5-2014

Permutation Groups and Puzzle Tile Configurations of Instant Insanity II

Amanda N. Justus
East Tennessee State University

Permutation Groups and Puzzle Tile Configurations of Instant Insanity II

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

---

by

Amanda Justus

May 2014

Robert A. Beeler, Ph.D., Chair

Robert Gardner, Ph.D.

Debra Knisley, Ph.D.

Rick Norwood, Ph.D.

Keywords: algebra, group theory, combinatorics

ABSTRACT

Permutation Groups and Puzzle Tile Configurations of Instant Insanity II

by

Amanda Justus

The manufacturer claims that there is only one solution to the puzzle Instant Insanity II. However, a recent paper shows that there are two solutions. Our goal is to find ways in which we only have one solution. We examine the permutation groups of the puzzle and use modern algebra to attempt to fix the puzzle. First, we find the permutation group for the case when there is only one empty slot at the top. We then examine the scenario when we add an extra column or an extra row to make the game a 4×5 puzzle or a 5×4 puzzle, respectively. We consider the possibilities when we delete a color to make the game a 3×3 puzzle and when we add a color, making the game a 5×5 puzzle. Finally, we determine if solution two is a permutation of solution one.

DEDICATION

I would like to dedicate my thesis in memory of the most wonderful mother anyone could ever have, Linda Johnson.

ACKNOWLEDGMENTS

I would first like to thank my thesis advisor, Dr. Robert Beeler, for all of his support, guidance, encouragement, and patience with me. I would like to thank the members of my committee, Dr. Robert Gardner, Dr. Debra Knisley, and Dr. Rick Norwood, for taking the time to help me and give me advice. I would also like to thank my father Robert "Cotton" Johnson for his continued love and support and for always believing in me.

TABLE OF CONTENTS

ABSTRACT 2

DEDICATION 4

ACKNOWLEDGMENTS 5

LIST OF FIGURES 7

1 INTRODUCTION 8

2 LITERATURE REVIEW 14

2.1 The Rubik's Cube 14

2.2 The Fifteen Puzzle 15

2.3 Instant Insanity 16

2.4 Instant Insanity II 17

3 PUZZLE GROUP 21

4 CONFIGURATION 29

4.1 4×5 Puzzle 29

4.2 Add a Row to Create a 5×4 Puzzle 37

4.3 Remove a Color to Create a 3×3 Puzzle 39

4.4 Add a Color to Create a 5×5 Puzzle 41

5 CONCLUDING REMARKS 68

BIBLIOGRAPHY 69

VITA 71

LIST OF FIGURES

1 Instant Insanity II 9

2 The two solutions of Instant Insanity II using our notation 10

3 An example of a vertical shift on column a 10

4 An example of an inverse vertical shift on column a 11

5 The basic $3 \times 3 \times 3$ Rubik's Cube 15

6 The fifteen puzzle 16

7 The original Instant Insanity 17

8 Tiles of Instant Insanity II using numbers 18

9 Tiles of Instant Insanity II using numbers 21

10 Tiles of Instant Insanity II after a series of vertical shifts 22

11 Tree diagram to obtain Set 1 on the 4×5 puzzle 30

12 Tree diagram to obtain Set 1 on the 5×4 puzzle 37

13 The new solution one after adding a fifth row with our missing tiles . 39

14 The new solution two after adding a fifth row with our missing tiles . 39

15 Solution to the 3×3 puzzle 41

16 Tree diagram to obtain Set 1 on the 5×5 puzzle 42

1 INTRODUCTION

The purpose of this thesis is to look at the permutations and configurations on a puzzle called Instant Insanity II. Instant Insanity II is a sliding combination puzzle on sixteen tiles, each of which consists of one of the following colors: blue (B), green (G), red (R), white (W), and yellow (Y). On each tile there is a large colored block with two smaller colored blocks on each side. We denote the colors on the smaller tiles using lowercase letters. The right window matches the central window. For this reason, we omit this window in our notation. For example, the tile with a green left window, white center window, and white right window is simply denoted gW. Instant Insanity II consists of the tiles

gB,	rB,	wB,	yB,
bG,	rG,	yG,	
bR,	gR,	wR,	
bW,	gW,	yW,	
bY,	rY,	wY.	

Notice that there are four tiles with the color Blue and each of the remaining four colors is only assigned three tiles. For instance, we have the tiles bR, gR, and wR, but we do not have the tile yR. We continue this process to find the remaining tiles that would yield four tiles per color. We claim that there are four "missing tiles," namely, gY, rW, yR, and wG. The sixteen tiles are broken up into four rows and four columns. There is a fifth row, which we call the "empty row," where each of the four columns can slide a tile into, as shown in Figure 1. Notice that the empty row only has two slots instead of four. The empty row and the bottom row can be rotated both clockwise and counter-clockwise, yet no other rows are able to rotate. The goal is to arrange the tiles in such a way that:

(i) no color on the large block arrears twice in any row or column and

(ii) on each row the preceding small tile shares the same color as the following large tile.

The manufacturers claim that there is only one solution to this puzzle up to permutations of the rows. However, a recent paper has shown that there are in fact two solutions to Instant Insanity II as shown in Figure 2 [14, 16]. Notice that these two solutions are both Latin Squares. The Latin square is derived from the Latin rectangle, which is an $m \times n$ matrix in which each row contains the numbers $1, 2, ..., n$. The numbers are ordered in such a way that no column will contain the same number twice [6]. The Latin square shares the same properties, that each row and each column will contain the numbers $1, 2, ..., n$, but each row and column contains each of the elements only once. A 2×2 matrix has $2! = 2$ possible orderings, and 3×3 matrix has $3!=6$ possible orderings. It follows that an $n \times n$ matrix will have $n!$ possible orderings of the n elements [15].

Figure 1: Instant Insanity II

With Instant Insanity II, we are interested in looking at the algebraic structure of the puzzle. We are also interested in trying to "fix" the puzzle. That is, we look at different ways in which we might get only one solution instead of two.

gW	wY	yB	bG
bY	yG	gR	rB
rG	gB	bW	wR
wB	bR	rY	yW

wY	yG	gB	bW
gW	wB	bR	rG
bG	gR	rY	yB
rB	bY	yW	wR

Figure 2: The two solutions of Instant Insanity II using our notation

Let the first column be column a, the second column b, third column c, and the last column be column d. Here, we can move the tiles through what we define as *vertical shifts*. One vertical shift on column a would be denoted as V_a. Similarly, we denote the inverse of a vertical shift on column a as V_a^{-1}. Figures 3 and 4 illustrate a vertical shift, V_a, and an inverse vertical shift, V_a^{-1}, respectively. Similarly, the inverse vertical shift on column a is illustrated as in Figure 4.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$\rightarrow$

1			
5	2	3	4
9	6	7	8
13	10	11	12
	14	15	16

$\rightarrow$

1			
5	2	3	4
9	6	7	8
13	10	11	12
16		14	15

$\rightarrow$

	1		
5	2	3	4
9	6	7	8
13	10	11	12
16		14	15

$\rightarrow$

5	1	3	4
9	2	7	8
13	6	11	12
16	10	14	15

$\rightarrow$

5	1	3	4
9	2	7	8
13	6	11	12
10	14	15	16

Figure 3: An example of a vertical shift on column a

To facilitate our main results it is useful to have some basic definitions from group theory. Our terminology will be consistent with that of [3, 4].

Definition 1.1 *A* group G *is a nonempty set* S *together with a binary operation* $*$

such that:

(i) *G is closed under $*$. That is, $\forall a, b \in G, a * b \in G$.*

(ii) *$*$ is associative. That is, $\forall a, b, c \in G, a * (b * c) = (a * b) * c$.*

(iii) *There exists an identity element, $e \in G$ such that $\forall a \in G$, $a * e = e * a = a$.*

(iv) *For every $a \in G$, we have an inverse element, $a^{-1} \in G$ such that $a^{-1} * a = a * a^{-1} = e$.*

5	1	3	4
9	2	7	8
13	6	11	12
10	14	15	16

$\rightarrow$

5	1	3	4
9	2	7	8
13	6	11	12
16	10	14	15

$\rightarrow$

	1		
5	2	3	4
9	6	7	8
13	10	11	12
16		14	15

$\rightarrow$

1			
5	2	3	4
9	6	7	8
13	10	11	12
16		14	15

$\rightarrow$

1			
5	2	3	4
9	6	7	8
13	10	11	12
	14	15	16

$\rightarrow$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Figure 4: An example of an inverse vertical shift on column a

The *order* of a group is the number of elements in that group. We denote the order of G as $o(G)$. A subset H of a group G is a *subgroup*, denoted $H \subseteq G$, of G if H is itself a group with respect to the operation on G. The following is a famous theorem relating the order of a group to that of its subgroups.

Theorem 1.2 (Lagrange's Theorem *[12]) If H is a subgroup of a finite group G, then the order of H divides the order of G.*

When a group G acts on a set X, it permutes the elements in that set in a particular order. The specific path that action takes is referred to as the *orbit*. It is denoted by $\sigma(x) = \{gx \in X : g \in G\}$. If we have an element $a \in G$ whose orbits contain all of the group G, then we say that a is a *generator* of G. If G is generated by a set $\{a_1, ..., a_n\}$, then every element in G can be written as a product of the a_i's. This situation is denoted $< a_1, ..., a_n >$.

Definition 1.3 *A* permutation *of a nonempty set S is a one-to-one mapping from S onto S. A permutation is a* cycle *if it has at most one orbit containing more than one element. A cycle of length n is typically called an n-cycle. A* transposition *is a cycle of length two.*

The following theorem results from the definition of permutations, cycles, and transpositions.

Theorem 1.4 *[4] Every permutation can be written as a product of disjoint cycles. Further, every permutation can be written as a product of transpositions. Note that a permutation is defined to be even provided that it can be expressed as an even number of transpositions.*

We will be representing group elements as products of disjoint cycles. Let A be the finite set $\{1, ..., n\}$. The group of all permutations of A is the *symmetric group* on n letters. This is denoted by S_n. The order of the symmetric group is $n!$. The subgroup of S_n consisting of the even permutations on n letters is the *alternating group* on n letters. This is denoted by A_n. The order of the alternating group is $\frac{n!}{2}$.

The following theorem is a result from the definition of the symmetric and alternating groups.

Theorem 1.5 *[7] The alternating group is generated by 3-cycles. In other words,* $A_n = \{(a, b, c) : a, b, c, \in [n]\}$.

The next theorem will be of great use to us when we observe the permutation group of Instant Insanity II.

Theorem 1.6 (Cayley's Theorem *[4]) Every group is isomorphic to a subgroup of* S_n*. An* isomorphism *is a one-to-one, onto mapping that preserves the group operation.*

2 LITERATURE REVIEW

2.1 The Rubik's Cube

The Rubik's cube is a $3 \times 3 \times 3$ cube whose components are arranged into one large cube. There are different generating sets that can be used to generate the solution to the cube by permuting the cube [11]. Each small colored tile is called a "cubie." There are two types of cubies, namely the edge cubies and corner cubies. The center tile is called the face and these never leave their location. There are six center faces, twelve edge cubies, and eight corner cubies. There are two different approaches in solving the Rubik's cube. The first approach is the algebraic approach, where long sequences of operators are derived from smaller ones. This method is risky but it is efficient. The second approach is the geometric, where there is a reason for each turn of the cube. This method is inefficient, however it is reliable [8].

Since there are eight corners, each with three orientations and twelve edges, each with two orientations, one would think there are $8! * 3^8 * 12! * 2^{12} \approx 5.2 \times 10^{20}$ different states to the cube. However, this would be overcounting. Hofstadter shows that any seven corner cubies can be arbitrary, implying that one corner cubie has three orientations. Similarly, he shows that any eleven edge cubies can be arbitrary, which leaves two edge cubies with two orientations each that are fixed [8]. So, we need to divide 5.2×10^{20} by a factor of $3 \times 2 \times 2 = 12$, resulting in 4.3×10^{19} configurations of the cube with only one solution [8, 13].

Morwen Thistlewaite derived a general algorithm for solving the Rubik's cube from any scrambled state within fifty-two turns. This number has since been reduced, with the use of computers, to fifty turns [9]. Each turn of a face on the cube will generate a 4-cycle, that is a cycle of length four. Like any permutation, any sequence of moves can by represented by a product of cycles. The minimum number of moves for solving the Rubik's cube from any scrambled state is referred to as *God's Number* [8]. *God's Algorithm* is the method for which we find this number [8]. Note that a move may not consist of one single turn, but a sequence of turns of the faces of the cube. God's Number for the cube has been computed to be twenty-three moves [8].

Figure 5: The basic $3 \times 3 \times 3$ Rubik's Cube

2.2 The Fifteen Puzzle

The fifteen puzzle is made of a shallow square shaped tray that is large enough to hold sixteen small tiles in four rows and four columns. There are only fifteen tiles and the sixteenth slot is left empty [1]. It was created by Sam Loyd and became popular in the nineteenth century. It is similar to the Rubik's Cube in that the goal for both is to unscramble tiles to get a solution [8]. The fifteen puzzle is similar to our puzzle since it looks at different permutations on a 4×4 matrix. It has been determined that some transpositions, such as $(14, 15)$, can not be obtained. That is, one cannot simply exchange the tiles 14 and 15 while leaving the rest of the puzzle fixed. Configurations of the puzzle that can be achieved are related to even permutations, while those that are not possible are related to odd permutations. Spitznagel shows that if three tiles are lined up next to each other, in a row or column, then it is possible to have a cyclic permutation of these three tiles while leaving the rest of the puzzle fixed [19]. He then continues to show that this is true for any three tiles. This leads to the conclusion that the permutation group on the fifteen puzzle is isomorphic to A_{15} and has no proper normal subgroups [1, 19].

Figure 6: The fifteen puzzle

2.3 Instant Insanity

Instant Insanity is a game that consists of four cubes whose sides are each assigned a different color: blue, green, red, or white. In order to achieve a solution, one must create a $1 \times 1 \times 4$ rectangular prism in which all four colors appear on each of the four faces of the prism, as shown in Figure 7 [5]. Brown estimated that there are

82,944 prisms that can be arranged. He continued to show that a solution can be more easily attained by looking at pairs of opposite faces rather than by looking at each individual face [2]. Schwartz also uses the paring principle, where he pairs up opposite faces of the cubes, to obtain a solution to this puzzle. However, he makes an improvement on Brown's original idea, and uses letters instead of numbers in his calculations. In doing this, he uses Brown's original paring principle idea with his own improvements to reduce the number of possible prisms down to eighty-one, with two solutions possible [18]. Grecos uses this method along with some graph theory concepts to obtain a graphical solution to the puzzle [2, 5, 10].

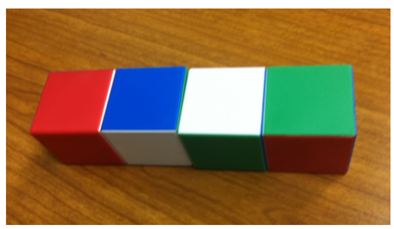

Figure 7: The original Instant Insanity

2.4 Instant Insanity II

As previously stated, Instant Insanity II is a sliding puzzle on sixteen tiles. Richmond and Young show in their paper that the puzzle group on Instant Insanity II is isomorphic to S_{16}. First, they show that any two tiles in the top row can be transposed and that any two vertically adjacent tiles can be transposed by sliding tiles into the empty slots at the top of the puzzle [14]. For clarity, we denote the two empty slots with an E as shown in Figure 8. They also showed that one can transpose two tiles in a column if they are not in the first two rows by doing a sequence of vertical shifts [14]. Note that their method is similar to the vertical shift and inverse vertical shift that we described above. They used the fact that any two tiles in the top row can be transposed together with the fact that any two tiles in a column can be transposed to show that any tile in the first row and be transposed with any other tile in the puzzle. Since there are both even and odd permutations, the puzzle group on Instant Insanity II is isomorphic to S_{16} [14]. A generalization for this states that for any $n \times k$ puzzle with two empty slots at the top, the puzzle group is isomorphic to S_{nk}.

	E	E	
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Figure 8: Tiles of Instant Insanity II using numbers

We can use Richmond and Young's method for transposing tiles to find the generators for Instant Insanity II. Recall that any two tiles in the first row can be transposed and that any two tiles in a column can also be transposed [14]. We use this fact, along with rotations of the puzzle, to derive the following generators for Instant Insanity II. We get our generator a by simply rotating the bottom row. The generator b is just a rotation of the entire puzzle. By completing one vertical shift, we get the generator c. Also, we get d by transposing tiles 1 and 5, which is a vertical transposition and e

by transposing tiles 1 and 2, which is a horizontal transposition.

$$a = (13, 14, 15, 16)$$

$$b = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16)$$

$$c = (1, 5, 9, 13, 12, 8, 4)$$

$$d = (1, 5)$$

$$e = (1, 2)$$

A recent paper by Richmond and Young and Jaap's website on puzzle games state that there are in fact two solutions to this mind boggling game [14, 16]. In their paper, Richmond and Young go through how they derive two unique solutions from the puzzle when the box claims there is only one solution. Richmond and Young use a directed graph to construct their cycles. Since there are five colors, there are five vertices labeled B, G, R, W, and Y [14]. Consider the tile gB. Since the small left window is green and the large window is blue, then there is an edge between G and B with an arrow showing that we are going from the color green to the color blue. They continue to do this for each of the remaining tiles in the puzzle. Then, they look at the vertex B and examine each of the 4-cycles that are forced from this process. Each row in a solution corresponds to a cycle in the directed graph. They find that there are 12 such cycles [14].

Notice that each of the four rows in the original solution is missing a distinct color, so they group their cycles according to the missing color. Using our notation, if the tile gB is in one cycle, then no other cycle can use gB to create a solution. Using this fact, Richmond and Young make an incompatibility graph and then take the complement of it to create a compatibility graph. The solution from the box lists four compatible cycles, one from each of the missing color groups. They then look at the possibilities from choosing the first cycle in the missing color group where the color red is absent. They continue this for the second and the third options that are in the same missing color group and find there are only two ways of doing this where the rows have a unique alignment. Thus, there are two possible solutions instead of one [14].

Recall the tile configuration of solution one. For simplicity, we can relabel this arrangement with the numbers one through sixteen. So, the tile gW is now referred to as 1, wY is 2, etcetera. Now, we look at where the tiles from solution one get sent to in solution two, as shown in Figure 9. So, 1 gets sent to 5, 5 gets sent to 14, 14 gets sent to 7 and so forth. If we continue this out, we get the permutation $(1, 5, 14, 7, 10, 3, 12, 16, 15, 11, 4, 9, 8, 13, 6, 2)$. Notice that this is a cycle of length sixteen. Since this is a cycle of even length, it is an odd permutation. Thus, solution two is an odd permutation on the tiles of solution one. If we can restrict the permutations to the alternating group, that is, only containing even cycles, then we would have a unique solution.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$\longrightarrow$

5	1	12	9
14	2	10	13
8	3	4	16
6	7	11	15

Figure 9: Tiles of Instant Insanity II using numbers

Definition 3.1 *Let $G_{n,k}$ be the permutation group on the tiles an Instant Insanity II type puzzle where n is the number of rows and k is the number of columns. Further, suppose that the puzzle has been left out in the sun too long and that one of the empty slots at the top has been melted shut. Now, there is only one open slot at the top to which we can move tiles.*

Theorem 3.2 *For the group $G_{n,k}$, we have the following:*

(i) If k is odd, then $G_{n,k} \approx A_{nk}$.

(ii) If k is even, then $G_{n,k} \approx S_{nk}$.

Proof. First we need to show that $A_{nk} \subseteq G_{n,k}$. By Theorem 1.5, we know that the alternating group is generated by 3-cycles. We begin by showing the cycle $(1, 2, 3) \in G_{n,k}$. Look at tiles 1, 2, and 3. We will denote the columns that tiles 1, 2, and 3 are in when the puzzle is in its original state as columns 'a, b,' and 'c,' respectively. We perform a series of vertical shifts on columns a, b, and c, namely by V_a, V_b, and V_c, respectively, until tiles 1, 2, and 3 are in the bottom row. Once the tiles are in these positions, we can rotate the bottom row until we get tile 1 in column b, tile 2 in column a, and tile 3 into column c. Then we can perform the inverses of the vertical shifts, V_a^{-1}, V_b^{-1}, and V_c^{-1} , to get the tiles back up to the top row. Figure 5 gives an example of V_a with our 4×4 Instant Insanity II type puzzle with one slot at the top.

2	1	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Figure 10: Tiles of Instant Insanity II after a series of vertical shifts

Notice in Figure 10 that (2, 1, 3) is the inverse of (1, 2, 3). Thus, the 3-cycle (1, 2, 3) is in the puzzle group $G_{n,k}$.

We now show that any 3-cycle is in $G_{n,k}$. Start by picking any three tiles to be in the 3-cycle. These will be referred to as our "target tiles." Move these tiles to the bottom row using the appropriate vertical shifts for each of the tiles. Note that when doing a vertical shift, only one tile on the bottom row gets moved. Now, we can rotate the bottom row to put the first target tile in column 'a.' Then we apply a series of vertical shifts to get the tile to the top row. We continue this process to get our second and third target tiles into their respective columns, followed by $n-1$ vertical shifts to get each of the tiles to the top row. Once our target tiles are in the top row, we do $n-1$ vertical shifts on column 'a,' $V_{a^{n-1}}$, n vertical shifts on column 'c,' denoted V_{c^n}, then rotate the bottom row twice, $R2$. Now we do $V_{c^n}^{-1}$, followed by $V_{a^{n-1}}^{-1}$, and $R2^{-1}$. This results in either a 3-cycle or a 3-cycle together with a set of 2-cycles, also known as transpositions. If we have a 3-cycle, then $A_{nk} \subseteq G_{n,k}$. If we get a 3-cycle together with a set of disjoint transpositions, then we can square the permutation, which will leave only a 3-cycle. Since every 3-cycle is in $G_{n,k}$, we have that $A_{nk} \subseteq G_{n,k}$.

Suppose k is odd. Thus, all generators are even permutations. This implies $G_{n,k}$ is a subgroup of A_{nk}. From above, we now have that $G_{n,k} \approx A_{nk}$.

Suppose k is even. This implies that we have at least one odd permutation, namely the rotation on the bottom row. By Cayley's Theorem, $G_{n,k} \subseteq S_{nk}$. Therefore, $o(G_{n,k})$ divides $o(S_{nk})$ by Lagrange's Theorem. So $o(G_{n,k})$ divides $(nk)!$. Further, we showed above that $A_{nk} \subseteq G_{n,k}$. Ergo, $o(G_{n,k}) \geq o(A_{nk}) = \frac{(nk)!}{2}$. Further, because $G_{n,k}$ has at least one odd permutation, $o(G_{n,k}) \geq \frac{(nk)!}{2}+1$. However, there are no integers between $\frac{(nk)!}{2}+1$ and $(nk)!$ that divide $(nk)!$ other than $(nk)!$ itself. Therefore, $o(G_{n,k}) = (nk)!$, which implies $G_{n,k} \approx S_{nk}$. $\square$

Theorem 3.3 *Solution two cannot be obtained as a permutation on the colors of solution one.*

Proof. For simplicity, we let B correspond to the color blue, G to green, R to red, W to white, and Y to yellow. Since there are five colors, there are $5! = 120$ permutations. Now we can begin looking at the 120 possible permutations. Since there are more blue tiles than any other color, we must leave the blue tiles fixed. This narrows our number of possible permutations down to twenty-four. We can further eliminate our possible number of permutations down to eighteen. Recall our missing tiles. Any permutation that contains the transpositions (G,Y), (G,W), (R,Y), or (R,W) will force our new solution to contain one of our missing tiles.

Case 1: (B)(G)(R)(W)(Y) is the identity permutation, e. This means that we leave everything fixed, which results in no change to the solution.

Case 2: (G,R) means we swap the colors green and red. Solution one then becomes

rW	wY	yB	bR
bY	yR	rG	gB
gR	rB	bW	wG
wB	bG	gY	yW

Notice that the tiles rY, yG, gW, and wR from our original solution are now missing from our new solution and that we have added the tiles rW, yR, wG, and gY. Thus, the permutation (G,R) does not result in us achieving solution two from solution one.

Case 3: (W,Y) will swap the colors white and yellow, making the first line of solution one become

gY yW wB bG

Since gY is not a tile in our original solution, then we cannot achieve solution two from solution one.

Case 4: (G,R)(W,Y) swaps green with red and swaps white with yellow. Now our solution looks like

rY	yW	WB	bR
bW	wR	rG	gB
gR	rB	bY	yG
yB	bG	gW	wY

Notice here that we have all of the same tiles from solution one in our new solution. However, if we look closely, we see that our new solution is really solution one with a permutation on the rows and columns.

Case 5: (G,R,W) will make the color green to to red, red to white and white back to green. So the first line of solution one will now look like

rG gY yB bR

Notice that gY is not a tile in our original solution. Thus, we cannot use the permutation (G,R,W).

Case 6: (G,W,R) will make green to go white, white to red, and red to green, making the first two lines of our new solution become

wR	rY	yB	bR
bY	yW	wG	gB

Since wG is not a tile, we cannot use this permutation.

Case 7: (G,R,Y) forces green to red, red to yellow, and yellow back to green. The first line of our solution is now of the form

yR rB bW wY

We know that yR is not a tile in our original solution, so we cannot use this permutation to achieve solution two from solution one.

Case 8: (G,Y,R) will make green go to yellow, yellow to red, and red cycles back to green. So the first three lines of our solution will now be

yW	wR	rB	bY
bR	rY	yG	gB
gY	yB	bW	wG

The tiles gY and wG are not in our original solution, so this permutation cannot be used.

Case 9: (G,W,Y) will make green go to white, white to yellow, and yellow to green, the first three lines of our solution being

wY	yG	gB	bW
bG	gW	wR	rB
rW	wB	bY	yR

The tiles rW and yR are not part of our original solution. Thus, we cannot use this permutation.

Case 10: (G,Y,W) forces green to go to yellow, yellow to white, and white back to green. Now the first two lines of our solution will look like

yG	gW	wB	bG
bW	wY	yR	rB

Since yR is not a tile in our original solution, we cannot use the permutation (G,Y,W).

Case 11: (R,W,Y) will make red go to white, white to yellow and yellow to red. The first line of our new solution will look like

gY	yR	rB	bG

The tiles gY and yR are not part of our original solution, so we cannot use this permutation.

Case 12: (R,Y,W) makes red go to yellow, yellow to white, and white to red. Now the first line of our solution will look like

gR	rW	wB	bG

Since rW is not a tile in our original solution, we cannot use this permutation.

Case 13: (G,R,W,Y) will make green go to red, red to white, white to yellow, and yellow to green. Now, the first three lines of our solution will look like

rY	yG	gB	bR
bG	gR	rW	wB
wB	bG	gY	yW

The tile rW is not in the original solution. Thus, we cannot use the permutation (G,R,W,Y).

Case 14: (G,R,Y,W) will make green go to red, red to yellow, yellow to white, and white back to green. The first three lines of our solution now becomes

rG	gW	wB	bR
bW	wR	rY	yB
yR	rB	bG	gY

Since we have the tiles yR and gY in our new solution and they are not in our original solution, we cannot use this permutation.

Case 15: (G,W,R,Y) will make green go to white, white to red, red to yellow, and yellow to green. Now our solution will look like

wR	rG	gB	bW
bG	gW	wY	yB
yW	wB	bR	rY
rB	bY	yG	gR

Notice that we have all of the tiles in our new solution that were in the original solution. However, this is just a permutation of the rows and columns of solution one.

Case 16: (G,W,Y,R) will make green go to white, white to yellow, yellow to red, and red will cycle back to green. Now the first line of our solution will be of the form

wY yR rB bW

Since the tile yR is not in our original solution, we cannot use this permutation to achieve solution two from solution one.

Case 17: (G,Y,R,W) forces green to go to yellow, yellow to red, red to white, and white cycles back around to green. Now, our solution looks like

yG	gR	rB	bY
bR	rY	yW	wB
wY	yB	bG	gW
gB	bW	wR	rG

This new configuration has the same tiles of our original solution. However it is just a rearrangement of the rows and columns of solution one. So, we cannot use this permutation.

Case 18: (G,Y,W,R) will force green to yellow, yellow to white, white to red, and red to green. Now the first line of our solution has the form

yR rW wB bY

Since yR and rW are not tiles of our original solution, we cannot use this permutation. Further, none of the twenty-four cases yields solution two from solution one. We now know that solution one and solution two are independent of each other. That is, solution two is not a permutation on the colors of solution one. □

4 CONFIGURATION

4.1 4×5 Puzzle

Recall our missing tiles, gY, rW, yR, and wG. If we choose to use these missing tiles, then we have either a 4×5 puzzle or a 5×4 puzzle. We now examine the possibilities for solutions if we make a 4×5 puzzle. Here we include the missing tiles mentioned above. Note that with a 4×5 puzzle, we will have four unique sets of 5-cycles, specifically:

<u>Set 1:</u>	<u>Set 2:</u>
a) bR, rY, yW, wG, gB	a) bY, yW, wR, rG, gB
b) bR, rY, yG, gW, wB	b) bY, yW, wG, gR, rB
c) bR, rG, gW, wY, yB	c) bY, yG, gR, rW, wB
d) bR, rG, gY, yW, wB	d) bY, yG, gW, wR, rB
e) bR, rW, wY, yG, gB	e) bY, yR, rG, gW, wB
f) bR, rW, wG, gY, yB	f) bY, yR, rW, wG, gB

<u>Set 3:</u>	<u>Set 4:</u>
a) bG, gY, yR, rW, wB	a) bW, wR, rG, gY, yB
b) bG, gY, yW, wR, rB	b) bW, wR, rY, yG, gB
c) bG, gW, wR, rY, yB	c) bW, wG, gR, rY, yB
d) bG, gW, wY, yR, rB	d) bW, wG, gY, yR, yB
e) bG, gR, rW, wY, yB	e) bW, wY, yG, gR, rB
f) bG, gR, rY, yW, wB	f) bW, wY, yR, rG, gB

These sets are obtained by taking a tile, say bR, and looking at all of the 5-cycles that correspond with it. We find the four sets by looking at tree diagrams. For example, Figure 11 shows how we obtain Set 1.

Theorem 4.1 *There is no possible solution for the* 4×5 *puzzle.*

Proof. Without loss of generality, we can assume that the element of Set 1 is fixed. Note that we will later refer to each of the cycles as 1a, 1b, etcetera for Set 1, 2a,

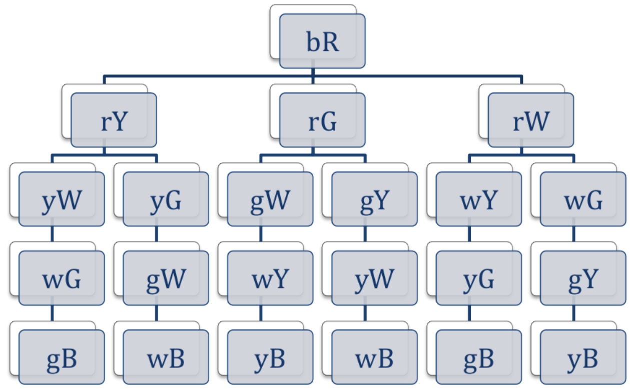

Figure 11: Tree diagram to obtain Set 1 on the 4×5 puzzle

2b, etcetera for Set 2; this will continue for each of the four sets. Further, lowercase roman numerals refer to rotations of that set. For example, 2d(i) refers to one rotation of 2d, hence we have yG, gW, wR, rB, bY. Similarly, 2d(ii) refers to two rotations of 2d, so we have gW, wR, rB, bY, yG.

In order to see if there is a unique solutions on the 4×5 puzzle, let us examine all of the possibilities. We start by fixing 1a: bR, rY, yW, wG, gB. Now we look at our possibilities from Set 2. We cannot use 2a or 2b because they share a common tile, namely, yW, with 1a. We cannot use 2f because it shares the tile wG. This minimizes our possibilities down to Rows c, d, and e from Set 2. We first look at all of the rotations of 2c together with 1a.

1a	bR	rY	yW	wG	gB
2c(i)	yG	gR	**rW**		
2c(ii)	**gR**				
2c(iii)	rW	wB	bY	**yG**	
2c(iv)	wB	**bY**			

Clearly, we cannot use any of the four rotations of 2c, since we we will have at least one of the main tiles lining up. Next, we look at the rotations of 2d with 1a.

1a	bR	rY	yW	wG	gB
2d(i)	yG	gW	wR	rB	bY
2d(ii)	gW	wR	rB	bY	yG
2d(iii)	**wR**				
2d(iv)	rB	**bY**			

From 2d, we have 2d(i) and 2d(ii) that would make for a possible solution to the 4×5 puzzle when we use 1a. Now consider possible rotations of 2e with 1a.

1a	bR	rY	yW	wG	gB
2e(i)	**yR**				
2e(ii)	rG	gW	wB	bY	yR
2e(iii)	gW	wB	bY	yR	rG
2e(iv)	wB	**bY**			

For 2e, we have 2e(ii) and 2e(iii) as possibilities with 1a. Hence, we need to fix each of these cases with 1a and examine them with Set 3. We start by fixing 1a and 2d(i).

1a	bR	rY	yW	wG	gB
2d(i)	yG	gW	wR	rB	bY

Just by examining the tiles in Set 3, we can see that we cannot use 3b because it shares tile yW with 1a. We cannot use 3c and 3d since they share tile gW with 2d(i). Also, 3f shares yW with 1a, leaving 3a and 3e as possible rows.

1a	bR	rY	yW	wG	gB
2d(i)	yG	gW	wR	rB	bY
3a(i)	gY	yR	**rW**		
3a(ii)	**yR**				
3a(iii)	**rW**				
3a(iv)	wB	bG	gY	yR	rW

We eliminate all rotations of 3a except 3a(iv). Next, we look at all rotations of 3e with 1a and 2d(i) fixed.

1a	bR	rY	yW	wG	gB
2d(i)	yG	gW	wR	rB	bY
3e(i)	**gR**				
3e(ii)	rW	**wY**			
3e(iii)	wY	yB	bG	gR	rW
3e(iv)	yB	bG	**gR**		

So the only rotation of 3e that we can use is 3e(iii). With 1a and 2d(i) fixed, from Set 3, we only have 3a(iv) and 3e(iii) as possible rotations.

1a	bR	rY	yW	wG	gB
2d(i)	yG	gW	wR	rB	bY
3a(iv)	wB	bG	gY	yR	rW

Here, we cannot use any of Set 4. This is because 4a and 4b share tile wR with 2d(i), 4d and 4e share tile wG with 1a, 4e shares yG with 2d(i), and 4f shares yR with 3a(iv). Now we look at the possibilities when we fix 1a, 2d(i) and 3e(iii).

1a	bR	rY	yW	wG	gB
2d(i)	yG	gW	wR	rB	bY
3e(iii)	wY	yB	bG	gR	rW

This also eliminates all of Set 4 since 4a and 4b share tile wR with 2d(i), 4c and 4d share tile wG with 1a, and 4e and 4f share tile wY with 3e(iii). Thus, there are no possible solutions using 1a and 2d(i). Next, we look at possible solutions when 1a and 2d(ii) are fixed.

1a	bR	rY	yW	wG	gB
2d(ii)	gW	wR	rB	bY	yG

Using this same argument, we can eliminate 2d(ii) since each element in Set 4 shares a tiles with 2d(ii). Hence, we have no possible solutions whenever we fix 1a and 2d(ii). Now consider when we fix 1a and 2e(ii).

1a	bR	rY	yW	wG	gB
2e(ii)	rG	gW	wB	bY	yR

We use the above process to eliminate 3a, 3b, 3c, and 3d from Set 3 and 4a, 4b, 4c, 4d, and 4f from Set 4. This leaves only 3e and 4e as possibilities from Sets 3 and 4, respectively. Thus, it suffices to only compare 3e and 4e as shown below.

3e	bG	gR	rW	**wY**	yB
4e	bW	**wY**	yG	gR	rB

Since 3e and 4e share tile wY, there are no possible solutions when we fix 1a and 2e(ii). The same is true for 1a and 2e(iii) since it consists of the same tiles, just in a different order. Thus, there are no possible solutions when we fix 1a.

We continue this same process to look at when each element in Set 1 is fixed. Since we eliminated 1a, we now redirect our attention onto 1b: bR, rY, yG, gW, wB. We can eliminate 2c, 2d, and 2e since 2c and 2d share tile yG with 1b, and 2e shares tiles gW and wB with 1b. So the only possibilities from Set 2 are 2a, 2b, and 2f. Now, we proceed to examine the rotations of 2a with 1b fixed.

1b	bR	rY	yG	gW	wB
2a(i)	yW	wR	**rG**		
2a(ii)	**wR**				
2a(iii)	rG	gB	bY	**yW**	
2a(iv)	gB	**bY**			

This shows that we cannot use any of the rotations of 2a when we fix 1b. Next we consider the rotations of 2b.

1b	bR	rY	yG	gW	wB
2b(i)	yW	wG	gR	rB	bY
2b(ii)	wG	gR	rB	bY	yW
2b(iii)	**gR**				
2b(iv)	rB	**bY**			

From this observation, we can use 2b(i) and 2b(ii) to look for a possible solution to the 4×5 puzzle. Now we consider the rotations of 2f when we leave 1b fixed.

1b	bR	rY	yG	gW	wB
2f(i)	**yR**				
2f(ii)	rW	wG	gB	bY	yR
2f(iii)	wG	gB	bY	yR	rW
2f(iv)	gB	**bY**			

So we can use 2f(ii) and 2f(iii) when looking for a solution while leaving 1b fixed. By fixing 1b and any rotation of 2b, we can eliminate the following: 3a because it shares tile wB with 1b, 3b since it shares yW and rB with 2b, 3c since it shares tiles rY and gW with 1b, 3d because it shares tile gW with 1b, 3e because it shares tile gR with 2b, and 3f since it shares tile rY with 1b. Thus, there are no possible solutions when 1b and 2b are fixed.

Now consider when we have 1b and any rotation of 2f fixed. When comparing tiles to 1b and 2f, we can eliminate all of Set 3 except 3b and all of Set 4 except 4a. So we only need to compare 3b and 4a. However, 3b and 4a share tile gY, so there are no solutions when we fix 1b. Now, we continue this process again, except this time leaving 1c fixed. Recall the tiles in 1c: bR, rG, gW, wY, yB. Notice that 2a and 2e share tile rG with 1c, 2d shares tile gW with 1c. Hence, the only rows from Set 2 that we can use are 2b, 2c, and 2f. We can also eliminate 3c, 3d, and 3e from Set 3 and 4a, 4c, 4e, and 4f from Set 4 because they each share a common tiles with 1c. Now we can fix 2b with 1c. Now we only need to look at the rotations of 2b, 3a, and 4b when 1c is fixed.

1c	bR	rG	gW	wY	yB
2b(i)	yW	**wG**			
2b(ii)	wG	gR	rB	**bY**	
2b(iii)	**gR**				
2b(iv)	rB	bY	wG	gR	**rB**

Clearly, we cannot use any rotation of 2b. Thus, we move on to consider the rotations of 2c.

1c	bR	rG	gW	wY	yB
2c(i)	yG	gR	**rW**		
2c(ii)	**gR**				
2c(iii)	rW	wB	bY	yG	gR
2c(iv)	wB	bY	yG	gR	rW

Notice that we can use 2c(iii) and 2c(iv). Using any rotation of 2c will eliminate 3a, 3f, and 4b, leaving only 3b and 4d that we need to check. But 3b and 4d share two common tiles, namely, gY and rB. Thus, there are no possible solutions when 2c is fixed with 1c. Next, we look at the possibilities when we fix 2f. Here, we eliminate 3a, 4b, and 4d by matching tiles. Since 4b and 4d were our only possibilities when we fix 1c, Set 4 is completely eliminated. Hence, there are no solutions when we fix 1c.

Now, we can repeat this process by fixing 1d: bR, rG, gY, yW, wB. By fixing 1d, we can eliminate 2a, 2b, 2c, and 2e from Set 2, 3a, 3b, and 3f from Set 3, and 4a, 4d and 4f from Set 4 by using the same matching tiles approach as before. If we choose to fix 2d with 1d, then all of Set 3 gets eliminated by matching tiles, which implies there is no possible solution when we use 2d. Since no rotation of 2d will conflict with 1d, we can now go back and start looking at the rotations of 2f, 3c, and 4e with 1d.

1d	bR	rG	gY	yW	wB
2f(i)	**yR**				
2f(ii)	rW	**wG**			
2f(iii)	wG	gB	**bY**		
2f(iv)	gB	bY	yR	**rW**	

It is clear to see that no rotation of 2f will go with 1d. Therefore, there are no possible solutions when we leave 1d fixed.

We now proceed to check for possible solutions when we fix 1e: bR, rW, wY, yG, gB. By looking for matching tiles, we can eliminate 2a, 2c, 2d, and 2f from Set 2, 3a, 3d, and 3e from Set 3, and 4b, 4e, and 4f from Set 4. This leaves only 2b and 2e from Set 2, 3b, 3c and 3f from Set 3, and 4a, 4c, and 4d from Set 4 that we need to focus on. If we choose to fix 2b, then we can further eliminate 3b and 3f, leaving only 3c from Set 3. We can also eliminate 4c and 4d, leaving only 4a from Set 4. However, 3c and 4a share a tile, namely, yB. So there are no possible solutions when we fix 2b with 1e. Our only option left from Set 2 that we can fix with 1e is 2e. So our next step is to look at the rotations of 2e when 1e is fixed.

1e	bR	rW	wY	yG	gB
2e(i)	**yR**				
2e(ii)	rG	**gW**			
2e(iii)	gW	wB	**bY**		
2e(iv)	wB	bY	yR	**rG**	

We see there are no rotations of 2e that we can use when we have 1e fixed. Since 2e was our last option from Set 2 to fix with 1e, there are no solutions when we have 1e fixed.

Our last option from Set 1 for obtaining a solutions to the 4×5 case is when we fix 1f: bR, rW, wG, gY, yB. By leaving 1f fixed, we can eliminate 2b, 2c, and 2f from Set 2, 3a, 3b, 3c, and 3e from Set 3, and 4a, 4c, and 4d from Set 4. This means that we only need to check 2a, 2d, and 2e from Set 2, 3d and 3f from Set 3, and 4b, 4e, and 4f from Set 4. If we choose to fix 2a with 1f, then we can further eliminate 3f from Set 3 and 4b and 4f from Set 4. This leaves only 3d from Set 3 and 4e from Set 4. Upon comparing the tiles in 3d and 4e, we find that they share tiles wY and rB. Hence, we cannot obtain a solutions if we use 2a. Our next option from Set 2 is to fix

2d with 1f. Thus, our next step is to look at all of the possible rotations of 2d with 1f.

1d	bR	rW	wG	gY	yB
2d(i)	yG	**gW**			
2d(ii)	gW	wR	rB	**bY**	
2d(iii)	**wR**				
2d(iv)	rB	bY	**yG**		

Upon reviewing the rotations of 2d, we find that none of the rotations will line up with 1f to give us a solution. The last option from Set 2 that could give us a solution to the 4x5 puzzle is 2e. When we compare tiles from Sets 3 and 4 to those in 2e, we eliminate 3d and 3f, resulting in no solutions when we fix 2e with 1f. Thus, the 4×5 puzzle is not solvable. □

4.2 Add a Row to Create a 5×4 Puzzle

Using the same tiles as we did in the 4×5 puzzle, we also look for possible solutions when we have a 5×4 puzzle instead. Here, we create tree diagrams to generate unique sets of 4-cycles. Figure 12 shows how we created each of our sets.

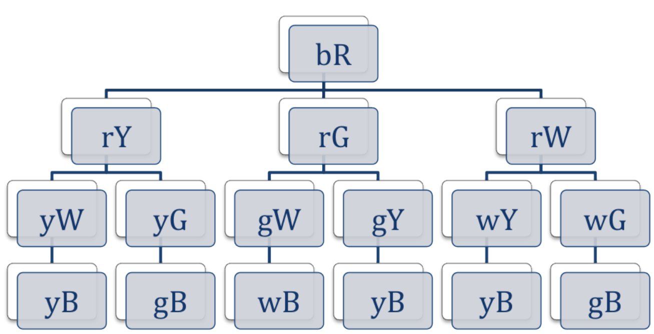

Figure 12: Tree diagram to obtain Set 1 on the 5×4 puzzle

By using the tree diagram above, we can generate Set 1. We use the same method to generate the remaining three sets as well. After completing our tree diagrams, we are left with four sets of 4-cycles.

Set 1:	Set 2:
a) bR, rY, yW, wB	a) bY, yW, wR, rB
b) bR, rY, yG, gB	b) bY, yW, wG, gB
c) bR, rG, gW, wB	c) bY, yG, gR, rB
d) bR, rG, gY, yB	d) bY, yG, gW, wB
e) bR, rW, wY, yB	e) bY, yR, rG, gB
f) bR, rW, wG, gB	f) bY, yR, rW, wB

Set 3:	Set 4:
a) bG, gY, yR, rB	a) bW, wR, rG, gB
b) bG, gY, yW, wB	b) bW, wR, rY, yB
c) bG, gW, wR, rB	c) bW, wG, gR, rB
d) bG, gW, wY, yB	d) bW, wG, gY, yB
e) bG, gR, rW, wB	e) bW, wY, yG, gB
f) bG, gR, rY, yB	f) bW, wY, yR, rB

Theorem 4.2 *The* 5×4 *puzzle has 240 solutions up to rotating the columns and permuting the colors.*

Proof. Recall that since there are five colors, there are $5! = 120$ ways to relabel the colors. We must show that for each solution of the 4×4 puzzle, we get a solution of the 5×4 puzzle. Recall the two solutions to Instant Insanity II in Figure 2. Also recall that our "missing tiles" for a 4×5 puzzle are yR, wG, rW, and gY. Since $4 * 5 = 20 = 5 * 4$, we know that there are also a total of twenty tiles for the 5×4 puzzle as well. Consider solution one. We can take our missing tiles and create a cycle with them and add the cycle in to make a fifth row as in Figure 13. Similarly, we can create a cycle with our missing tiles to add in a fifth row on solution two as shown in Figure 14.

gW	wY	yB	bG
bY	yG	gR	rB
rG	gB	bW	wR
wB	bR	rY	yW
yR	rW	wG	gY

Figure 13: The new solution one after adding a fifth row with our missing tiles

wY	yG	gB	bW
gW	wB	bR	rG
bG	gR	rY	yB
rB	bY	yW	wR
rW	wG	gY	yR

Figure 14: The new solution two after adding a fifth row with our missing tiles

So, it is clear that we can take each of the two solutions of the 4×4 puzzle and add in a fifth row using the our "missing tiles" and generate solutions for the 5×4 puzzle. Since for each solution of the 4×4 puzzle, we have a solution of the 5×4 puzzle, we have that the 5×4 puzzle has twice as many solutions as ways to relabel the colors. Further, since solution two cannot be obtained from solution one by permuting the colors, all these solutions are distinct. $\square$

4.3 Remove a Color to Create a 3×3 Puzzle

Recall that there are five colors to the 4×4 puzzle: blue, greed, red, white, and yellow. What happens if we remove one of the colors to create a 3×3 puzzle? Without loss of generality, we will eliminate the color white. This leaves us with only the colors blue, green, red, and yellow. Since we are working with a 3×3 puzzle, we will need a total of nine tiles. Recall that in all of our other cases, we have more blue tiles than any other color, so we will keep the same pattern and have one more blue tile than the other colors. Hence, we will have three blue tiles, two green tiles, two red tiles, and two yellow tiles. So our new set of tiles for this case are as follows:

gB,	rB,	yB,
bG,	yG,	
bR,	gR,	
bY,	rY.	

Theorem 4.3 *There is a unique solution to the* 3×3 *puzzle.*

Proof. As before, we must now create our sets. Since we have a 3×3 puzzle, this means that we will have three sets of 3-cycles. Let us begin with the tile rB to create Set 1. The tile rB has the options to go to bR, bY, or bG. It cannot go to bR since that forces a 2-cycle. Now we look at what happens when we have rB going to bY. We would need the tile yR to create a 3-cycle, however yR is not one of our nine tiles. Thus, we cannot let rB go to bY. Our only option left is for rB to go to bG. If we do this, then we need the tile gR to create a 3-cycle. Since gR is one of our nine tiles, we have that Set 1 only has one 3-cycle, namely rB, bG, gR.

Next, we look at the tile gB to find our Set 2. The tile gB has the option to go to bG, bR, or bY. It cannot go to bG, as this would only generate a 2-cycle. If it goes to bR, then we would need the tile rG in order to create a 3-cycle. Since we do not have the tile rG for the 3×3 puzzle, then gB cannot go to bR. This leaves only one option for Set 2, namely gB, bY, yG.

The last thing that we need to do is to generate Set 3. The only tile that we have not yet used is yB. Starting with yB, it can go to either bY, bG, or bR. The tile yB cannot go to tile bY since it will force a 2-cycle. We cannot have the tile yB go to bG. This is because bG would need to go to gY in order to create a 3-cycle, but we do not have the tile gY. So the only option for yB to go to is bR and our Set 3 is precisely yB, bR, rY.

Is it possible to find a solution with our three 3-cycles? Since no two colors can be in the same row or column and we have three blue tiles, then we need to set blue in a diagonal, similar to that of a Latin square.

rB		
	gB	
		yB

This will force our solution to be in Figure 15.

rB	bG	gR
yG	gB	gY
bR	rY	yB

Figure 15: Solution to the 3×3 puzzle

As in the 5×4 puzzle, we can permute the rows and columns but we will still have the same solution. Thus, there is only one unique solution for the 3×3 puzzle. $\square$

4.4 Add a Color to Create a 5×5 Puzzle

We know that we do not have a unique solution with the standard 4×4 puzzle. We also know that there is no solution if we change the puzzle to a 4×5 puzzle with only one slot at the top, and that there are numerous solutions with the 5×4 case. We next examine the case where we have a 5×5 puzzle, which leads to our next result. Here, we add a sixth color, say purple, and look for possible solutions.

Recall, with the standard 4×4 case, there are four tiles with one color and each of the remaining colors have three tiles each. We will keep the same patten here with our 5×5 case. As before, we assign "blue" to five tiles, and each of the other colors will be assigned to four tiles each. So our tiles for the 5×5 case are as follows:

rB,	gB,	yB,	wB,	pB,
bR,	gR,	wR,	pR,	
bG,	rG,	yG,	pG,	
bY,	wY,	rY,	pY,	
bP,	gP,	yP,	wP.	

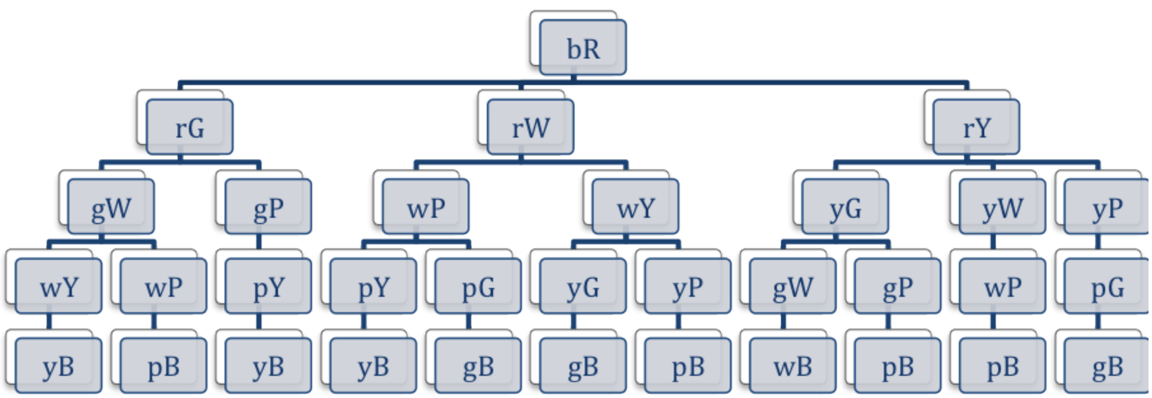

Figure 16: Tree diagram to obtain Set 1 on the 5×5 puzzle

As with the previous cases, we partition our sets based on the blue tiles, to obtain five sets of 5-cycles. We construct tree diagrams to find each of our 5-cycles, as shown in Figure 16.

Set 1:

a) bR, rG, gW, wY, yB
b) bR, rG, gW, wP, pB
c) bR, rG, gP, pY, yB
d) bR, rW, wP, pY, yB
e) bR, rW, wP, pG, gB
f) bR, rW, wY, yG, gB
g) bR, rW, wY, yP, pB
h) bR, rY, yG, gW, wB
i) bR, rY, yG, gP, pB
j) bR, rY, yW, wP, pB
k) bR, rY, yP, pG, gB

Set 2:

a) bG, gR, rY, yW, wB
b) bG, gR, rY, yP, pB
c) bG, gR, rW, wY, yB
d) bG, gR, rW, wP, pB
e) bG, gW, wR, rY, yB
f) bG, gW, wY, yP, pB
g) bG, gW, wP, pR, rB
h) bG, gW, wP, pY, yB
i) bG, gP, pR, rY, yB
j) bG, gP, pR, rW, wB
k) bG, gP, pY, yW, wB

Set 3:

a) bY, yG, gR, rW, wB
b) bY, yG, gW, wR, rB
c) bY, yG, gW, wP, pB
d) bY, yG, gP, pR, rB
e) bY, yW, wR, rG, gB
f) bY, yW, wP, pR, rB
g) bY, yW, wP, pG, gB
h) bY, yP, pR, rG, gB
i) bY, yP, pR, rW, wB
j) bY, yP, pG, gR, rB
k) bY, yP, pG, gW, wB

Set 4:

a) bW, wR, rG, gP, pB
b) bW, wR, rY, yG, gB
c) bW, wR, rY, yP, pB
d) bW, wY, yG, gR, rB
e) bW, wY, yG, gP, pB
f) bW, wY, yP, pR, rB
g) bW, wY, yP, pG, gB
h) bW, wP, pG, gR, rB
i) bW, wP, pR, rG, gB
j) bW, wP, pR, rY, yB
k) bW, wP, pY, yG, gB

Set 5:

a) bP, pR, rG, gW, wB
b) bP, pR, rY, yG, gB
c) bP, pR, rY, yW, wB
d) bP, pR, rW, wY, yB
e) bP, pG, gR, rY, yB
f) bP, pG, gR, rW, wB
g) bP, pG, gW, wR, rB
h) bP, pG, gW, wY, yB
i) bP, pY, yG, gR, rB
j) bP, pY, yG, gW, wB
k) bP, pY, yW, wR, rB

Theorem 4.4 *There are no possible solutions for the* 5×5 *puzzle.*

Proof. Without loss of generality, we can assume the element from Set 1 is fixed.

Case 1: Fix 1a. When we fix 1a, we have the tiles bR, rG, gW, wY, and yB to use for a possible solution. We look for matching tiles, as we did in our 4×5 case, we can eliminate 3e and 3h from Set 3, 4a and 4i from Set 4, and 5a from Set 5 since they all share tile rG with 1a. Next, we move on to tile gW. We can eliminate 2e, 2f, 2g, and 2h from Set 2, 3b, 3c, and 3k from Set 3, and 5a, 5g, 5h, and 5j from Set 5. The next tile we look at is wY. We can eliminate 2c and 2f from Set 2, 4d, 4e, 4f, and 4g from Set 4, and 5d and 5h from Set 5. The final tile we look at is yB. We can eliminate 2c, 2e, 2h, and 2i from Set 2, 4j from Set 4, and 5d, 5e, and 5h from Set 5. This leaves us with 2a, 2b, 2d, 2j, and 2k from Set 2, 3a, 3d, 3f, 3g, 3i, and 3j from Set 3, 4b, 4c, 4h, and 4k from Set 4, and 5b, 5c, 5f, 5i, and 5k from Set 5 as possibilities when we have 1a fixed.

We start by fixing 2a with 1a. This automatically eliminates 3a, 3f, 3g, 3i, and 3j from Set 3, 4b, 4c, and 4h from Set 4, and 5b, 5c, 5f, 5i, and 5k from Set 5 by using the matching tiles approach from above. This leaves us with no possibilities from Set 5 when we fix 2a with 1a.

Our next possible solution with 1a fixed is to look at what happens when we fix 2b. By fixing 2b with 1a, we can eliminate 3a, 3c, 3h, 3i, 3j, and 3k from Set 3, 4b, 4c, 4d, 4f, 4g, 4h, and 4j from Set 4, and 5b, 5c, 5e, 5f, an 5i from Set 5. This eliminates all of our remaining possibilities from Set 4. So we cannot obtain a solution when 2b is fixed with 1a.

The next possibility from Set 2 with 1a fixed is 2d. When we fix 2d with 1a, we can eliminate 3a, 3c, 3f, 3g, 3i, and 3j from Set 3, 4a, 4c, 4d, 4e, 4h, 4i, 4j, and 4k from Set 4, and 5d, 5e, 5f, and 5i from Set 5. This leaves us with 3d from Set 3, 4b from Set 4, and 5b, 5c, and 5k from Set 5 as possibilities when 2d is fixed with 1a. Since there is only one option from Set 3 and only one option from Set 4, we compare 3d and 4b. Comparing 3d and 4b, we see that they share tile yG. Thus, we cannot use them together. This implies that we cannot use 2d with 1a.

The next element from Set 2 that we look for a possible solution with 1a fixed is 2j. Fixing 2j with 1a, we can eliminate 3a, 3d, 3f, 4h, 3i, and 3k from Set 3, 4a, 4e, 4f, 4i, and 4j from Set 4, and 5a, 5b, 5c, 5d, 5f, and 5j from Set 5. This means that our possibilities from the remaining three sets gets narrowed down to 3g and 3j from Set 3, 4b, 4c, 4h, and 4k from Set 4, and 5i and 5k from Set 5. Looking for matching tiles when we fix 3g with 2j and 1a, we can eliminate 4b, 4c, and 4k from Set 4, and 5k from Set 5. This leaves us with no possibilities from Set 4 to use when finding a solution with 3g, 2d, and 1a fixed. Hence, we cannot use 3g. Next, we look at fixing 3j with 2j and 1a. Doing this, we can eliminate 4c and 4h from Set 4, and 5i and 5k from Set 5, leaving us with no options from Set 5 to use for a possible solution. So we cannot use 3j with 2j and 1a, which implies that we cannot use 2j when looking for a possible solution with 1a.

Our only other option from Set 2 to fix with 1a is 2k. When we fix 2k, we can eliminate 3a, 3d, 3e, 3f, 3g, 3i, and 3k from Set 3, 4a, 4e, and 4k from Set 4, and 5a, 5c, 5f, 5i, 5j, and 5k from Set 5. This leaves us with 3j from Set 3, 4b, 4c, and 4h from Set 4, and 5b from Set 5 as possibilities. Since there is only one option from Set 3 and only one option from Set 5, we can compare their tiles to see if they have any in common. Upon doing so, we find that 3j and 5b have tile rB in common, which means that we cannot use 3j and 5b in the same solution. Since they were the only options from Set 3 and Set 5 when 2k is fixed with 1a, we cannot use 2k with 1a when looking for a possible solution. Thus, there is no solution with 1a fixed.

<u>Case 2: Fix 1b.</u> Next, we look at what happens when we fix 1b, which is made up of tiles bR, rG, gW, wP, and pB. Fixing 1b allows us to eliminate 2b, 2d, 2f, 2g, and 2h from Set 2, 3b, 3c, 3e, 3f, 3g, 3h, and 3k from Set 3, 4a, 4c, 4e, 4h, 4i, 4j, and 4k from Set 4, and 5a, 5g, 5h, and 5j from Set 5. This leaves us with 2a, 2c, 2i, 2j, and 2k from Set 2, 3a, 3d, 3i, and 3j from Set 3, 4b, 4d, 4f, and 4g from Set 4, and 5b,

5c, 5d, 5e, 5f, 5i, and 5k from Set 5 as possibilities when 1b is fixed.

Our first possibility from Set 2 with 1b fixed is from 2a. Fixing 2a with 1b eliminates 3a, 3e, 3f, 3g, 3i, 3j, and 3k from Set 3, 4b, 4c, 4d, 4h, and 4j from Set 4, and 5a, 5b, 5c, 5e, 5f, 5i, 5j, and 5k from Set 5. This leaves us with 3d from Set 3, 4f and 4g from Set 4, and 5d from Set 5 as possibilities with 2a and 1b fixed. Since Set 3 and Set 5 each only have one possibility, we can compare their tiles. Upon comparing 3d and 5d, we see that they have tile pR in common. Hence, there is no possible solution when we have 2a fixed with 1b.

Next, we fix 2c with 1b. Using the matching tiles approach, we can eliminate 3a, 3i, and 3j from Set 3, 4d, 4e, 4f, 4g, 4h, and 4j from Set 4, and 5d, 5e, 5f, 5h, and 5i from Set 5. This leaves us with 3d, 4b, 5b, 5c, 5d, 5e, 5f, and 5k as possibilities from Sets 3, 4, and 5, respectively. Since there is only one possibility from Set 3, namely 3d, we can fix it and look at the possibilities we have from Sets 4 and 5. By fixing 3d with 2c and 1b, we can eliminate 4b from Set 4, and 5d, 5e, and 5k from Set 5. This eliminates all of Set 4 since 4b was our only option. So there is no solution when we fix 2c with 1b.

Next, we look at the possibilities we have when we fix 2i with 1b. Recall that 2i has the tiles bG, gP, pR, rY, and yB. This eliminates 3d and 3i from Set 3, 4b and 4f from Set 4 and 5b, 5c, 5d, and 5e from Set 5. This leaves us with 3a and 3j from Set 3, 4d and 4g from Set 4, and 5f, 5i, and 5k from Set 5 as possibilities when we fix 2i with 1b. We now fix 3a and look for matching tiles. Recall that 3a has the tiles bY, yG, gR, rW, and wB. When we fix 3a with 2i and 1b, we can eliminate 4d and 4g from Set 4, and 5f and 5i from Set 5. This eliminates all of Set 4, which implies that we cannot use 3a with 2i and 1b fixed. Now we see what happens when we fix 3j, which has the tiles bY, yP, pG, gR, and rB. This eliminates 4d and 4g from Set 4 and 5f, 5i, and 5k from Set 5, eliminating all of Set 4 and all of Set 5. Thus, we cannot use 2i with 1b.

We now fix 2j with 1b and look for matching tiles. Notice that 2j contains the tiles bG, gP, pR, rW, and wB. This eliminates 2a, 2d, and 2i from Set 2, 4f from Set 4, and 5b, 5c, 5d, and 5f from Set 5. Our possibilities to find a solution are 3j from Set 3, 4b, 4d, and 4g from Set 4, and 5e, 5i, and 5k from Set 5. Since 3j is our only possibility from Set 3 when we fix 1b and 2j, we fix 3j and look for matching tiles from Set 4 and Set 5. Recall that 3j is made up of the tiles bY, yP, pG, gR, and rB. Looking for matching tiles, we can eliminate 4d and 4g from Set 4, and 5e, 5i, and 5k from Set 5, eliminating all of Set 5. Thus, we cannot use 3j with 2j and 1b fixed. This implies that we cannot use 2j when we have 1b fixed.

Our last option when we have 1b fixed is from 2k. Recall that 2k is made up of the tiles bG, gP, pY, yW, and wB. We can eliminate 3a, 3d, and 3i from Set 3, and 5c, 5f, 5i, and 5k from Set 5. Our possibilities from the remaining three sets are 3j from Set 3, 4b, 4d, 4f, and 4g from Set 4, and 5b, 5d, and 5e from Set 5. Since there is only one possibility from Set 3, we fix 3j. This eliminates 4d, 4f, and 4g from Set 4 and 5e from Set 5, leaving 4b, 5b, and 5d from Sets 4 and 5, respectively, as possibilities. We now compare the tiles in 4b to those in 5b and 5d. Upon comparison, we find that we cannot use 5b with 4b since they share tiles rY, yG, and gB but we can use 5d with 4b, 3j, 2k, and 1b fixed. Now we look at rotations of 2k with 1b fixed.

1b	bR	rG	gW	wP	pB
2k(i)	gP	pY	**yW**		
2k(ii)	pY	yW	wB	bG	gP
2k(iii)	yW	wB	bG	**gP**	
2k(iv)	wB	**bG**			

The only rotation of 2k that we can use with 1b fixed is 2k(ii). We now look at the rotations of 3j with 1b and 2k(ii) fixed.

1b	bR	rG	gW	wP	pB
2k(ii)	pY	yW	wB	bG	gP
3j(i)	yP	**pG**			
3j(ii)	pG	gR	**rB**		
3j(iii)	**gR**				
3j(iv)	rB	bY	yP	pG	gR

The only rotation of 3j that we can use with 1b and 2k(ii) fixed is 3j(iv). Now, we fix 3j(iv) with 2k(ii) and 1b and look at rotations of 4b.

1b	bR	rG	gW	wP	pB
2k(ii)	pY	yW	wB	bG	gP
3j(iv)	rB	bY	yP	pG	gR
4b(i)	**wR**				
4b(ii)	**rY**				
4b(iii)	yG	gB	**bW**		
4b(iv)	**gB**				

No rotation of 4b works with 1b, 2k(ii), and 3j(iv) fixed. Thus, there are no possible solutions with 1b fixed.

<u>Case 3: Fix 1c.</u> We now look at what happens when we fix 1c, which contains the tiles bR, rG, gP, pY, and yB. This eliminates 2c, 2e, 2h, 2i, 2j, and 2k from Set 2, 3d, 3e, and 3h from Set 3, 4a,4e, 4i, 4j, and 4k from Set 4, and 5a, 5d, 5e, 5h, 5i, 5j, and 5k from Set 5. This leaves 2a, 2b, 2d, 2f, and 2g from Set 2, 3a, 3b, 3c, 3f, 3g, 3i, 3j, and 3k from Set 3, 4b, 4c, 4d, 4f, 4g, and 4h from Set 4 and 5b, 5c, 5f, and 5g from Set 5 as possibilities.

The first possibility from Set 2 is by fixing 2a, which contains the tiles bG, gR, rY, yW, and wB. This eliminates 3a, 3e, 3f, 3g, 3i, 3j, and 3k from Set 3, 4b, 4c, ,4d, 4h, and 4j from Set 4, and 5a, 5b, 5c, 5e, 5f, 5i, 5j, and 5k from Set 5. We have 3b and 3c from Set 3, 4f and 4g from Set 4 and 5g from Set 5 as possibilities. Since there is only one option from Set 5, we will fix 5g. This eliminates 3b and 3c from Set 3 and 4g from Set 4, leaving no more possibilities from Set 3 to use. This implies that there is no solutions when 2a is fixed with 1c.

Next, we fix 2b with 1a. Recall that 2b is made up of tiles bG, gR, rY, yP, and pB. This allows us to eliminate 3a, 3c, 3i, 3j, and 3k from Set 3, 4b, 4c, 4d, 4f, 4g, and 4h from Set 4, and 5b, 5c, and 5f from Set 5. Notice that this eliminates all of Set 4, preventing us from having a solution when 2b is fixed with 1c.

Now we look at what happens when we fix 2d, whose tiles are bG, gR, rW, wP, and pB. We can eliminate 3a, 3c, 3f, 3g, 3i and 3j from Set 3, 4c, 4d, and 4h from Set 4, and 5f from Set 5. Now we have 3b and 3k from Set 3, 4b, 4f, and 4g from Set 4, and 5b, 5c, and 5g from Set 5 as possibilities. We now fix 3b with with 2d and 1c. This allows us to eliminate 4b and 4f from Set 4, and 5b and 5g from Set 5, leaving only 4g and 5c as possibilities from Sets 4 and 5, respectively. Upon comparing the tiles in 4g and 5c, we see there are no tiles in common. So we now go back and look at the rotations of 2d with 1c fixed.

1c	bR	rG	gP	pY	yB
2d(i)	**gR**				
2d(ii)	rW	wP	pB	bG	gR
2d(iii)	wP	pB	bG	gR	rW
2d(iv)	pB	**bG**			

We can use the rotations 2d(ii) and 2d(iii) with 1c. So now we fix 2d(ii) and look at the rotations of 3b.

1c	bR	rG	gP	pY	yB
2d(ii)	rW	wP	pB	bG	gR
3b(i)	yG	gW	wR	rB	bY
3b(ii)	**gW**				
3b(iii)	**wR**				
3b(iv)	rB	bY	yG	gW	**wR**

The only rotation of 3b that we can use with 1c and 2d(ii) is 3b(i). Now we look at the rotations of 4g when we have 1c, 2d(ii), and 3b(i) fixed.

1c	bR	rG	gP	pY	yB
2d(ii)	rW	wP	pB	bG	gR
3b(i)	yG	gW	wR	rB	bY
4g(i)	wY	**yP**			
4g(ii)	yP	**pG**			
4g(iii)	**pG**				
4g(iv)	gB	**bW**			

No rotation of 4g works with 1c, 2d(ii), and 3b(i) fixed. Now, we go back and look at the possible rotations of 3b when 2d(iii) is fixed with 1c.

1c	bR	rG	gP	pY	yB
2d(iii)	wP	pB	bG	gR	rW
3b(i)	yG	gW	wR	rB	bY
3b(ii)	gW	wR	rB	**bY**	
3b(iii)	**wR**				
3b(iv)	rB	bY	**yG**		

The only rotation of 3b that works here is 3b(i). Now we look at the rotations of 4g with 3b(i), 2d(iii), and 1c fixed.

1c	bR	rG	gP	pY	yB
2d(iii)	wP	pB	bG	gR	rW
3b(i)	yG	gW	wR	rB	bY
4g(i)	wY	yP	**pG**		
4g(ii)	**yP**				
4g(iii)	**pG**				
4g(iv)	gB	**bW**			

Since we cannot use any rotation of 4g with 3b(i), 2d(iii), and 1c, we have no solution when we use 3b with 2d and 1c. Now we look for a solution when we fix 3k with 1c and 2d. This eliminates 4f and 4g from Set 4, and 5g, and 5c from Set 5. Our only option from Set 4 is 4b and our only option from Set 5 is 5b. Upon comparing 4b and 5b, we find that they have tile rY in common, which implies that we cannot use 3k with 2d and 1c. Thus, there is no solution when 2d is fixed with 1c.

We now look at when we fix 2f, whose tiles are bG, gW, wY, yP, and pB. Doing this eliminates 3b, 3c, 3h, 3i, 3j, and 3k from Set 3, and 4c, 4d, 4f, and 4g from Set 4. Our options for a possible solution come from 3a, 3f, and 3g from Set 3, 4b and 4g from Set 4, and 5b, 5c, 5f, and 5g from Set 5. Since there are the least amount of options from Set 4, we look at what happens when we fix 4b with 1c and 2f. This eliminates 3a and 3g from Set 3, and 5b, 5c, and 5g from Set 5, leaving only 3f and 5f from Sets 3 and 5, respectively. Upon comparing tiles of 3f to those of 5f, we find that there are no common tiles. Now we look at the rotations of 2f with 1c fixed.

1c	bR	rG	gP	pY	yB
2f(i)	gW	wY	**yP**		
2f(ii)	wY	yP	pB	bG	gW
2f(iii)	yP	pB	bG	gW	wY
2f(iv)	pB	**bG**			

We can use the rotations 2f(ii) and 2f(iii) with 1c fixed. Now we fix 2f(ii) and look at rotations of 4b.

1c	bR	rG	gP	pY	yB
2f(ii)	wY	yP	pB	bG	gW
4b(i)	**wR**				
4b(ii)	**rY**				
4b(iii)	yG	gB	bW	wR	rY
4b(iv)	gB	bW	wR	**rY**	

So the only rotation of 4b that we can use is 4b(iii). Now, we fix 1c, 2f(ii), and 4b(iii) and look at the rotations of 3f.

1c	bR	rG	gP	pY	yB
2f(ii)	wY	yP	pB	bG	gW
4b(iii)	yG	gB	bW	wR	rY
3f(i)	yW	**wP**			
3f(ii)	wP	pR	**rB**		
3f(iii)	**pR**				
3f(iv)	rB	bY	**yW**		

No rotation of 3f will work with 1c, 2f(ii), and 4b(iii) fixed. So now we fix 2f(iii) and look at the rotations of 4b.

1c	bR	rG	gP	pY	yB
2f(iii)	yP	pB	bG	gW	wY
4b(i)	**wR**				
4b(ii)	rY	**yG**			
4b(iii)	yG	**gB**			
4b(iv)	gB	bW	wR	**rY**	

No rotation of 4b works when 1c and 2f(iii) are fixed, implying that we cannot use 4b with 1c and 2f. Now we can look at when we fix 4h with 1c and 2f. This eliminates 3a, 3f, and 3g from Set 3, and 5g from Set 5, which eliminates all of Set 3. This implies that there is no solutions when we fix 2f with 1c.

We now look at our last possibility with 1c fixed. When we fix 2g, we have the tiles bG, gW, wP, pR, and rB. This eliminates 3b, 3c, 3f, 3h, and 3i from Set 3, 4d, 4f, and 4h from Set 4, and 5b, 5c, and 5g from Set 5. Our possibilities are 3a from Set 3, 4b, 4c, and 4g from Set 4, and 5f from Set 5. Upon comparing the tiles in 3a to those in 5f, we find that they share the tiles gR, rW, and wB, thus eliminating any possibility of fixing 2g with 1c. Hence, there is no solution when we fix 1c.

Case 4: Fix 1d. Since we cannot use 1a, 1b, or 1c when looking for possible solutions to the 5×5 puzzle with one open slot at the top, we proceed by now investigating what happens when we fix 1d. Recall, the tiles of 1d are bR, rW, wP, pY, and yB. Just by fixing 1d, we eliminate 2c, 2d, 2e, 2g, 2h, 2i, 2j, and 2k from Set 2, 3a, 3c, 3f, 3g, and 3i from Set 3, 4h, 4i, 4j, and 4k from Set 4, and 5d, 5e, 5f, 5h, 5i, 5j, and 5k from Set 5. Our remaining elements from Sets 2, 3, 4 and 5, respectively, are 2a, 2b, 2f, 3b, 3d, 3e, 3h, 3j, 3k, 4a, 4b, 4c, 4d, 4e, 4f, 4g, 5a, 5b, 5c, and 5g.

We begin by fixing 2a with 1d. This eliminates 3e, 3j, and 3k from Set 3, 4b, 4c, and 4d from Set 4, and 5a, 5b, and 5c from Set 5, leaving 3b, 3d, and 3h from Set 3, 4a, 4e, 4f, and 4g from Set 4, and 5g from Set 5 as possibilities. for a solution. Notice that there is only one possibility from Set 5, namely 5g, so we set that fixed with 1d and 2a to look for a solution. By fixing 5g, we eliminate 3b and 3d from Set 3, and 4a, 4f, and 4g from Set 4. We are left with only one element from each of the two remaining sets, 3h from Set 3 and 4e from Set 4. Upon comparing the tiles in 3h and 4e, we find that there are no tiles in common. Now we look at the rotations of 2a when 1d is fixed.

1d	bR	rW	wP	pY	yB
2a(i)	**gR**				
2a(ii)	rY	**yW**			
2a(iii)	yW	wB	bG	gR	rY
2a(iv)	wB	bG	gR	**rY**	

Clearly, 2a(iii) is the only rotation of 2a that works when we have 1d fixed. Now, we can fix 2a(iii) with 1d and look at the rotations of 3h.

1d	bR	rW	wP	pY	yB
2a(iii)	yW	wB	bG	gR	rY
3h(i)	yP	pR	**rG**		
3h(ii)	**pR**				
3h(iii)	rG	**gB**			
3h(iv)	gB	bY	**yP**		

No rotation of 3h works when we have 1d and 2a(iii) fixed. This implies that we cannot use 2a with 1d fixed.

Now, we fix 1d with 2b, which is composed of the tiles bG, gR, rY, yP, and pB. This allows us to eliminate 3h, 3j, and 3k from Set 3, 4a, 4b, 4c, 4d, 4f, and 4g from Set 4, and 5b and 5c from Set 5. This leaves us with no options from Set 4 to use to construct a possible solution. Hence, there is no solution when we fix 2b with 1d.

Our last option from Set 2 that we can fix with 1d is 2f. Recall that 2f is made up of the tiles bG, gW, wY, yP, and pB. Fixing 2f with 1d lets us further eliminate 3b, 3h, 3j, and 3k from Set 3, 4a, 4c, 4d, 4e, 4f, and 4g from Set 4, and 5a, 5c, and 5g from Set 5. We are left with 3d and 3e from Set 3, 4b from Set 4, and 5b from Set 5. Since there is only one option from Set 4 and only one option from Set 5, we can compare their tiles. We find that 4b and 5b share the tiles rY, yG, and gB. This means that we cannot use 2f with 1d. Since 2f was our last option from Set 2 that we could fix with 1d, we now know that there is no solution when 1d is fixed.

Case 5: Fix 1e. We proceed down the list of elements in Set 1 to look for a possible solution. Now, we will look for a solution when we fix 1e, whose tiles are bR, rW, wP, pG, and gB. By fixing 1e, we can eliminate 2c, 2d, 2g, 2h, and 2j from Set 2, 3a, 3c, 3e, 3f, 3g, 3h, 3i, 3j, and 3k from Set 3, 4b, 4g, 4h, 4i, 4j, and 4k from Set 4, and 5b, 5d, 5e, 5f, 5g, and 5h from Set 5. This leaves us with possibilities of 2, 2b, 2e, 2f, 2i, and 2k from Set 2, 3b and 3d from Set 3, 4a, 4c, 4d, 4e, and 4f from Set 4, and 5a 5c, 5i, 5j, and 5k from Set 5 when looking for a possible solution. Since Set 3 has the least amount of possibilities, we look at each of the cases when we fix 3b and 3d with 1e.

By fixing 3b with 1e, we can eliminate 2e and 2f from Set 2, 4a, 4c, 4d, 4e, and 4f from Set 4, and 5a, 5i, 5j, and 5k from Set 5. Notice that this eliminates all of our remaining options from Set 4. Thus, we cannot choose 3b to be fixed with 1e when looking for a possible solution.

Our only other option is to fix 3d with 1e. By doing this, we eliminate 2i and 2k from Set 2, 4a, 4d, 4e, and 4f from Set 4, and 5a, 5c, 5i, 5j, and 5k from Set 5. When we fix 3d with 1e, we end up eliminating all of Set 5. This implies that we cannot use 3d with 1e when looking for a possible solution. Hence, there is no solutions with 1e fixed since our only options from Set 3 were 3b and 3d.

Case 6: Fix 1f. We now look for a possible solution to the 5×5 puzzle when we fix 1f. Recall that 1f contains the tiles bR, rW, wY, yG, and gB. This allows us to eliminate 2c, 2d, 2f, and 2j from Set 2, 3a, 3b, 3c, 3d, 3e, 3g, 3h, and 3i from Set 3, 4b, 4d, 4e, 4f, 4g, 4i, and 4k from Set 4, and 5b, 5d, 5f, 5h, 5i, and 5j from Set 5. This leaves us with options of 2a, 2b, 2e, 2g, 2h, 2i, and 2k from Set 2, 3f, 3j, and 3k from Set 3, 4a, 4c, 4h, and 4j from Set 4, and 5a, 5c, 5e, 5g, and 5k from Set 5. Again, Set 3 has the least amount of options.

We begin by fixing 3f with 1f. This allows us to further eliminate 2a, 2g, 2h, 2i, and 2k from Set 2, 4h and 4j from Set 4, and 5a, 5c, 5g, and 5k from Set 5. Since Set 5 only has one option left, we now fix 5e with 1f and 3f. By doing this, we can eliminate 2b and 2e from Set 2, and 4c from Set 4. This does not leave us with any more options from Set 2 to look for a possible solution. This implies that we cannot use 3f when 1f is fixed.

Our next option from Set 3 with 1f fixed is 3j. If we fix 3j with 1f, then we eliminate 2a, 2b, and 2g from Set 2, 4c and 4h from Set 4, and 5e, 5g, and 5k from Set 5. This leaves us with 2e, 2h, 2i, and 2k from Set 2, 4a and 4j from Set 4, and 5a and 5c from Set 5 as options when looking for a possible solution. Since there are only two options from Set 4, we look at what happens when we fix 4a with 3j and 1f. Using the matching tiles approach, we find that we can eliminate 2e, 2i, and 2k from Set 2, and 5a from Set 5. This only leaves us with one option from Set 2 and one option from Set 5. Now we can compare the tiles in 2h to those in 5c. We find that 2h and 5c share no common tiles. Now we look at the rotations of 2h with 1f fixed.

1f	bR	rW	wY	yG	gB
2h(i)	gW	wP	**pY**		
2h(ii)	wP	pY	yB	**bG**	
2h(iii)	pY	yB	bG	gW	wP
2h(iv)	yB	bG	gW	wP	pY

We see that the rotation 2h(iii) and the rotation 2h(iv) both will work when 1f is fixed. Now we set 2h(iii) fixed with 1f and look at the rotations of 3j.

1f	bR	rW	wY	yG	gB
2h(iii)	pY	yB	bG	gW	wP
3j(i)	yP	pG	gR	rB	bY
3j(ii)	pG	gR	rB	bY	**yP**
3j(iii)	**gR**				
3j(iv)	rB	bY	yP	**pG**	

The only rotation of 3j that will work when we fix 1f and 2h(iii) is 3j(i). Now we keep 3j(i) fixed with 1f and 2j(iii) and look at the rotations of 4a.

1f	bR	rW	wY	yG	gB
2h(iii)	pY	yB	bG	gW	wP
3j(i)	yP	pG	gR	rB	bY
4a(i)	**wR**				
4a(ii)	rG	gP	pB	**bW**	
4a(iii)	**gP**				
4a(iv)	pB	**bW**			

We cannot use any rotation of 4a whenever 1f, 2h(iii), and 3j(i) are fixed. Now we go back and look at the rotations of 3j when we have 2h(iv) fixed.

1f	bR	rW	wY	yG	gB
2h(iv)	yB	bG	gW	wP	pY
3j(i)	yP	**pG**			
3j(ii)	pG	gR	rB	bY	yP
3j(iii)	**gR**				
3j(iv)	**rB**				

The only rotation of 3j that works when we fix 2h(iv) with 1f is 3j(ii). Now we look at the rotations of 4a whenever we have 1, 2h(iv), and 3j(ii) fixed.

1f	bR	rW	wY	yG	gB
2h(iv)	yB	bG	gW	wP	pY
3j(ii)	pG	gR	rB	bY	yP
4a(i)	**wR**				
4a(ii)	**rG**				
4a(iii)	gP	pB	**bW**		
4a(iv)	**pB**				

No rotation of 4a works whenever we have 1f, 2h(iv), and 3j(ii) fixed. This means that we cannot use 4a with 1f and 3j. Now we look at whenever we fix 4j with 1f and 3j. We can eliminate 2e, 3h, and 3i from Set 3, and 5a and 5c from Set 5, leaving no more options from Set 5. So, we cannot use 4j with 3j and 1f, which implies that we cannot use 3j whenever we have 1f fixed.

Our last option is to fix 3k with 1f. This eliminates 2a, 2b, 2e, 2g, 2h, and 2k from Set 2, 4c and 4h from Set 4, and 5a, 5c, 5e, 5g, and 5k from Set 5. Now we are left with 2i from Set 2, 4a and 4j from Set 4 and 5k from Set 5. Since there is only one option from Set 2 and only one option from Set 5, we can compare the tiles in 2i and 5k. Upon doing so, we find that 2i and 5k have no tiles in common. Now we compare the tiles in 4a to those in 1f, 2i, 3k, and 5k. We find that 4a shares wR with

5k and gP with 2i. So we cannot use 4a with 1f, 2i, 3k, and 5k fixed. Our last option is to compare the tiles in 4j to those in 1f, 2i, 3k, and 5k. We find that 4j shares tiles pR, rY, and yB with 2i. This implies that we cannot use 4j, which implies that there is no solution when we have 1f fixed.

Case 7: Fix 1g. Since we cannot obtain a solution from using 1a through 1f. We now turn our attention on 1g, which contains the tiles bR, rW, wY, yP, and pB. This eliminates 2b, 2c, 2d, 2f, and 2j from Set 2, 3a, 3c, 3h, 3i, 3j, and 3k from Set 3, 4a, 4c, 4d, 4e, 4f, and 4g from Set 4, and 5d, 5f, and 5h from Set 5. We are then left with 2a, 2e, 2g, 2h, 2i, 2k, 3b, 3d, 3e, 3f, 3g, 4b, 4h, 4i, 4j, 4k, 5a, 5b, 5c, 5e, 5g, 5i, 5j, and 5k from Sets 2, 3, 4, and 5, respectively, as possibilities when looking for a solution.

Notice that Set 3 and Set 4 have the same amount of options, so we will fix 3b from Set 3. Doing so eliminates 2e, 2g, and 2h from Set 2, 4b, 4h, and 4k from Set 4, and 5a, 5b, 5g, 5i, 5j and 5k from Set 5. This leaves only 2a from Set 2, along with 4i and 4j from Set 4, and 5c and 5e from Set 5. Since there is only one possibility from Set 2, we fix 2a with 1g and 3b. When we fix 2a with 1g and 3b, we eliminate 4j from Set 4 and 5c and 5e from Set 5 leaving no options from Set 5, which implies that we cannot use 2a with 1g and 3b fixed. Thus, there is no solution when we fix 3b with 1g.

We now look at what happens when we fix 3d with 1g. Recall that 3d contains the tiles bY, yG, gP, pR, and rB. By fixing 3d with 1g, we can eliminate 2g, 2i, and 2k from Set 2, 4b, 4h, 4i, 4j, and 4k from Set 4, and 5a, 5b, 5c, 5g, 5i, 5j, and 5k from Set 5. This eliminates all of our options from Set 4. So we cannot use 3d with 1g when looking for a possible solution.

Our next option from Set 3 is to fix 3e with 1g. By doing this, we eliminate 2a, 2e, and 2k from Set 2, 4b, 4i, and 4k from Set 4, and 5a, 5b, 5c, 5g, and 5k from Set 5, leaving us with 2g, 2h, and 2i from Set 2, 4h and 4j from Set 4, and 5e, 5i, and 5j from Set 5 as possibilities. By fixing 4h with 3e and 1g, we eliminate all of Sets 2 and 5 except 2i and 5j, respectively. Upon comparing the tiles in 2i and 5j, we see there are no common tiles. So we look at the rotations of 2i with 1g fixed.

1g	bR	rW	wY	yP	pB
2i(i)	gP	pR	**rY**		
2i(ii)	**pR**				
2i(iii)	rY	yB	bG	**gP**	
2i(iv)	yB	bG	gP	pR	rY

The only rotation of 2i that will work with 1g fixed is 2i(iv). We now proceed to fix 2i(iv) with 1g and look at the rotations of 3e.

1g	bR	rW	wY	yP	pB
2i(iv)	yB	bG	gP	pR	rY
3e(i)	yW	wR	rG	gB	**bY**
3e(ii)	**wR**				
3e(iii)	rG	gB	**bY**		
3e(iv)	**gB**				

Clearly, there is no rotation of 3e that with work when 1g and 2i(iv) are fixed. Hence, we cannot use 4h when looking for a possible solution. Now, we fix 4j with 1g and 3e. This eliminates 2g, 2h, and 2i from Set 2 and 5e from Set 5, leaving no more options from Set 2 when looking for a possible solution. This implies that we cannot use 4j with 3e and 1g fixed, which also implies that we cannot use 3e with 1g when looking for a possible solution.

The next possibility from Set 3 with 1g is 3f. This eliminates 2a, 2g, 2h, 2i, and 2k from Set 2, 4h, 4i, 4j, and 4k from Set 4, and 5a, 5b, 5c, 5g, 5i, and 5k from Set

5. This leaves us with only 2e from Set 2, 4b from Set 4, and 5e and 5j from Set 5. Since there is only one option from Set 2 and only one option from Set 4, we compare 2e and 4b. Upon comparing the tiles, we find that 2e and 4b share the tile wR, which implies that we cannot use 3f when looking for a possible solution with 1g.

Our last option from Set 3 to use when we have 1g fixed is 3g. By fixing 3g, we eliminate 2a, 2g, 2h, and 2k from Set 2, 4b, 4h, 4i, 4j, and 4k from Set 4, and 5b, 5c, 5e, 5g, and 5k. This leaves no possible options from Set 4 to use when looking for a solution. This implies that we cannot use 3f with 1g. Thus, there is no solution when 1g is fixed.

<u>Case 8: Fix 1h.</u> We now proceed to look for a possible solution by fixing 1h, which made up of the tiles bR, rY, yG, gW, and wB. This automatically eliminates 2a, 2b, 2e, 2f, 2g, 2h, 2i, 2j, and 2k from Set 2, 3a, 3b, 3c, 3d, 3i, and 3k from Set 3, 4b, 4c, 4d, 4e, 4j, and 4k from Set 4, and 5a, 5b, 5c, 5e, 5f, 5g, 5h, 5i, and 5j from Set 5. This leaves us with 2c and 2d from Set 2, 3e, 3f, 3g, 3h, and 3j from Set 3, 4a, 4f, 4g, 4h, and 4i from Set 4, and 5d and 5k from Set 5 as options to look for a possible solution.

First, we see what happens when we fix 2c with 1h. This eliminates 3j from Set 3, 4f, 4g, and 4h from Set 4, and 5d from Set 5. Our options are 3e, 3f, 3g, and 3h from Set 3, 4a and 4i from Set 4, and 5k from Set 5. Since there is only one option left from Set 5, we fix 5k with 1h and 2c. This eliminates 3e, 3f, and 3g from Set 3, and 4a from Set 4. This reduces our options down to only 3h and 4i from Sets 3 and 4, respectively. When we compare the tiles in 3h to those in 4i, we find that they share pR, rG, and gB. This means that we cannot use 3h with 4i, which implies that we cannot use 2c with 1h.

Our only other option from Set 2 to fix with 1h is 2d. Recall that 2d is composed of the tiles bG, gR, rW, wP, and pB. This eliminates 3f, 3g, and 3j from Set 3, 4a, 4h, and 4i from Set 4, and 5d from Set 5. Our options from Sets 3, 4, and 5 are 3e, 3h, 4f, 4g, and 5k. Since 5k is the only option from Set 5, we choose to fix 5k with 2d and 1h. This eliminates 3e from Set 3 and 4f from Set 4, leaving only 3h from Set 3 and 4g from Set 4 as possibilities. Upon comparing the tiles of 3h and 4g, we find that they share yP and gB. Hence, we cannot use 2d with 1g. This implies that there is no solution when 1h is fixed.

<u>Case 9: Fix 1i.</u> Next, we look at 1i, which contains the tiles bR, rY, yG, gP, and pB, to look for a possible solution. Just by comparing tiles, we can eliminate 2a, 2b, 2d, 2e, 2f, 2i 2j, and 2k from Set 2, 3a, 3b, 3c, and 3d, from Set 3, 4a, 4b, 4c, 4d, 4e, 4j, and 4k from Set 4, and 5b, 5c, 5e, 5i, and 5j from Set 5. This leaves us with 2c,2g, and 2h from Set 2, 3, 3f, 3g, 3h, 3i, 3j, and 3k from Set 3, 4f, 4g, 4h, and 4i from Set 4, and 5a, 5d, 5f, 5g, 5h, and 5k as options to look for a possible solution.

First, we set 2c fixed with 1i. We can further eliminate 3i and 3j from Set 3, 4f, 4g, and 4h from Set 4, and 5d, 5f, and 5h from Set 5, leaving 3e, 3f, 3g, 3h, and 3k from Set 3, 4i from Set 4, and 5a, 5g, and 5k from Set 5. Since there is only one option from Set 4, 4i, we set it fixed with 2c and 1i. This eliminates all of Set 3 except 3k and all of Set 5 except 5g and 5k. Since there is only one option from Set 3 and two options from Set 5, we compare the tiles in 3k to those in 5g and 5k. We cannot use 5g with 3k since they have tiles pG and gW in common. However, we can use 5k with 3k as they have no tiles in common. We now look at the rotations of 2c with 1i being fixed.

1i	bR	rY	yG	gP	pB
2c(i)	**gR**				
2c(ii)	rW	**wY**			
2c(iii)	wY	yB	**bG**		
2c(iv)	yB	bG	gR	rW	wY

The only rotation of 2c that works when 1i is fixed is 2c(iv). Now, we look at the rotations of 3k with 1i and 2c(iv) are fixed.

1i	bR	rY	yG	gP	pB
2c(iv)	yB	bG	gR	rW	wY
3k(i)	yP	**pG**			
3k(ii)	pG	gW	wB	bY	yP
3k(iii)	gW	wB	bY	**yP**	
3k(iv)	**wB**				

When we have 1i and 2c(iv) fixed, the only rotation of 3k that will work is 3k(ii). We now proceed to look at the rotations of 4i with 1i, 2c(iv), and 3k(ii) fixed.

1i	bR	rY	yG	gP	pB
2c(iv)	yB	bG	gR	rW	wY
3k(ii)	pG	gW	wB	bY	yP
4i(i)	wP	pR	**rG**		
4i(ii)	**pR**				
4i(iii)	**rG**				
4i(iv)	**gB**				

No rotation of 4i works when we have 1i, 2c(iv), and 3k(ii) fixed. Thus, there is no solution when 2c is fixed with 1i.

Now, we see what happens when we fix 2g with 1i. This eliminates 3f, 3g, 3h, 3i, 3j, and 3k from Set 3, 4f, 4i, and 4h from Set 4, and 5a, 5d, 5g, 5h, and 5k from Set 5. This leaves us with only 3e from Set 3, 4g from Set 4, and 5f from Set 5. Upon comparing the tiles in 3e, 4g, and 5f, we find that 3e and 4g share tile gB and 4g and 5f share tiles pG. So there is no solution when we set 2g fixed with 1i.

Our last option from Set 2 that we can fix with 1i is 2h. By doing this, we eliminate 3f, 3g, and 3k from Set 3, 4h and 4i from Set 4, and 5a, 5d, 5g, 5h, and 5k from Set 5. This leaves us with 3e, 3h, 3i, and 3j from Set 3, 4f and 4g from Set 4, and 5f from Set 5. This leaves us with only one option left from Set 5, so we set 5f fixed with 2h and 1i. We can now eliminate 3i and 3j from Set 3, and 4g from Set 4. Our options from Set 3 are now 3e and 3h, while our only option left from Set 4 is 4f. We now compare the tiles in 3e and 3h to those in 4f. We cannot use 3h since it shares tiles yP and pR with 4f, but since 3e and 4f have no tiles in common we can use 3e. Now we look at the rotations of 2h when 1i is fixed.

1i	bR	rY	yG	gP	pB
2h(i)	gW	wP	pY	yB	bG
2h(ii)	wP	**pY**			
2h(iii)	pY	yB	**bG**		
2h(iv)	yB	bG	gW	**wP**	

The only rotation of 2h that works with 1i fixed is 2h(i). Now we look at the rotations of 3e with 1i and 2h(i) fixed.

1i	bR	rY	yG	gP	pB
2h(i)	gW	wP	pY	yB	bG
3e(i)	**yW**				
3e(ii)	**wR**				
3e(iii)	rG	gB	**bY**		
3e(iv)	gB	**bY**			

No rotation of 3e works when we have 1i and 2h(i) fixed. Thus, we cannot use 2h with 1i. Therefore, there is no solution when 1i is fixed.

Case 10: Fix 1j. We now look for possible solutions when we have 1j fixed. Note that 1j contains the tiles bR, rY, yW, wP, and pB. This allows us to eliminate 2a, 2b, 2d, 2e, 2f, 2g, 2h, and 2i from Set 2, 3c, 3e, 3f, and 3g from Set 3, 4a, 4b, 4c, 4e, 4h, 4i, 4j, and 4k from Set 4, and 5b, 5c, 5e, and 5k from Set 5. This leaves us with 2c and 2j from Set 2, 3a, 3b, 3d, 3h, 3i, 3j, and 3k from Set 3, 4d, 4f, and 4g from Set 4, and 5a, 5d, 5f, 5g, 5h, 5i, and 5j from Set 5.

We begin by fixing 2c with 1j. We can further eliminate 3a, 3i and 3j from Set 3, 4d, 4f, and 4g from Set 4, and 5d, 5f, 5h, and 5i from Set 5. Notice that this eliminates all of Set 4, which implies that there is no solution when we fix 2c with 1j.

Now, we set 2j fixed with 1j to look for possible solutions. We can eliminate 3a, 3d, 3h, 3i, and 3k from Set 3, 4f from Set 4, and 5a, 5d, 5f, and 5j from Set 5. Now our only possibilities are 3b, 3j, 4d, 4g, 5g, 5h, and 5i, which come from Sets 3, 4, and 5, respectively. There are two options from Set 3, so we first fix 3b with 2j and 1j. This allows us to eliminate 4d from Set 4, and 5g, 5h, and 5i from Set 5, eliminating all of our options from Set 5. So we cannot use 3b with 1j and 2j. Our next option from Set 3 is 3j to fix with 1j and 2j. This allows us to eliminate 4d and 4g from Set 4 and 5g, 5h, and 5i from Set 5. Notice that this eliminates all of Set 4 and all of Set 5. Thus, we cannot use 2j with 1j. Hence, there is no solution when we fix 1j.

Case 11: Fix 1k. The last possibility when looking for a solution comes by fixing 1k, which contains the tiles bR, rY, yP, pG, and gB. We can eliminate 2a, 2b, 2e, 2f, and 2i from Set 2, 3e, 3g, 3h, 3i, 3j, and 3k from Set 3, 4b, 4c, 4f, 4g, 4h, 4i, 4j, and 4k from Set 4, and 5b, 5c, 5e, 5f, and 5g from Set 5. This leaves us with 2c, 2d, 2g, 2h, 2j, and 2k from Set 2, 3a, 3b, 3c, 3d, and 3f from Set 3, 4a, 4d, and 4e from Set 4, and 5a, 5d, 5i, 5j, and 5k from Set 5 as options when looking for a possible solution.

Set 4 contains the smallest number of possibilities, so we begin by fixing 4a with 1k. We can eliminate 2d, 2j, and 2k from Set 2, 3b, 3c, and 3d from Set 3, and 5a and 5k from Set 5. This reduces our options down to 2c, 2g, and 2h from Set 2, 3a and 3f from Set 3, and 5d, 5i, and 5j from Set 5. Since Set 3 contains the smallest number of possibilities, we set 3a fixed with 4a and 1k. This eliminates 2c from Set 2, leaving us with 2g and 2h as possibilities. It also eliminates 5d, 5i, and 5j from Set 5, which in turn is all of Set 5, leaving us with no solution when we fix 3a with 4a and 1k. Next, we fix 3f with 4a and 1k. This eliminates 2g and 2h from Set 2, and 5d and 5i from Set 5. Our only possible options from Sets 2 and 5 are 2c and 5j, respectively. Upon comparing the tiles in 2c and 5j, we find there are no common tiles. So we now look at the rotations of 2c when 1k is fixed.

1k	bR	rY	yP	pG	gB
2c(i)	**gR**				
2c(ii)	rW	**wY**			
2c(iii)	wY	yB	bG	gR	rW
2c(iv)	yB	bG	gR	rW	wY

We find that we can use rotations 2c(iii) and 2c(iv) when we have 1k fixed. Now we look at the rotations of 3f when we have 1k and 2c(iii) fixed.

1k	bR	rY	yP	pG	gB
2c(iii)	wY	yB	bG	gR	rW
3f(i)	yW	wP	pR	rB	bY
3f(ii)	wP	pR	rB	bY	**yW**
3f(iii)	**pR**				
3f(iv)	rB	**bY**			

The only rotation of 3f that works when we have 1k and 2c(iii) fixed is 3f(i). We can now set 3f(i) fixed with 1k and 2c(iii) and look at the rotations of 4a.

1k	bR	rY	yP	pG	gB
2c(iii)	wY	yB	bG	gR	rW
3f(i)	yW	wP	pR	rB	bY
4a(i)	**wR**				
4a(ii)	rG	**gP**			
4a(iii)	gP	**pB**			
4a(iv)	pB	bW	**wR**		

No rotation of 4a works when we have 1k, 2c(iii), and 3f(i) fixed. This implies that we cannot use the rotation 2c(iii) with 1k. We have one other option from Set 2, 2c(iv), that we can fix with 1k. We need to now look at the rotations of 3f with 2c(iv) and 1k fixed.

1k	bR	rY	yP	pG	gB
2c(iv)	yB	bG	gR	rW	wY
3f(i)	yW	wP	**pR**		
3f(ii)	wP	pR	rB	bY	yW
3f(iii)	**pR**				
3f(iv)	**rB**				

The only rotation of 3f that works when we have 2c(iv) fixed with 1k is 3f(ii). We now look at the rotations of 4a when we have 1k, 2c(iv), and 3f(ii) fixed.

1k	bR	rY	yP	pG	gB
2c(iv)	yB	bG	gR	rW	wY
3f(ii)	wP	pR	rB	bY	yW
4a(i)	**wR**				
4a(ii)	rG	gP	**pB**		
4a(iii)	**gP**				
4a(iv)	**pB**				

No rotation of 4a works when we fix 1k, 2c(iv), and 3f(ii). This implies that we cannot use 4a with 1k.

We now look set 4d fixed with 1k and look for options from Sets 2, 3, and 5. By fixing 4d, we eliminate 2c, 2d, and 2g from Set 2, 3a, 3b, 3c, 3d, and 3f, from Set 3, and 5d, 5i, 5j, and 5k from Set 5. Notice that while we are left with some options from Sets 2 and 5, we have eliminated all of Set 3. Thus, we cannot use 4d with 1k.

Our last possibility is to set 4e fixed with 1k. This eliminates 2c, 2d, 2j, and 2k from Set 2, 3a, 3b, 3c, and 3d from Set 3, and 5d, 5i, and 5j from Set 5. Our options are now 2g and 2h from Set 2, 3f from Set 3, and 5a and 5k from Set 5. We now fix

3f with 1k and 4e. This eliminates 2g and 2h from Set 2, and 5a and 5k from Set 5, eliminating all of Set 2 and all of Set 5. So we cannot use 3f with 1k and 4e, which implies that we cannot use 1k when looking for a possible solution. Therefore, the 5×5 puzzle is not solvable. $\square$

5 CONCLUDING REMARKS

Earlier, we discussed what it means to be a generator. We also discussed the difference between the alternating group and the symmetric group on n letters. Theorem 1.5 tells us that the alternating group is generated by 3-cycles. So, what is a minimum generating set for S_n?

Section 2.1 describes that God's Algorithm is the minimum number of moves to solve the Rubik's cube from any scrambled state. Does God's Algorithm exists for Instant Insanity II? If so, what is God's Algorithm for Instant Insanity II? Also, for what other values of n and k does an Instant Insanity II puzzle on $n \times k$ tiles have a unique solution?

We know that the fifteen puzzle is a two-dimensional puzzle that is on a 4×4 grid. We also observed that the original Instant Insanity is a three-dimensional puzzle on a $1 \times 1 \times 4$ grid, and the Rubik's cube is also a three-dimensional puzzle but is on a $3 \times 3 \times 3$ grid. Looking at these three different puzzles, one can claim that the fifteen puzzle may be "easier" than Instant Insanity or the Rubik's cube since it is only a two-dimensional puzzle. So we must ask the question "How do we quantify the difficulty of a puzzle?" The following lists some things one might want to consider in trying to find the difficulty of a puzzle.

- The number of combinations.
- The average "word" length. That is, what is the minimum sequence of moves required to swap any two tiles of Instant Insanity II?
- The existence of a "good" algorithm.

BIBLIOGRAPHY

[1] W. W. Rouse Ball and H. S. M. Coxeter. *Mathematical recreations and essays*. Dover Publications Inc., New York, thirteenth edition, 1987.

[2] T. A. Brown. A Note on “Instant Insanity”. *Math. Mag.*, 41(4):167–169, 1968.

[3] J. R. Durbin. *Modern algebra*. John Wiley & Sons Inc., New York, sixth edition, 2009. An introduction.

[4] J. B. Fraleigh. *A first course in abstract algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1967.

[5] A. P. Grecos and R. W. Gibberd. A Diagrammatic Solution to “Instant Insanity” Problem. *Math. Mag.*, 44(3):119–124, 1971.

[6] M. Hall, Jr. *Combinatorial theory*. Blaisdell Publishing Company, 1967.

[7] I. N. Herstein. *Topics in algebra*. Blaisdell Publishing Co. Ginn and Co. New York-Toronto-London, 1964.

[8] D. R. Hofstadter. Magic cubology. In *Metamagical Themas*, pages 301–328. Basic Books, 1981.

[9] D. R. Hofstadter. On crossing the rubicon. In *Metamagical Themas*, pages 329–363. Basic Books, 1982.

[10] A. Jebasingh and A. Simoson. Platonic solid insanity. In *Proceedings of the Thirty-third Southeastern International Conference on Combinatorics, Graph*

Theory and Computing (Boca Raton, FL, 2002), volume 154, pages 101–111, 2002.

[11] O. Knill, R. E. Mäder, and ETH Zürich. The rotation group of Rubik's cube. *Sigsam Bulletin*, 21:33–43, 1987.

[12] J. L. Lagrange. Reflections on the algebraic solution of equations. *New Memoirs of the Royal Academy of Sciences and Belled-Lettres in Berlin*, pages 202–203, 1771.

[13] J. Mulholland. Lecture 12: Puzzle groups. http://people.math.sfu.ca/~jtmulhol/math302/notes/12-PuzzleGroups.pdf, 2011. Accessed: June 20, 2013.

[14] T. Richmond and A. Young. Instant Insanity II. *College Math. J.*, 44(4):265–272, 2013.

[15] F. S. Roberts. *Applied Combinatorics*. Prentice-Hall, Inc., New Jersey, 1984.

[16] J. Scherphuis. Instant Insanity II. http://www.jaapsch.net/puzzles/insanity2.htm, 2013. Accessed: July 10, 2013.

[17] W. Schmidt. This and that. http://www.waynesthisandthat.com/15puzzle.htm, 2013. Accessed: September 15, 2013.

[18] B. L. Schwartz. An Improved Solution to "Instant Insanity". *Math. Mag.*, 43(1):20–23, 1970.

[19] E. L. Spitznagel, Jr. A New Look at the Fifteen Puzzle. *Math. Mag.*, 40(4):171–174, 1967.

VITA

AMANDA JUSTUS

Education:	B.S. Mathematics, King College, Bristol, Tennessee 2012
	M.S. Mathematical Sciences, East Tennessee State University, Johnson City, Tennessee 2014
Professional Experience:	Graduate Assistant, East Tennessee State University, Johnson City, Tennessee, 2012–2014
Awards/Honors:	Features story “Solving Puzzles with Algebra” in ETSU School of Graduate Studies graduate student research magazine, “Illuminated,” Volume 3(1), pp. (14-15).