $\gamma_{vc\epsilon}(T)$ in terms of $\gamma(T)$ for trees T? We show that $2\gamma(T)-2$ is in fact an upper bound on $\gamma_{vc\epsilon}(T)$, and we also show that every value of $\gamma_{vc\epsilon}(T)$ between $\gamma(T)$ and $2\gamma(T)-2$ is realizable.

In [10], the authors provided a useful algorithm for building a cost effective dominating set from a $\gamma(T)$-set of a tree T. We note that we use a slightly modified version of this algorithm to prove Theorem 4.1, and so, our proof is very similar to the one used to prove Theorem 2.6 in [10].

Theorem 4.1 *If T is a tree with $\gamma(T) \geq 3$, then $\gamma(T) \leq \gamma_{vc\epsilon}(T) \leq 2\gamma(T)-2$. Further, these bounds are sharp.*

Proof. Corollary 2.3 yields the lower bound. For the upper bound, let S be a $\gamma(T)$-set. If S is very cost effective, then we are finished. Thus, we may assume that S is not very cost effective. Let $U = \{u_1, u_2, ..., u_k\}$ be the vertices in S that are not very cost effective with respect to S. Let $s_i = |N(u_i) \cap S|$ and $o_i = |N(u_i) \cap (V \setminus S)|$, for $1 \leq i \leq k$. Thus for each $u_i \in U$, $o_i \leq s_i$. Let $U' \subseteq V \setminus S$ be the vertices in $V \setminus S$ whose only neighbors in S are in U. Note that since each u_i is not very cost effective, u_i has a neighbor in S, that is, $s_i \geq 1$. Hence, the minimality of S implies that u_i has at least one S-external private neighbor in U'. Thus, $|U'| \geq \Sigma_{i=1}^{k}|\text{epn}(u_i, S)| \geq k$.

We next prove that $\Sigma_{i=1}^{k} s_i \leq \gamma(T)+k-2$. To see this, we establish the bound on the degree sum in $T[S]$ by considering the possible edges of $T[S]$ incident to a vertex in U. If one endvertex of an edge in $T[S]$ is in U and the other is in $S \setminus U$, then we say the edge is a Type-1 edge. If both endvertices of an edge in $T[S]$ are in U, then we say the edge is of Type-2. Thus, each Type-1 edge adds 1 to the degree sum in $T[S]$, and each Type-2 edge adds 2. Let t_i be the number of Type-i edges.

Note that if a pair of vertices in U are connected by a path in $T[U]$, then they have no common neighbor in $S \setminus U$, for otherwise a cycle is formed. Let $T[U]$ have c components. Since T is a tree, $t_2 = k - c$. Moreover, no two vertices in the same component of $T[U]$ have a common neighbor in $S \setminus U$. Also, there are at most $c - 1$ vertices in $S \setminus U$ adjacent to more than one vertex in U; for otherwise, a cycle is formed. On the other hand, by the Pigeonhole Principle, there are at least $t_1 - |S \setminus U|$ vertices in $S \setminus U$ adjacent to more than one vertex in U. Thus, $t_1 - |S \setminus U| \leq c - 1$.

Hence, $\Sigma_{i=1}^{k} s_i = t_1 + 2t_2 \leq 2(k - c) + |S \setminus U| + c - 1 = 2k - 2c + \gamma(T) - k + c - 1 = \gamma(T) + k - c - 1 \leq \gamma(T) + k - 2$. Since $s_i \geq o_i$ for each $1 \leq i \leq k$, we have $\Sigma_{i=1}^{k} o_i \leq \Sigma_{i=1}^{k} s_i \leq \gamma(T) + k - 2$. Hence, $|U'| \leq \gamma(T) + k - 2$.

Next we give an algorithm to recursively build a very cost effective dominating set S_k from a $\gamma(T)$-set S. As before, let $U = \{u_1, u_2, ..., u_k\}$ be the subset of vertices in S that are not very cost effective, and let U' be the set of vertices in $V \setminus S$ whose only neighbors in S are in U.

begin

let $S_0 = S$.

for $i = 1$ **to** k **do**

    if u_i is very cost effective in S_{i-1}

        then let $S_i = S_{i-1}$

else if $epn(u_i, S_{i-1}) = \emptyset$

then let $S_i = S_{i-1} \setminus \{u_i\}$

else let $S_i = (S_{i-1} \setminus \{u_i\}) \cup epn(u_i, S_{i-1})$

fi

fi

end

We next prove that the algorithm produces a very cost effective dominating set with cardinality at most $2\gamma(T) - 2$.

By definition the set $S_0 = S$ is a dominating set and the vertices of $S \setminus \{u_1, u_2, ..., u_k\}$ are very cost effective in S. We define the loop invariant: for $1 \leq i \leq k$, the set S_i is a dominating set and all of the vertices in $S_i \setminus \{u_{i+1}, ..., u_k\}$ are very cost effective in S_i.

To see that S_i is a dominating set, we note that S_{i-1} is a dominating set, so if u_i is very cost effective and $S_i = S_{i-1}$, clearly S_i is a dominating set. If u_i is not very cost effective in S_{i-1}, then u_i has at least one neighbor in S_{i-1}, implying that u_i is dominated by S_i. Moreover, the external private neighbors of u_i with respect to S_{i-1} are added to form S_i, so S_i is a dominating set.

To see that the set $S_i \setminus \{u_{i+1}, ..., u_k\}$ is very cost effective, note if u_i is not very cost effective in S_{i-1}, then $S_i = (S_{i-1} \setminus \{u_i\}) \cup$ epn(u_i, S_{i-1}). Let $X =$ epn(u_i, S_{i-1}). Since T is a tree and each vertex in X is adjacent to u_i, X is an independent set. Moreover, since each vertex $x \in X$ is a private neighbor of u_i, x has no neighbors in $S_{i-1} \setminus \{u_i\}$. In other words, X is independent in $T[S_i]$, and so the vertices of X are very cost effective with respect to S_i. Hence, the vertices that are not very cost effective in S_i are at most the ones that are not very cost effective in $S_{i-1} \setminus \{u_i\}$. On iteration k, the algorithm terminates with the very cost effective dominating set S_k.

It remains to be shown that $|S_k| \leq 2\gamma(T) - 2$. To do this, we count the maximum possible vertices being added to form the set S_k. Since U' consists of the vertices whose only neighbors in S are in U, we have that $\mathrm{epn}(u_i, S) \subseteq U'$ for $1 \leq i \leq k$.

Consider the construction of S_k. At iteration i, if u_i is very cost effective in S_{i-1}, we let $S_i = S_{i-1}$. Since $u_i \in U$, it is not very cost effective in S, so we have that $|\mathrm{epn}(u_i, S)| \geq 1$. Hence, for our counting purposes, letting $S_i = S_{i-1}$ is essentially the same as removing u_i and replacing it with a vertex from $\mathrm{epn}(u_i, S) \subseteq U'$.

If u_i is not very cost effective in S_{i-1}, then we remove u_i and add the set $\mathrm{epn}(u_i, S_{i-1})$ to form S_i. To show that at most $|U'|$ vertices are added to S to form S_k, it suffices to show that $\mathrm{epn}(u_i, S_{i-1}) \subseteq U'$. To see this, suppose to the contrary that $x \in \mathrm{epn}(u_i, S_{i-1})$ and $x \notin U'$. By the definition of U', it follows that x has a neighbor in $S \setminus U$. Since $S \setminus U \subseteq S_{i-1}$, x has a neighbor in $S_{i-1} \setminus U$. But $u_i \in U$, contradicting that $x \in \mathrm{epn}(u_i, S_{i-1})$. Hence, $\mathrm{epn}(u_i, S_{i-1}) \subseteq U'$, and so we may conclude that every vertex added to form S_k is in the set U'.

It follows that to form S_k from our original set S, we add at most $|U'|$ vertices, while for the purposes of counting, we "remove" $|U| = k$ vertices. Since $|U'| \leq \gamma(T) + k - 2$, we have $|S_k| \leq |S| - |U| + |U'| \leq \gamma(T) - k + \gamma(T) + k - 2 = 2\gamma(T) - 2$, the desired upper bound.

We now show that the two bounds given are sharp. The corona $T \circ K_1$ of any tree T has $\gamma(T \circ K_1) = \gamma_{vce}(T \circ K_1) = \frac{n}{2}$. For the upper bound, let T be the corona $K_{1,t} \circ \overline{K}_t$ for $t \geq 2$. Then $\gamma(T) = t + 1$ and $\gamma_{vce}(T) = 2\gamma(T) - 2$. $\square$

Before we characterize the trees attaining the upper bound of Theorem 4.1, we prove a useful lemma. As a point of interest, we notice that Lemma 4.2 only guarantees the existence of one such $\gamma(T)$-set. After the characterization is complete, however, we see that this is indeed the only $\gamma(T)$-set.

Lemma 4.2 *If T is a tree with $\gamma(T) \geq 3$ and $\gamma_{vc\epsilon}(T) = 2\gamma(T) - 2$, then some $\gamma(T)$-set has exactly one vertex which is not very cost effective.*

Proof. Let T be a tree with $\gamma(T) \geq 3$ and $\gamma_{vc\epsilon}(T) = 2\gamma(T) - 2$. Let S be a $\gamma(T)$-set that minimizes the number of vertices which are not very cost effective. Let S_k be a very cost effective dominating set of T formed by the algorithm given in Theorem 4.1 from S and $U = \{u_1, u_2, ..., u_k\}$ be the vertices in S which are not very cost effective. Let U' be the vertices in $V \setminus S$ whose only neighbors in S are in U. Further, let $T[U]$ have c components. Let o_i and s_i be defined as before also. We will show that S has exactly one vertex which is not very cost effective.

Notice that $2\gamma(T) - 2 = \gamma_{vc\epsilon}(T) \leq |S_k| \leq |S| + |U'| - k \leq \gamma(T) + \gamma(T) + k - 2 - k = 2\gamma(T) - 2$, giving equality throughout. This implies $|U'| = \gamma(T) + k - 2$. Further, $\gamma(T) + k - 2 = |U'| \leq \sum_{i=1}^{k} o_i \leq \sum_{i=1}^{k} s_i \leq \gamma(T) + k - c - 1 \leq \gamma(T) + k - 2$, and we have equality throughout. This implies that $\sum_{i=1}^{k} o_i = \sum_{i=1}^{k} s_i$ and $c = 1$. Thus, $T[U]$ is a tree. Since $o_i \leq s_i$ for all $1 \leq i \leq k$ (as each u_i is not very cost effective by assumption) and $\sum_{i=1}^{k} o_i = \sum_{i=1}^{k} s_i$, we conclude that $o_i = s_i$ for all $1 \leq i \leq k$.

To see that $k = 1$, assume not. Let $k \geq 2$. Then $T[U]$ is a nontrivial tree. By definition, every vertex in U' has at least one neighbor in U. Since $T[U]$ is a nontrivial tree, if any vertex in U' has more than one neighbor in U, a cycle would be formed. Thus, we conclude that every vertex in U' has exactly one neighbor in U.

Recall that $o_i = |N(u_i) \cap (V \setminus S)|$ and $\sum_{i=1}^{k} o_i = \gamma(T) + k - 2 = |U'|$. Since every vertex in U' has exactly one neighbor in U and $\sum_{i=1}^{k} o_i = |U'|$, we conclude that, for every $u_i \in U$, $(N(u_i) \cap (V \setminus S)) \subseteq U'$. So, for every vertex in $u_i \in U$, $N(u_i) \subseteq S \cup U'$.

We now show that each u_i has at least two neighbors in U'. To see this, assume not. Let $u_i \in U$ have exactly one neighbor in U', say y. Then the set $S' = (S \setminus \{u_i\}) \cup \{y\}$ is a $\gamma(T)$-set with no more than $k-1$ vertices which are not very cost effective, contradicting our choice of S. We conclude that u_i has at least two neighbors in U'.

For the sake of simplicity for the remainder of the argument, we will consider K_2 as having one leaf and one support vertex.

Since $T[U]$ is a nontrivial tree, it contains at least one leaf and one support vertex. Let $U_\ell \subset U$ be the leaves in $T[U]$ and $\ell = |U_\ell|$. Note that $\ell \geq 1$. Let $U'_T \subset U'$ be the vertices in U' whose neighbor in U is in $U \setminus U_\ell$. Consider the set $S' = (S \setminus U) \cup U_\ell \cup U'_T$. Clearly, S' is a dominating set. To see that S' is a very cost effective set, note that, by hypothesis, every vertex in $S \setminus U$ is very cost effective. Further, every vertex in U_ℓ has one neighbor in $U \setminus U_\ell$. However, this neighbor is not in S'. Since $o_i = s_i$ for every vertex in U, it follows that every vertex in U_ℓ has more neighbors in $V \setminus S'$ than in S', so each vertex in U_ℓ is very cost effective in S'. Further, since U' is an independent set (for otherwise a cycle is formed), every vertex in U'_T has no neighbors in S' and precisely one neighbor in $V \setminus S'$. Hence, each vertex in U'_T is very cost effective in S'. Thus, S' is a very cost effective dominating set. Further, since every vertex in U has at least two neighbors in U' and every vertex in U' has exactly one neighbor in U, it follows that $|U'_T| \leq |U'| - 2|U_\ell| = \gamma(T) + k - 2 - 2\ell$. Hence, $\gamma_{vce}(T) \leq |S'| = |S| - |U| + |U_\ell| + |U'_T| \leq \gamma(T) - k + \ell + \gamma(T) + k - 2 - 2\ell = 2\gamma(T) - 2 - \ell \leq 2\gamma(T) - 3$.

However, by assumption, $\gamma_{vc\epsilon}(T) = 2\gamma(T) - 2$, a contradiction. Hence, $k = 1$. $\square$

We now characterize the trees attaining the upper bound given in Theorem 4.1. Let $\mathcal{F}$ be a family of trees obtained from the star $K_{1,t}$ with center x and leaves $x_1, x_2, ..., x_t, t \geq 2$, as follows. Append precisely t new vertices to x, and append at least t new vertices to $x_i, 1 \leq i \leq t$. Notice that both the corona $K_{1,t} \circ \overline{K}_t$ given to show the bound is sharp and the graph H in Figure 3 are in $\mathcal{F}$.

Theorem 4.3 *Let T be a tree with $\gamma(T) \geq 3$. Then $\gamma_{vc\epsilon}(T) = 2\gamma(T) - 2$ if and only if $T \in \mathcal{F}$.*

Proof. Let $T \in \mathcal{F}$. Clearly, $\gamma(T) = t + 1$ and $\gamma_{vc\epsilon}(T) = 2\gamma(T) - 2$.

For the converse, let T be a tree with $\gamma(T) \geq 3$ and $\gamma_{vc\epsilon}(T) = 2\gamma(T) - 2$. Let S be the $\gamma(T)$-set guaranteed by Lemma 4.2. Let S_k be a very cost effective dominating set of T formed by the algorithm given above from S. Thus, $2\gamma(T) - 2 = \gamma_{vc\epsilon}(T) \leq |S_k| \leq 2\gamma(T) - 2$. We therefore have equality throughout, and it follows that S_k is a $\gamma_{vc\epsilon}(T)$-set.

Let $x \in S$ be the only vertex which is not very cost effective. Also, $S_k = S_1 = (S \setminus \{x\}) \cup \text{epn}(x, S)$. Thus, $2\gamma(T) - 2 = \gamma_{vc\epsilon}(T) \leq |S_k| = |S| - 1 + |\text{epn}(x, S)| = \gamma(T) - 1 + |\text{epn}(x, S)| \leq 2\gamma(T) - 2$, and again we have equality throughout. It follows that $|\text{epn}(x, S)| = \gamma(T) - 1$. Further, since x is not very cost effective in S, it follows that x must have at least $\gamma(T) - 1$ neighbors in S. Since x is also in S, clearly x has exactly $\gamma(T) - 1$ neighbors in S. Since T is a tree, we know that the subgraph induced by S is the star $K_{1,\gamma(T)-1}$, wherein x is the central vertex, and every vertex in $V(T) \setminus S$ is a leaf in T. Further, x is a support vertex to precisely $\gamma(T) - 1$ leaves which are in $V \setminus S$. To show that $T \in \mathcal{F}$, it suffices to show that every vertex in $S \setminus \{x\}$ is adjacent to at least $\gamma(T) - 1$ vertices in $V \setminus S$.

Suppose, for the sake of contradiction, that there exists a vertex $y \in (S \setminus \{x\})$ such that y is a support vertex to at most $\gamma(T) - 2$ leaves. Note then that $S_1 = (S \setminus \{y\}) \cup$ epn(y, S) is a very cost effective dominating set. Thus, $\gamma_{vc\epsilon}(T) \leq |S_1| \leq |S| - 1 + \gamma(T) - 2 = 2\gamma(T) - 3 < 2\gamma(T) - 2 = \gamma_{vc\epsilon}(T)$, a contradiction. We therefore deduce that every vertex in $S \setminus \{x\}$ is a support vertex to at least $\gamma(T) - 1$ leaves. We conclude that $T \in \mathcal{F}$. $\square$

We now show that all values between the lower and upper bounds of Theorem 4.1 are realizable. Let $K^v_{1,t}$ be the star with center v and leaves $v_1, v_2, ..., v_t$.

Theorem 4.4 *Given positive integers a and b such that $3 \leq a \leq b \leq 2a - 2$, there exsists a tree T having $\gamma(T) = a$ and $\gamma_{vc\epsilon}(T) = b$.*

Proof. We first consider the case of $a = 3$. If $a = 3$, then $b = 3$ or $b = 4$. If $b = 3$, let T be $P_3 \circ K_1$. Then clearly $a = \gamma(T) = \gamma_{vce}(T) = b = 3$. If $b = 4$, let T be $P_3 \circ \overline{K}_2$. Then $\gamma(T) = 3 = a$ and $\gamma_{vc\epsilon}(T) = 4 = b$. We now turn our attention to the case of $a \geq 4$.

To construct a tree T having $\gamma(T) = a$ and $\gamma_{vc\epsilon}(T) = b$, we begin with the forest $(K^x_{1,a-2} \circ \overline{K}_{a-1}) \cup K^y_{1,b-a+1}$ and add the edge xy. Then T has a support vertices.

We first show that $\gamma(T) = a$. Since T has a support vertices, and each leaf or its support is in any dominating set, we have $\gamma(T) \geq a$. Also note that the set of all support vertices is a dominating set, so $\gamma(T) \leq a$. Then, $\gamma(T) = a$.

We now show $\gamma_{vc\epsilon}(T) = b$. Let $S = \{x, x_1, x_2 ..., x_{a-2}, y_1, y_2, ..., y_{b-a+1}\}$. Clearly, S is a dominating set of T. To see that S is very cost effective, note that y_i is independent in $T[S]$, and thus is very cost effective in S. Further, $|N(x_i) \cap S| = 1$ and $|N(x_i) \cap (V \setminus S)| = a - 1 \geq 3$, so each x_i is very cost effective in S. Finally, notice $|N(x) \cap S| = a - 2$ and $|N(x) \cap (V \setminus S)| = a$, so x is very cost effective in S. Thus, S is a very cost effective dominating set, and $\gamma_{vc\epsilon}(T) \leq |S| = 1 + a - 2 + b - a + 1 = b$.

Now, let S' be a $\gamma_{vc\epsilon}(T)$-set. To dominate T, each leaf or its support vertex must be in S'. We show that at least one of the support vertices is not in S'. Assume to the contrary that S' contains all the support vertices of T, that is, $\{x, x_1, x_2, ..., x_{a-2}, y\} \subseteq S'$. Then $|N(x) \cap (V \setminus S')| = a - 1 = |N(x) \cap S'|$, and so S' is not very cost effective. Hence, at least one support vertex is not in S', call it w. Thus, since S' is a very cost effective dominating set that does not contain a support vertex w, S' contains the leaves adjacent to w. Let l_w be the number of leaves adjacent to w. Recall that T has a support vertices, so $a - 1 + l_w \leq |S'| = \gamma_{vc\epsilon}(T) \leq b$. Therefore, $l_w \leq b - a + 1$. Since $b \leq 2a - 2$, we have $b - a + 1 \leq 2a - 2 - a + 1 = a - 1$. Now each support vertex in T is adjacent to either $a - 1$ or $b - a + 1$ leaves. Since $b - a + 1 \leq a - 1$, we conclude each support vertex is adjacent to at least $b - a + 1$ leaves, specifically, $l_w \geq b - a + 1$. Thus, $l_w = b - a + 1$. Also, $b \geq \gamma_{vc\epsilon}(T) = |S'| \geq a - 1 + l_w = a - 1 + b - a + 1 = b$, so $\gamma_{vc\epsilon}(T) = b$ as desired. $\square$

Once a bound is given for trees, a natural next question is whether or not the same bound works for graphs in general? If not, since trees are bipartite, does it work for bipartite graphs? We address this in the next section.

4.2 General Graphs

In [10], it was left as an open question as to whether or not the bound given in Theorem 2.6 held for graphs in general. In the following result, we show that, even when restricted to bipartite graphs, there is no upper bound on $\gamma_{c\epsilon}(G)$ in terms of $\gamma(G)$, allowing us to conclude that the same is true of $\gamma_{vc\epsilon}(G)$.

Theorem 4.5 *For every integer $k \geq 2$, there exists a connected, bipartite graph G with $\gamma_{c\epsilon}(G) > k\gamma(G)$.*

Proof. Consider the following construction of G. Begin with the complete bipartite graph on $4k + 1$ vertices, with partite sets V_1, V_2 such that $|V_1| = 2k, |V_2| = 2k + 1$. To every vertex in V_1, append at least $4k^2 - 3k + 1$ leaves, and to every vertex in V_2, append exactly $2k - 1$ leaves. Let the resulting graph be G.

Note that G is bipartite and every vertex in G is either a support vertex or a leaf vertex. Further, every support vertex has at least three leaves. Then, $S = V_1 \cup V_2$ (all of the support vertices in G) is $\gamma(G)$-set and $|S| = 4k + 1$. Therefore, $\gamma(G) = 4k + 1$.

We now show that $\gamma_{c\epsilon}(G) > k\gamma(G)$. Notice that in S, every vertex in V_1 is cost effective, while no vertices in V_2 are cost effective. Since every support vertex or all of its leaf neighbors are in every dominating set, to form a cost effective dominating set from S, we must either remove one vertex from V_1 and add its external private neighbors, or we must remove all the vertices from V_2 and add their external private neighbors. If we remove one vertex from V_1 and add its external private neighbors, we have a cost effective dominating set S' such that $|S'| \geq 4k + 1 - 1 + 4k^2 - 3k + 1 = 4k^2 + k + 1 > 4k^2 + k = k\gamma(G)$. Also, if we remove every vertex from V_2 and add their external private neighbors, we have a cost effective dominating set S^* such that $|S^*| = 4k+1-(2k+1)+(2k+1)(2k-1) = 4k^2+k+k-1 \geq 4k^2+k+1 > k\gamma(G)$. We conclude that $\gamma_{ce}(G) > k\gamma(G)$, as desired. $\square$

Noting that $\gamma_{vc\epsilon}(G) \geq \gamma_{c\epsilon}(G)$ for any graph G, we have the following result.

Corollary 4.6 *For every integer $k \geq 2$, there exists a connected, bipartite graph G with $\gamma_{vc\epsilon}(G) > k\gamma(G)$.*

5 CONCLUSION

We have found an upper bound on the very cost effective domination number of trees and characterized the trees which obtain this bound. It was also shown that all possible values between the domination number and the upper bound are attainable. We also showed that no such bound exists for graphs in general, even when restricted to bipartite graphs.

As a point of interest, we note a slight variation on the notion of a cost effective set. A set $S \subseteq V$ is *set-wise cost effective* if there are more edges between vertices in S and $V \setminus S$ than between vertices in S. A set $S \subseteq V$ is a *set-wise cost effective dominating set* if it is both dominating and set-wise cost effective. Let the *set-wise domination number* $\gamma_{sc\epsilon}(G)$ of a graph G be the minimum cardinality of all set-wise cost effective dominating sets.

We finish with some open problems suggested by this work:

1. Characterize the trees for which $\gamma(T) = \gamma_{vc\epsilon}(T)$.

2. Characterize the trees for which $\gamma_{c\epsilon}(T) = \gamma_{vc\epsilon}(T)$.

3. Characterize the trees for which $\gamma_{vc\epsilon}(T) = i(T)$.

4. What, if anything, can be said about $\gamma_{c\epsilon}(G \square H)$ and $\gamma_{vc\epsilon}(G \square H)$ in terms of $\gamma(G)$ and $\gamma(H)$? In terms of $\gamma_{c\epsilon}(G), \gamma_{c\epsilon}(H)$ and $\gamma_{vc\epsilon}(G), \gamma_{vc\epsilon}(H)$, respectively?

5. It can be easily shown that $\gamma(T) = \gamma_{sc\epsilon}(T)$ for trees T. What, if anything, can be said about $\gamma_{sc\epsilon}(G)$ in terms of $\gamma(G)$? In other words, prove or disprove: For all graphs G, there exists a constant k such that $k\gamma(T) \geq \gamma_{sc\epsilon}(T)$.

6. Investigate bounds on the parameters $\Gamma_{c\epsilon}(G)$ and $\Gamma_{vc\epsilon}(G)$.

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