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Decomposition, Packings and Coverings of Complete Digraphs with a Transitive-Triple and a Pendant Arc.

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by

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ABSTRACT

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In the study of design theory, there are eight orientations of the complete graph on three vertices with a pendant edge, $K_3 \bigcup \{e\}$. Two of these are the 3-circuit with a pendant arc and the other six are transitive triples with a pendant arc. Necessary and sufficient conditions are given for decompositions, packings and coverings of the complete digraph with each of the six transitive triples with a pendant arc.

DEDICATION

This thesis is dedicated to my father, Frank Hierholzer, for all those science kits and educational toys that he bought for me and for the many hours he spent helping me with my homework. His own academic example (M.S. in E.E and over 50 graduate credits beyond that) and love of learning set the stage for me to excel.

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CONTENTS

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1 INTRODUCTION

Design theory, an intriguing branch of combinatorial mathematics, has applications in many fields. Some applications in the area of computer science and electronics are "the theory of parallel algorithms, the design of file organization schemes, the design of hardware switches, and the analysis of algorithms[17]." In design theory, using graphs to represent structures, we study decompositions, packings and coverings with a smaller structure in order to better understand the whole structure.

Graphs provide a visible link between theory and application that makes them ideal for design theory. A graph G consists of a set of elements together with a binary relation defined on the set. In a graph, the elements are represented by points (*vertices*) and the binary relation is represented by lines (*edges*) joining pairs of points. A directed graph (*digraph*) D is simply a graph where the edges (*arcs*) have been assigned a direction. If two vertices have an edge between them, we say that they are *adjacent*. A *complete graph* on v vertices, K_v, is a graph where every vertex is adjacent to every other vertex in the graph. The complete digraph on v vertices, D_v, is formed by replacing each edge in K_v with two arcs of opposite orientation. As an example, see Figure 1 which shows K_3, the complete graph on three vertices and D_3, the complete digraph on three vertices.

The *degree* of a vertex u in a graph, G, is defined as the number of edges that are adjacent to u. Directed graphs, however, have out degrees and in degrees for each vertex. The *out degree*, $od(u)$, of vertex u in D is defined as the number of vertices of D that are adjacent from u. The *in degree*, $id(u)$ of vertex u in D refers to the number of vertices of D that are adjacent to u. The *total degree* of vertex u in D is

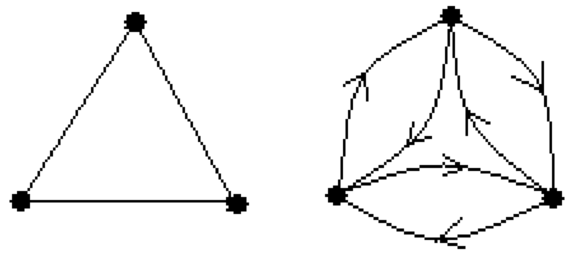

Figure 1: A Complete Graph, K_3 and a Complete Digraph, D_3

$od(u) + id(u)$.

A *decomposition* of a *digraph* with *isomorphic* copies of digraph d is a set $\{d_1, d_2, ..., d_n\}$ where $d_i \cong d$ and $V(d_i) \subset V(D)$ for all i and $A(d_i) \cap A(d_j) = \emptyset$ for $i \neq j$ and the union over all d_i's gives the digraph D. The d_i's are called the *blocks* of the decomposition while $V(D)$ is the *vertex set* of D and $A(D)$ is the *arc set*.

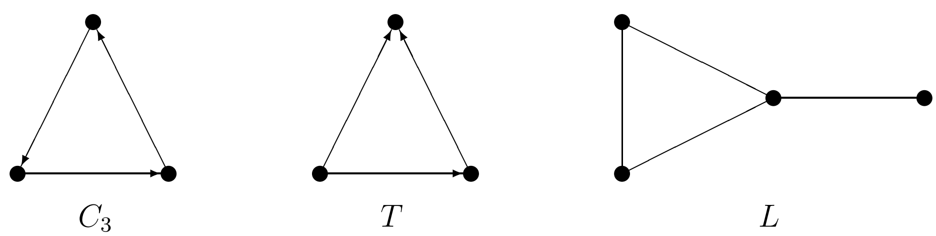

Figure 2: The 3-Circuit C_3, the Transitive Triple T, and L (K_3 with a Pendant Edge)

Many types of decompositions have been studied that led to this research. A *triple system* is a graph (or digraph) decomposition into isomorphic copies of a graph (or digraph) on three vertices. *Steiner Triple systems*, denoted $STS(v)$, are decompositions of K_v into K_3 's. Figure 2 shows the two orientations of D_3 (called directed triples) which are known as the 3-*circuit* and the *transitive triple*. The 3-*circuit* and the *transitive triple* decompositions of D_v are called *Mendelsohn triple systems* ($MTS(v)$) and *directed triple systems* ($DTS(v)$), respectively. In addition, L (from Figure 2) decompositions of D_v have been studied.

The concentration of this thesis is the three *transitive triple* orientations, applied to the K_3 subgraph in the L and the two different orientations on the pendent arc. We will decompose, pack and cover complete digraphs with these six orientations. These orientations are labeled d_1, d_2, d_3, d_4, d_5, and d_6 as shown in Figure 3.

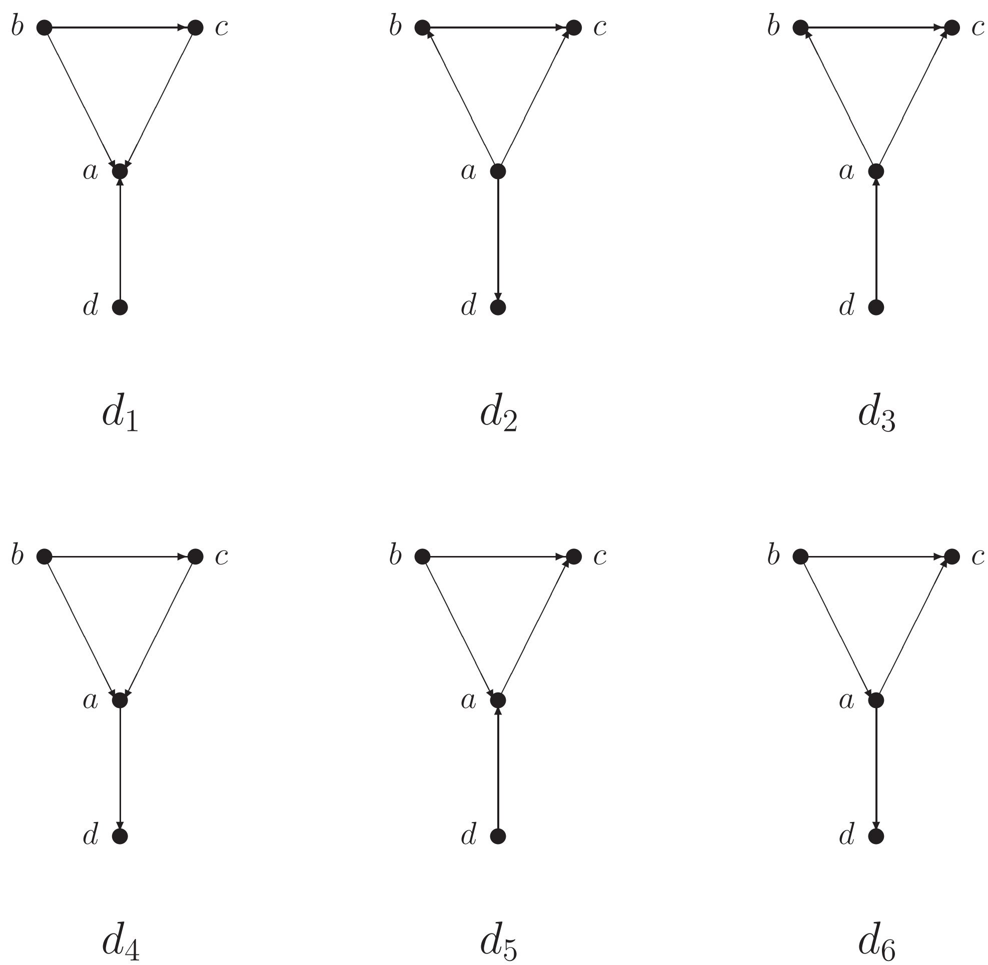

Figure 3: Six Orientations of $K_3 \cup \{e\}$.

These six orientations are denoted as $(a, b, c) - (d)_{d1}, (a, b, c) - (d)_{d2}, \ldots, (a, b, c) - (d)_{d6}$, respectively.

2 DECOMPOSITIONS

2.1 INTRODUCTION

Many mathematicians have studied decompositions of a complete graph. These are some of the results of their studies that motivated the research for this thesis:

- A $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6) [13],
- A $MTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [12],
- A $DTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3) [8], and
- A decomposition of K_v into copies of K_3 with a pendant edge (the graph L of Figure 2) exists if and only if $v \equiv 0$ or 1 (mod 8) [1].

Figure 4 shows the decomposition of D_3 into two isomorphic copies of transitive triples.

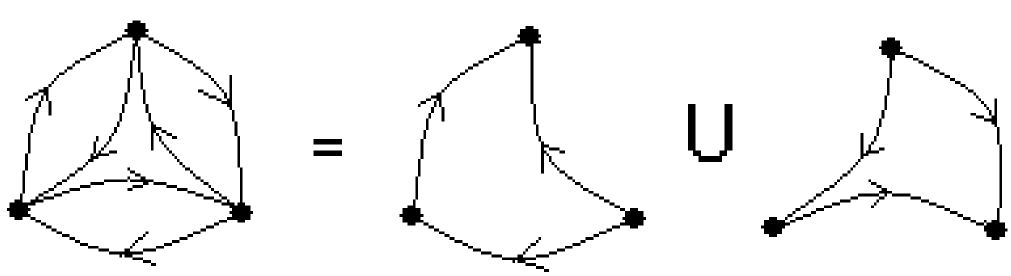

Figure 4: D_3 Decomposition with Transitive Triples.

2.2 DIFFERENCE METHOD

The *difference method* is a method of decomposing a complete graph using the distances (differences) between vertices. We define *differences* as follows. Suppose we have a complete graph with v vertices. We start by labelling the vertices 0 through $v-1$. The *difference* of the arc from vertex a to vertex b, is defined as $b-a$ (mod $v-1$). The complete graph, D_v, has $N = v-1$ differences.

We define *orbit* as follows. The *orbit* of a block b under permutation π is the set $\{\pi^n(b)|n = 0, 1, 2, ...\}$. A set $\{b_1, b_2, ..., b_n\}$ is a set of *base blocks under permutation* π of a b-decomposition of graph G if each b_i is isomorphic to b and the orbits of the blocks of the b_i generate a b-decomposition of G and the orbit of b_i is disjoint from the orbit of b_j when $i \neq j$.

To illustrate the difference method and the use of fixed points, we will decompose the complete digraph, D_v, with $v \equiv 0$ (mod 4). Suppose we have a complete digraph D_v of 24 vertices (see Figure 5). This implies that $v \equiv 0$ (mod 4) or $v = 4k$. Since $v = 24$, we have 23 differences. We are decomposing with d_1 which has four arcs so the number of differences has to be divisible by 4. We should then use one fixed point because that will bring the number of differences down to 22. Then, we use two differences with our base block containing the fixed point, making the number of differences 20. So, we use fixed points when 4 does not divide v. Since each fixed point uses up three differences, we keep adding fixed points until the number of differences left (after we have subtracted three differences per fixed point) is a multiple of 4.

Now we let vertex c in d_1 equal ∞ for the base block containing the fixed point. Next we write all the differences down (1 through 22 in this case). We use the last

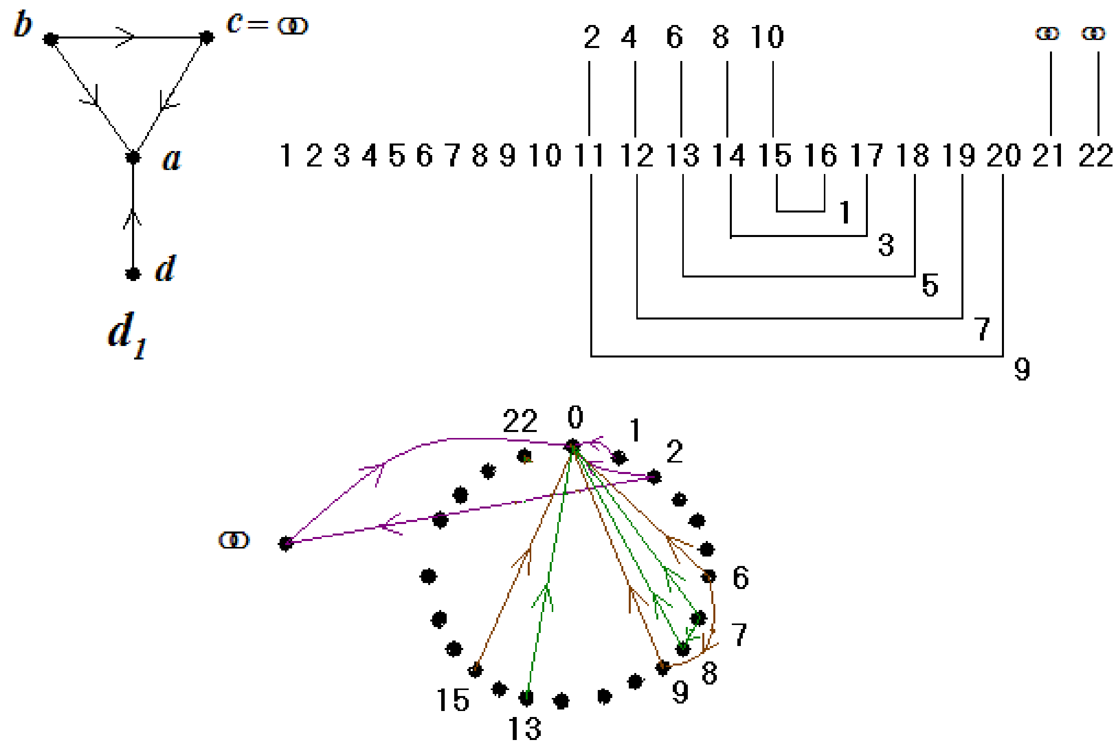

Figure 5: Example of a d_1 Decomposition of D_{24} Using the Difference Method.

two differences in the base block containing the fixed point. We write our base blocks as (a, b, c) - $(d)_{d1}$. The base block containing the fixed point is (0, 2, ∞) -(1) which takes care of the arcs $(\infty, 0)$, (2,∞), (1,0), and (2,0). Then we permute this base block as follows $\{(0+j, 2+j, \infty) - (1+j)_{d1} | j = 0, 1, \ldots, 22\}$ in order to cover all the arcs to and from ∞ and the differences of 21 and 22.

Next, we write the blocks for the differences of 10, 15, 16, and 1. We chose these using the difference method for transitive triples which is: if a, b and c are vertices of the base block, then we choose our differences such that $(c-b)$+$(a-c)$=$(a-b)$. So our base block for the differences of 10, 15, 16, and 1 is $(0, 7, 8) - (13)_{d1}$. We then take this base block and permute it around the circle shown in Figure 5 in order to use up all these differences. In mathematical notation, this is $\{(0+j, 7+j, 8+j) - (13+j)_{d1} | j =$

$0, 1, \ldots, 22\}$. Similarly, the rest of our blocks are $\{(j, 6+j, 9+j) - (15+j)_{d1}$, $(j, 5+j, 10+j) - (17+j)_{d1}$, $(j, 4+j, 11+j) - (19+j)_{d1}$, $(j, 3+j, 12+j) - (21+j)_{d1}$ $\mid j = 1, 2, \ldots, 22\}$.

Now, we check to see if we have covered all the arcs in D_{24}. In our base block with the fixed point, we have used 23 times 4 arcs which equals 92. In the other five blocks, we've used 23 times 4 times 5 arcs which equals 460. When we add these together, we get 552 arcs which is 24 times 23 which equals the number of arcs in D_{24}.

This process is then extended to all the complete digraphs, D_v, with $v \equiv 0 \pmod 4$, i.e. $v = 4k$. Figure 6 shows the differences marked for the general case where $v = 4k$. Next we write the blocks which are annotated in Theorem 2.1, case 4.

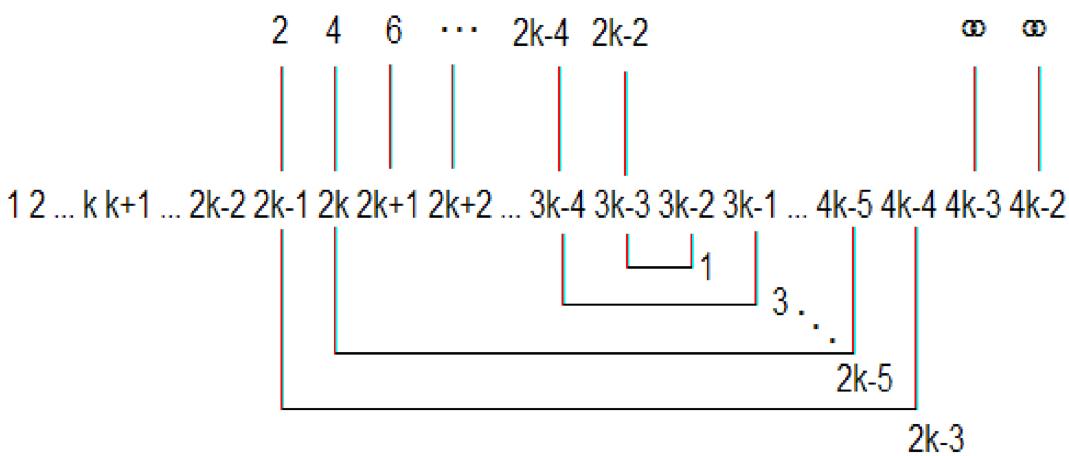

Figure 6: A d_1 Decomposition of D_v with $v = 4k$ Vertices Using the Difference Method.

2.3 RESULTS

Theorem 2.1 *A d_1-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4).*

Proof. The necessary condition for a d_1-decomposition of D_v to exist is $v \equiv 0$ or 1 (mod 4) since 4 divides v if and only if $v \equiv 0$ or 1 (mod 4). We show sufficiency in four cases.

Case 1. Suppose $v \equiv 1 \pmod{12}$, say $v = 12k + 1$. Consider the blocks:

$$\{(j, 6k - i + j, 12k - 2i + j) - (3k + 1 + i + j)_{d1},$$

$$(j, 5k - i + j, 10k - 2i + j) - (8k + 1 + 2i + j)_{d1}$$

$$\mid i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 12k\}$$

$$\bigcup\{(j, k - 1 - i + j, 12k - 3 - 2i + j) - (2k + 2 + i + j)_{d1}$$

$$\mid i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 12k\}$$

$$\bigcup\{(j, k + j, 12k - 1 + j) - (k + 1 + j)_{d1} \mid j = 0, 1, \ldots, 12k\}.$$

Case 2. Suppose $v \equiv 5 \pmod{12}$, say $v = 12k + 5$. Consider the blocks:

$$\{(j, 6k + 2 - i + j, 12k + 4 - 2i + j) - (3k + 1 + i + j)_{d1},$$

$$(j, 5k + 1 - i + j, 10k + 2 - 2i + j) - (8k + 5 + 2i + j)_{d1}$$

$$\mid i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 12k + 4\}$$

$$\bigcup\{(j, k - 1 - i + j, 12k + 1 - 2i + j) - (2k + 2 + i + j)_{d1}$$

$$\mid i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 12k + 4\}$$

$$\bigcup\{(j, 5k+2+j, 10k+4+j) - (4k+1+j)_{d1},$$

$$(j, k+j, 12k+3+j) - (k+1+j)_{d1} \mid j = 0, 1, \ldots, 12k+4\}.$$

Case 3. Suppose $v \equiv 9 \pmod{12}$, say $v = 12k+9$. Consider the blocks:

$$\{(j, 6k+4-i+j, 12k+8-2i+j) - (3k+4+i+j)_{d1},$$

$$(j, 5k+3-i+j, 10k+6-2i+j) - (8k+7+2i+j)_{d1},$$

$$(j, k-i+j, 12k+5-2i+j) - (2k+4+i+j)_{d1}$$

$$\mid i = 0, 1, \ldots, k-1, j = 0, 1, \ldots, 12k+8\}$$

$$\bigcup\{(j, 5k+4+j, 10k+8+j) - (8k+6+j)_{d1},$$

$$(j, k+1+j, 12k+7+j) - (k+2+j)_{d1} \mid j = 0, 1, \ldots, 12k+8\}.$$

In each of Cases 1–3, the given set of blocks forms a decomposition of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v.

Case 4. Suppose $v \equiv 0 \pmod 4$, say $v = 4k$. Consider the blocks:

$$\{(j, 2+j, \infty) - (1+j)_{d1}\}\bigcup\{(j, k+1-i+j, k+2+i+j) - (2k+1+2i+j)_{d1}$$

$$\mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 4k-2\}.$$

In Case 4, the given set of blocks forms a decomposition of D_v where $V(D_v) = \{\infty, 0, 1, \ldots, v-2\}$ and numerical vertex labels in the blocks are reduced modulo $v-1$. □

Corollary 2.2 *A d_2-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4).*

Proof. The necessary condition follows as in Theorem 2.1. Since the converse of d_1 is d_2 and the D_v is self converse, the result follows trivially from Theorem 2.1. □

Corollary 2.3 *A d_3-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4).*

Proof. The necessary condition follows as in Theorem 2.1. In the case $v \equiv 1$ (mod 4), blocks for such a system can be constructed from the d_1 system of Theorem 2.1 by replacing every block of the form $(j, a+j, b+j) - (c+j)_{d1}$ with a block of the form $(a-b+j, a+j, j)-(a-b+c+j)_{d3}$. In the case $v \equiv 0$ (mod 4), blocks for such a system can be constructed from the d_1 system of Theorem 2.1 by replacing every block of the form $(j, a+j, b+j)-(c+j)_{d1}$ with a block of the form $(a-b+j, a+j, j)-(a-b+c+j)_{d3}$ and by replacing every block of the form $(j, a+j, \infty) - (c+j)_{d1}$ with a block of the form $(a+j, \infty, j) - (a+c+j)_{d3}$. $\square$

Corollary 2.4 *A d_4-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4).*

Proof. The necessary condition follows as in Theorem 2.1. Since the converse of d_4 is d_3 and the D_v is self converse, the result follows trivially from Corollary 2.3. $\square$

Corollary 2.5 *A d_5-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4).*

Proof. As in Theorem 2.1, one necessary condition is that $v \equiv 0$ or 1 (mod 4). Notice that the vertices of d_5 are of in-degree 0, 0, 2, and 2. Therefore another necessary condition for a d_5-design on D_v is that each vertex of D_v is of in-degree even — that is, v must be odd. Therefore $v \equiv 1$ (mod 4) is necessary.

Blocks for such a system can be constructed from the d_1 system of Theorem 2.1 by replacing every block of the form $(j, a+j, b+j) - (c+j)_{d1}$ with a base block of the form $(b+j, j, a+j) - (b+c+j)_{d5}$. $\square$

Corollary 2.6 *A d_6-decomposition of D_v exists if and only if* $v \equiv 1 \ (mod\ 4)$.

Proof. The necessary condition follows as in Corollary 2.5. Since the converse of d_5 is d_4 and the D_v is self converse, the result follows trivially from Corollary 2.5. □

3 PACKINGS

3.1 INTRODUCTION

When a decomposition does not exist, we ask, "How close to a decomposition can we get?" In this section, we will consider one way to answer this question which is with packings. In a *g-packing* of D_v, we remove isomorphic copies of g without repeating any arcs until we cannot remove more copies of g. The packing is said to be *maximal* if the number of arcs left over or not used (*the leave*) in D_v is minimal. The formal definition for a maximal packing follows.

A *maximal packing* of a directed graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, ..., g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i and $A(g_i) \cap A(g_j) = \phi$ for $i \neq j$ and $\bigcup_i^n g_i \subset G$ and

$$|A(L)| = |A(G)/\bigcup_i^n g_i|$$

is minimal , where $V(G)$ is the vertex set, $A(G)$ is the arc set of the graph G and L represents the leave of the packing.

The following lemma is an example of a maximal d_1 packing of D_6. Figure 7 illustrates the maximal d_1 packing of D_6 which consists of six d_1 's.

Lemma 3.1 *A d_1 packing of D_6 with minimal leave L exists and consists of exactly six d_1's with $|A(L)| = 6$.*

Proof.

Since $|A(D_6)| = 30$ and $|A(d_1)| = 4$, seven d_1's should fit in D_6. However, since each vertex in D_6 has an in-degree of five, and vertex a of d_1 has an in-degree of three, only one vertex a of d_1 will fit per vertex of D_6. Therefore the maximum packing is six d_1's and not seven. Here are the blocks and the leave of a maximal packing:

$$\{(0,2,4)-(3)_{d1}, (1,2,3)-(0)_{d1}, (2,4,3)-(1)_{d1},$$

$$(3,5,1)-(0)_{d1}, (4,5,6)-(3)_{d1}, (5,1,0)-(4)_{d1}\}$$

$$A(L) = \{(3,5), (0,2), (2,5), (5,2), (1,4), (4,1)\}$$

□

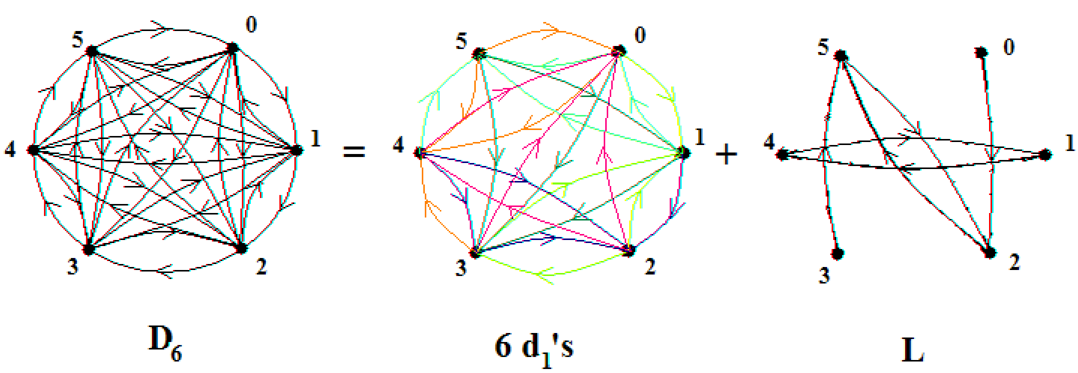

Figure 7: The Packing and Leave of a Maximal d_1-Packing of D_6.

3.2 RESULTS

Theorem 3.1 *A maximal d_1-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 0$ *or* 1 $(mod\ 4)$, *and*

(ii) $|A(L)| = 2$ *if* $v \equiv 2$ *or* 3 $(mod\ 4)$, $v \neq 6$, *and*

(iii) $|A(L)| = 6$ *if* $v = 6$.

Proof. Clearly it is necessary that $|A(L)| \equiv |A(D_v)|(\text{mod } 4)$. We show that, with the exception of $v = 6$, $|A(L)| = |A(D_v)|(\text{mod } 4)$.

Case 1. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. If $v \equiv 2$ (mod 4), say $v = 4k + 2$, where $k \neq 1, 2, 3, 4, 5$ then consider the blocks in $A \bigcup B$ where

$$A = \{j, k + 9 - i + j, k + 10 + i + j) - (2k + 5 + 2i + j)_{d1}$$

$$| \ i = 0, 1, \ldots, k - 6, j = 0, 1, \ldots, 4k - 6\}$$

$$\bigcup\{(j, 2i + 2 + j, \infty_i) - (2i + 1 + j)_{d1} \mid i = 0, 1, \ldots, 6, j = 0, 1, \ldots, 4k - 6\}$$

and

$$B = \{(\infty_0, \infty_1, \infty_3) - (\infty_6)_{d1}, (\infty_1, \infty_2, \infty_4) - (\infty_0)_{d1}, (\infty_2, \infty_5, \infty_3) - (\infty_1)_{d1},$$

$$(\infty_3, \infty_6, \infty_4) - (\infty_2)_{d1}, (\infty_4, \infty_5, \infty_0) - (\infty_3)_{d1}, (\infty_5, \infty_1, \infty_6) - (\infty_4)_{d1},$$

$$(\infty_6, \infty_2, \infty_0) - (\infty_5)_{d1}, (\infty_2, \infty_4, \infty_0) - (\infty_6)_{d1},$$

$$(\infty_1, \infty_3, \infty_6) - (\infty_5)_{d1}, (\infty_5, \infty_0, \infty_3) - (\infty_2)_{d1}\}.$$

Then $A \bigcup B$ is a maximal d_1-packing of D_v with leave L where $A(L) = \{(\infty_1, \infty_4), (\infty_4, \infty_6)\}$, and so the packing is maximal. The given set of blocks forms a packing of D_v where $V(D_v) = \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, 0, 1, \ldots, v-8\}$ and vertex labels in the blocks are reduced modulo $v - 7$.

Case 3. If $v \equiv 3 \pmod 4$, say $v = 4k + 3$, $k \neq 1$ then consider the blocks in $A \bigcup B$ where

$$A = \{j, k+3-i+j, 4k-2i+j) - (2k+3+2i+j)_{d1} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 4k\}$$

and

$$B = \{(j, 1 + 2i + j, \infty_i) - (2 + 2i + j)_{d1} \mid i = 0, 1, j = 0, 1, \ldots, 4k\}$$

Then $A \bigcup B$ is a maximal d_1-packing of D_v with leave L where $A(L) = \{(\infty_0, \infty_1), (\infty_1, \infty_0)\}$, and so the packing is maximal. The given set of blocks forms a packing of D_v where $V(D_v) = \{\infty_0, \infty_1, 0, 1, \ldots, v-3\}$ and vertex labels in the blocks are reduced modulo $v - 2$.

Case 4. If $v = 6$ then Lemma 3.1 applies.

Case 5. If $v = 7$ then consider the blocks:

$$\{(0, 1, 3) - (6)_{d1}, (1, 2, 4) - (0)_{d1}, (2, 5, 3) - (1)_{d1}, (3, 6, 4) - (2)_{d1},$$

$$(4, 5, 0) - (3)_{d1}, (5, 1, 6) - (4)_{d1}, (6, 2, 0) - (5)_{d1}, (2, 4, 0) - (6)_{d1},$$

$$(1, 3, 6) - (5)_{d1}, (5, 0, 3) - (2)_{d1}\}.$$

The leave L is $A(L) = \{(1, 4), (4, 6)\}$ so the packing is maximal.□

Corollary 3.2 *A maximal d_2-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 0$ *or* 1 *(mod 4), and*

(ii) $|A(L)| = 2$ *if* $v \equiv 2$ *or* 3 *(mod 4),* $v \neq 6$*, and*

(iii) $|A(L)| = 6$ *if* $v = 6$.

Proof. The necessary condition follows as in Theorem 3.1. Since the converse of d_1 is d_2 and the D_v is self converse, the result follows trivially from Theorem 3.1. □

Theorem 3.3 *A maximal d_3-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 0$ *or* 1 *(mod 4), and*

(ii) $|A(L)| = 2$ *if* $v \equiv 2$ *or* 3 *(mod 4).*

Proof. The necessary condition follows as in Theorem 3.1. We consider sufficiency in five cases.

Case 1. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. Suppose $v \equiv 2$ (mod 4), blocks for such a system can be constructed from the d_1 system of Theorem 3.1 by replacing every block of the form $(j, a+j, b+j)-(c+j)_{d1}$ with a block of the form $(a-b+j, a+j, j)-(a-b+c+j)_{d3}$.

Case 3. Suppose $v \equiv 3$ (mod 4), blocks for such a system can be constructed from the d_1 system of Theorem 3.1 by replacing every block of the form $(j, a+j, b+j)-(c+j)_{d1}$ with a block of the form $(a-b+j, a+j, j)-(a-b+c+j)_{d3}$ and by replacing every block of the form $(j, a+j, \infty)-(c+j)_{d1}$ with a block of the form $(a+j, \infty, j)-(a+c+j)_{d3}$.

Case 4. Suppose $v=6$, consider the following blocks:

$$\{(4,1,3)-(0)_{d3},(4,5,2)-(3)_{d3},(5,3,0)-(1)_{d3},(3,1,2)-(0)_{d3},$$

$$(0,5,1)-(2)_{d3},(1,4,0)-(2)_{d3},(2,5,4)-(0)_{d3}\}.$$

The leave L is $A(L)=\{(2,3),(3,5)\}$ so the packing is maximal.

Case 5. Suppose $v=7$, consider the following blocks:

$$\{(0,5,1)-(3)_{d3},(1,5,2)-(4)_{d3},(2,5,3)-(0)_{d3},(3,5,4)-(1)_{d3},$$

$$(4,5,0)-(2)_{d3},(0,6,3)-(1)_{d3},(1,6,4)-(2)_{d3}(2,6,0)-(3)_{d3},$$

$$(3,6,1)-(4)_{d3}(4,6,2)-(0)_{d3}\}.$$

The leave L is $A(L)=\{(5,6),(6,5)\}$ so the packing is maximal. □

Corollary 3.4 *A maximal d_4-packing of D_v with leave L satisfies*

(i) $|A(L)|=0$ *if* $v\equiv 0$ *or* 1 $(mod\ 4)$, *and*

(ii) $|A(L)|=2$ *if* $v\equiv 2$ *or* 3 $(mod\ 4)$.

Proof. The necessary condition follows as in Theorem 3.1. Since the converse of d_3 is d_4 and the D_v is self converse, the result follows trivially from Theorem 3.3. □

Lemma 3.5 *A maximal d_5-packing of D_v with leave L has $|A(L)|\geq v$ if $v\equiv 0$ or 2 $(mod\ 4)$.*

Proof. Each vertex of D_v is of in-degree $v-1$ (which is odd) and each vertex of d_5 is of in-degree even. Therefore, in a maximal packing, we are left with each vertex of D_v of in-degree at least 1. Thus, $|A(L)| \geq v$. □

Theorem 3.5 *A maximal d_5-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 1$ $(mod\ 4)$,

(ii) $|A(L)| = v$ *if* $v \equiv 0$ *or* 2 $(mod\ 4)$, *and*

(iii) $|A(L)| = 2$ *if* $v \equiv 3$ $(mod\ 4)$.

Proof. The necessary conditions follow as in Theorem 3.1 when $v \equiv 1$ or 3(mod 4) and follow from Lemma 3.5 when $v \equiv 0$ or 2(mod 4). We consider sufficiency in four cases.

Case 1. If $v \equiv 1$ (mod 4), then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. If $v \equiv 0$ (mod 4), then by Lemma 3.5, $|A(L)| \geq v$. Consider the following blocks in $A \bigcup B$ where:

$$A = \{(2i, 4k-1+2i, 1+2i) - (4k-2+2i)_{d5} | i = 0, 1, \ldots, 2k-1\}$$

and

$$B = \{(j, 3k-3+j, 4k-2+j) - (3k-2+j)_{d5} | j = 0, 1, ..., 4k-1\}$$

$$\bigcup \{(j, 2k-1+i+j, 2k+2+2i+j) - (2k-3-2i+j)_{d5} | i = 0, 1, \ldots, k-3, j = 0, 1, ..., 4k-1\}.$$

Then $A \bigcup B$ is a maximal d_5-packing of D_v with leave L where

$$A(L) = \{(j-1, j) | j = 0, 1, \ldots, 4k-1\}.$$

The given set of blocks forms a packing of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v.

Case 3. If $v \equiv 2 \pmod 4$, then by Corollary 3.5, $|A(L)| \geq v$. Consider the following blocks:

$$A = \{(j, k+2+i+j, 1+2i+j) - (2k+2+2i)_{d5} | i = 0, 1, \ldots, k-1, j = 0, 1, ..., 4k+1\}$$

Then A is a maximal d_5-packing of D_v with leave L where

$$A(L) = \{(j-1, j) | j = 0, 1, \ldots, 4k-1\}.$$

The given set of blocks forms a packing of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v.

Case 4. If $v \equiv 3 \pmod 4$ Consider the following blocks in $A \bigcup B$ where:

$$A = \{(2i, 4k+2+2i, 1+2i) - (4k+1+2i)_{d5} | i = 0, 1, \ldots, 2k\}$$

and

$$B = \{(j, 3k-1+j, 4k+2+j) - (4k+j)_{d5} | j = 0, 1, ..., 4k+2\}$$

$$\bigcup \{(j, 2k+i+j, 2k+4+2i+j) - (2+2i+j)_{d5} | i = 0, 1, \ldots, k-2, j = 0, 1, ..., 4k-1\}.$$

Then $A \bigcup B$ is a maximal d_5-packing of D_v with leave L where

$$A(L) = \{(4k+1, 4k+2), (4k, 4k+2)\}.$$

The given set of blocks forms a packing of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v. $\square$

Corollary 3.6 *A maximal d_6-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 1$ $(mod\ 4)$,

(ii) $|A(L)| = v$ *if* $v \equiv 0$ *or* 2 $(mod\ 4)$, *and*

(iii) $|A(L)| = 2$ *if* $v \equiv 3$ $(mod\ 4)$.

Proof. The necessary condition follows as in Theorem 3.5. Since the converse of d_5 is d_6 and the D_v is self converse, the result follows trivially from Theorem 3.5. □

4 COVERINGS

4.1 INTRODUCTION

In answering the question, "How close to a decomposition can we get?", our second response is to consider coverings. In a *g-covering* of D_v, we cover D_v with isomorphic copies of g until every arc of D_v is covered. The covering is said to be *minimal* if the number of arcs repeated (*the padding*) in D_v is minimal. The formal definition for a minimal covering follows.

A *minimal covering* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $G \subset \cup_{i=1}^{n} g_i$, and

$$|A(P)| = |\cup_{i=1}^{n} A(g_i) \setminus A(G)|$$

is minimal (the graph $\cup_{i=1}^{n} g_i$ may not be simple and $\cup_{i=1}^{n} E(g_i)$ may be a multiset). The graph P is called the *padding* of the covering.

The following lemma is an example of finding the minimal covering of D_7 with isomorphic copies of d_5. Figure 8 shows d_5 and blocks A and B from the proof of Lemma 4.1. Block A is permuted by adding 2 to each vertex three times. It produces all arcs with differences of 1 and 2 and it produces the padding. Block B is permuted by adding 1 to each vertex (six times) and produces all arcs with differences of 4, 5 and 6. Block A is another example of the difference method in that a difference of 1 plus a difference of 1 equals a difference of 2. Block B also demonstrates the difference method because a difference of 5 plus a difference of 6 equals a difference of 4 (mod 7).

Lemma 4.1 *A d_5 covering of D_7 with minimal padding P exists and consists of exactly eleven d_5's with $|A(L)| = 2$.*

Proof.

If $v = 7$, then consider the blocks in $A \bigcup B$ where (See Figure 8):

$$A = \{(1 + 2i, 0 + 2i, 2 + 2i) - (6 + 2i)_{d5} | i = 0, 1, 2, 3\}$$

and

$$B = \{(1 + j, 3 + j, 0 + j) - (5 + j)_{d5} | j = 0, 1, ..., 6\}.$$

Then $A \bigcup B$ is a minimal d_5-covering of D_7 with padding P where

$$A(P) = \{(0, 1), (6, 1)\}.$$

□

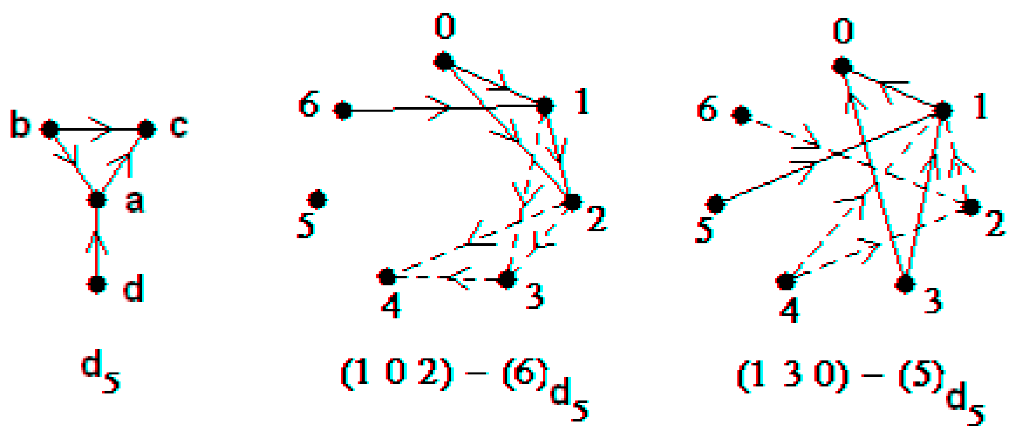

Figure 8: Blocks for a Minimal d_5-Covering of D_7.

4.2 RESULTS

Theorem 4.1 *A minimal d_1-covering of D_v with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* 1 $(mod\ 4)$, *and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* 3 $(mod\ 4)$.

Proof. Clearly it is necessary that $|A(D_v)| - |A(P)| \equiv 0(\text{mod } 4)$. We show that, $|A(P)|$ is the smallest value possible, namely $|A(P)| = |A(D_v)|(\text{mod } 4)$.

Case 1. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. If $v \equiv 2$ (mod 4), say $v = 4k + 2$, where $k \neq 1, 2, 3, 4, 5$ then consider the blocks in $A \bigcup B \bigcup \{(\infty_6, \infty_1, \infty_4) - (\infty_3)_{d1}\}$ where sets A and B are defined in Theorem 3.1. This is a minimal covering of D_v with padding P where $A(P) = \{(\infty_1, \infty_6), (\infty_3, \infty_6)\}$.

Case 3. If $v = 6$, then consider the blocks

$$\{(5,0,1) - (4)_{d1}, (1,5,4) - (2)_{d1}, (3,1,0) - (5)_{d1}, (2,4,3) - (1)_{d1},$$

$$(4,3,1) - (0)_{d1}, (0,2,4) - (3)_{d1}, (5,2,3) - (4)_{d1}, (2,5,0) - (3)_{d1}\}.$$

The padding P is $A(P) = \{(3,2),(4,5)\}$ so the covering is minimal.

Case 4. If $v \equiv 3$ (mod 4), say $v = 4k + 3$, where $k \neq 1$ then consider the blocks in $A \bigcup B \bigcup \{(\infty_1, 3, 0) - (\infty_0)_{d1} \cup (0, \infty_1, \infty_0) - (4)_{d1}\ /(0, 3, \infty_1) - (4)_{d1}\}$ where sets A and B are defined in Theorem 3.1. This is a minimal covering of D_v with padding P

where $A(P) = \{(\infty_0, 0), (\infty_1, 0)\}$.

Case 5. If $v = 7$, then consider the blocks

$$\{(0,1,3)-(6)_{d1}, (1,2,4)-(0)_{d1}, (2,5,3)-(1)_{d1}, (3,6,4)-(2)_{d1},$$

$$(4,5,0)-(3)_{d1}, (5,1,6)-(4)_{d1}, (6,2,0)-(5)_{d1}, (2,4,0)-(6)_{d1},$$

$$(1,3,6)-(5)_{d1}, (5,0,3)-(2)_{d1}, (6,1,4)-(3)_{d1}\}.$$

The padding P is $A(P) = \{(1,6), (3,6)\}$ so the covering is minimal. □

Corollary 4.2 *A minimal d_2-covering of D_v with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* 1 $(mod\ 4)$, *and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* 3 $(mod\ 4)$.

Proof. The necessary condition follows as in Theorem 4.1. Since the converse of d_1 is d_2 and the D_v is self converse, the result follows trivially from Theorem 4.1. □

Theorem 4.3 *A minimal d_3-covering of D_v with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* 1 $(mod\ 4)$, *and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* 3 $(mod\ 4)$.

Proof. The necessary condition follows as in Theorem 4.1. We consider sufficiency in five cases.

Case 1. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. Suppose $v \equiv 2 \pmod 4$, blocks for such a system can be constructed from the d_1 system of Corollary 4.1 by replacing every block of the form $(j, a+j, b+j)-(c+j)_{d1}$ with a block of the form $(a-b+j, a+j, j)-(a-b+c+j)_{d3}$.

Case 3. Suppose $v \equiv 3 \pmod 4$, blocks for such a system can be constructed from the d_1 system of Corollary 4.1 by replacing every block of the form $(j, a+j, b+j)-(c+j)_{d1}$ with a block of the form $(a-b+j, a+j, j)-(a-b+c+j)_{d3}$ and by replacing every block of the form $(j, a+j, \infty)-(c+j)_{d1}$ with a block of the form $(a+j, \infty, j)-(a+c+j)_{d3}$.

Case 4. Suppose $v = 6$, consider the following blocks:

$$\{(4,1,3)-(0)_{d3}, (4,5,2)-(3)_{d3}, (5,3,0)-(1)_{d3}, (3,1,2)-(0)_{d3},$$

$$(0,5,1)-(2)_{d3}, (1,4,0)-(2)_{d3}, (2,5,4)-(0)_{d3}, (2,3,5)-(1)_{d3}\}.$$

The padding P is $A(P) = \{(1,2),(2,5)\}$ so the covering is minimal.

Case 5. Suppose $v = 7$, consider the following blocks:

$$\{(0,5,1)-(3)_{d3}, (1,5,2)-(4)_{d3}, (2,5,3)-(0)_{d3}, (3,5,4)-(1)_{d3},$$

$$(4,5,0)-(2)_{d3}, (1,6,4)-(2)_{d3}(2,6,0)-(3)_{d3},$$

$$(3,6,1)-(4)_{d3}, (4,6,2)-(0)_{d3}, (6,0,3)-(5)_{d3}, (0,6,5)-(1)_{d3},\}.$$

The padding P is $A(P) = \{(6,0),(0,5)\}$ so the covering is minimal. □

Corollary 4.4 *A minimal d_4-covering of D_v with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* $1 \pmod 4$*, and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* $3 \pmod 4$*.*

Proof. The necessary condition follows as in Theorem 4.1. Since the converse of d_3 is d_4 and the D_v is self converse, the result follows trivially from Theorem 4.3. □

Lemma 4.5 *A minimal d_5-covering of D_v with padding P has $|A(P)| \geq v$ if $v \equiv 0$ or 2 (mod 4).*

Proof. Each vertex of D_v is of in-degree $v-1$ (which is odd) and each vertex of d_5 is of in-degree even. Therefore, in a minimal covering, each vertex of the covering is of in-degree at least 1. Thus, $|A(P)| \geq v$. □

Theorem 4.5 *A minimal d_5-covering of D_v with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 1 \ (mod\ 4)$, *and*

(ii) $|A(P)| = v$ *if* $v \equiv 0$ *or* $2 \ (mod\ 4)$, *and*

(iii) $|A(P)| = 2$ *if* $v \equiv 3 \ (mod\ 4)$.

Proof. The necessary conditions follow as in Theorem 4.1 when $v \equiv 1$ or 3(mod 4) and follow from Lemma 4.5 when $v \equiv 0$ or 2(mod 4). We consider sufficiency in four cases.

Case 1. If $v \equiv 1$ (mod 4), then there is a decomposition by Theorem 2.1 and the result follows.

Case 2. If $v \equiv 0$ (mod 4), then by Lemma 4.5, $|A(P)| \geq v$. Consider the following blocks in $A \bigcup B$ where:

$$A = \{(j, 2k+j, 2k-1+j) - (4k-1+j)_{d5} | j = 0, 1, \ldots, 4k-1\}$$

and

$$B = \{(j, k+1+i+j, 1+2i+j) - (4k-2+j)_{d5}$$

$$|i = 0, 1, \ldots, k-1, j = 0, 1, ..., 4k-1\}.$$

Then $A \bigcup B$ is a minimal d_5-covering of D_v with padding P where

$$A(P) = \{(j, j+1) | j = 0, 1, \ldots, 4k-1\}.$$

The given set of blocks forms a covering of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v.

Case 3. If $v \equiv 2 \pmod 4$, then by Lemma 4.5, $|A(P)| \geq v$. Consider the following blocks in $A \bigcup B$ where:

$$A = \{(2i, 4k+1+2i, 1+2i) - (4k+2i)_{d5} | i = 0, 1, \ldots, 2k\}$$

and

$$B = \{(j, k+1+j, 1+j) - (2k+1+j)_{d5} | j = 0, 1, ..., 4k+1\}$$

$$\bigcup \{(j, k+2+i+j, 3+2i+j) - (4k-2-2i+j)_{d5} | i = 0, 1, \ldots, k-2, j = 0, 1, ..., 4k+1\}.$$

Then $A \bigcup B$ is a minimal d_5-covering of D_v with padding P where

$$A(P) = \{(j, j+1) | j = 0, 1, \ldots, 4k+1\}.$$

The given set of blocks forms a packing of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v.

Case 4. If $v \equiv 3 \pmod 4$ Consider the following blocks in $A \bigcup B$ where:

$$A = \{(2i, 4k+2+2i, 1+2i) - (4k+1+2i)_{d5} | i = 0, 1, \ldots, 2k+1\}$$

and

$$B = \{(j, 3k-1+j, 4k+2+j) - (4k+j)_{d5} | j = 0, 1, ..., 4k+2\}$$

$$\bigcup \{(j, 2k+i+j, 2k+4+2i+j) - (2+2i+j)_{d5} | i = 0, 1, \ldots, k-2, j = 0, 1, ..., 4k-1\}.$$

Then $A \bigcup B$ is a minimal d_5-packing of D_v with padding P where

$$A(P) = \{(4k+1, 0), (4k+2, 0)\}.$$

The given set of blocks forms a covering of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v. □

Corollary 4.6 *A minimal d_6-covering of D_v with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 1$ *(mod 4), and*

(ii) $|A(P)| = v$ *if* $v \equiv 0$ *or* 2 *(mod 4), and*

(iii) $|A(P)| = 2$ *if* $v \equiv 3$ *(mod 4).*

Proof. The necessary conditions follow as in Theorem 4.5. Since the converse of d_5 is d_6 and the D_v is self converse, the result follows trivially from Theorem 4.5. □

5 CONCLUSION

Here is a summary of our decomposition results:

(i) d_1, d_2, d_3, or d_4 decompositions of D_v exist if and only if $v \equiv 0$ or 1 (mod 4), and

(ii) d_4 or d_5 decompositions of D_v exist if and only if $v \equiv 1$ (mod 4).

The most similar decomposition result to compare this to is the transitive triple decompositions of D_v as studied by Hung and Mendelsohn [8]. That is, a $DTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3) [8]. We have similar results for d_1 through d_4 decompositions — these decompositions exist if and only if v is equivalent to 0 or 1 modulo the number of arcs in the figure with which we are decomposing. This makes sense because the number of arcs in D_v is divisible by 3 if and only if $v \equiv 0$ or 1 (mod 3), i.e. $v = 3k$ or $v = 3k + 1$. (Recall that $|A(D_v)| = v(v - 1)$.) So if $v = 3k$, then $|A(D_v)| = 3k(3k - 1)$ and if $v = 3k + 1$, then $|A(D_v)| = 3k + 1(3k)$. Similarly, the number of arcs in D_v is divisible by 4 if and only if $v \equiv 0$ or 1 (mod 4).

What was surprising though, was that d_5 or d_6 decompositions of D_v did not exist if $v \equiv 0$ (mod 4). A closer look at the structure of d_5 and d_6 revealed the reason for this. The vertices of d_5 were of in-degree even and D_v had an even number of vertices which meant that the in-degree of each vertex in D_v was odd. Thus a decomposition was not possible because the sum of any number of even numbers can never equal an odd number.

Here is a summary of our packing results:

(i) Maximal d_1 or d_2 packings of D_v with leave L has $|A(L)| = 0$ if $v \equiv 0$ or 1 (mod 4) and $|A(L)| = 2$ if $v \equiv 2$ or 3 (mod 4), $v \neq 6$, and $|A(L)| = 6$ if $v = 6$.

(ii) Maximal d_3 or d_4 packings of D_v with leave L has $|A(L)| = 0$ if $v \equiv 0$ or 1 (mod 4) and $|A(L)| = 2$ if $v \equiv 2$ or 3 (mod 4).

(iii) Maximal d_5 or d_6 packings of D_v with leave L has $|A(L)| = 0$ if $v \equiv 1$ (mod 4), and $|A(L)| = v$ if $v \equiv 0$ or 2 (mod 4), and $|A(L)| = 2$ if $v \equiv 3$ (mod 4).

Again, we want to compare this to the transitive triple packing results of D_v. Here are the results from R. Gardner's research in [5]:

Theorem 5.1 [5] A maximal packing of D_v, where $v \neq 6$, with copies of the transitive triple, T, and a leave L satisfies:

1) $|A(L)| = 0$ if $v \equiv 0$ or 1 (mod 3), or

2) $|A(L)| = 2$ and L=C_2 if $v \equiv 2$ (mod 3) .

Our results were very similar to R. Gardner's but were dependent on which graph configuration with which we were packing. In packing with d_1 or d_2, we found that $|A(L)| = 6$ when $v = 6$. The reason for this was that one of the vertices in d_1 had an in-degree of three (See Lemma 3.1). Dr. Gardner also had an exception when $v = 6$, but this would not be for the same reason since there are no vertices of in-degree three in a transitive triple.

Another unusual result was that in packing with d_5 or d_6, we found $|A(L)| = v$ if $v \equiv 0$ or 2 (mod 4). Again the reason for this was the structure of d_5 and d_6 as discussed above in the decomposition section (See Lemma 3.5).

Finally, here is the summary of our covering results:

(i) Minimal d_1, d_2, d_3 or d_4 coverings of D_v with padding P has $|A(P)| = 0$ if $v \equiv 0$ or 1 (mod 4) and $|A(P)| = 2$ if $v \equiv 2$ or 3 (mod 4).

(ii) Minimal d_5 or d_6 coverings of D_v with padding P has $|A(P)| = 0$ if $v \equiv 1$ (mod 4), and $|A(L)| = v$ if $v \equiv 0$ or 2 (mod 4), and $|A(L)| = 2$ if $v \equiv 3$ (mod 4).

And here are R. Gardner's results from [5]:

Theorem 5.2 [5] A minimal covering of D_v, where $v \neq 6$, with copies of the transitive triple, T, and padding, P, satisfies:

1) $|A(P)| = 0$ if $v \equiv 0$ or 1 (mod 3), or

2) $|A(P)| = 4$ if $v \equiv 2$ (mod 3) and P may be two disjoint copies of C_2, any orientation of a 4-cycle or two osculating 2-circuits OC_2.

Dr. Gardner's transitive triple results were quite different from minimal d_1 through d_4 coverings of D_v because we found $|A(P)| = 2$ if $v \equiv 2$ or 3 (mod 4) instead of $|A(P)| = 4$ if $v \equiv 2$ (mod 3). Again, minimal d_5 or d_6 coverings of D_v differed because of the structure of d_5 and d_6. We found $|A(P)| = v$ if $v \equiv 0$ or 2 (mod 4). These results differ significantly and could warrant further study.

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