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Tricyclic Steiner Triple Systems with 1-Rotational Subsystems.

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by

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ABSTRACT

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A Steiner triple system of order v, denoted $STS(v)$, is said to be *tricyclic* if it admits an automorphism whose disjoint cyclic decomposition consists of three cycles. In this thesis we give necessary and sufficient conditions for the existence of a tricyclic $STS(v)$ when one of the cycles is of length one. In this case, the $STS(v)$ will contain a subsystem which admits an automorphism consisting of a fixed point and a single cycle. The subsystem is said to be 1-*rotational*.

DEDICATION

To my family whom I love and value most.

ACKNOWLEDGMENTS

I would like to thank Dr. Robert Gardner, my advisor and mentor, for his advice, support and patience during preparation of this thesis. I am deeply grateful for all his help throughout my time at ETSU.

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1 INTRODUCTION

There are many real world situations which can be described best by a diagram consisting of points and connecting lines. For example, by means of using points to represent people and lines to connect relatives, we can construct family trees for anyone. The mathematical abstraction behind such situations leads to the concept of a graph and eventually to the field of graph theory. It was not a surprise that, when the first computer system was invented, graph theoretic tools immediately arose in every computational study: network design and analysis, database theory, artificial intelligence, complexity theory, and matrix computations to name a few.

As a branch of graph theory, combinatorial designs arose together with the appearance of computer science and operations research. According to Colbourn and Van Oorschot in their article "Applications of Combinatorial Designs in Computer Science," [5] the application of combinatorial designs varies from file organization, sorting in rounds to probabilistic and deterministic algorithms, authentication codes, and lower bounds for algorithms. Designs [not only] provide balanced set systems, [but they also] are minimum coverings and maximum packings [5]. In this thesis, we will explore the combinatorial design for the model of tricyclic Steiner triple systems with 1-rotational subsystems. However, many basic definitions are needed; hence, this chapter serves the purpose of providing enough background so that readers will have a thorough understanding of this research.

1.1 Basic Definitions

Given a graph G on V vertices and E edges, denoted $G = (V, E)$, we can define a *simple graph* to be a graph with no multiple edges. For any two vertices $x, y \in G$, if (x, y) is an edge of G, then we say x and y are adjacent to each other. We call a graph G a *complete graph* if all of its vertices are pairwise adjacent, and we denote a complete graph on n vertices by K_n. For example: K_2 is a segment (line); K_3 is a triangle; K_4 is a rectangle with its diagonals, etc.

As expected, the concept of graph isomorphism is quite similar to many other isomorphism concepts that we have encountered. Given two graphs $G = (V, E)$ and $G' = (V', E')$, we say G and G' are *isomorphic*, denoted $G \cong G'$, if there exists a bijection $\phi : V \to V'$ with $(x, y) \in E$ if and only if $(\phi(x), \phi(y)) \in E'$ for all $x, y \in V$. Such a map ϕ is called an *isomorphism*. Furthermore, if $G \cong G'$ then ϕ is called an *automorphism*.

A *decomposition* of a graph $G = (V, E)$ into isomorphic copies of a graph $g < G$ is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$, $V(g_i) \subset V(G)$ (for all i), $E(g_i) \cap E(g_j) = \emptyset$ (where $i \neq j$) and $\bigcup E(g_i) = E(G)$ (for all i). Such decomposition will be referred to as "g-decomposition of G".

We may define a *Steiner Triple System* (denoted STS) to be a K_3-decomposition of a complete graph K_v of order v. Let G be a graph and let $\beta = \{g_1, g_2, \ldots, g_n\}$ be a K_3-decomposition of G. An automorphism of this decomposition is a permutation of the vertex set $V(G)$ which fixes the set β. That is, if g_i is a block of the triple and π is a permutation, then $\pi(g_i)$ also forms a block. A permutation π of a v-element set is said to be of type $[\pi]_v = [\pi_1, \pi_2, ..., \pi_v]$ if the disjoint cyclic decomposition of π

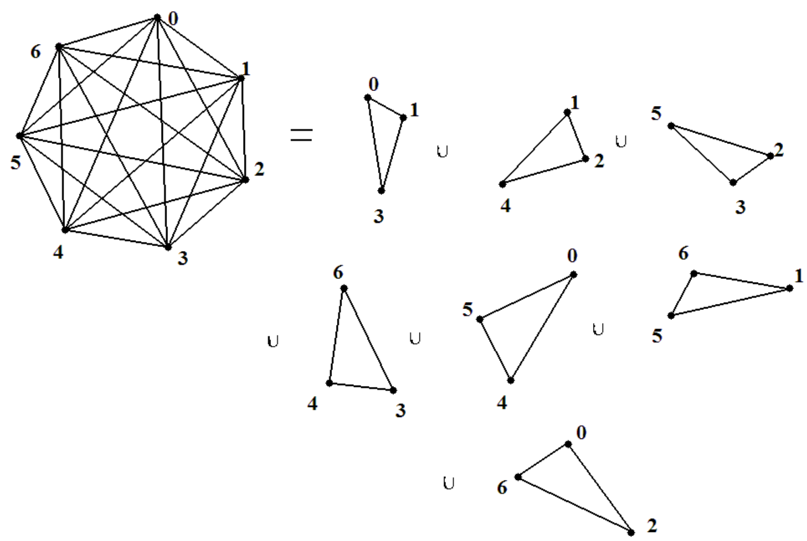

Figure 1: An Illustration of Steiner Triple System: STS for K_7.

contains π_i cycles of length i.

The *orbit* of a block under an automorphism π is the image of the block under the powers of π. A collection of blocks B is said to be a collection of base blocks for a STS under the permutation π if the orbits of the blocks of B produce the STS and exactly one block of B occurs in each orbit.

Consider a permutation π on a v element set. We say a $STS(v)$ is a *cyclic* $STS(v)$ if it admits an automorphism of type $[0, 0, ..., 0, 1]$ and such a system exists if and only if $v \equiv 1$ or $3 \pmod 6$ and $v \neq 9$ [8]. For example: the permutation $\pi = (0, 1, 2, 3, 4, 5, 6)$ is the automorphism of a $STS(7)$ which consists of the blocks $\{(013), (124), (235), (346), (450), (561), (602)\}$ as shown in Figure 1. A *reverse STS(v)* admits an automorphism of type $[1, (v-1)/2, 0, \ldots, 0]$. Reverse $STS(v)$s exist if and only if $v \equiv 1, 3, 9, 19 \pmod{24}$ [9].

1.2 Previous Study of 1-Rotational Steiner Triple Systems

A *k-rotational* $STS(v)$ admits an automorphism of type $[1, 0, 0, \ldots, 0, k, 0, \ldots, 0]$. That is, the disjoint cyclic decomposition of π consists of a single point and precisely k cycles of length $\frac{v-1}{k}$ as shown in Figure 2.

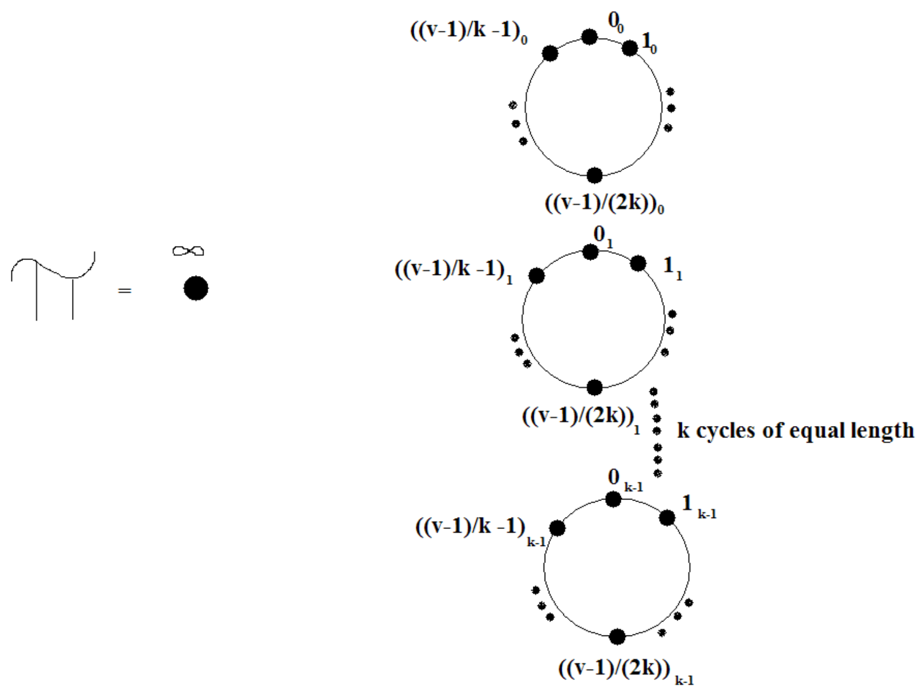

Figure 2: A k-Rotational Steiner Triple System.

The k-rotational $STS(v)$ was first introduced by Phelps and Rosa in their paper "Steiner Triple Systems with Rotational Automorphisms." [9] They were able to prove the existence of such system for the cases where $k = 1, 2, 6$ [9]. Later, Cho proved the existence of cases where $k = 3, 4$ in his paper "Rotational Steiner Triple Systems" which appeared in *Discrete Mathematics* [3]. However, it was not until 1996, in his paper "The Spectrum for Rotational Steiner Triple Systems," which appeared in *Journal of Combinatorial Designs*, that Colbourn and Jiang were able to show the necessary conditions of a general case of k-rotational $STS(v)$ to exist which settled, once and for all, the existence problem for a k-rotational $STS(v)$ [4].

Theorem 1.1 [4] *Let v, k be positive integers such that $1 \leq k \leq (v-1)/2$. Then a k-rotational STS(v) exists if and only if*

i) $v \equiv 1, 3$ (mod 6)

ii) $v \equiv 3$ (mod 6) if $k = 1$

iii) $v \equiv 1$ (mod k)

iv) $v \neq 7, 13, 15, 21$ (mod 24) if $(v-1)/k$ is even.

In this thesis, we are much interested in the *1-rotational STS(v)* which leads us back to the pioneering work of Phelps and Rosa [9]. Following the definition of a k-rotational $STS(v)$, a *1-rotational STS(v)* admits an automorphism of type $\pi = [1, 0, 0, \ldots, 0, 1, 0, \ldots, 0]$. We can describe a 1-rotational STS(v) as a disjoint cyclic decomposition of π consisting of a single point and a cycle of length $v - 1 > 1$ as shown in Figure 3.

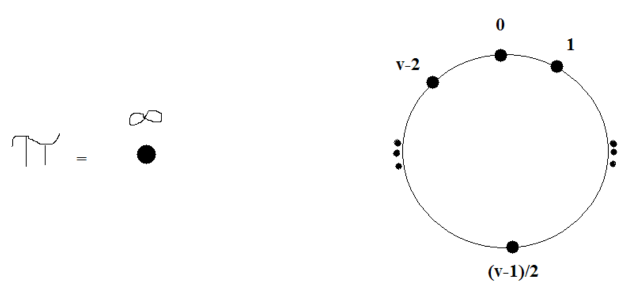

Figure 3: A 1-Rotational Steiner Triple System.

The condition in which 1-rotational $STS(v)$ exists along with proposed base blocks are shown in the following theorem and proved by Phelps and Rosa in their paper "Steiner Triple Systems with Rotational Automorphisms" which appeared in *Discrete Mathematics* in 1981 [9].

Theorem 1.2 [9] *A 1-rotational STS(v) exists if and only if* $v \equiv 3$ *or* 9 *(mod* 24*).*

Proof. To prove the necessary condition, let $V = \mathbb{Z}_{v-1} \cup \{\infty\}$, and let $\alpha = (\infty)(0, 1, \ldots, v-2)$ be an automorphism of a 1-rotational $STS(v)$. Since $\{\infty, i, j\} \in B$ [where B is the base block of 1-rotational $STS(v)$] implies $\{\infty, i+1, j+1\} \in B$, it follows that $\{\infty, i, j\} \in B$ if and only if $i - j \equiv \frac{1}{2}(v-1)(\text{mod } v-1)$; in other words, any 1-rotational $STS(v)$ contains $\frac{1}{2}(v-1)$ triples of the form $\left\{\infty, i, i+\frac{1}{2}(v-1)\right\}$ (mod v). All 3-subsets of V not containing the element ∞ are partitioned into orbits under α all of which are of length $v-1$ except possibly a single orbit Q_0 of length $\frac{1}{3}(v-1)$ of triples $\left\{0, \frac{1}{3}(v-1), \frac{2}{3}(v-1)\right\}$. It is easily seen that no 1-rotational $STS(v)$ contains a triple of Q_0: this would require $v \equiv 1$ (mod 6), and at the same time, there would be need for further $\frac{1}{6}v(v-1) - \frac{1}{2}(v-1) - \frac{1}{3}(v-1) = \frac{1}{6}(v-1)(v-5)$ triples in B which would then necessarily have to be partitioned into $\frac{1}{6}(v-5)$ orbits of length $v-1$; this is obviously impossible as $\frac{1}{6}(v-5)$ is not an integer. Thus, the remaining $\frac{1}{6}v(v-1) - \frac{1}{2}(v-1) = \frac{1}{6}(v-1)(v-3)$ triples of B fall into $\frac{1}{6}(v-3)$ orbits of length $v-1$. If $\{a, b, c\}$ is a triple in one such orbit then clearly the six differences $\pm(a-b), \pm(b-c), \pm(a-c)$ are all distinct, and if $\{a_1, b_1, c_1\}, \{a_2, b_2, c_2\}$ are two triples from two orbits in B then the corresponding 12 differences are all distinct. Since there are still $v-3$ non-zero differences "available" it follows that $\frac{1}{6}(v-3)$ must be an integer, and so we must have

$$v \equiv 3 \pmod{6}. \tag{1}$$

On the other hand, since v is odd, the automorphism $\alpha^{(v-1)/2}$ is a permutation of type $[j] = [1, \frac{1}{2}(v-1), 0 \ldots, 0]$, and so (V, B) is a reverse $STS(v)$. Hence, we should have

$$v \equiv 1, 3, 9, 19 \pmod{24}. \tag{2}$$

The congruences (1) and (2) together yield the necessary condition. Now, we will show the sufficiency of the condition. So let $v \equiv 3, 9$ $(mod\ 24)$. Define a set B of triples on V as follows:

$$B = B_1 \cup B_2$$

where

$$B_1 = \left\{ \left\{ \infty, i, i + \frac{1}{2}(v-1) \right\} | i = 0, \ldots, \frac{1}{2}(v-3) \right\},$$

$$B_2 = \{\{i, i+r, i+b_r+k\} \, | i = 0, \ldots, v-2; r = 1, \ldots, k\}$$

where $\{(a_r, b_r)|r = 1, \ldots, k\}$ is any (A, k)-system with $k = \frac{1}{6}(v-3)/6$. Note: an (A, k)-system is a set of order pairs $\{(a_r, b_r)|r = 1, \ldots, k\}$ such that $b_r - a_r = r$ for $r = 1, \ldots, k$, and $\bigcup_{r=1}^{k} \{a_r, b_r\} = \{1, 2, \ldots, 2k\}$; an (A, k)-system exists if and only if $k \equiv 0, 1$ (mod 4) [8]. Now, since $v \equiv 3, 9$ (mod 24), $k \equiv 0, 1$ (mod 4) and so an (A, k)-system exists. We claim that (V, B) is a 1-rotational $STS(v)$. Indeed, given a pair of elements ∞, i where $i \in \mathbb{Z}_{v-1}$, it is contained in exactly one triple of B_1, since clearly the set $\left\{\{i, i + \frac{1}{2}(v-1)\} \, | i = 0, 1, \ldots, \frac{1}{2}(v-3)\right\}$ partitions $\mathbb{Z}_{v-1}$. Given a pair $i, j \in \mathbb{Z}_{v-1}, i \neq j$, we look at their difference Δ_{ij}: if $\Delta_{ij} = \frac{1}{2}(v-1)$ then $\{i, j\}$ is contained in a unique triple of B_1. If $\Delta_{ij} = s \neq \frac{1}{2}(v-1)$ we may assume w.l.o.g $1 \leq$

$s \leq \frac{1}{2}(v-3)$. The six differences between the elements of a triple in B_2 are $\pm r, \pm(a_r + k), \pm(b_r + k)$, and since the difference triples $\{\{r, a_r + k, b_r + k\} \,|r = 1, \ldots, k\}$ cover the set $\left\{1, 2, \ldots, \frac{1}{2}(v-3)\right\}$ it follows that the pair i, j with $\Delta_{ij} = s$ is contained in exactly one triple of $B - 2$. Thus, (V, B) is a $STS(v)$. Observing that $\alpha = (\infty)(0, 1, \ldots, v-2)$ is an automorphism of (V, B) completes the proof. ∎

1.3 Bicyclic Steiner Triple Systems and the Motivation

We may define a *bicyclic Steiner triple system* as a $STS(v)$ admitting an automorphism of type $[\pi] = [0, 0, \ldots, 0, \pi_M, 0, \ldots, 0, \pi_N, 0, \ldots, 0]$ where $\pi_M = \pi_N = 1, M < N$ and $M + N = v$. In other words, a *bicyclic Steiner triple systems* has an automorphism π with a disjoint cyclic decomposition of π consisting of one cycle of length M and another (larger) cycle of length N as shown in Figure 4. It is easily seen that a 1-rotational $STS(v)$ is just a special case of a bicyclic $STS(v)$ where $M = 1$.

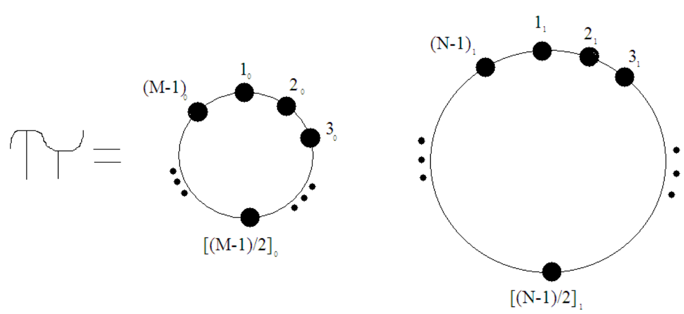

Figure 4: A Bicyclic Steiner Triple System.

In fact, the existence problem for a bicyclic $STS(v)$ was proposed and first studied by Calahan and Gardner in their paper "A Special Case of Bicyclic Steiner Triple Systems" in 1992 [1] . In their paper "Bicyclic Steiner Triple Systems", they were able to determine and prove the existence condition for bicyclic $STS(v)$s [2].

Theorem 1.3 [2] *A bicyclic STS(v) where $v = M + N$ admitting an automorphism whose disjoint cyclic decomposition is a cycle of length M, where $M > 1$, and a cycle of length N exists if and only if $M \equiv 1$ or 3 (mod 6), $M \neq 9$, $M|N$ and $v = M + N \equiv 1$ or 3 (mod 6).*

Since the existence problem for bicyclic $STS(v)$ has been solved, a reasonable new research approach should be *tricyclic Steiner triple systems* and its existence condition. Up until this point, our readers might already have guessed what a tricyclic permutation should be or should look like. In fact, tricyclic $STS(v)$ has a disjoint cyclic decomposition consisting of three cycles of the same length or different lengths. The ideal general case for a tricyclic $STS(v)$ should be a disjoint cyclic decomposition of an automorphism π consisting of a cycle of length X, another cycle of length Y, and another cycle of length Z where $X < Y < Z$.

We can certainly take advantage of the previous results from bicyclic $STS(v)$ to attack the existence problem for a general case of tricyclic $STS(v)$; that is, the general case may be catergorized into tricyclic $STS(v)$ with bicyclic $STS(v)$ subsystem and tricyclic $STS(v)$ with no bicyclic $STS(v)$ subsystem. Since 1-rotational $STS(v)$ is a special case of bicyclic $STS(v)$, in order to study the existence condition of tricyclic $STS(v)$, we must first investigate the existence condition for the case of tricyclic $STS(v)$ with 1-rotational subsystem. Therefore, this thesis will not address the general case of tricyclic $STS(v)$s; however, it will address some of the basic special cases of such system, especially the case of tricyclic $STS(v)$ with 1-rotational subsystem.

2 TRICYCLIC STEINER TRIPLE SYSTEM WITH 1-ROTATIONAL SUBSYSTEMS

In this chapter, we shall dicuss several tricyclic Steiner triple systems. The following theorem 2.1 and 2.2, along with lemma 2.1 to lemma 2.5 introduced in this chapter are following results of work of Calahan and Gardner from the early 1990s [1]. The main result introduced in this chapter is the block design of tricyclic Steiner triple systems with 1-rotational subsystems which leads to the highlighted theorem 4.1 in chapter 4. Furthermore, the method of construction will be explained in more detail in chapter 3 where we introduce the *difference method*.

2.1 Some Tricyclic Steiner Triple Systems

We define a *tricyclic* $STS(v)$ to be one that admits an automorphism either of type $[0,\ldots,0,3,0,\ldots,0]$ (as shown in Figure 5), $[0,\ldots,0,1,0,\ldots 0,2,0,\ldots,0]$, or of type $[0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0]$. From the existence of a cyclic $STS(v)$ we readily have:

Theorem 2.1 *A tricyclic* $STS(v)$ *admitting an automorphism of type* $[0,\ldots,0,3,0,\ldots,0]$ *exists if and only if* $v \equiv 3 \ (mod\ 6)$.

Proof. Of course the condition $v \equiv 3 \pmod 6$ is necessary. For all such v, except $v = 9$, there is a cyclic $STS(v)$. Simply by cubing the cyclic automorphism, we see that the systems are also tricyclic. For $v = 9$, consider the collection of blocks: $(0_0, 0_1, 2_0)$, $(0_0, 0_2, 2_2)$, $(0_0, 1_2, 2_1)$, and $(0_1, 1_1, 1_2)$. This is a collection of base blocks for a tricyclic $STS(9)$ under the automorphism $\pi = (0_0, 1_0, 2_0)(0_1, 1_1, 2_1)(0_2, 1_2, 2_2)$ where the point set is $\mathbb{Z}_3 \times \mathbb{Z}_3$. Here, and throughout, we represent the ordered pair (x, y) as the subscripted pair x_y. ∎

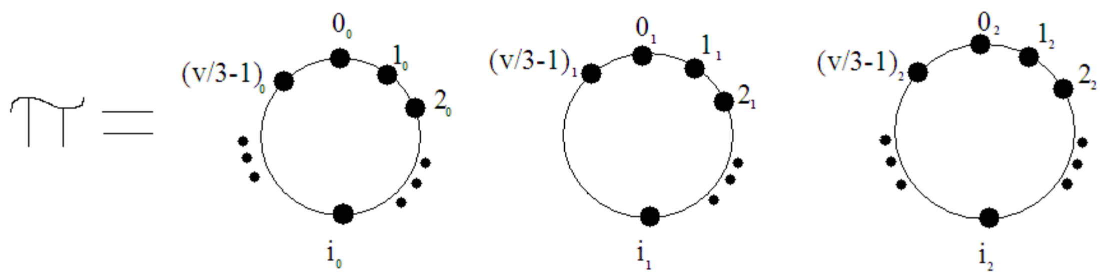

Figure 5: A Tricyclic Steiner Triple System Where Every Cycle Have the Same Length.

Similarly, we can establish the existence of a large class of tricyclic STSs from the existence of the bicyclic STSs.

Theorem 2.2 *A tricyclic $STS(v)$ admitting an automorphism of type $[\pi] = [0, \ldots, 0, 1, 0, \ldots, 0, 2, 0, \ldots, 0]$ where $\pi_M = 1$, $\pi_N = 2$ and $M > 1$ exists if and only if $M \equiv 1$ or 3 $(mod\ 6)$, $M \neq 9$, $M \mid N$ and $v = N + 2M \equiv 1$ or 3 $(mod\ 6)$.*

Proof. First, suppose there is such a system with the point set $\mathbb{Z}_M \bigcup \mathbb{Z}_N \times \mathbb{Z}_2$ admitting the automorphism $\pi = (0, 1, \ldots, M-1)(0_0, 1_0, \ldots, (N-1)_0)(0_1, 1_1, \ldots, (N-1)_1)$.

The fixed points of an automorphism form a subsystem of a STS. That is, if we fix two vertices of a base block then the remaining vertex has to be fixed too. By considering π^M we see, therefore, that such a $STS(v)$ has a cyclic subsystem of order M. Therefore, $M \equiv 1$ or 3 (mod 6) and $M \neq 9$ is necessary. Also, such a STS must contain some block of the form (x, y_i, z_j) where $x \in \mathbb{Z}_M$ and $y_i, z_j \in \mathbb{Z}_N \times \mathbb{Z}_2$. By applying π^N to this block, we see that $(\pi^N(x), y_i, z_j)$ must also be a block of the STS and therefore $\pi^N(x) = x$ and $M \mid N$ is necessary.

To establish sufficiency, suppose M and N satisfy the stated conditions. Then there is a bicyclic $STS(v)$ admitting an automorphism consisting of a cycle of length M and a cycle of length $2N$. By considering the square of this automorphism, we see that the bicyclic $STS(v)$ is also tricyclic and admits an automorphism of the desired type. ∎

Notice that 2-rotational and 1-transrotational STSs are also examples of tricyclic STSs.

2.2 Tricyclic Steiner Triple Systems with 1-Rotational Subsystems

We now turn our attention to STSs admitting automorphisms of type $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = \pi_M = \pi_N = 1$, $v = M + N + 1$, and $M < N$. In our discussion, we will let the point set of such a system be $\{\infty\} \bigcup \mathbb{Z}_M \times \{0\} \bigcup \mathbb{Z}_N \times \{1\}$ and let the automorphism be $\pi = (\infty)(0_0, 1_0, \ldots, (M-1)_0)(0_1, 1_1, \ldots, (N-1)_1)$ as shown in Figure 6.

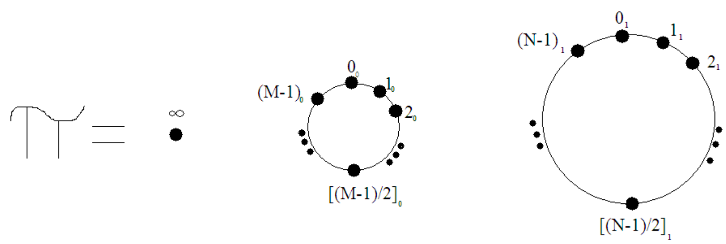

Figure 6: A Tricyclic Steiner Triple System with 1-Rotational Subsystem.

As in the proof of Theorem 2.2, by considering π^M, we see that the $STS(v)$ contains a 1-rotational subsystem of order $M+1$. Therefore we have:

Lemma 2.1 *If a tricyclic $STS(v)$ exists admitting an automorphism of the type $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = \pi_M = \pi_N = 1$ then $M \equiv 2$ or 8 $(mod\ 24)$.*

Also, in such a STS there must be some block of the form (x_0, y_1, z_1). By considering the image of this block under π^N, as in Theorem 2.2, we have:

Lemma 2.2 *If a tricyclic $STS(v)$ as described in Lemma 2.1 exists, then $M \mid N$.*

We have a final necessary condition:

Lemma 2.3 *If a tricyclic $STS(v)$ as described in Lemma 2.1 exists, then $N = kM$ where $k \equiv 2, 3, 6$ or 11 $(mod\ 12)$ whenever $M \equiv 2$ $(mod\ 24)$. If $M \equiv 8$ $(mod\ 24)$, then $k \equiv 0$ or 2 $(mod\ 3)$.*

Proof. A base block of the form (x_0, y_1, z_1) covers two mixed differences and one pure difference of type 1. One of the mixed differences must be congruent to the sum of the other two differences modulo M. Since M is even, either zero or two of these differences is/are odd. If $3 \mid N$, then a possible base block is one of the form $(x_1, (x+N/3)_1, (x+2N/3)_1)$. A block of this type is said to be a *short orbit block* since the length of its orbit under π is precisely one-third the length of the orbit of any other block on the points $\mathbb{Z}_N \times \{1\}$. A short orbit block covers the pure difference of type 1 of $N/3$ only, and $N/3$ is even. A base block of the form (x_1, y_1, z_1) (other than a short orbit block) covers three distinct pure differences of type 1. These three differences satisfy either the condition that one is the sum of the other two, or the condition that all three sum to 0 modulo N. In either case, either zero or two of these differences is/are odd. So, a collection of blocks of the form (x_0, y_1, z_1) or (x_1, y_1, z_1) covers an even number of odd differences. Therefore, the number of odd differences in the set $\{0, 1, \ldots, M-1\} \bigcup \{1, 2, \ldots, N/2-1\}$ must be even. From this, the lemma follows. ∎

We now show that the necessary conditions of Lemmas 2.1–2.3 are sufficient in a series of constructions.

Lemma 2.4 *If $M \equiv 2$ (mod 24) and $k \equiv 2, 3, 6$ or 11 (mod 12), then there exists a tricyclic $STS(v)$ as described above.*

Proof. Consider the given collections of blocks.

<u>Case 1.</u> Suppose that $M \equiv 2$ (mod 24) and $k \equiv 2$ (mod 12).

If $M = 26$ and $k = 2$, consider the following collection of blocks:

$(0_1, 7_1, 18_1)$, $(0_1, 8_1, 17_1)$, $(0_1, 13_1, 25_1)$, $(0_1, 14_1, 24_1)$, $(\infty, 0_1, 26_1)$,

$(0_0, 0_1, 15_1)$, $(0_0, 1_1, 17_1)$, $(0_0, 8_1, 28_1)$, $(0_0, 7_1, 29_1)$, $(0_0, 11_1, 30_1)$, $(0_0, 10_1, 31_1)$,

$(0_0, 9_1, 32_1)$, $(0_0, 12_1, 18_1)$, $(0_0, 13_1, 16_1)$, $(0_0, 14_1, 19_1)$, $(0_0, 20_1, 24_1)$, $(0_0, 23_1, 25_1)$, $(0_0, 21_1, 22_1)$.

Otherwise, consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k-1)M+10}{6} - 2r\right)_1, \left(\frac{(k-1)M}{2} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-2}{12},$$

$$\left(0_1, \left(\frac{(k-1)M-8}{6} - 2r\right)_1, \left(\frac{(k-1)M-5}{3} - r\right)_1\right)$$
$$\text{for } r = 1, 2, \ldots, \frac{(k-1)M-50}{24} \text{ (omit if } M = 26\text{)},$$

$$\left(0_1, \left(\frac{(k-1)M-14}{12} - 2r\right)_1, \left(\frac{7(k-1)M-14}{24} - r\right)_1\right)$$
$$\text{for } r = 1, 2, \ldots, \frac{(k-1)M-50}{24},$$

$$\left(0_1, \left(\frac{(k-1)M+10}{12}\right)_1, \left(\frac{7(k-1)M-14}{24}\right)_1\right), \left(0_1, 1_1, \left(\frac{5(k-1)M+14}{24}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M-14}{12}\right)_1, \left(\frac{(k-1)M-5}{3}\right)_1\right) \text{ (omit if } M = 26\text{)},$$

$$\left(0_1, \left(\frac{(k-1)M-8}{6}\right)_1, \left(\frac{5(k-1)M-10}{12}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_0, (M-r)_1, \left(\frac{(k+1)M-2}{2} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{4},$$

$$\left(0_0, \left(\frac{M-2}{2} - r\right)_1, \left(\frac{kM-2}{2} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{8},$$

$$\left(0_0, \left(\frac{3M-6}{8} - r\right)_1, \left(\frac{(4k+1)M+6}{8} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-18}{8},$$

$$\left(0_0, \left(\frac{M+6}{8}\right)_1, \left(\frac{(4k+1)M-2}{8}\right)_1\right), \left(0_0, \left(\frac{3M-2}{4}\right)_1, \left(\frac{(2k+1)M-2}{4}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{2}\right)_1, \left(\frac{(2k+1)M+2}{4}\right)_1\right).$$

Case 2. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 3 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_1, \left(\frac{kM}{6} - 2r\right)_1, \left(\frac{kM}{2} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, \left(\frac{kM+6}{6} - 2r\right)_1, \left(\frac{kM}{3} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_0, \left(\frac{M+4}{3} - r\right)_1, \left(\frac{(3k+3)M-12}{12} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+10}{12},$$

$$\left(0_0, \left(\frac{M+1}{3} + r\right)_1, \left(\frac{(3k+5)M+8}{12} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{12},$$

$$\left(0_0, \left(\frac{(3k+5)M-4}{12} + r\right)_1, \left(\frac{(8k+6)M+12}{12} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{12},$$

$$\left(0_0, \left(\frac{(4k+3)M}{6} + r\right)_1, \left(\frac{(13k+7)M+4}{12} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{24},$$

$$\left(0_0, \left(\frac{(16k+13)M-2}{24} + r\right)_1, \left(\frac{(16k+17)M+14}{24} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{12},$$

$$\left(0_0, \left(\frac{(16k+17)M-10}{24} + r\right)_1, \left(\frac{(16k+21)M+30}{24} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{24},$$

$$\left(0_0, \left(\frac{(16k+18)M-12}{24} + r\right)_1, \left(\frac{(26k+18)M}{24} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{24},$$

$$\left(0_0, \left(\frac{(26k+15)M+6}{24} + r\right)_1, \left(\frac{(26k+17)M+2}{24} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-26}{24},$$

$$\left(0_0, \left(\frac{(13k+7)M+4}{12}\right)_1, \left(\frac{(26k+16)M+4}{24}\right)_1\right).$$

Case 3. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 6 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{4}\right)_1, \left(\frac{5kM}{12}\right)_1\right), \left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k+1)M-8}{6}+r\right)_1, \left(\frac{(2k-1)M+8}{6}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-2)M+4}{12},$$

$$\left(0_1, \left(\frac{kM}{3}+r\right)_1, \left(\frac{kM-2}{2}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{kM-24}{12},$$

$$\left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right), \left(0_0, 0_1, \left(\frac{5kM}{12}-1\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{2}+r\right)_1, \left(\frac{(k+3)M-6}{6}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-8}{6},$$

$$\left(0_0, \left(\frac{M-5}{3}\right)_1, \left(\frac{M-2}{3}\right)_1\right), \left(0_0, \left(\frac{M-6}{4}\right)_1, \left(\frac{(4k+1)M-2}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-6}{4}-r\right)_1, \left(\frac{(2k+3)M-18}{12}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-14}{12},$$

$$\left(0_0, \left(\frac{M-2}{12}+r\right)_1, \left(\frac{(4k+1)M-2}{12}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-14}{12},$$

$$\left(0_0, \left(\frac{9M-18}{12}-r\right)_1, \left(\frac{(2k+9)M-30}{12}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{12},$$

$$\left(0_0, \left(\frac{11M-46}{12}+r\right)_1, \left(\frac{(4k+11)M-34}{12}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-14}{12},$$

$$\left(0_0, (M-3)_1, \left(\frac{(k+6)M-24}{6}\right)_1\right), \text{ and } \left(0_0, (M-1)_1, \left(\frac{(k+2)M-4}{2}\right)_1\right).$$

Case 4. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 11 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k-1)M-8}{6}-2r\right)_1, \left(\frac{(k-1)M-2}{2}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-1)M-20}{12},$$

$$\left(0_1, \left(\frac{(k-1)M-2}{6}-2r\right)_1, \left(\frac{(k-1)M-2}{3}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-1)M-44}{24},$$

$$\left(0_1, \left(\frac{(k-1)M-8}{12}-2r\right)_1, \left(\frac{7(k-1)M+4}{24}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-1)M-44}{24},$$

$$\left(0_1, \left(\frac{(k-1)M+16}{12}\right)_1, \left(\frac{7(k-1)M+4}{24}\right)_1\right), \left(0_1, 1_1, \left(\frac{5(k-1)M+20}{24}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M-8}{12}\right)_1, \left(\frac{(k-1)M-2}{3}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M-8}{6}\right)_1, \left(\frac{5(k-1)M-4}{12}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M-2}{6}\right)_1, \left(\frac{(k-1)M}{2}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_0, 0_1, \left(\frac{(k-1)M-2}{2}\right)_1\right), \left(0_0, \left(\frac{M-10}{8}\right)_1, \left(\frac{4k-1)M-6}{8}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{8}\right)_1, \left(\frac{(4k-3)M+6}{8}\right)_1\right), \left(0_0, \left(\frac{M-2}{4}\right)_1, \left(\frac{kM-4}{2}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+2}{4}\right)_1, \left(\frac{(2k+1)M-6}{4}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-4}{2}+r\right)_1, \left(\frac{(k+1)M-2}{2}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{4},$$

$$\left(0_0, r_1, \left(\frac{kM-4}{2}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-18}{8}, \text{ and}$$

$$\left(0_0, \left(\frac{M+6}{8}+r\right)_1, \left(\frac{(4k-1)M-6}{8}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-18}{8}.$$

In each case, the given collection of blocks, along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\} \bigcup \mathbb{Z}_M \times \{0\}$ under the automorphism $(\infty)(0_0, 1_0, \ldots, (M-1)_0)$, forms a collection of base blocks for a STS of the desired type. ∎

Lemma 2.5 *If $M \equiv 8 \pmod{24}$ and $k \equiv 0$ or $2 \pmod 3$, then there exists a tricyclic $STS(v)$ as described above.*

Proof. Consider the given collections of blocks.

Case 1. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 0 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{6}+r\right)_1, \left(\frac{kM}{3}-r\right)_1\right) \text{ for } r = 1,2,\ldots,\frac{kM-12}{12},$$

$$\left(0_1, \left(\frac{(2k+1)M-8}{6}+r\right)_1, \left(\frac{(3k-1)M+2}{6}-r\right)_1\right)$$
$$\text{for } r = 1,2,\ldots,\frac{(k-2)M-8}{12},$$

$$\left(0_1, \left(\frac{kM}{4}\right)_1, \left(\frac{5kM}{12}\right)_1\right), \left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-8}{6}\right)_1, \left(\frac{M-2}{6}\right)_1\right), \left(0_0, \left(\frac{7M-20}{12}\right)_1, \left(\frac{(6k+5)M-16}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{6}+r\right)_1, \left(\frac{(k+1)M-8}{6}-r\right)_1\right) \text{ for } r = 1,2,\ldots,\frac{M-8}{6},$$

$$\left(0_0, \left(\frac{5M-16}{12}-r\right)_1, \left(\frac{(4k+5)M-16}{12}+r\right)_1\right) \text{ for } r = 1,2,\ldots,\frac{M-8}{12},$$

$$\left(0_0, \left(\frac{7M-20}{12}+r\right)_1, \left(\frac{(6k+7)M-20}{12}-r\right)_1\right) \text{ for } r = 1,2,\ldots,\frac{M-8}{12},$$

$$\left(0_0, \left(\frac{8M-28}{12}+r\right)_1, \left(\frac{(4k+10)M-32}{12}-r\right)_1\right) \text{ for } r = 1,2,\ldots,\frac{M-8}{12},$$

$$\left(0_0, \left(\frac{11M-40}{12}+r\right)_1, \left(\frac{(6k+11)M-52}{12}-r\right)_1\right) \text{ for } r = 1,2,\ldots,\frac{M-20}{12},$$

$$\left(0_0, \left(\frac{11M-40}{12}\right)_1, \left(\frac{(2k+11)M-52}{12}\right)_1\right), \left(0_0, (M-3)_1, \left(\frac{(k+2)M-8}{2}\right)_1\right),$$

$$\left(0_0, (M-1)_1, \left(\frac{(5k+12)M-24}{12}\right)_1\right).$$

Case 2. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 2 \pmod{6}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k-1)M+4}{6}-2r\right)_1, \left(\frac{(k-1)M}{2}-r\right)_1\right) \text{ for } r = 1,2,\ldots,\frac{(k-1)M-8}{12},$$

$$\left(0_1, \left(\frac{(k-1)M-2}{6} - 2r\right)_1, \left(\frac{(k-1)M-2}{3} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-8}{24},$$

$$\left(0_1, \left(\frac{(k-1)M-20}{12} - 2r\right)_1, \left(\frac{7(k-1)M-8}{24} - r\right)_1\right)$$

$$\text{for } r = 1, 2, \ldots, \frac{(k-1)M-56}{24} \text{ (omit if } M = 8\text{)},$$

$$\left(0_1, 1_1, \left(\frac{5(k-1)M+32}{24}\right)_1\right), \left(0_1, \left(\frac{(k-1)M-2}{6}\right)_1, \left(\frac{5(k-1)M-4}{12}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M-20}{12}\right)_1, \left(\frac{(k-1)M-2}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_0, (M-r)_1, \left(\frac{(k+1)M-4}{2} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M}{4},$$

$$\left(0_0, \left(\frac{M-4}{2} - r\right)_1, \left(\frac{kM-2}{2} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-16}{8},$$

$$\left(0_0, \left(\frac{3M-8}{8} - r\right)_1, \left(\frac{(4k+1)M}{8} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-16}{8},$$

$$\left(0_0, \left(\frac{M}{8}\right)_1, \left(\frac{(4k+1)M-8}{8}\right)_1\right), \left(0_0, \left(\frac{M-4}{2}\right)_1, \left(\frac{(2k+1)M-4}{4}\right)_1\right) \text{ (omit if } M = 8\text{)},$$

$$\left(0_0, \left(\frac{3M-4}{4}\right)_1, \left(\frac{(2k+1)M}{4}\right)_1\right), \text{ and } \left(0_0, \left(\frac{3M-8}{8}\right)_1, \left(\frac{(4k+1)M-16}{8}\right)_1\right).$$

Case 3. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 3 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_1, \left(\frac{kM}{6} - 2r\right)_1, \left(\frac{kM}{2} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, \left(\frac{kM+6}{6} - 2r\right)_1, \left(\frac{kM}{3} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, \left(\frac{M+1}{3} - r\right)_1, \left(\frac{(3k+3)M-12}{12} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{M-8}{12}+r\right)_1, \left(\frac{(5k+2)M-4}{12}-r\right)_1\right) \text{ for } r=1,2,\dots,\frac{M-8}{12},$$

$$\left(0_0, \left(\frac{(5k+3)M+12}{12}-r\right)_1, \left(\frac{(8k+2)M-16}{12}+r\right)_1\right)$$
$$\text{for } r=0,1,2,\dots,\frac{M+4}{12},$$

$$\left(0_0, \left(\frac{(5k+2)M-4}{12}\right)_1, \left(\frac{(10k+1)M+4}{12}\right)_1\right)$$

$$\left(0_0, \left(\frac{(5k+3)M}{12}+r\right)_1, \left(\frac{(10k+4)M+16}{12}-r\right)_1\right) \text{ for } r=1,2,\dots,\frac{M-8}{12},$$

$$\left(0_0, \left(\frac{(10k+4)M+16}{12}\right)_1, \left(\frac{(10k+2)M+32}{12}\right)\right),$$

$$\left(0_0, \left(\frac{(10k+3)M+12}{12}\right)_1, \left(\frac{(10k+5)M+8}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(10k+4)M+16}{12}+r\right)_1, \left(\frac{(10k+6)M}{12}-r\right)_1\right) \text{ for } r=1,2,\dots,\frac{M-20}{12}.$$

If $M=32$, also take the two blocks:

$$\left(0_0, \left(\frac{(10k+1)M+16}{12}+r\right)_1, \left(\frac{(10k+1)M+52}{12}-r\right)_1\right) \text{ for } r=0,1.$$

If $M>32$, instead of the last two blocks, take the blocks:

$$\left(0_0, \left(\frac{(10k+3)M+12}{12}-r\right)_1, \left(\frac{(10k+1)M+16}{12}+r\right)_1\right) \text{ for } r=1,2,\dots,\frac{M-32}{24},$$

$$\left(0_0, \left(\frac{(20k+5)M+56}{24}-r\right)_1, \left(\frac{(20k+3)M+48}{24}+r\right)_1\right) \text{ for } r=1,2,\dots,\frac{M-32}{24},$$

$$\left(0_0, \left(\frac{(10k+1)M+16}{12}\right)_1, \left(\frac{(10k+2)M+20}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(20k+3)M+24}{24}\right)_1, \left(\frac{(20k+3)M+48}{24}\right)_1\right).$$

Case 4. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 5 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k-1)M+10}{6}+r\right)_1, \left(\frac{(k-1)M+7}{3}-r\right)_1\right) \text{ for } r=1,2,\dots,\frac{(k-1)M-8}{24},$$

$$\left(0_1, \left(\frac{5(k-1)M+56}{24}+r\right)_1, \left(\frac{7(k-1)M+64}{24}-r\right)_1\right)$$
$$\text{for } r=1,2,\dots,\frac{(k-1)M-32}{24},$$

$$\left(0_1, \left(\frac{5(k-1)M+56}{24}\right)_1, \left(\frac{3(k-1)M+24}{8}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M+8}{4}\right)_1, \left(\frac{5(k-1)M+44}{12}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M+12}{4}\right)_1, \left(\frac{5(k-1)M+32}{12}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M+7}{3}\right)_1, \left(\frac{5(k-1)M+20}{12}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M+7}{3}+r\right)_1, \left(\frac{(k-1)M+4}{2}-r\right)_1\right) \text{ for } r=1,2,\dots,\frac{(k-1)M-8}{24},$$

$$\left(0_1, \left(\frac{3(k-1)M+24}{8}+r\right)_1, \left(\frac{11(k-1)M+56}{24}-r\right)_1\right)$$
$$\text{for } r=1,2,\dots,\frac{(k-1)M-56}{24},$$

$$\left(0_0, \left(\frac{M-2}{2}+r\right)_1, \left(\frac{(k+1)M-6}{2}-r\right)_1\right) \text{ for } r=1,2,\dots,\frac{M-8}{4},$$

$$\left(0_0, (r-1)_1, \left(\frac{kM-4}{2}-r\right)_1\right) \text{ for } r=1,2,\dots,\frac{M-16}{8},$$

$$\left(0_0, \left(\frac{M}{8}+r\right)_1, \left(\frac{(4k-1)M-8}{8}-r\right)_1\right) \text{ for } r=1,2,\dots,\frac{M-16}{8},$$

$$\left(0_0, \left(\frac{M-16}{8}\right)_1, \left(\frac{(4k-1)M-8}{8}\right)\right)_1, \left(0_0, \left(\frac{M-8}{8}\right)_1, \left(\frac{M}{8}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-4}{4}\right)_1, \left(\frac{kM-4}{2}\right)_1\right), \left(0_0, \left(\frac{M}{4}\right)_1, \left(\frac{(2k+1)M-8}{4}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{2}\right)_1, \left(\frac{(k+1)M-4}{2}\right)_1\right), (0_0, (M-3)_1, (M-1)_1),$$

and $\left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right)$.

Case 5. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 6 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right), \left(0_1, \left(\frac{kM-6}{6}\right)_1, \left(\frac{kM-3}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k+1)M-2}{6}+r\right)_1, \left(\frac{kM-3}{3}-r\right)_1\right) \text{ for } r = 1, 2, \ldots \frac{(k-1)M-16}{12},$$

$$\left(0_1, \left(\frac{kM}{3}+r\right)_1, \left(\frac{(3k-1)M+2}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12}.$$

If $M = 8$, also take these four blocks:

$$\left(0_0, 1_1, \left(\frac{(3k+1)M+4}{12}\right)_1\right), \left(0_0, \left(\frac{(3k+1)M-8}{12}\right)_1, \left(\frac{(9k+1)M-20}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(9k+1)M-20}{12}\right)_1, \left(\frac{(11k+1)M-44}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(3k+1)M+16}{12}\right)_1, \left(\frac{(5k+1)M+28}{12}\right)_1\right).$$

If $M = 32$, instead of the last four blocks, take these blocks:

$$\left(0_0, \left(\frac{-kM+4}{4}\right)_1, \left(\frac{M+4}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+10}{6}-r\right)_1, \left(\frac{kM-6}{6}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{M+7}{3}-r\right)_1, \left(\frac{(3k+1)M+4}{6}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{(-3k+1)M+16}{12}-r\right)_1, \left(\frac{(3k-1)M+8}{12}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{9M+48}{24}-r\right)_1, \left(\frac{(4k+9)M+24}{24}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{24},$$

$$\left(0_0, \left(\frac{(-3k+1)M+16}{12}\right)_1, \left(\frac{(-k+3)M+12}{12}\right)_1\right),$$

$$\left(0_0, (M-1)_1, \left(\frac{(k+5)M+8}{6}\right)_1\right),$$

$$\left(0_0, \left(\frac{(-3k+1)M+40}{12}\right)_1, \left(\frac{(-k+3)M+24}{12}\right)_1\right),$$

$$\left(0_0, (M-3)_1, \left(\frac{(k+5)M-16}{6}\right)_1\right),$$

$$\left(0_0, \left(\frac{(-3k+1)M+28}{12}\right)_1, \left(\frac{(-k+1)M+52}{12}\right)_1\right),$$

$$\left(0_0, (M-5)_1, \left(\frac{(k+6)M-12}{6}\right)_1\right).$$

If $M > 32$, instead of the last two collections of blocks, take these blocks:

$$\left(0_0, \left(\frac{-kM+4}{4}\right)_1, \left(\frac{M+4}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+10}{6}-r\right)_1, \left(\frac{kM-6}{6}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{M+7}{3}-r\right)_1, \left(\frac{(3k+1)M+4}{6}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{(-3k+1)M+16}{12}-r\right)_1, \left(\frac{(3k-1)M+8}{12}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{9M+48}{24}-r\right)_1, \left(\frac{(4k+9)M+24}{24}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{24},$$

$$\left(0_0, \left(\frac{(-3k+1)M+4}{12}+r\right)_1, \left(\frac{(-5k+3)M+24}{12}-r\right)_1\right)$$
$$\text{for } r = 1, 2, \ldots, \frac{M+16}{24},$$

$$\left(0_0, \left(\frac{(-6k+3)M+24}{24}+r\right)_1, \left(\frac{(-2k+7)M+64}{24}-r\right)_1\right)$$
$$\text{for } r = 1, 2, \ldots, \frac{M+16}{24},$$

$$\left(0_0, \left(\frac{(-6k+4)M+40}{24}\right)_1, \left(\frac{(-2k+8)M+56}{24} - r\right)_1\right)$$
for $r = 1, 2, \ldots, \frac{M-32}{24}$,

$$\left(0_0, \left(\frac{(-2k+7)M+64}{24}\right)_1, \left(\frac{(2k+9)M+96}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{(2k+9)M+72}{24}\right)_1, \left(\frac{(6k+11)M+8}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{(2k+9)M+48}{24}\right)_1, \left(\frac{(6k+11)M+32}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{(-2k+8)M+56}{24}\right)_1, \left(\frac{(2k+8)M+104}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{(k+4)M+40}{12}\right)_1, \left(\frac{(-k+5)M+20}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(6k+11)M+32}{24} + r\right)_1, \left(\frac{(10k+11)M+8}{24} - r\right)_1\right)$$
for $r = 1, 2, \ldots, \frac{M-56}{24}$,

$$\left(0_0, \left(\frac{(2k+9)M+48}{24} - r\right)_1, \left(\frac{(6k+9)M+96}{24} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-80}{24}.$$

<u>Case 6.</u> Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 9 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k+1)M-8}{6} + r\right)_1, \left(\frac{kM}{3} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, \left(\frac{(2k+1)M-2}{6} + r\right)_1, \left(\frac{kM}{2} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, (-1+r)_1, \left(\frac{kM+6}{6} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{M+4}{12}\right)_1, \left(\frac{(3k+2)M-4}{12}\right)_1\right), \left(0_0, \left(\frac{(-4k+5)M+8}{24}\right)_1, \left(\frac{(4k+9)M}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+4}{12} + r\right)_1, \left(\frac{(2k+1)M+16}{12} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{2M+8}{12}+r\right)_1, \left(\frac{(-k+1)M+4}{6}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-8}{12},$$

$$\left(0_0, \left(\frac{3M}{12}+r\right)_1, \left(\frac{(4k+5)M-4}{12}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-8}{24},$$

$$\left(0_0, \left(\frac{7M-8}{24}+r\right)_1, \left(\frac{(-4k+7)M+16}{24}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-8}{12},$$

$$\left(0_0, \left(\frac{(-k+1)M-2}{6}+r\right)_1, \left(\frac{(k+2)M+2}{6}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-8}{24},$$

$$\left(0_0, \left(\frac{(4k+9)M}{24}-r\right)_1, \left(\frac{(12k+9)M}{24}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-8}{24},$$

$$\left(0_0, \left(\frac{(12k+10)M-8}{24}+r\right)_1, \left(\frac{(5k+3)M}{6}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-8}{24}, \text{ and}$$

$$\left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right).$$

Case 7. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 11 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k-1)M+4}{6}+r\right)_1, \left(\frac{(2k-2)M+2}{6}-r\right)_1\right)$$
$$\text{for } r=1,2,\ldots,\frac{(k-1)M-32}{24},$$

$$\left(0_1, \left(\frac{5(k-1)M+32}{24}+r\right)_1, \left(\frac{7(k-1)M+40}{24}-r\right)_1\right)$$
$$\text{for } r=1,2,\ldots,\frac{(k-1)M-32}{24},$$

$$\left(0_1, \left(\frac{8(k-1)M+32}{24}+r\right)_1, \left(\frac{12(k-1)M}{24}-r\right)_1\right)$$
$$\text{for } r=1,2,\ldots,\frac{(k-1)M-20}{12},$$

$$\left(0_1, \left(\frac{5(k-1)M+8}{24}\right)_1, \left(\frac{5(k-1)M+32}{24}\right)_1\right),$$

$$\left(0_1, \left(\frac{6(k-1)M+24}{24}\right)_1, \left(\frac{8(k-1)M+32}{24}\right)_1\right),$$

$$\left(0_1, \left(\frac{6(k-1)M+48}{24}\right)_1, \left(\frac{10(k-1)M+16}{24}\right)_1\right),$$

$$\left(0_1, \left(\frac{8(k-1)M+8}{24}\right)_1, \left(\frac{(k-1)M}{2}\right)_1\right),$$

$$\left(0_0, \left(\frac{M}{2}+r\right)_1, \left(\frac{(k+1)M}{2}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{6},$$

$$\left(0_0, \left(\frac{2M-4}{6}+r\right)_1, \left(\frac{(3k+2)M+2}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{6}-1,$$

$$\left(0_0, 0_1, \left(\frac{(k-1)M+4}{6}\right)_1\right), \left(0_0, \left(\frac{2M-4}{6}\right)_1, \left(\frac{3kM-6}{6}\right)_1\right),$$

$$\left(0_0, \left(\frac{2M-10}{6}\right)_1, \left(\frac{3kM}{6}\right)_1\right),$$

$$\left(0_0, \left(\frac{(24k-17)M-56}{24}\right)_1, \left(\frac{(36k-29)M-32}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+4}{12}\right)_1, \left(\frac{(6k-3)M-24}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(3k-2)M+4}{6}\right)_1, \left(\frac{(12k-9)M-12}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(3k-2)M+4}{6}+r\right)_1, \left(\frac{(6k-4)M-10}{6}-r\right)_1\right) \text{ for } r = 0, 1, 2, \ldots, \frac{M-32}{24},$$

$$\left(0_0, \left(\frac{(6k-3)M-24}{12}-r\right)_1, \left(\frac{(12k-9)M-12}{12}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-56}{24},$$

$$\left(0_0, (r)_1, \left(\frac{(3k-2)M+4}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12}, \text{ and } \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right).$$

In each case, the given collection of blocks, along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\}\bigcup\mathbb{Z}_M\times\{0\}$ under the automorphism $(\infty)(0_0, 1_0, \ldots, (M-1)_0)$, forms a collection of base blocks for a STS of the desired type. ∎

Lemmas 2.1–2.5 combine to give us necessary and sufficient conditions for the existence of the desired type of STS. That is, a $STS(v)$ admitting an automorphism of type $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = \pi_M = \pi_N = 1$, $M < N$, exists if and only if $M \equiv 2$ or $8 \pmod{24}$ and $N = kM$ where

1. if $M \equiv 2 \pmod{24}$ then $k \equiv 2, 3, 6$ or $11 \pmod{12}$,

2. if $M \equiv 8 \pmod{24}$ then $k \equiv 0$ or $2 \pmod{3}$.

3 THE DIFFERENCE METHOD AND EXAMPLES

In this chapter, we shall discribe the method used to arrive at the results of chapter 2. The difference method introduces two distinguished types of differences between vertices which are *pure difference* and *mixed differences*. This method allows us to make use of the the "differences" (or distances) between vertices to design our desired base blocks. Furthermore, we will also take a close look at one of our cases in order to understand and verify the difference method thoroughly.

3.1 The Difference Method

This section will introduce the difference method which plays an important role in finding the base block for our tricyclic Steiner triple systems with 1-rotational subsystems. For an edge $(m, n) \in E(K_v)$ where the vertex set is $V(K_v) = \{0, 1, 2, \ldots, v-1\}$, we define the difference as

$$d = |m - n| = \min\{(m-n)\ (mod\ v), (n-m)\ (mod\ v)\}$$

Notice that $d \leq \frac{v}{2}$.

Given any $STS(v)$ and its differences d_i, we may obtain up to 3 possible types of differences in a $STS(v)$; that is,

Type 1. $d_1 + d_2 = d_3$

Type 2. $d_1 + d_2 + d_3 \equiv 0 \pmod v$

Type 3. $d = \frac{v}{3}$.

In our tricyclic Steiner triple systems with 1-rotational subsystems model, with a pair of points of the form (x_1, y_1) we associate a *pure difference of type* 1 of $\min\{(x -$

$y)(\text{mod } N), (y-x)(\text{mod } N)\}$. With a pair of points of the form (x_0, y_1) we associate the *mixed difference* $(y-x)$ (mod M). The set of pure differences of type 1 is $\{1, 2, \ldots, N/2\}$ and the set of mixed differences is $\{0, 1, \ldots, M-1\}$. A collection of base blocks for the desired type of STS must contain a block of the form $(\infty, x_1, (x+N/2)_1)$. Notice that this block contains a pair of points with the associated pure difference of type 1 of $N/2$. Therefore, constructing the desired type of STS is equivalent to presenting a collection of blocks on the point set $\mathbb{Z}_M \times \{0\} \bigcup \mathbb{Z}_N \times \{1\}$ such that the differences associated with the pairs of points of these blocks precisely cover the set of pure differences of type 1 of $\{1, 2, \ldots, N/2-1\}$ and the set of mixed differences of $\{0, 1, \ldots, M-1\}$. Such a collection of blocks along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\} \bigcup \mathbb{Z}_M \times \{1\}$ (under the obvious automorphism) and the block $(\infty, 0_1, (N/2)_1)$ from a collection of base blocks for a tricyclic $STS(1+M+N)$ with a 1-rotational subsystem under π.

For a full understanding of the proposed constructions in section 2, we will look at one of the construction, the case of Lemma 2.5 Case 2, $M \equiv 8 \pmod{24}$ and $k \equiv 2$ or $8 \pmod 6$, in detail.

3.2 An Example: Lemma 2.5 Case 2, $M \equiv 8 \pmod{24}$ and $k \equiv 2$ or $8 \pmod 6$

Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 2 \pmod 6$. Consider the following collection of blocks:

The first block

$$\left(0_1, \left(\frac{(k-1)M+4}{6} - 2r\right)_1, \left(\frac{(k-1)M}{2} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-8}{12}$$

corresponds to the following differences

$$\begin{bmatrix} Pure: & \frac{(k-1)M+4}{6} - 2r: & \frac{(k-1)M-8}{6}, \ldots, 2 \\ Pure: & \frac{(k-1)M}{2} - r: & \frac{(k-1)M-2}{2}, \ldots, \frac{(5k-5)M+8}{12} \\ Pure: & \frac{(2k-2)M-4}{6} + r: & \frac{(2k-2)M+2}{6}, \ldots, \frac{(5k-5)M-16}{12} \end{bmatrix}$$

The second block

$$\left(0_1, \left(\frac{(k-1)M-2}{6} - 2r\right)_1, \left(\frac{(k-1)M-2}{3} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-8}{24}$$

corresponds to the following differences

$$\begin{bmatrix} Pure: & \frac{(k-1)M-2}{6} - 2r: & \frac{(k-1)M-14}{6}, \ldots, \frac{(k-1)M+4}{12} \\ Pure: & \frac{(k-1)M-2}{3} - r: & \frac{(k-1)M-5}{3}, \ldots, \frac{7(k-1)M-8}{24} \\ Pure: & \frac{(k-1)M-2}{6} + r: & \frac{(k-1)M+4}{6}, \ldots, \frac{5(k-1)M-16}{24} \end{bmatrix}$$

The third block

$$\left(0_1, \left(\frac{(k-1)M-20}{12} - 2r\right)_1, \left(\frac{7(k-1)M-8}{24} - r\right)_1\right)$$
$$\text{for } r = 1, 2, \ldots, \frac{(k-1)M-56}{24} \text{ (omit if } M = 8)$$

corresponds to the following differences

$$\begin{bmatrix} Pure: & \frac{(k-1)M-20}{12} - 2r: & \frac{(k-1)M-44}{12}, \ldots, 3 \\ Pure: & \frac{7(k-1)M-8}{24} - r: & \frac{7(k-1)M-32}{24}, \ldots, \frac{6(k-1)M+48}{24} \\ Pure: & \frac{5(k-1)M+32}{24} + r: & \frac{5(k-1)M+56}{24}, \ldots, \frac{6(k-1)M-24}{24} \end{bmatrix}$$

The 4th block

$$\left(0_1, 1_1, \left(\frac{5(k-1)M+32}{24}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Pure: & 1 \\ Pure: & \frac{5(k-1)M+32}{24} \\ Pure: & \frac{5(k-1)M+8}{24} \end{bmatrix}$

The 5th block

$$\left(0_1, \left(\frac{(k-1)M-2}{6}\right)_1, \left(\frac{5(k-1)M-4}{12}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Pure: & \frac{(k-1)M-2}{6} \\ Pure: & \frac{5(k-1)M-4}{12} \\ Pure: & \frac{3(k-1)M}{12} \end{bmatrix}$

The 6th block

$$\left(0_1, \left(\frac{(k-1)M-20}{12}\right)_1, \left(\frac{(k-1)M-2}{3}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Pure: & \frac{(k-1)M-20}{12} \\ Pure: & \frac{(k-1)M-2}{3} \\ Pure: & \frac{3(k-1)M+12}{12} \end{bmatrix}$

The 7th block

$$\left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right)$$

corresponds to the following difference: $\begin{bmatrix} Pure: & \frac{kM}{2} \end{bmatrix}$

The 8th block

$$\left(0_0, (M-r)_1, \left(\frac{(k+1)M-4}{2}+r\right)_1\right) \text{ for } r = 1, 2, \dots, \frac{M}{4}$$

corresponds to the following differences

$$\begin{bmatrix} Mix: & M-r: & M-1, \dots, \frac{3M}{4} \\ Mix: & \frac{(k+1)M-4}{2}+r: & \frac{M-2}{2}\ (mod\ M), \dots, \frac{3M-8}{4}\ (mod\ M) \\ Pure: & \frac{(k-1)M-4}{2}+2r: & \frac{(k-1)M}{2}, \dots, \frac{2kM-8}{4} \end{bmatrix}$$

The 9th block

$$\left(0_0, \left(\frac{M-4}{2}-r\right)_1, \left(\frac{kM-2}{2}+r\right)_1\right) \text{ for } r = 1, 2, \dots, \frac{M-16}{8}$$

corresponds to the following differences

$$\begin{bmatrix} Mix: & \frac{M-4}{2}-r: & \frac{M-6}{2}, \dots, \frac{3M}{8} \\ Mix: & \frac{kM-2}{2}+r: & 0\ (mod\ M), \dots, \frac{M-24}{8}\ (mod\ M) \\ Pure: & \frac{(k-1)M+2}{2}+2r: & \frac{(k-1)M+6}{2}, \dots, \frac{(2k-1)M-12}{4} \end{bmatrix}$$

The 10th block

$$\left(0_0, \left(\frac{3M-8}{8} - r\right)_1, \left(\frac{(4k+1)M}{8} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-16}{8}$$

corresponds to the following differences

$$\left[\begin{array}{lll} Mix: & \frac{3M-8}{8} - r: & \frac{3M-16}{8}, \ldots, \frac{2M+8}{8} \\ Mix: & \frac{(4k+1)M}{8} + r: & \frac{M+8}{8} \ (mod\ M), \ldots, \frac{2M-16}{8} \ (mod\ M) \\ Pure: & \frac{(4k-2)M+8}{8} + 2r: & \frac{(4k-2)M+24}{8}, \ldots, \frac{4kM-24}{8} \end{array}\right]$$

The 11th block

$$\left(0_0, \left(\frac{M}{8}\right)_1, \left(\frac{(4k+1)M-8}{8}\right)_1\right)$$

corresponds to the following differences $\left[\begin{array}{lc} Mix: & \frac{M}{8} \\ Mix: & \frac{M-8}{8} \ (mod\ M) \\ Pure: & \frac{4kM-8}{8} \end{array}\right]$

The 12th block

$$\left(0_0, \left(\frac{M-4}{2}\right)_1, \left(\frac{(2k+1)M-4}{4}\right)_1\right) \text{ (omit if } M = 8)$$

corresponds to the following differences $\left[\begin{array}{lc} Mix: & \frac{M-4}{2} \\ Mix: & \frac{M-4}{4} \ (mod\ M) \\ Pure: & \frac{(2k-1)M+4}{4} \end{array}\right]$

The 13th block

$$\left(0_0, \left(\frac{3M-4}{4}\right)_1, \left(\frac{(2k+1)M}{4}\right)_1\right)$$

corresponds to the following differences $\left[\begin{array}{lc} Mix: & \frac{3M-4}{4} \\ Mix: & \frac{M}{4} \ (mod\ M) \\ Pure: & \frac{(k-1)M+2}{2} \end{array}\right]$

The 14th block

$$\left(0_0, \left(\frac{3M-8}{8}\right)_1, \left(\frac{(4k+1)M-16}{8}\right)_1\right)$$

corresponds to the following differences $\left[\begin{array}{lc} Mix: & \frac{3M-8}{8} \\ Mix: & \frac{M-16}{8} \ (mod\ M) \\ Pure: & \frac{(4k-2)M-8}{8} \end{array}\right]$

It is easy to check that the pure differences run from $1, \ldots, \frac{kM}{2}$ and the mixed differences run from $0, 1, \ldots, M-1$ which is exactly what we expected. However, let's also check one specific example, say, $M = 32, k = 20$, and $N = 640$. Consider the following collection of blocks: The first block

$$\left(0_1, \left(\frac{(k-1)M+4}{6} - 2r\right)_1, \left(\frac{(k-1)M}{2} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-8}{12}$$

corresponds to the following differences

$$\begin{bmatrix} Pure: & 2, 4, 6 \ldots, 98, 100 \\ Pure: & 254, 255, 256 \ldots, 302, 303 \\ Pure: & 203, 204, 205 \ldots, 251, 252 \end{bmatrix}$$

The second block

$$\left(0_1, \left(\frac{(k-1)M-2}{6} - 2r\right)_1, \left(\frac{(k-1)M-2}{3} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-8}{24}$$

corresponds to the following differences

$$\begin{bmatrix} Pure: & 51, 53, 55 \ldots, 97, 99 \\ Pure: & 177, 178, 179, \ldots, 200, 201 \\ Pure: & 102, 103, 104 \ldots, 125, 126 \end{bmatrix}$$

The third block

$$\left(0_1, \left(\frac{(k-1)M-20}{12} - 2r\right)_1, \left(\frac{7(k-1)M-8}{24} - r\right)_1\right)$$
$$\text{for } r = 1, 2, \ldots, \frac{(k-1)M-56}{24}$$

corresponds to the following differences

$$\begin{bmatrix} Pure: & 3, 5, 7 \ldots, 45, 47 \\ Pure: & 154, 155, 156 \ldots, 176 \\ Pure: & 129, 130, 132, \ldots, 150, 151 \end{bmatrix}$$

The 4th block

$$\left(0_1, 1_1, \left(\frac{5(k-1)M+32}{24}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Pure: & 1 \\ Pure: & 128 \\ Pure: & 127 \end{bmatrix}$

The 5th block

$$\left(0_1, \left(\frac{(k-1)M-2}{6}\right)_1, \left(\frac{5(k-1)M-4}{12}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Pure: & 101 \\ Pure: & 253 \\ Pure: & 152 \end{bmatrix}$

The 6th block

$$\left(0_1, \left(\frac{(k-1)M-20}{12}\right)_1, \left(\frac{(k-1)M-2}{3}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Pure: & 49 \\ Pure: & 202 \\ Pure: & 153 \end{bmatrix}$

The 7th block

$$\left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right)$$

corresponds to the following difference: $\begin{bmatrix} Pure: & 320 \end{bmatrix}$

The 8th block

$$\left(0_0, (M-r)_1, \left(\frac{(k+1)M-4}{2}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M}{4}$$

corresponds to the following differences

$$\begin{bmatrix} Mix: & 24, 25, 26, \ldots, 30, 31 \\ Mix: & 15, 16, 17, \ldots, 21, 22 \\ Pure: & 304, 306, 308, \ldots, 316, 318 \end{bmatrix}$$

The 9th block

$$\left(0_0, \left(\frac{M-4}{2}-r\right)_1, \left(\frac{kM-2}{2}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-16}{8}$$

corresponds to the following differences $\begin{bmatrix} Mix: & 12, 13 \\ Mix: & 0, 1 \\ Pure: & 307, 309 \end{bmatrix}$

The 10th block

$$\left(0_0, \left(\frac{3M-8}{8} - r\right)_1, \left(\frac{(4k+1)M}{8} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-16}{8}$$

corresponds to the following differences $\begin{bmatrix} Mix: & 9, 10 \\ Mix: & 5, 6 \\ Pure: & 315, 317 \end{bmatrix}$

The 11th block

$$\left(0_0, \left(\frac{M}{8}\right)_1, \left(\frac{(4k+1)M-8}{8}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Mix: & 4 \\ Mix: & 3 \\ Pure: & 319 \end{bmatrix}$

The 12th block

$$\left(0_0, \left(\frac{M-4}{2}\right)_1, \left(\frac{(2k+1)M-4}{4}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Mix: & 14 \\ Mix: & 7 \\ Pure: & 313 \end{bmatrix}$

The 13th block

$$\left(0_0, \left(\frac{3M-4}{4}\right)_1, \left(\frac{(2k+1)M}{4}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Mix: & 23 \\ Mix: & 8 \\ Pure: & 305 \end{bmatrix}$

The 14th block

$$\left(0_0, \left(\frac{3M-8}{8}\right)_1, \left(\frac{(4k+1)M-16}{8}\right)_1\right)$$

corresponds to the following differences $\begin{bmatrix} Mix: & 11 \\ Mix: & 2 \\ Pure: & 7 \end{bmatrix}$

It is easily seen that the pure differences run from $1, \ldots, 320$, and the mixed differences run from $0, \ldots, 31$.

4 CONCLUSION

We have shown the necessary and sufficient conditions for the existence of the 1-rotational $STS(v)$ which may be concluded as the following theorem,

Theorem 4.1 *A* $STS(v)$ *admitting an automorphism of type* $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ *where* $\pi_1 = \pi_M = \pi_N = 1$*,* $M < N$*, exists if and only if* $M \equiv 2$ *or* 8 $(mod\ 24)$ *and* $N = kM$ *where*

1. *if* $M \equiv 2$ $(mod\ 24)$ *then* $k \equiv 2, 3, 6$ *or* 11 $(mod\ 12)$*,*
2. *if* $M \equiv 8$ $(mod\ 24)$ *then* $k \equiv 0$ *or* 2 $(mod\ 3)$*.*

However, the case of 1-rotational $STS(v)$ is just an initial case of studying tricyclic $STS(v)$. Future research will concentrate on solving the existence problem for tricyclic $STS(v)$ with bicyclic $STS(v)$ subsystem and tricyclic $STS(v)$ with no bicyclic $STS(v)$ subsystem.

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