12-2008

Restrained and Other Domination Parameters in Complementary Prisms.

Wyatt Jules DesOrmeaux
East Tennessee State University

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A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

Wyatt J. DesOrmeaux

December 2008

Teresa W. Haynes, Ph.D., Chair

Robert B. Gardner, Ph.D.

Debra J. Knisley, Ph.D.

Keywords: graph theory, complementary prism, restrained domination,

stratification, 2-*step* domination, *distance*-k domination, chromatic number.

ABSTRACT

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In this thesis, we will study several domination parameters of a family of graphs known as complementary prisms. We will first present the basic terminology and definitions necessary to understand the topic. Then, we will examine the known results addressing the domination number and the total domination number of complementary prisms. After this, we will present our main results, namely, results on the restrained domination number of complementary prisms. Subsequently results on the $distance-k$ domination number, 2-step domination number and stratification of complementary prisms will be presented. Then, we will characterize when a complementary prism is Eulerian or bipartite, and we will obtain bounds on the chromatic number of a complementary prism. We will finish the thesis with a section on possible future problems.

DEDICATION

I would like to dedicate this thesis to my father, Robert Jules Desormeaux, for always stressing that education comes before anything else, and for the wisdom that he taught me to always question those things that I want to believe harder than those things which I don't. To my mother, Lorraine Desormeaux, for always supporting me in my educational endeavors and keeping me well fed throughout the writing of this thesis. Without her love and support, I not only couldn't finish this thesis, but I'd starve. Finally, to the memory of my late brother, Robert John Desormeaux.

ACKNOWLEDGMENTS

First, I would like to thank my committee chair Dr. Teresa Haynes for being the awesome teacher that she is. She has instilled me with an all-consuming love for graph theory that I am certain I'll never be able to lose. I can't get the subject out of my mind, I think about it all the time and for that I am eternally grateful, God bless you Dr. Haynes. Next, I would like to thank Dr. Debra Knisley for suggesting to me the first book I ever read on graph theory, Buckley and Lewinter's *A Friendly Introduction To Graph Theory* and for serving on my committee. I would like to thank my third committee member Dr. Robert Gardner (aka. Dr. Bob, Old Bob, Moe) for being the awesome analysis teacher that he is, and for the encouragement that he has given throughout the years, especially his encouragement to come back to school after such a long hiatus. Thank you Dr. Bob. I would also like to thank Dr. Robert Beeler for his help with drawing figures in PC-Tex and for his guidance and wisdom throughout this process. Finally, I would like to thank Assistant Professor Kaylee. Her cold-nosed approach to problem solving made Graph Theory I and II a shear joy.

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1 INTRODUCTION

The purpose of this thesis is to study selected domination parameters of a family of graphs known as complementary prisms. In Section 1.1, we introduce the basic terminology of graph theory utilized in this paper. In Section 1.2, we introduce the definitions of each of the domination parameters discussed in this paper. In Section 1.3, we define the complementary prism of a graph and give examples.

1.1 Basic Terminology of Graph Theory

As defined in [3], a graph $G = (V(G), E(G))$ is a nonempty, finite set of elements called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. In Figure 1, we have an example of a graph.

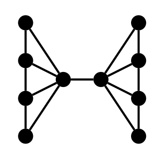

Figure 1: A Graph G

In this paper, we will be studying *simple graphs*, which are graphs for which there exists, at most, 1 edge between any two vertices, and for which the endpoints of any edge are distinct. Given any graph G, the *order* of G, denoted $n(G) = |V(G)|$, is the number of vertices in G. The *size* of G, denoted $m(G) = |E(G)|$, is the number of edges in G. For example, for the graph G in Figure 1, the order $n(G) = 10$ and the

Figure 2: A Graph G and $\overline{G}$

size $m(G) = 15$. The *complement* of G, denoted $\overline{G}$, is a graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{ab | ab \notin E(G)\}$. For example, consider the graphs G and $\overline{G}$ seen in Figure 1.1.

For any vertices $v, u \in V(G)$, u and v are *adjacent* if $uv \in E(G)$. A *u-v path* is a finite alternating sequence $\{u = v_0, e_1, v_1, e_2 \ldots e_k, v_k = v\}$ of vertices and edges such that $e_i = v_{i-1}v_i$ for $i = 1...k$ and $e_i = e_j$ if, and only if, $i = j$. Among all u-v paths, the number of edges in a shortest length u-v path is known as the *distance* from u to v, denoted by $dist(u, v)$. For any vertex $v \in V(G)$, the *open neighborhood* of v is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, its *open neighborhood* $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* $N[S] = N(S) \cup S$. The degree of a vertex v is $deg_G(v) = |N(v)|$. The *minimum degree* of G is $\delta(G) = \min\{deg_G(v) | v \in V(G)\}$. The *maximum degree* of G is $\Delta(G)$ $= \max\{deg_G(v) | v \in V(G)\}$. A vertex of degree zero is an *isolated vertex.* A vertex of degree one is called a *leaf* or an *endvertex*, and its neighbor is called a *support vertex.* For any leaf vertex v and support vertex w, the edge vw is called a *pendant edge.*

Given $S \subseteq V(G)$, and $v \in S$, a vertex $w \in V(G)$ is an *S-private neighbor of v* if $N_G(w) \cap S = \{v\}$. The *S-external private neighborhood of v*, denoted $epn(v, S)$, is the set of all S-private neighbors of v in $V(G) \setminus S$. For any $S \subseteq V(G)$, the subgraph of G *induced* by S is denoted $\langle S \rangle$. If $S \subseteq V(G)$ and $uv \in E(G)$ for every $u, v \in S$, then S forms a *clique* of order $|S|$, and $\langle S \rangle$ is called a *complete graph* of order $|S|$. If $uv \notin E(G)$ for every $u, v \in S$, then S is an independent set of order $|S|$ and $\langle S \rangle$ is called an *empty graph* of order $|S|$. For any set $S \subseteq V(G)$, and any vertex $v \in V(G)$, $dist(v, S) = \min\{d(v, u) | u \in S\}$. A graph G whose vertex set $V(G)$ can be partitioned into a clique and an independent set is called a *split graph*. For any graph G, the *corona* of G, denoted $G \circ K_1$, is formed by adding for each $v \in V(G)$, a new vertex v' and a pendant edge vv'. A set $P \subseteq V(G)$ is a *packing* if $N[u] \cap N[v] = \emptyset$ for every $u, v \in P$.

Given a graph G with vertex set $V(G)$, a *proper coloring* of G is a partitioning of $V(G)$ into independent sets. These sets are called *color classes*. A proper coloring of G that has a minimum number of color classes is called a $\chi(G)$*-coloring* and the number of color classes in such a coloring is $\chi(G)$. For other definitions and terminology related to graph theory, the interested reader is referred to [3, 4, 9].

1.2 Domination Parameters

A set $S \subseteq V(G)$ is a *dominating set* (abbreviated DS) if $N[S] = V(G)$ and is a *total dominating set* (abbreviated TDS) if $N(S) = V(G)$. The minimum cardinality of any DS (respectively, TDS) of G is the *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$). A DS of G with cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*, and a $\gamma_t(G)$*-set* is defined similarly. A set $S \subset V(G)$ is a *restrained dominating set* (abbreviated RDS) of G, if for every vertex $v \in V(G) \setminus S$, v is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The *restrained domination number* $\gamma_r(G)$ is the cardinality of a minimum RDS of G. A RDS of G with cardinality $\gamma_r(G)$ is called a $\gamma_r(G)$-*set*. A set $S \subseteq V(G)$ is a *distance-k* dominating set if for every u in $V(G) \setminus S$, u is within a distance k of some vertex in S, that is for each $u \in V(G) \setminus S$, there exists a vertex $v \in S$ such that $dist(u, v) \leq k$. The $distance - k$ *domination number*, denoted $\gamma_{\leq k}(G)$, is the cardinality of a minimum $distance - k$ dominating set of G. A $distance - k$ dominating set with cardinality $\gamma_{\leq k}(G)$ is called a $\gamma_{\leq k}(G)$-*set*. A set $S \subseteq V(G)$ is a $k - step$ dominating set if for every u in $V(G) \setminus S$, u is exactly a distance k from some vertex in S, that is, for each $u \in V(G) \setminus S$, there exists a vertex $v \in S$ such that $dist(u, v) = k$. The *k-step domination number*, denoted $\gamma_{=k}(G)$, is the cardinality of a minimum *k-step* dominating set of G. A $k - step$ dominating set of cardinality $\gamma_{=k}(G)$ is called a $\gamma_{=k}(G)$- set. For further information related to domination in graphs, the interested reader is referred to [5, 6].

1.3 Stratification and Domination

In Chapter 4 of this paper, we will study stratification and domination in complementary prisms. The concept of stratification and domination in graphs was introduced by Chartrand, Haynes, Henning and Zhang in [2]. A *stratification* of a graph G is a partitioning of its vertex set. If the partition has two equivalence classes $\{V_1, V_2\}$, then G is a *2-stratified* graph. We restrict our attention in this paper to 2-stratified graphs, where we treat the equivalence classes V_1 and V_2 as color classes. We will color V_1 red and V_2 blue. Let F be a 2-stratified graph rooted at one blue vertex, and containing at least one red vertex. Given a graph G, an *F-coloring* of G is a red-blue coloring of the vertices of $V(G)$ done in such a manner that every blue vertex $v \in V(G)$ is contained in a copy of F rooted at v. The *F-domination* number, denoted $\gamma_F(G)$, is the minimum number of red vertices in an F-coloring of G. An $F-$*coloring* that colors $\gamma_F(G)$ vertices of G red is a γ_F-*coloring* of G. The set of red vertices in a γ_F coloring is called a $\gamma_F(G)$-set. If G contains no copy of F and if G has order n, then $\gamma_F(G) = n$. In this thesis, we will examine the $F-$*domination* number of a complementary prism $G\overline{G}$ when F is a 2-stratified P_3 with 1 blue rooted vertex v. This will give us 5 possibilities for the graph F.

1.4 Complementary Prisms

Complementary prisms were first introduced by Haynes, Henning, Slater and Van der Merwe in [7]. For a graph G, its *complementary prism*, denoted $G\overline{G}$, is formed from a copy of G and a copy of $\overline{G}$ by adding a perfect matching between corresponding vertices. For each $v \in V(G)$, let $\overline{v}$ denote the vertex v in the copy of $\overline{G}$. Formally $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$. For any graph G, we denote its complementary prism by $G\overline{G}$. Many well known graphs can be realized as complementary prisms. For instance, the corona $K_n \circ K_1$ is the complementary prism $K_n\overline{K}_n$. Another example, is the Petersen graph, which is the complementary prism $C_5\overline{C}_5$. See Figure 1.4.

To aid in the discussion of complementary prisms, we will use the following terminology. For a set $P \subseteq V(G)$, let $\overline{P}$ be the corresponding set of vertices in $V(\overline{G})$. For a vertex $v \in V(G)$, let $\overline{v}$ represent the corresponding vertex in $V(\overline{G})$.

In this thesis, we will explore the domination parameters defined in this introduction for complementary prisms. We will also characterize graphs for which the

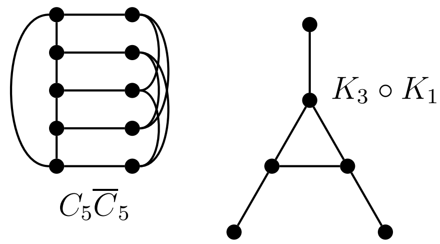

Figure 3: Examples of Complementary Prisms

complementary prism is Eulerian or bipartite and obtain bounds on the chromatic number of complementary prisms.

2 LITERATURE REVIEW

In this chapter, we review the literature dealing with complementary prisms. In Section 2.1, we will examine the complementary product first introduced in [7] and we will see how complementary prisms are a subset of this. In Section 2.2, we will review the work on domination and total domination in complementary prisms seen in [7, 8].

2.1 The Complementary Product of Two Graphs

In [7], Haynes, Henning, Slater and Van der Merwe introduced a generalization of the Cartesian product of two graphs. Let G_1 and G_2 be graphs with $V(G_1) = \{u_1, u_2, ..., u_n\}$ and $V(G_2) = \{v_1, v_2, ..., v_p\}$. The *Cartesian product* of the graphs G_1 and G_2, symbolized by $G_1 \Box G_2$, is the graph formed from G_1 and G_2 in the following manner.

The graph $G_1 \Box G_2$ has np vertices. Each of these vertices has a label taken from $V(G_1) \times V(G_2)$. In $G_1 \Box G_2$, 2 vertices (u_i, v_j) and (u_r, v_s) are adjacent if, and only if 1 of the following conditions hold:

(1.) $i = r$, and $v_j v_s \in E(G_2)$.

(2.) $j = s$, and $u_i u_r \in E(G_1)$.

For each i, the induced subgraph on the vertices (u_i, v_j) for $1 \leq j \leq p$ is a copy of G_2, and for each j, the induced subgraph on the vertices (u_i, v_j) for $1 \leq i \leq n$ is a copy of G_1. In less formal terms, $G_1 \Box G_2$ can either be viewed as the graph formed by taking each vertex of G_1, replacing it with a copy of G_2 and matching the

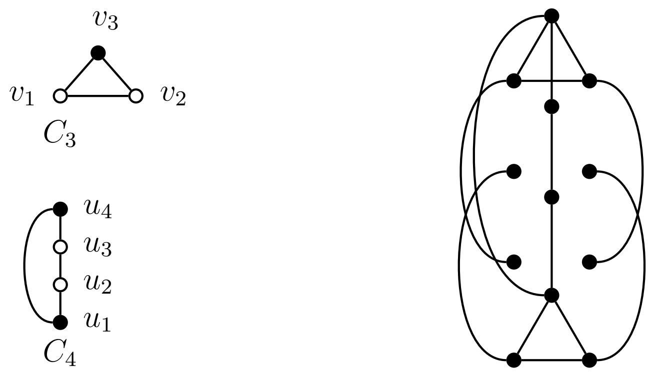

Figure 4: $C_4(\{u_1, u_4\}) \Box C_3(\{v_3\})$

corresponding vertices and taking each vertex of G_2, replacing it with a copy of G_1 and matching the corresponding vertices.

In [7], the *complementary product* of two graphs is defined as follows: Let R be a subset of $V(G)$ and S be a subset of $V(H)$. The complementary product (symbolized by $G(R)\Box H(S)$) is constructed as follows. The vertex set $V(G(R)\Box H(S))$ is $\{(u_i, v_j) \ : \ 1 \leq i \leq n, 1 \leq j \leq p\}$. And the edge $(u_i, v_j)(u_h, v_k)$ is in $E(G(R)\Box H(S))$ if one of the following conditions hold.

1. If $i = h$, $u_i \in R$, and $v_j v_k \in E(H)$, or if $i = h$, $u_i \notin R$ and $v_j v_k \notin E(H)$.

2. If $j = k$, $v_j \in S$, and $u_i u_h \in E(G)$, or if $j = k$, $v_j \notin S$, and $u_i u_h \notin E(G)$.

In other words, for each $u_i \in V(G)$, we replace u_i with a copy of H if u_i is in R and with a copy of its complement $\overline{H}$ if u_i is not in R, and for each $v_j \in V(H)$, we replace each v_j with a copy of G if $v_j \in S$ and a copy of $\overline{G}$ if $v_j \notin S$.

In the case where $R = V(G)$ (respectively, $S = V(H)$), the complementary product $G(R)\Box H(S)$ is written $G\Box H(S)$ (respectively, $G(R)\Box H$). To put it more informally, $G\Box H(S)$ is the graph obtained by replacing each vertex $v \in V(H)$ with a copy of G if $v \in S$ and by a copy of $\overline{G}$ if $v \notin S$, and replacing each u_i with a copy of H. In the extreme case where $R = V(G)$, and $S = V(H)$, the complementary product $G(V(G))\Box H(V(H)) = G\Box H$ is simply the same as the Cartesian product $G\Box H$. See Figure 4 for an illustration of $C_4(\{u_1, u_4\}) \Box C_3(\{v_3\})$.

A complementary prism $G\overline{G}$ is the complementary product $G\Box K_2(S)$ with $|S| = 1$.

2.2 Domination and Total Domination in Complementary Prisms

In [8], Haynes, Henning and Van der Merwe studied domination and total domination in complementary prisms, they obtained the following results.

When G is a complete graph K_n, the graph tK_2, the corona $K_t \circ K_1$, a cycle C_n, or a path P_n, they obtained the exact values of $\gamma(G\overline{G})$ and $\gamma_t(G\overline{G})$.

Proposition 1 *[8]*

If $G = K_n$, then $\gamma(G\overline{G}) = n$.

If $G = tK_2$, then $\gamma(G\overline{G}) = t + 1$.

If $G = K_t \circ K_1$ and $t \geq 3$, then $\gamma(G\overline{G}) = \gamma(G) = t$.

If $G = C_n$ and $n \geq 3$, then $\gamma(G\overline{G}) = \lceil (n + 4)/3 \rceil$.

If $G = P_n$ and $n \geq 2$, then $\gamma(G\overline{G}) = \lceil (n + 3)/3 \rceil$.

Proposition 2 *[8]*

If $G = K_n$, then $\gamma_t(G\overline{G}) = n$.

If $G = tK_2$, then $\gamma_t(G\overline{G}) = n = 2t$.

If $G = K_t \circ K_1$ and $t \geq 3$, then $\gamma_t(G\overline{G}) = \gamma_t(G) = t$.

If $G \in \{C_n, P_n\}$ with order $n \geq 5$, then

$$\gamma_t(G\overline{G}) = \begin{cases} \gamma_t(G), & \text{if } n \equiv 2 \pmod{4} \\ \gamma_t(G)+2, & \text{if } G = C_n, \text{ and } n \equiv 0 \pmod{4} \\ \gamma_t(G)+1, & \text{otherwise.} \end{cases}$$

They characterized graphs G for which the domination number $\gamma(G\overline{G})$ and the total domination number $\gamma_t(G\overline{G})$ of a complementary prism are small.

Proposition 3 *[8] Let G be a graph of order n. Then,*

$\gamma(G\overline{G}) = 1$ if, and only if, $G = K_1$.

$\gamma(G\overline{G}) = 2$ if, and only if, $n \geq 2$ and G has a support vertex that dominates $V(G)$ or $\overline{G}$ has a support vertex that dominates $V(\overline{G})$.

Proposition 4 *[8] Let G be a graph of order $n \geq 2$, with $|E(G)| = |E(\overline{G})|$. Then*

$\gamma_t(G\overline{G}) = 2$ if, and only if, $G = K_2$.

$\gamma_t(G\overline{G}) = 3$ if, and only if, $n \geq 3$ and $G = K_3$ or G has a support vertex that dominates $V(G)$ or $\overline{G}$ has a support vertex that dominates $V(\overline{G})$.

They found upper and lower bounds on the parameters $\gamma(G\overline{G})$ and $\gamma_t(G\overline{G})$. Proposition 5 will be needed later in this work.

Proposition 5 *[8] For any graph G,* $\max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma(G\overline{G}) \leq \gamma(G) + \gamma(\overline{G})$.

Proposition 6 *[8] If G and $\overline{G}$ are without isolates, then* $\max\{\gamma_t(G), \gamma_t(\overline{G})\} \leq \gamma_t(G\overline{G}) \leq \gamma_t(G) + \gamma_t(\overline{G})$.

Finally, they characterized graphs G for which $\gamma(G\overline{G}) = max\{\gamma(G), \gamma(\overline{G})\}$ and $\gamma_t(G\overline{G}) = max\{\gamma_t(G), \gamma_t(\overline{G})\}$.

Proposition 7 *[8] A graph G satisfies $\gamma(G\overline{G}) = \gamma(G) \geq \gamma(\overline{G})$ if, and only if, G has an isolated vertex or there exists a packing P of G such that $|P| \geq 2$ and $\gamma(G \setminus P) = \gamma(G) - |P|$.*

Proposition 8 *[8] Let G be a graph such that neither G nor $\overline{G}$ has an isolated vertex. Then $\gamma_t(G\overline{G}) = \gamma_t(G) \geq \gamma_t(\overline{G})$ if, and only if, $G = n/2K_2$ or there exists an open packing $P = P_1 \cup P_2$ in G satisfying the following conditions:*

$|P| \geq 2$;

$P_1 \cap P_2 = \emptyset$;

if $P_1 \neq \emptyset$, then P_1 is a packing in G;

if $P_1 = \emptyset$, then $|P| \geq 3$ or $G[P] = \overline{K}_2$;

$\gamma_t(G \setminus N[P_1] \setminus P_2) = \gamma_t(G) - 2|P_1| - |P_2|$.

3 RESTRAINED DOMINATION IN COMPLEMENTARY PRISMS

In this chapter, we present some of the major results of this thesis. We will parallel the work done in [8] for domination and total domination and will obtain analogous results for the restrained domination number $\gamma_r(G\overline{G})$ of a complementary prism.

3.1 Restrained Domination Number of $G\overline{G}$ for a Specific Graph G

In this section, we determine the restrained domination number of the complementary prism $G\overline{G}$ for selected graphs G. We begin with a useful observation.

Observation 9 *Every RDS of a graph G must include all of the isolated and end-vertices of G.*

First, we find the restrained domination number of $G\overline{G}$, when G is a complete graph.

Proposition 10 *If G is the non-trivial complete graph K_n, then $\gamma_r(G\overline{G}) = n$.*

Proof. Let $G = K_n$ with order $n \geq 2$. Since for every $\overline{u} \in V(\overline{G})$, $deg_{G\overline{G}}\,(\overline{u}) = 1$; by Observation 9 it follows that $\overline{u}$ is in every $\gamma_r(G\overline{G})$-set. Thus $\gamma_r(G\overline{G}) \geq n$. Since $n \geq 2$, every vertex in $V(G)$ has a neighbor in $V(G)$ and a neighbor in $V(\overline{G})$. Therefore, $V(\overline{G})$ is a RDS for $G\overline{G}$. Hence, $\gamma_r(G\overline{G}) \leq |V(\overline{G})| = n$ and so $\gamma_r(G\overline{G}) = n$. $\square$

We now give the restrained domination number for the complementary prism of paths and cycles. We need the following observation from [8].

Observation 11 *For the path P_n, $\gamma(P_n) = \lceil n/3 \rceil$.*

Proposition 12 *If $G \in \{C_n, P_n\}$ and $n \geq 3$, then*

$$\gamma_r(G\overline{G}) = \begin{cases} \lceil (n+7)/3 \rceil, & \text{if } n \in \{4,5\} \\ \lceil (n+4)/3 \rceil, & \text{otherwise.} \end{cases}$$

Proof. It is a simple exercise to verify the cases when $n \in \{3,4,5\}$. Let $G \in \{C_n, P_n\}$, and assume that $n \geq 6$. Let $\{u_1 \dots u_n\}$ be the n vertices of G labeled sequentially, where if $G = P_n$, then u_1 and u_n are the two endvertices of the path. Let $G^* = \langle V(G) \setminus \{u_1, u_n\} \rangle$. Then G^* is a path on $n-2$ vertices. Let T be any $\gamma(G^*)$-set that does not include vertices u_2 and u_{n-1} (such a set always exists since if an endvertex is in a DS, we can replace it by its neighboring support vertex). By Observation 11, $|T| = \lceil (n-2)/3 \rceil$. We note that the set $T \cup \{\overline{u}_1, \overline{u}_n\}$ forms a RDS for $G\overline{G}$. Therefore, $\gamma_r(G\overline{G}) \leq |T| + 2 = \lceil (n-2)/3 \rceil + 2 = \lceil (n+4)/3 \rceil$. If $G = C_n$, Proposition 1 states that $\gamma(G\overline{G}) = \lceil (n+4)/3 \rceil$. Hence, if $G = C_n$, then $\gamma_r(G\overline{G}) \geq \gamma(G\overline{G}) = \lceil (n+4)/3 \rceil$, and so $\gamma_r(G\overline{G}) = \lceil (n+4)/3 \rceil$.

If $G = P_n$, then by Proposition 1, $\gamma(G\overline{G}) = \lceil (n+3)/3 \rceil$. If $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$, then $\gamma(G\overline{G}) = \lceil (n+4)/3 \rceil$. Thus $\gamma_r(G\overline{G}) \geq \gamma(G\overline{G}) = \lceil (n+4)/3 \rceil$, and so $\gamma_r(G\overline{G}) = \lceil (n+4)/3 \rceil$.

Hence, the only remaining case to consider is for $n \equiv 0 \pmod 3$. Let $n = 3k$ for some integer $k \geq 2$. Now, by Observation 11, $\gamma(G\overline{G}) = k+1$ hence, $k+1 \leq \gamma_r(G\overline{G}) \leq \lceil (n+4)/3 \rceil = k+2$. Assume (for purposes of contradiction) that $\gamma_r(G\overline{G}) = k+1$, that is, there exists a $\gamma(G\overline{G})$-set S that is also a $\gamma_r(G\overline{G})$-set.

Claim $|S \cap V(G)| \geq k$.

Proof of Claim Suppose not, then $k - |S \cap V(G)| = t \geq 1$. Since each vertex in $V(G)$ dominates at most three vertices of $V(G)$, it follows that at least $3t$ vertices of $V(G)$ are not dominated by $S \cap V(G)$. Moreover, each vertex in $V(\overline{G})$ dominates exactly one vertex of $V(G)$, implying that $|S \cap V(\overline{G})| \geq 3t$. Hence, $|S| = |S \cap V(G)|$ $+|S \cap V(\overline{G})| \geq k - t + 3t = k + 2t > k + 1$, a contradiction. Thus $|S \cap V(G)| \geq k$. (end of proof of claim)

If $S \subseteq V(G)$, then to dominate $V(\overline{G})$, $|V(\overline{G})| = |S| = k + 1$, a contradiction. Thus $|S \cap V(G)| = k$, and $|S \cap V(\overline{G})| = 1$. Let $\{\overline{v}\} = V(\overline{G}) \cap S$. Assume that v is an endvertex of G. Without loss of generality let $v = u_1$. Then S must equal $\{u_{2+3i} | 0 \leq i \leq (n-3)/3\} \cup \{\overline{u}_1\}$ in order to dominate V(G). But then $N(u_1) \subseteq S$, so S is not a RDS of $G\overline{G}$, a contradiction. Assume next that v is not an endvertex of G, that is $v = u_i,\ 2 \leq i \leq n-1$. In order for S to dominate $\overline{u}_{i-1}$ *and* $\overline{u}_{i+1}$, it is necessary for both $u_{i-1} \in S$ *and* $u_{i+1} \in S$. This fails to make S a DS for $G\overline{G}$ a contradiction. Therefore, $\gamma_r(G\overline{G}) = k + 2 = \lceil (n+4)/3 \rceil$. □

Next, we find the restrained domination number of the corona $K_t \circ K_1$.

Proposition 13 *If G is the corona $K_t \circ K_1$ then*

$$\gamma_r(G\overline{G}) = \begin{cases} t, & \text{if } t \geq 4 \\ 2, & \text{if } t = 1 \\ 4, & \text{if } t \in \{2,3\}. \end{cases}$$

Proof. Let $G = K_t \circ K_1$. If t=1, then $G = K_2$, $G\overline{G} = P_4$ and $\gamma_r(G\overline{G}) = 2$. If $t = 2$, then $G = P_4$, and by Proposition 11, $\gamma_r(G\overline{G}) = 4$. It is a straightforward exercise to check the case where $t = 3$. Assume $t \geq 4$, and label the vertices of G as follows: let

$A = \{a_i | 1 \leq i \leq t\}$ be the set of t vertices that induce the K_t subgraph of G, and let $B = \{b_j, | 1 \leq j \leq t\}$ be the endvertices in G adjacent to the vertices in A such that $a_i b_i \in E(G)$. Let $S = \{\overline{B} \setminus \{\overline{b}_1, \overline{b}_2\}\} \cup \{a_1, b_2\}$. Then S is a RDS of $G\overline{G}$ with $|S| = t$. Hence, $\gamma_r(G\overline{G}) \leq t$. Now let S be any $\gamma_r(G\overline{G})$-set. Since S must dominate $G\overline{G}$, it is necessary that for each $b_i \in B$, $N_{G\overline{G}}[b_i] \cap S \neq \emptyset$. However, for each $b_i, b_j \in B$, $N_{G\overline{G}}[b_i] \cap N_{G\overline{G}}[b_j] = \emptyset$; therefore, $\gamma_r(G\overline{G}) = |S| \geq |B| = t$. Thus $\gamma_r(G\overline{G}) = t$.□

Finally, we find the restrained domination number of $G = tK_2$.

Proposition 14 *If $G = tK_2$, then $\gamma_r(G\overline{G}) = t + 1$ unless $t = 2$, then $\gamma_r(G\overline{G}) = 4$.*

Proof. If $t = 1$, then $G\overline{G} = P_4$ and $\gamma_r(G\overline{G}) = \gamma_r(P_4) = 2 = t + 1$. If $t = 2$, then $\overline{G}$ $=C_4$, and by Proposition 12, $\gamma_r(G\overline{G}) = 4$. Let $t \geq 3$. Label the $2t$ vertices of $V(G)$ as u_i, v_i, $1 \leq i \leq t$, such that $u_i v_i \in E(G)$. The set $\{\overline{u}_1, \overline{v}_1, u_2 \ldots u_t\}$ is a RDS for $G\overline{G}$ with cardinality $t + 1$. Hence, $\gamma_r(G\overline{G}) \leq t + 1$. By Proposition 1 $\gamma(G\overline{G}) \geq t + 1$, and therefore, $\gamma_r(G\overline{G}) \geq \gamma(G\overline{G}) \geq t + 1$ so $\gamma_r(G\overline{G}) = t + 1$. □

3.2 Complementary Prisms with Small Restrained Domination Number

In this section, we consider complementary prisms with small restrained domination numbers.

Proposition 15 *For a graph G and its complementary prism $G\overline{G}$,*

(*1*) $\gamma_r(G\overline{G}) \neq 1$ *for any graph G.*

(*2*) $\gamma_r(G\overline{G}) = 2$ *if, and only if, $|V(G)| \in \{1, 2\}$.*

(3) $\gamma_r(G\overline{G}) = 3$ *if, and only if,* $|V(G)| = 3$.

Proof. **(1)** Assume to the contrary that $\gamma_r(G\overline{G}) = 1$ for some graph G. Without loss of generality, assume that $S \subseteq V(G)$ and $S = \{u\}$ is a $\gamma_r(G\overline{G})$-set. The only vertex in $\overline{G}$ dominated by u is $\overline{u}$. Therefore, $|V(\overline{G})|$ =1, and $G = K_1$. Hence, $G\overline{G}$ $=K_2$. By Observation 9, $\gamma_r(G\overline{G}) = 2$, a contradiction.

(2) If $|V(G)| = 1$, then $G\overline{G} = K_2$ and $\gamma_r(G\overline{G}) = 2$. If $|V(G)| = 2$, then $G\overline{G} = P_4$ and $\gamma_r(G\overline{G}) = 2$. Assume that $\gamma_r(G\overline{G}) = 2$, and let S be any $\gamma_r(G\overline{G})$-set. If $S \subseteq V(G)$, then in order for S to dominate $V(\overline{G})$, it is necessary for $|V(\overline{G})| = |S| = 2$. Similarly if $S \subseteq V(\overline{G})$, then $|V(G)| = 2$.

Assume that $S \cap V(G) \neq \emptyset$ and $S \cap V(\overline{G}) \neq \emptyset$. Let $S = \{u, \overline{v}\}$. Vertex u must dominate $V(G) \setminus \{v\}$ and $\overline{v}$ must dominate $\overline{G} \setminus \{\overline{u}\}$. If $u = v$, then $G = K_1$ and $G\overline{G} = K_2$. Assume that $u \neq v$. If u is adjacent to v, then $deg_G(u) = n - 1$ and $\overline{u}$ is a leaf in $G\overline{G}$. By Observation 9, it follows that $\overline{u} \in S$ but this forces $u = v$, a contradiction. If u is not adjacent to v, then $\overline{u}$ is adjacent to $\overline{v}$. This implies that $deg_{\overline{G}}(\overline{v}) = n - 1$. Hence, v is a leaf in $G\overline{G}$. By Observation 9, it follows that $v \in S$ hence, $u = v$, a contradiction.

(3) It is an easy observation to see that if $|V(G)| = 3$, then $G \in \{K_3, \overline{K}_3, P_3, \overline{P}_3\}$ and by Propositions 10 and 12, $\gamma_r(K_3\overline{K}_3) = \gamma_r(P_3\overline{P}_3) = 3$.

For the converse, assume that $\gamma_r(G\overline{G}) = 3$, and let S be any $\gamma_r(G\overline{G})$-set. If $S \subseteq V(G)$, then in order to dominate $\overline{G}$, it is necessary for $|V(\overline{G})| = 3$. Likewise, if $S \subseteq V(\overline{G})$, then in order to dominate G, it is necessary for $|V(G)| = 3$. In either case, it follows that $|V(G)| = 3$.

Assume that $S \cap V(G) \neq \emptyset$ and $S \cap V(\overline{G}) \neq \emptyset$. Without loss of generality, let $S = \{u, \overline{v}, \overline{w}\}$, where $u \in V(G)$, and $\overline{v}, \overline{w} \in V(\overline{G})$. If $\overline{u} \notin \{\overline{v}, \overline{w}\}$, then u dominates $G \setminus \{v, w\}$. Hence, in $\overline{G}$, the only neighbors of $\overline{u}$ are in $\{\overline{v}, \overline{w}\}$. This implies that $N_{G\overline{G}}(\overline{u}) \subseteq S$. But $\overline{u} \in V(G\overline{G}) \setminus S$, contradicting the fact that S is a RDS of $G\overline{G}$. If $\overline{u} \in \{\overline{v}, \overline{w}\}$, then without loss of generality, let $\overline{u} = \overline{v}$. This implies that u dominates $V(G) \setminus \{w\}$, thus $\overline{u}$ has no neighbor in $V(\overline{G}) \setminus \{\overline{w}\}$. Since S is a RDS for $G\overline{G}$, it follows that $\overline{w}$ dominates $V(\overline{G}) \setminus \{\overline{u}\}$. Hence, $N_{G\overline{G}}(w) \subseteq S$, again contradicting that S is a RDS of $G\overline{G}$. □

3.3 Bounds on the Restrained Domination Number for $G\overline{G}$

In contrast to the bounds seen in Proposition 5 and Proposition 6, $\gamma_r(G\overline{G})$ is not bounded below by $\max\{\gamma_r(G), \gamma_r(\overline{G})\}$. For example, let G=$K_{1,m}$ where $m \geq 4$. Then $\gamma_r(G)$ =$m+1$, $\gamma_r(\overline{G})$ =2, but $\gamma_r(G\overline{G}) = 4 < \max\{\gamma_r(G), \gamma_r(\overline{G})\}$. However, we give the following bounds for $\gamma_r(G\overline{G})$.

Proposition 16 *For any graph G,* $\max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma_r(G\overline{G}) \leq \gamma_r(G) + \gamma_r(\overline{G})$ *and these bounds are sharp.*

Proof. Let S be any $\gamma_r(G)$-set, and let T be any $\gamma_r(\overline{G})$ set. Then $S \cup T$ is a RDS for $G\overline{G}$. Therefore, $\gamma_r(G\overline{G}) \leq |S \cup T| = \gamma_r(G) + \gamma_r(\overline{G})$. For sharpness of the upper bound, consider the family of split graphs formed from a clique of order $n \geq 3$, and an independent set of order $r \geq 3$ with all possible edges between the clique vertices and the independent set vertices.

By Proposition 5, $\gamma(G\overline{G}) \geq \max\{\gamma(G), \gamma(\overline{G})\}$. Since $\gamma_r(G\overline{G}) \geq \gamma(G\overline{G})$, it follows that $\gamma_r(G\overline{G}) \geq \max\{\gamma(G), \gamma(\overline{G})\}$. For sharpness, consider the $K_t \circ K_1$ (see Proposition

13). $\square$

Next, we characterize the graphs G attaining the lower bound $\gamma_r(G\overline{G}) = \max\{\gamma(G)\gamma(\overline{G})\}$.

Theorem 17 *Let G be a graph of order n, and assume without loss of generality, that $\gamma(\overline{G}) \leq \gamma(G)$. Then $\gamma_r(G\overline{G}) = \gamma(G)$ if, and only if, $G = \overline{K}_n$ or G has a maximal packing P, such that for every $v \in P$, $deg_G(v) \geq 1$, $2 \leq |P| \leq \gamma(G) - 2$ and $2 \leq \gamma(G \setminus P) = \gamma(G) - |P|$.*

Proof. First, assume that $G = \overline{K}_n$. Then $\gamma(G) = n > \gamma(\overline{G}) = 1$ and $\gamma_r(G\overline{G}) = n = \gamma(G)$.

Assume that $G \neq \overline{K}_n$, has a maximal packing satisfying the hypothesis. Let S be any $\gamma(G \setminus P)$- set. Consider the set of vertices $S \cup \overline{P}$.

Claim $S \cup \overline{P}$ is an RDS of $G\overline{G}$.

Proof of Claim: Clearly $\overline{P}$ dominates P in $G\overline{G}$. Since P is a packing in G and $|P| \geq 2$, it follows that $\overline{P}$ dominates $V(\overline{G})$. By definition, S dominates $V(G \setminus P)$ so $S \cup \overline{P}$ is a DS of $G\overline{G}$. Now it is necessary to show that every vertex in $V(G\overline{G}) \setminus (S \cup \overline{P})$ has a neighbor in $V(G\overline{G}) \setminus (S \cup \overline{P})$. For every $v \in V(G) \setminus (S \cup P)$, its neighbor $\overline{v} \in V(\overline{G}) \setminus (\overline{S} \cup \overline{P})$ is in $V(G\overline{G}) \setminus (S \cup \overline{P})$. In like manner, for every $\overline{v} \in V(\overline{G}) \setminus (\overline{S} \cup \overline{P})$, its neighbor $v \in V(G) \setminus (S \cup P)$ is in $V(G\overline{G}) \setminus (S \cup \overline{P})$. Now, we need only to consider the vertices in $\overline{S} \cup P$. Since every $v \in P$ has $deg_G(v) \geq 1$, and P is a packing, v must have a neighbor in $V(G) \setminus P$. If v has a neighbor in S, then $S \cup (P \setminus \{v\})$ is a DS of

G and $\gamma(G) \leq |S \cup (P \setminus \{v\})| = |S| + |P| - 1 = \gamma(G \setminus P) + |P| - 1$, contradicting the hypothesis. Hence, every $v \in P$, has a neighbor in $V(G) \setminus (P \cup S) \subseteq V(G\overline{G}) \setminus (S \cup \overline{P})$. Finally, every $\overline{v} \in \overline{S}$ must have a neighbor in $V(\overline{G}) \setminus \overline{P} \subseteq V(G\overline{G}) \setminus (S \cup \overline{P})$ else if this is not the case, then in G, vertex v dominates $G \setminus P$ contradicting the hypothesis that $\gamma(G \setminus P) \geq 2$. Therefore, $S \cup \overline{P}$ is an RDS for $G\overline{G}$. (end of proof of claim)

Hence,$\gamma_r(G\overline{G}) \leq |S \cup \overline{P}| = |S| + |\overline{P}| = |S| + |P| = \gamma(G) - |P| + |P| = \gamma(G)$. Since by Proposition 16, $\gamma_r(G\overline{G}) \geq \gamma(G)$, it follows that $\gamma_r(G\overline{G}) = \gamma(G)$.

Now, assume that for a graph G of order n, $\gamma_r(G\overline{G}) = \gamma(G) \geq \gamma(\overline{G})$. Let S be any $\gamma_r(G\overline{G})$-set. Let $S_1 = S \cap V(G)$ and $S_2 = S \cap V(\overline{G})$. If, without loss of generality, $S = S_1$, then $S \subseteq V(G)$. In order to dominate $\overline{G}$, it would be necessary for $|S| = \gamma(G) = n$ and this implies that $G = \overline{K}_n$. Thus, assume that $|S_1| \geq 1$ and $|S_2| \geq 1$. Since $|S_1| < |S| = \gamma(G)$, it follows that S_1 does not dominate G. Let P be the set of vertices of G not dominated by S_1. Since S is a DS of $G\overline{G}$, P is dominated by $\overline{P} \subseteq S_2$. Since $S_1 \cup P$ dominates G, it follows that $|S_1| + |P| \geq \gamma(G) = \gamma_r(G\overline{G}) = |S| = |S_1| + |S_2| \geq |S_1| + |\overline{P}| = |S_1| + |P|$, implying that $|S_2| = |\overline{P}|$. Since $\overline{P} \subseteq S_2$, it follows that $S_2 = \overline{P}$. The set P is an independant set since if $u, v \in P$ are adjacent, then $S_1 \cup P \setminus \{v\}$ is a DS of G with cardinality less than $\gamma(G)$, a contradiction. The set P must be a packing in G, since if there exists a vertex $w \in V(G)$ that is a neighbor of $v, u \in P$, then $S_1 \cup (P \setminus \{u, v\}) \cup \{w\}$ is a DS of G with cardinality $|S_1| + |P| - 1 = \gamma(G) - 1$, a contradiction. Moreover, no vertex $x \in P$ has a neighbor in S_1 for otherwise $S_1 \cup (P \setminus \{x\})$ dominates G, a contradiction. Therefore, P is a packing in G.

Let D be a $\gamma(G\backslash P)$-set. If $\gamma(G\backslash P) < |S_1|$, then $D \cup P$ is a DS of G with cardinality $|D \cup P| = |D| + |P| < |S_1| + |P| = |S_1| + |\overline{P}| = |S_1| + |S_2| = \gamma_r(G\overline{G}) = \gamma(G)$, a contradiction. Hence, $\gamma(G \setminus P) = |S_1| = \gamma(G) - |P|$. To see that $\gamma(G \setminus P) = |S_1| \geq 2$, assume to the contrary that $\gamma(G \setminus P) = 1$. Then, $S_1 = \{v\}$ and $N_{G\overline{G}}(\overline{v}) \subseteq \overline{P} \cup \{v\} = S_2 \cup \{v\}$, contradicting that S is a $\gamma_r(G\overline{G})$-set. To see that for every vertex $u \in P$, $deg_G(u) \geq 1$, assume to the contrary that there exists some vertex $u \in P$, such that $deg_G(u) = 0$. Then $deg_{G\overline{G}}(u) = 1$ hence, by Observation 9, u is in every $\gamma_r(G\overline{G})$-set. Hence, $u \in S_1$, and thus $u \notin P$, a contradiction. Finally, to see that $|P| \geq 2$. Assume to the contrary that $P = \{v\}$. Since $deg_G(v) \geq 1$, it follows that v is adjacent to some vertex $w \in V(G \setminus P \cup S_1)$. In $\overline{G}$, $\overline{v}$ does not dominate $\overline{w}$, but then $\overline{w}$ is not dominated by $S_1 \cup S_2$ a contradiction. □

Next, we give another upper bound for $\gamma_r(G\overline{G})$.

Theorem 18 *For any graph G, $\gamma_r(G\overline{G}) \leq 2\max\{\gamma(G), \gamma(\overline{G})\}$ and this bound is sharp.*

Proof: Let $R = \max\{\gamma(G), \gamma(\overline{G})\}$. If $R = 1$, then $G = K_1$ and $\gamma_r(G\overline{G}) = 2 \leq 2R$. Assume $R \geq 2$. Let D_1 be a $\gamma(G)$-set, and let D_2 be a $\gamma(\overline{G})$-set. If $D_1 \cup D_2$ forms a RDS for $G\overline{G}$, then $\gamma_r(G\overline{G}) \leq |D_1| + |D_2| \leq 2R$. Assume $D_1 \cup D_2$ is not a RDS of $G\overline{G}$. Then, without loss of generality, there exists a vertex $w \in V(G) \setminus D_1$, such that $N_{G\overline{G}}(w) \subseteq D_1 \cup D_2$. Thus $\overline{w} \in D_2$. Since D_1 dominates $V(G)$, it follows that $|N_{G\overline{G}}(w) \cap D_1| \geq 1$. We consider two cases:

Case 1. $|N_{G\overline{G}}(w) \cap D_1| = 1$. Let $v \in D_1$ be a neighbor of w. Since $N_G(w) \subseteq D_1$, we have $deg_G(w) = 1$. Hence, $\overline{w}$ dominates $\overline{G} \setminus \{\overline{v}\}$. If $epn(v, D_1) = \{w\}$, then $D_1 \setminus \{v\}$

dominates $G \setminus \{w, v\}$. Thus $S = D_1 \setminus \{v\} \cup \{\overline{w}\} \cup \overline{D}_1$ is a DS of $G\overline{G}$. Furthermore, each vertex $x \in V(G)\setminus(D_1 \cup \{w\})$ has a neighbor $\overline{x} \in V(\overline{G})\setminus(\overline{D}_1 \cup \{\overline{w}\}) \subseteq V(G\overline{G})\setminus S$. Also, each $\overline{x} \in V(\overline{G})\setminus(\overline{D}_1 \cup \{\overline{w}\})$, has a neighbor $x \in V(G)\setminus(D_1 \cup \{w\}) \subseteq V(G\overline{G})\setminus S$. Both $v, w \in V(G\overline{G}) \setminus S$ and v is adjacent to w so S is an RDS of $G\overline{G}$, and $\gamma_r(G\overline{G}) \leq |S| = |D_1 \setminus \{v\}| + 1 + |\overline{D}_1| = 2|D_1| \leq 2R$.

If $|epn(v, D_1)| > 1$, then let $y \neq w$ and $y \in epn(v, D_1)$. If $\gamma(G) = 1$, then $\overline{v}$ is an isolate in $\overline{G}$. Since $\overline{w}$ dominates $V(\overline{G}) \setminus \{\overline{v}\}$, it follows that $\gamma(\overline{G}) = 2$. The set $\{\overline{v}, \overline{w}\}$ dominates $\overline{G}$. The set $\{v, w\}$ dominates $V(G)$, so $S = \{v, \overline{v}, w, \overline{w}\}$ is a DS for $G\overline{G}$. Every $u \in V(G) \setminus S$ has a neighbor $\overline{u} \in V(\overline{G}) \setminus S \subseteq V(G\overline{G}) \setminus S$. Likewise every $\overline{u} \in V(\overline{G}) \setminus S$ has a neighbor $u \in V(G) \setminus S \subseteq V(G\overline{G}) \setminus S$. Therefore, S is an RDS for $G\overline{G}$ and $\gamma_r(G\overline{G}) \leq |S| = 4 \leq 2R$. Assume that $\gamma(G) \geq 2$. The set D_1 dominates $V(G) \cup \overline{D}_1$. The vertex $\overline{w}$ dominates $V(\overline{G}) \setminus \overline{D}_1$. Therefore, the set $D_1 \cup \{\overline{w}\}$ is a DS for $G\overline{G}$. To see that the set $S = D_1 \cup \{w, \overline{w}, \}$ is a RDS for $G\overline{G}$, consider the following. Every $u \in V(G) \setminus S$ has a neighbor $\overline{u} \in V(\overline{G}) \setminus S \subseteq V(G\overline{G}) \setminus S$, and every $\overline{u} \in V(\overline{G}) \setminus (S \cup \overline{D}_1)$ has a neighbor $u \in V(G) \setminus S \subseteq V(G\overline{G}) \setminus S$. Every $\overline{u} \in \overline{D}_1 \setminus \{\overline{v}\}$ is adjacent to $\overline{y} \in V(G\overline{G}) \setminus S$. Since $\gamma(G) \geq 2$, it follows that there exists a vertex $\overline{q} \in V(\overline{G}) \setminus S \subseteq V(G\overline{G}) \setminus S$ such that $\overline{q}$ is adjacent to $\overline{v}$. Therefore, S is an RDS for $G\overline{G}$ and $\gamma_r(G\overline{G}) \leq |S| = |D_1| + |\{w, \overline{w}\}| = \gamma(G) + 2 \leq 2R$.

Case 2. $|N_{G\overline{G}}(w) \cap D_1| \geq 2$. Let u and v be in $N_{G\overline{G}}(w) \cap D_1$. Assume first that epn(u,D_1)= $\emptyset$. Since $\{\overline{w}\}$ dominates $V(\overline{G}) \setminus \overline{D}_1$, $D_1 \setminus \{u\}$ dominates $V(G) \setminus \{u\}$ and $\overline{D}_1$ dominates D_1, it follows that the set $S = D_1 \setminus \{u\} \cup \overline{D}_1 \cup \{\overline{w}\}$ is a DS for $G\overline{G}$. Furthermore, each vertex $x \in V(G) \setminus (D_1 \cup \{w\})$ has a neighbor $\overline{x} \in$

$V(\overline{G}) \setminus (\overline{D}_1 \cup \{\overline{w}\}) \subseteq V(G\overline{G}) \setminus S$. In like manner, every vertex $\overline{x} \in V(\overline{G}) \setminus (\overline{D}_1 \cup \{\overline{w}\})$ has a neighbor $x \in V(G) \setminus (D_1 \cup \{w\}) \subseteq V(G\overline{G}) \setminus S$. Also, u is adjacent to w. Therefore, S is an RDS for $G\overline{G}$, and hence, $\gamma_r(G\overline{G}) \leq |S| = 2|D_1| \leq 2R$.

Assume that $epn(u, D_1) \neq \emptyset$ and $epn(v, D_1) \neq \emptyset$. Let $y \in epn(u, D_1)$ and let $z \in epn(v, D_1)$. Since D_1 dominates $V(G) \cup \overline{D}_1$ and $\overline{w}$ dominates $V(\overline{G}) \setminus \overline{D}_1$, it follows that the set $S = D_1 \cup \{w\} \cup \{\overline{w}\}$ is a DS for $G\overline{G}$. Furthermore, every vertex $x \in V(G) \setminus (D_1 \cup \{w\})$ has a neighbor $\overline{x} \in V(\overline{G}) \setminus (\overline{D}_1 \cup \{\overline{w}\}) \subseteq V(G\overline{G}) \setminus S$. In like manner, every vertex $\overline{x} \in V(\overline{G}) \setminus (\overline{D}_1 \cup \{\overline{w}\})$ has a neighbor $x \in V(G) \setminus (D_1 \cup \{w\}) \subseteq V(G\overline{G}) \setminus S$. Finally, every vertex $\overline{x} \in \overline{D}_1$ is adjacent to $\overline{y} \in V(G\overline{G}) \setminus S$ or $\overline{z} \in V(G\overline{G}) \setminus S$. Therefore, S is an RDS for $G\overline{G}$ hence, $\gamma_r(G\overline{G}) \leq |D_1| + 2 \leq 2R$ □

This bound is sharp, consider the graph $K_{1,m}$. For $m \geq 3$, $\gamma_r(K_{1,m}\overline{K}_{1,m}) = 4 = 2 \max\{\gamma(K_{1,m}), \gamma(\overline{K}_{1,m})\}$.

We conclude this chapter with a theorem on the realizability of the parameter $\gamma_r(G\overline{G})$.

Theorem 19 *The restrained domination number $\gamma_r(G\overline{G})$ is realizable for every possible integer k where $2 \leq \max\{\gamma(G), \gamma(\overline{G})\} \leq k \leq 2\max\{\gamma(G), \gamma(\overline{G})\}$.*

Proof: Let $R = \max\{\gamma(G), \gamma(\overline{G})\}$. We consider three cases.

Case 1. $R = 2$. Let $G = K_2$. Then $\gamma(G) = 1$, $\gamma(\overline{G}) = 2$ and $G\overline{G} = P_4$, for which $\gamma_r(G\overline{G}) = \gamma_r(P_4) = 2 = R$. If $G = P_3$, then $\gamma(G) = 1$, $\gamma(\overline{G}) = 2$ and by Proposition 12, $\gamma_r(G\overline{G}) = 3 = R + 1$. If G is a star of order at least 5, then $\gamma(G) = 1$, $\gamma(\overline{G}) = 2$ and $\gamma_r(G\overline{G}) = 4 = 2R$.

Case 2. $R \geq 3$ and $\gamma_r(G\overline{G}) = R$ or $R + 1$. Let G= K_R, then $\gamma(G) = 1$, $\gamma(\overline{G}) = R$ and $\gamma_r(G\overline{G}) = R$ by Proposition 10. Now let G be a split graph formed by a clique of order $R - 1$ and an independent set of order two with all possible edges between the vertices of the clique and the vertices of the independent set. Thus $\gamma(G) = 1$, $\gamma(\overline{G}) = R$ and $\gamma_r(G\overline{G}) = R + 1$.

Case 3. $R \geq 3$ and $\gamma_r(G\overline{G}) = R + i$, $2 \leq i \leq R$. The following construction is similar to the graph constructed in [1]. Define graph G as follows. Let $V(G) = A \cup B \cup C$ where A consists of $\binom{R^2}{i-1}$ vertices labeled by subsets of cardinality $i - 1$ taken from the first R^2 integers $\{1, 2, \ldots R^2\}$. Let $B = \{b_l | 1 \leq l \leq R\}$, and let $C = \{c_j, | 1 \leq j \leq R^2\}$. Add edges as follows to obtain G. Make $\langle A \cup B \rangle$ a complete subgraph. For $1 \leq j \leq R^2$, c_j is adjacent to the $\binom{R^2-1}{i-2}$ vertices in A which contain j in their labels. Make vertex b_t $1 \leq t \leq R$ adjacent to vertex c_s, $1 \leq s \leq R^2$ if, and only if, s and t are congruent modulo R, denoted $s \equiv t (\text{mod } R)$.

The set B forms a DS for G since any element of B dominates $A \cup B$, and every element of C is adjacent to an element of B. Hence, $\gamma(G) \leq R$. Now if there exists a $\gamma(G)$-set S that is missing at least one element of B, say element b_w, $1 \leq w \leq R$, then since C forms an independent set, the only way to dominate the R elements of C of the form c_j where $j \equiv w (\text{mod } R)$ requires at least $\lceil R/(i-1) \rceil$ elements of $A \cup C$. Thus for every element $b_w \in B$ such that $b_w \notin S$, S must contain at least one unique element of $A \cup C$ to dominate the R elements of C of the form c_j where $j \equiv w (\text{mod } R)$. This implies that $|S| \geq R$. Therefore, $\gamma(G) = R$.

In the graph $\overline{G}$, $\langle \overline{A} \cup \overline{B} \rangle$ is an empty graph and $\langle \overline{C} \rangle$ is a complete graph. Any vertex $\overline{c}_j \in \overline{C}$ is adjacent to those elements of $\overline{A}$ which do not include j in their label. As such, any subset of $\overline{C}$ with cardinality of at least i dominates the set $\overline{A}$. Since $\langle \overline{C} \rangle$ is complete, any vertex of $\overline{C}$ dominates $\overline{C}$. Any two vertices $\overline{c}_j, \overline{c}_q \in \overline{C}$ such that $j \not\equiv q (\text{mod } R)$ dominate $\overline{B}$ therefore, the set $\{\overline{c}_1 \ldots \overline{c}_i\}$ dominates $\overline{G}$. Now, assume that there exists a $\gamma(\overline{G})$-set S of $\overline{G}$ that contains fewer than i vertices of $\overline{C}$. If S contains no vertices of $\overline{C}$, then $|S| = \binom{R^2}{i-1} + R$ since $\langle \overline{A} \cup \overline{B} \rangle$ is an empty subgraph and $|\langle \overline{A} \cup \overline{B} \rangle| = \binom{R^2}{i-1} + R$. For any t such that $1 \leq t < i-1$, if S contains exactly t vertices of $\overline{C}$, then it must also contain the $\binom{R^2-t}{i-1-t}$ vertices of $\overline{A}$ that contain the indices of the t elements of $\overline{C}$ in their label. This implies that $|S| \geq t + R^2 - t = R^2$ $> i$. If S contains exactly $i-1$ elements of $\overline{C}$, then S must contain the one element of A which has the indices of each of the $i-1$ elements of $\overline{C}$ in its label. This means that $|S| \geq i$. In conclusion if S is a $\gamma(\overline{G})$-set, then $|S| \geq i$. Since $\{\overline{c}_1 \ldots\ \overline{c}_i\}$ dominates $\overline{G}$, it follows that $\gamma(\overline{G}) = i$.

Let D = $\{\overline{c}_1 \ldots\ \overline{c}_i\}$. Since B dominates G and D dominates $\overline{G}$, $B \cup D$ is a DS for $G\overline{G}$ thus $\gamma(G\overline{G}) \leq |B \cup D|$ $= R + i$. Assume (for purposes of contradiction), that there exists a set S such that S dominates $G\overline{G}$, and $|S| \leq R + i - 1$.

Note that every $c_i \in C$ dominates exactly one element of C, every $b_t \in B$ dominates R elements of C, every $a_r \in A$ dominates $i-1$ elements of C, and every $\overline{c}_j \in \overline{C}$ dominates exactly 1 element of C. Since $|C| = R^2$, it follows that a minimum of R elements from A, B, C and $\overline{C}$ are needed in S to dominate C. Therefore, $|S| \geq R$. Since $2 \leq i \leq R$, it follows that $|S| \leq R + R - 1$ by assumption.

Claim $|S \cap V(G)| \geq R$.

Proof of Claim: Assume (for purposes of contradiction) that $|S \cap V(G)| = R - t,\ 1 < t$. Then the largest number of vertices of C that S can dominate is $R + t - 1 + (R - t)R = R^2 - tR + R + t - 1 < R^2$. This is a contradiction since $|C| = R^2$. If $|S \cap V(G)| = R - 1$, then $|S \cap V(\overline{G})| \leq R$. If $|S \cap V(\overline{G})| = K < R$ then the largest number of vertices of C that S can dominate is $R(R-1) + K < R^2$, a contradiction. Assume that $|S \cap V(\overline{G})| = R$. Then it is necessary that $S \cap V(G)$ contain all except one element of B in order to dominate C. Call this missing element b_w. In order to dominate C, it is necessary that $S \cap V(\overline{G}) = \{\overline{c}_j | j \equiv w (\mathrm{mod}\ R)\}$. Then the set S will dominate C, but the vertex $\overline{b}_w$ will be undominated, a contradiction. (end of proof of claim)

Claim $|S \cap V(\overline{G})| \geq i$.

Proof of Claim: Assume (for purposes of contradiction) that $|S \cap \overline{C}| = t, 1 \leq t < i - 1$. Since $\langle \overline{A} \cup \overline{B} \rangle$ is empty in $\overline{G}$, it necessarily follows that S must contain at least $\binom{R^2-t}{i-t-1}$ vertices of $A \cup \overline{A}$ to dominate the vertices of $\overline{A}$ which have in their label all the indices of $S \cap \overline{C}$. Hence, $|S| \geq R^2 - t + t = R^2 > R + i$, a contradiction. If S contains exactly $i - 1$ vertices of $\overline{C}$, then let T= $\{\overline{c}_{j^1} \ldots \overline{c}_{j^{i-1}}\}$ represent these vertices. The set T dominates all of $V(\overline{G})$ except the one vertex in $\overline{A}$ which has all of the indices of the vertices of T in its label. Call this vertex $\overline{a}^*$. If $\overline{a}^* \in S \cap V(\overline{G})$, then by the preceding claim, $|S| = |S \cap V(G)| + |S \cap V(\overline{G})| \geq R + i$, a contradiction. If $\overline{a}^* \notin S \cap V(\overline{G})$, then $a^* \in S \cap V(G)$. Vertex a^* and the set T both dominate the same $i - 1$ elements of C. Since $|C| = R^2$, and $i - 1 < R$, it necessarily follows that $|S \cap V(G)| \geq R + 1$. Therefore, $|S| = |S \cap V(G)| + |S \cap V(\overline{G})| \geq R + 1 + i - 1 = R + i$, a contradiction. Finally, if S contains exactly i elements of $\overline{C}$, then by the preceding claim, $|S| = |S \cap V(G)| + |S \cap V(\overline{G})| \geq R + i$, a contradiction. (end of proof of claim)

The above two claims imply that if S is a DS for $G\overline{G}$ and $|S| < R + i$ then $|S| = |S \cap V(G)| + |S \cap V(\overline{G})| \geq R + i$. Hence, a contradiction has been reached thus $\gamma(G\overline{G}) \geq R + i$. Since $\gamma(G\overline{G}) \leq |B \cup D| = R + i$, it follows that $\gamma(G\overline{G}) = R + i$. Hence, $B \cup D$ is a $\gamma(G\overline{G})$-set.

The set $B \cup D$ is also an RDS for $G\overline{G}$. To see this, consider the following. Every vertex $a \in A$ is adjacent to its counterpart $\overline{a} \in \overline{A} \subseteq V(G\overline{G}) \setminus (B \cup D)$. Every vertex $\overline{a} \in \overline{A}$ is adjacent to its counterpart $a \in A \subseteq V(G\overline{G}) \setminus (B \cup D)$. Every vertex $c_i \in C$ is adjacent to a vertex of $A \subseteq V(G\overline{G}) \setminus (B \cup D)$ that includes i in its label. Every $\overline{b}_j \in \overline{B}$ is adjacent to some $\overline{c}_k \in \overline{C}$ such that $k > R$ and $k \not\equiv j (\text{mod } R)$. (Note: By definition, $\overline{c}_k \notin D \cup B$). Every $\overline{c}_j \in \overline{C}$ is adjacent to $c_j \in C \subseteq V(G\overline{G}) \setminus (B \cup D)$. Therefore, $B \cup D$ is also a RDS for $G\overline{G}$ and thus $\gamma_r(G\overline{G}) \leq |B \cup D| = R + i$. However, since $\gamma_r(G\overline{G}) \geq \gamma(G\overline{G}) = R + i$, it follows that $\gamma_r(G\overline{G}) = R + i$. $\square$

4 OTHER DOMINATION PARAMETERS AND MISCELLANEOUS RESULTS

In this chapter, we study the $distance-k$ domination number, 2-step domination number, stratification and domination, chromatic number and planarity of complementary prisms. We will also characterize Eulerian and bipartite complementary prisms.

4.1 The $Distance-k$ Domination Number of $G\overline{G}$

In this section, we will determine the $distance-k$ domination number of a complementary prism for all possible values of k. We begin with a simple observation for the case when $k=1$.

Observation 20 *For any graph G, $\gamma_{\leq 1}(G\overline{G}) = \gamma(G\overline{G})$.*

Now, we proceed to the case when $k=2$.

Proposition 21 *For any graph G, $\gamma_{\leq 2}(G\overline{G}) = 1$ if, and only if, $\gamma_{\leq 2}(G)$ =1 or $\gamma_{\leq 2}(\overline{G})$ =1 and $\gamma_{\leq 2}(G\overline{G}) = 2$ otherwise.*

Proof: Let $u \in V(G)$. Consider the set S= $\{u, \overline{u}\}$. Without loss of generality, let v be any vertex in $V(G) \setminus S$. If v is adjacent to u, then $dist(v,S) = 1$. If not, then $\overline{v}$ is adjacent to $\overline{u}$ in $V(\overline{G})$. The path $P : v, \overline{v}, \overline{u}$ is of length two, so $dist(v,S) = 2$. A symmetric argument holds for every $\overline{v} \in V(\overline{G}) \setminus S$. Therefore, S is a distance-2 dominating set for $G\overline{G}$. Hence, $\gamma_{\leq 2}(G\overline{G}) \leq \mid S \mid= 2$.

Assume that $\gamma_{\leq 2}(G\overline{G}) = 1$. Without loss of generality, let $S = \{u\} \subseteq V(G)$ be a $\gamma_{\leq 2}(G\overline{G})$-set. Since $\gamma_{\leq 2}(G\overline{G})$ =1, it follows that for every $v \in V(G)$, $dist_{G\overline{G}}(v,u) \leq 2$.

No v-u path of length at most two can contain any elements of $V(\overline{G})$, therefore, every v-u path of length at most two is contained in $E(G)$. Thus $\gamma_{\leq 2}(G) = 1$. Now, assume without loss of generality, that $\gamma_{\leq 2}(G) = 1$. Let $S = \{u\}$ be any $\gamma_{\leq 2}(G)$-set. From this it follows that for every $v \in V(G)$, $dist_G(v, u) \leq 2$. Let $\overline{v}$, be any vertex of $V(\overline{G})$. If $\overline{v}$ is adjacent to $\overline{u}$ or if $\overline{v} = \overline{u}$, then the path P : $\overline{v}$, $\overline{u}$, u has length at most two. Thus $dist_{G\overline{G}}(\overline{v}$, S$) \leq 2$. If $\overline{v}$ is not adjacent to $\overline{u}$, then v is adjacent to u in $V(G)$. The path P : $\overline{v}$, v, u has length 2, therefore, $dist_{G\overline{G}}(\overline{v}$, S$) = 2$. Hence, $S = \{u\}$ forms a distance-2 dominating set for $G\overline{G}$, and since $| S |= 1$, it follows that $\gamma_{\leq 2}(G\overline{G}) = 1$. □

We now proceed to the case where $k = 3$.

Proposition 22 *For any graph G, $\gamma_{\leq 3}(G\overline{G}) = 1$.*

Proof:. Let $u \in V(G)$, and let $S = \{u\}$. Let $v \in V(G)$ and $u \neq v$. If v is adjacent to u, then $dist(v, S) = 1$. If not, then $\overline{v}$ is adjacent to $\overline{u}$ in $V(\overline{G})$. The path P: v, $\overline{v}$, $\overline{u}$, u has length 3. Therefore, dist(v, S) ≤ 3. Now let $\overline{v}$ be any vertex of $V(\overline{G})$. If $\overline{v}$ is adjacent to $\overline{u}$ or if $\overline{v} = \overline{u}$, then the path P :$\overline{v}$, $\overline{u}$, u has length at most 2. Thus dist ($\overline{v}$, S) ≤ 2. If $\overline{v}$ is not adjacent to $\overline{u}$, then v is adjacent to u in $V(G)$. The path P : $\overline{v}$, v, u has length 2, therefore, $dist(\overline{v}$, S$) = 2$. In conclusion, $S = \{u\}$ forms a distance-3 dominating set for $G\overline{G}$, and since $| S |$ =1, it follows that $\gamma_{\leq 3}(G\overline{G})$ =1. □

The following corollary is obvious.

Corollary 23 *For any graph G, if $k \geq 3$ then $\gamma_{\leq k}(G\overline{G}) = 1$.*

4.2 The 2-*step* Domination Number of $G\overline{G}$

In this section we will restrict our attention solely to the study of 2-*step* domination in complementary prisms. We need the following lemma before proceeding.

Lemma 24 *For any graph G, $\gamma_{=2}(G\overline{G}) \neq 1$.*

Proof: Assume that there exists a graph G such that $\gamma_{=2}(G\overline{G}) = 1$. Without loss of generality, let $S = \{u\} \subseteq V(G)$ be any $\gamma_{=2}(G\overline{G})$-set. Consider the vertex $\overline{u} \in V(\overline{G})$. The $dist(u, \overline{u}) = 1$; therefore, S is not a $\gamma_{=2}(G\overline{G})$-set, hence, a contradiction. $\square$

The next proposition gives the 2-step domination number for the complementary prism of any graph G.

Proposition 25 *For any graph G, $\gamma_{=2}(G\overline{G}) = 2$.*

Proof: Let $u \in V(G)$, and let $S = \{u, \overline{u}\}$. For every $\overline{v} \in V(\overline{G}) \setminus S$, if $\overline{v}$ is adjacent to $\overline{u}$, then P :$\overline{v}$, $\overline{u}$, u is a $\overline{v}$-u path of length 2, and $dist(\overline{v}, u) = 2$. If $\overline{v}$ is not adjacent to $\overline{u}$, then v is adjacent to u. Accordingly, P : $\overline{v}$, v, u is a $\overline{v}$-u path of length 2 thus $dist(\overline{v}, u) = 2$. For every $v \in V(G) \setminus S$, if v is adjacent to u, then P : v, u, $\overline{u}$ is a v-$\overline{u}$ path of length 2 thus $dist(v, \overline{u})$ =2. If v is not adjacent to u, then $\overline{v}$ is adjacent to $\overline{u}$. Thus P : v, $\overline{v}$, $\overline{u}$ is a v-$\overline{u}$ path of length 2, hence, $dist(v, \overline{u})$ =2.

Therefore, $\gamma_{=2}(G\overline{G}) \leq |S| = 2$. However, by Lemma 24, $\gamma_{=2}(G\overline{G}) \neq 1$ for any graph G. So $\gamma_{=2}(G\overline{G}) = 2$. $\square$

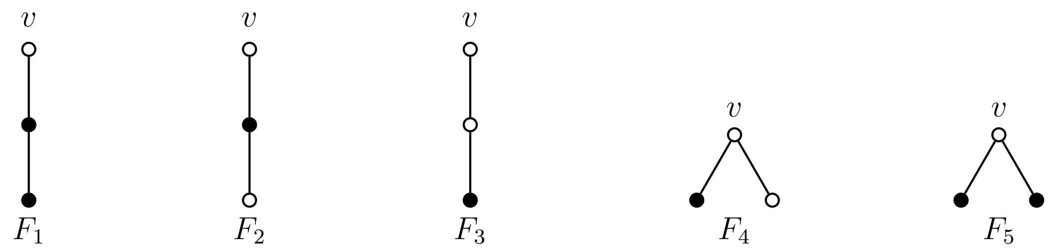

Figure 5: The Five 2-Stratified Graphs P_3

4.3 Stratification and Domination in Graphs

In this section, we study the F-domination number of a complementary prism $G\overline{G}$ when F is a 2-stratified P_3 with 1 blue rooted vertex v. This gives 5 possibilities for the graph F.

4.4 The F_1 Domination Number of Complementary Prisms

The graph F_1, is a 2-stratified P_3, with one endvertex a rooted blue vertex, the center vertex a red vertex and the second endvertex a red vertex. In [2], the following proposition equates for a given graph G the F_1 domination number γ_{F_1}, with the total domination number γ_t.

Proposition 26 *[2] If G is a graph without isolated vertices, then $\gamma_{F_1}(G)$ $=\gamma_t(G)$.*

Every complementary prism $G\overline{G}$ is isolate free for any graph G. Hence, $\gamma_{F_1}(G\overline{G})$ $=\gamma_t(G\overline{G})$ for any graph G. The total domination number $\gamma_t(G\overline{G})$ of complementary prisms was studied extensively in [8].

4.5 The F_2 Domination Number of Complementary Prisms

The graph F_2, is a 2-stratified P_3, with both endvertices colored blue, one of them rooted, and the center vertex colored red. In [2], the following proposition equates for a given graph G the F_2 domination number γ_{F_2}, with the domination number γ.

Proposition 27 *[2] If G is a connected graph of order at least 3, then $\gamma_{F_2}(G) = \gamma(G)$.*

In the case of complementary prisms, the only one with order less than three is $K_1\overline{K}_1$. Since this is a graph of order two, it cannot contain a copy of F_2 hence, $\gamma_{F_2}(K_1\overline{K}_1) = 2$. All other complementary prisms have order greater than or equal to four and are connected. The following corollary is obvious:

Corollary 28 *If $| V(G) | > 1$, then $\gamma_{F_2}(G\overline{G}) = \gamma(G\overline{G})$.*

The domination number $\gamma(G\overline{G})$ of complementary prisms was studied extensively in [8].

4.6 The F_3 Domination Number of Complementary Prisms

The graph F_3, is a 2-stratified P_3, with one endvertex rooted and colored blue, the center vertex colored blue and the other endvertex colored red. The following lemma is needed before proceeding.

Lemma 29 *For any graph G, $\gamma_{F_3}(G\overline{G}) \neq 1$.*

Proof: Assume that there exists a graph G, such that $\gamma_{F_3}(G\overline{G}) = 1$. Without loss of generality, let $S = \{u\} \subseteq V(G)$ be any $\gamma_{F_3}(G\overline{G})$-set. The vertex $\overline{u}$ is not rooted in a copy of F_3, a contradiction. □

Proposition 30 *For any graph G,*

$$\begin{cases} \gamma_{F_3}(G\overline{G}) = 2, & \text{if } \gamma(G) \neq 2 \text{ and } \gamma(\overline{G}) \neq 2. \\ 2 \leq \gamma_{F_3}(G\overline{G}) \leq 4, & \text{otherwise.} \end{cases}$$

Proof: We consider four cases:

Case 1. Assume that $\gamma(G) \geq 3$ and $\gamma(\overline{G}) \geq 3$. Let $u \in V(G)$, and let $S = \{u, \overline{u}\}$. Color the vertices of S red. Color the vertices of $V(G\overline{G}) \setminus S$ blue. It is claimed that S forms a $\gamma_{F_3}(G\overline{G})$-set. Consider the following.

Let $A = N_G(u)$ and $B = V(G) \setminus N_G[u]$. Note that $\overline{A} = V(\overline{G}) \setminus N_{\overline{G}}[\overline{u}]$ and $\overline{B} = N_{\overline{G}}(\overline{u})$.

Let $\overline{x}$ be any element of $\overline{A}$. The subgraph $\langle\{\overline{x},\, x,\, u\ \}\rangle$ is a copy of F_3 rooted at $\overline{x}$. Let v be any element of B. Vertex $\overline{v}$ is adjacent to $\overline{u}$ and the subgraph $\langle\{v, \overline{v}, \overline{u}\}\rangle$ is a copy of F_3 rooted at v. Let $w \in A$. Vertex w must have a neighbor $v \in A$ else if w is isolated in $\langle A\rangle$, then $\overline{w}$ dominates $\overline{A}$ hence, $\{\overline{u}, \overline{w}\}$ forms a DS for $V(\overline{G})$ implying that $\gamma(\overline{G}) \leq 2$, a contradiction. Therefore, vertex w has a neighbor v in A. The subgraph $\langle\{w, v, u\}\rangle$ contains a copy of F_3 rooted at w. Let $\overline{w}$ be any element of $\overline{B}$. Then, $\overline{w}$ must have a neighbor $\overline{v} \in \overline{B}$ else if $\overline{w}$ is an isolated vertex in $\langle\overline{B}\rangle$, then w dominates B. Therefore, $\{u, w\}$ forms a DS for $V(G)$ implying that $\gamma(G) \leq 2$, a contradiction. Therefore, $\overline{w}$ has a neighbor $\overline{v}$ in $\overline{B}$, and the subgraph $\langle\{\overline{w}, \overline{v}, \overline{u}\}\rangle$ contains a copy of F_3 rooted at $\overline{w}$.

Thus, the red-blue coloring described is an F_3-coloring. Therefore, $\gamma_{F_3}(G\overline{G}) \leq |\ S\ | = 2$. By Lemma 29, $\gamma_{F_3}(G\overline{G}) \neq 1$ for any graph G. Therefore, $\gamma_{F_3}(G\overline{G}) = 2$.

Case 2. Assume, without loss of generality, that $\gamma(G) = 1$ and $\gamma(\overline{G}) \neq 2$. If $\gamma(G) = \gamma(\overline{G}) = 1$, then $G\overline{G} = K_2$ and $\gamma_{F_3}(G\overline{G}) = 2$. Assume that $\gamma(\overline{G}) \geq 3$. Let $u \in V(G)$, such that u dominates $V(G)$. Let $S = \{u, \overline{u}\}$. Color the vertices of S red, and the vertices of $V(G\overline{G}) \setminus S$ blue. It is claimed that S forms a $\gamma_{F_3}(G\overline{G})$-set. Let $A = N_G(u)$. Then $\overline{A} = V(\overline{G}) \setminus \{\overline{u}\}$.

Let $\overline{w} \in \overline{A}$. The subgraph $\langle\{\overline{w}, w, u\}\rangle$ is a copy of F_3 rooted at $\overline{w}$. Now, let $w \in A$. There must exist some vertex $v \in A$ that is a neighbor of w. If not, then $\overline{w}$ dominates $\overline{A}$. Thus $\{\overline{w}, \overline{u}\}$ is a DS for $V(\overline{G})$. This implies that $\gamma(\overline{G}) \leq 2$, a contradiction. Therefore, vertex w has a neighbor $v \in A$. The subgraph $\langle\{w, v, u\}\rangle$ contains a copy of F_3 rooted at w.

Thus, the red-blue coloring described is an F_3-coloring. Therefore, $\gamma_{F_3}(G\overline{G}) \leq \mid S \mid = 2$. By Lemma 29, $\gamma_{F_3}(G\overline{G}) \neq 1$ for any graph G. Therefore, $\gamma_{F_3}(G\overline{G}) = 2$.

Case 3. Assume, without loss of generality, that $\gamma(G) = 2$ and $\gamma(\overline{G}) \neq 2$.

There are 2 possible sub-cases to consider:

Case 3a. If $\gamma(\overline{G}) = 1$, then let $\overline{u} \in V(\overline{G})$ be such that $\overline{u}$ dominates $V(\overline{G})$. Let T be a $\gamma(G)$-set. Since vertex u is isolated in G, it follows that $u \in T$. Hence, $T = \{u, x\}$ for some $x \in V(G)$. Let $S = \{u, x, \overline{u}, \overline{x}\}$. Color the vertices of S red and the vertices of $V(G\overline{G}) \setminus S$ blue. Let $\overline{w} \in V(\overline{G}) \setminus S$. The subgraph $\langle\overline{w}, w, x\rangle$ is a copy of F_3 rooted at $\overline{w}$. Let $v \in V(G) \setminus S$. The subgraph $\langle\{v, \overline{v}, \overline{u}\}\rangle$ is a copy of F_3 rooted at v. Hence, the red-blue coloring is an F_3-coloring, therefore, $\gamma_{F_3}(G\overline{G}) \leq |S| = 4$.

Case 3b. If $\gamma(\overline{G}) \geq 3$, then let $\overline{u} \in V(\overline{G})$. Let $S = \{u, \overline{u}\}$. Let $A = N_G(u)$, and $B = V(G) \setminus N_G[u]$. Note that $\overline{A} = V(\overline{G}) \setminus N_{\overline{G}}[\overline{u}]$ and $\overline{B} = N_{\overline{G}}(\overline{u})$. Color the vertices of S red, and color the vertices of $V(G\overline{G}) \setminus S$ blue. Let $w \in B$. Since $\overline{w}$ is adjacent to $\overline{u}$, it follows that the subgraph $\langle\{w, \overline{w}, \overline{u}\}\rangle$ contains a copy of F_3 rooted at w. Let $\overline{v} \in \overline{A}$. Then it follows that v is adjacent to u, and the subgraph $\langle\{\overline{v}, v, u\}\rangle$ is a copy of F_3 rooted at $\overline{v}$. Let $w \in A$. Since $\gamma(\overline{G}) \geq 3$, it is necessary that w has at least one neighbor in A. If this is not the case, then $\{\overline{w}, \overline{u}\}$ forms a DS for $V(\overline{G})$, a contradiction. Therefore, there exists some vertex $v \in A$, such that v is adjacent to w. The subgraph $\langle\{w, v, u\}\rangle$ contains a copy of F_3 rooted at w. Let $\overline{w} \in \overline{B}$. If $\langle\overline{B}\rangle$ is isolate free, then there exists a vertex $\overline{p} \in \overline{B}$ such that $\overline{p}$ is adjacent to $\overline{w}$. The subgraph $\langle\{\overline{w}, \overline{p}, \overline{u}\}\rangle$ contains a copy of F_3 rooted at $\overline{w}$. The red-blue coloring described is an F_3-coloring therefore, $\gamma_{F_3}(G\overline{G}) \leq 2$.

If, on the other hand, $\langle\overline{B}\rangle$ contains isolates, then let $\overline{T}$ represent a maximum independent set of $\langle\overline{B}\rangle$. The subgraph $\langle T\rangle$ forms a clique. Let x be any element of T. Let $R = \{u, \overline{u}, x, \overline{x}\}$. Color the vertices of R red, and the vertices of $V(G\overline{G}) \setminus R$ blue. Let $w \in B \setminus R$. Since $\overline{w}$ is adjacent to $\overline{u}$, it follows that the subgraph $\langle\{w, \overline{w}, \overline{u}\}\rangle$ contains a copy of F_3 rooted at w. Let $\overline{v} \in \overline{A} \setminus R$. Then it follows that v is adjacent to u, and the subgraph $\langle\{\overline{v}, v, u\}\rangle$ contains a copy of F_3 rooted at $\overline{v}$. Let $w \in A \setminus R$. Since $\gamma(\overline{G}) \geq 3$, it is necessary that w has at least one neighbor in $A \setminus R$. If this is not the case, then $\{\overline{w}, \overline{u}\}$ forms a DS for $\overline{G}$, a contradiction. Therefore, there exists some vertex $v \in A \setminus R$, such that v is adjacent to w. The subgraph $\langle\{w, v, u\}\rangle$ contains a copy of F_3 rooted at w. Let $\overline{w} \in \overline{B} \setminus R$. If $\overline{w} \in \overline{T}$, then the subgraph $\langle\{\overline{w}, w, x\}\rangle$ contains a copy of F_3 rooted at $\overline{w}$. If $\overline{w} \notin \overline{T}$, then there exists some vertex $\overline{p} \in \overline{B} \setminus R$, such that $\overline{p}$ is adjacent to $\overline{w}$. Therefore, the subgraph $\langle\{\overline{w}, \overline{p}, \overline{u}\}\rangle$ contains a copy of F_3 rooted at $\overline{w}$.

In summary, the red-blue coloring described is an F_3-coloring therefore, $\gamma_{F_3}(G\overline{G}) \leq 4$. By Lemma 29, $\gamma_{F_3}(G\overline{G}) \neq 1$ for any graph G, so $2 \leq \gamma_{F_3}(G\overline{G}) \leq 4$.

Case 4 The final case to consider is for $\gamma(G) = \gamma(\overline{G}) = 2$. Let $R = \{u, v\}$ be any $\gamma(G)$-set, and let $\overline{T} = \{\overline{x}, \overline{y}\}$ be any $\gamma(\overline{G})$-set. Let $A = V(G) \setminus R$ and $B = V(\overline{G}) \setminus \overline{T}$. Color the vertices of $R \cup \overline{T}$ red and the vertices of $A \cup B$ blue. If every $w \in A \cup B$, has a neighbor $q \in A \cup B$, then one of the four subgraphs $\langle\{w, q, u\}\rangle\langle\{w, q, v\}\rangle$ $\langle\{w, q, \overline{x}\}\rangle$ or $\langle\{w, q, \overline{y}\}\rangle$ either is or contains a copy of F_3 rooted at w. Hence, the given coloring is an F_3-coloring and therefore, $\gamma_{F_3}(G\overline{G}) \leq 4$. If this is not the case, then there exists some vertex $p \in A \cup B$, such that $N_{G\overline{G}}(p) \subseteq R \cup \overline{T}$. Without loss of generality, assume that $p \in V(G)$. Then it follows that $N_G(p) \subseteq R$ and $\overline{p} \in \overline{T}$. If vertex u is not adjacent to vertex v, or if each of $\overline{u}$ and $\overline{v}$ has a neighbor in $\overline{A} \setminus \{\overline{p}\}$, then $R \cup \{p, \overline{p}\}$ is an F_3-set for $G\overline{G}$. To see this, color the vertices $R \cup \{p, \overline{p}\}$ red, color the vertices of $V(G\overline{G}) \setminus \{R \cup \{p, \overline{p}\}\}$ blue, and consider the following. For each vertex $w \in A \setminus \{p\}$, since $\overline{p}$ dominates $\overline{A}$, the subgraph $\langle\{w, \overline{w}, \overline{p}\}\rangle$ is a copy of F_3 rooted at w. For each vertex $\overline{w} \in \overline{A} \setminus \{\overline{p}\}$, the subgraph $\langle\{\overline{w}, w, u\}\rangle$ or $\langle\{\overline{w}, w, v\}\rangle$ is a copy of F_3 rooted at $\overline{w}$. Finally, if u is not adjacent to v, the subgraphs $\langle\{\overline{u}, \overline{v}, v\}\rangle$ and $\langle\{\overline{v}, \overline{u}, u\}\rangle$ are copies of F_3 rooted at $\overline{u}$ and $\overline{v}$ respectively. If u and v are adjacent, and if $\overline{u}$ has some neighbor $\overline{s} \in \overline{A} \setminus \{\overline{p}\}$ and $\overline{v}$ has some neighbor $\overline{r} \in \overline{A} \setminus \{\overline{p}\}$, then the subgraphs $\langle\{\overline{u}, \overline{s}, \overline{p}\}\rangle$ and $\langle\{\overline{v}, \overline{r}, \overline{p}\}\rangle$ contain a copy of F_3 rooted at $\overline{u}$ and $\overline{v}$, respectively. The coloring is an F_3 coloring, and therefore, $\gamma_{F_3}(G\overline{G}) \leq 4$.

The only case left to consider is when u and v are adjacent in G and at least one of the vertices $\overline{u}$ or $\overline{v}$ has no neighbors in $\overline{A} \setminus \{\overline{p}\}$. Assume, without loss of generality, that this is the case with vertex $\overline{u}$. The vertex $\overline{u}$ must be adjacent to $\overline{p}$, else $\gamma(G) = 1$ a contradiction. Since R must dominate $V(G)$, it follows that v is adjacent to p. Color the vertices of S= $\{u, \overline{u}, \overline{v}, \overline{p}\}$ red and the vertices of $V(G\overline{G}) \setminus S$ blue. To see that this coloring is an F_3 coloring, consider the following. For each vertex $w \in A \setminus \{p\}$, the subgraph $\langle\{w, \overline{w}, \overline{p}\}\rangle$ contains a copy of F_3 rooted at w. For every vertex $\overline{w} \in \overline{A} \setminus \{\overline{p}\}$, the subgraph $\langle\{\overline{w}, w, u\}\rangle$ is a copy of F_3 rooted at $\overline{w}$. Finally, the subgraphs $\langle\{p, v, \overline{v}\}\rangle$ and $\langle\{v, p, \overline{p}\}\rangle$ are copies of F_3 rooted at p and v, respectively. In summary, the coloring is an F_3 coloring, hence, $\gamma_{F_3}(G\overline{G}) \leq 4$. By Lemma 29, $\gamma_{F3}(G\overline{G}) \neq 1$ therefore, $2 \leq \gamma_{F_3}(G\overline{G}) \leq 4$. □

4.7 The F_4 Domination Number of Complementary Prisms

The graph F_4, is a 2-stratified P_3, with one endvertex colored blue, the center vertex rooted and colored blue and the other endvertex colored red. This parameter is not new. For any graph G, $\gamma_{F_4}(G) = \gamma_r(G)$. The parameter $\gamma_{F_4}(G\overline{G})$ is the restrained domination number for a complementary prism, and has been studied extensively in this work.

4.8 The F_5 Domination Number of Complementary Prisms

The graph F_5, is a 2-stratified P_3, with both endvertices colored red and the center vertex rooted and colored blue. The parameter $\gamma_{F_5}(G)$ is not new. Some terminology is in order. For any graph G, a set $S \subseteq V(G)$ is a *k-dominating set* if for every vertex $v \notin S$, v is adjacent to at least k vertices in S. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a k-dominating set of G.

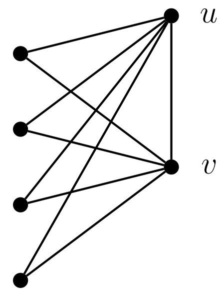

Figure 6: $\overline{K}_4 + P_2$

An F_5-coloring of a graph G requires every blue vertex to be adjacent to at least 2 red vertices. The parameter γ_{F_5} is the minimum number of red vertices required in such a coloring. This is the same as the *2-domination number* of G. Therefore, we make the following observation.

Observation 31 *For any graph G, $\gamma_2(G) = \gamma_{F_5}(G)$.*

This leads to the following proposition that establishes the upper and lower bounds on $\gamma_{F_5}(G\overline{G})$ for any graph G.

Proposition 32 *For any graph G, $\max\{\gamma_{F_5}(G), \gamma_{F_5}(\overline{G})\} \leq \gamma_{F_5}(G\overline{G}) \leq \gamma_{F_5}(G) + \gamma_{F_5}(\overline{G})$, and these bounds are sharp.*

Proof: Let S be any $\gamma_{F_5}(G\overline{G})$-set. Without loss of generality, assume that $\gamma_{F_5}(G) \geq \gamma_{F_5}(\overline{G})$. Let $S_1 = S \cap V(G)$ and let $\overline{S}_2 = S \cap V(\overline{G})$. Assume (for purposes of contradiction), that $\mid S \mid < \max\{\gamma_{F_5}(G), \gamma_{F_5}(\overline{G})\}$. The set $S_1 \cup S_2$ forms a 2 dominating set for $V(G)$, but $\mid S_1 \cup S_2 \mid \leq \mid S \mid < \gamma_{F_5}(G)$ a contradiction. Therefore, $\gamma_{F_5}(G\overline{G}) \geq \max\{\gamma_{F_5}(G), \gamma_{F_5}(\overline{G})\}$. For sharpness, consider the graph $G = tK_2$ for $t \geq 3$. In this case, $\gamma_{F_5}(G) = 2t$, $\gamma_{F_5}(\overline{G}) = 2$ and $\gamma_{F_5}(G\overline{G}) = 2t$.

Let S_1 be any $\gamma_{F_5}(G)$ -set and let S_2 be any $\gamma_{F_5}(\overline{G})$-set. The set $S = S_1 \cup S_2$ forms a 2-*dominating* set for $G\overline{G}$ therefore, $\gamma_{F_5}(G\overline{G}) \leq \mid S \mid = \mid S_1 \mid + \mid S_2 \mid = \gamma_{F_5}(G) + \gamma_{F_5}(\overline{G})$. For sharpness, consider the following graph G. Take an empty graph $\overline{K}_n$ $n \geq 4$. Join two new vertices u and v to this graph, each vertex adjacent to every vertex in the $\overline{K}_n$, and adjacent to each other. From this, it follows that $\gamma_{F_5}(G)$ =2, $\gamma_{F_5}(\overline{G})$ =4 and $\gamma_{F_5}(G\overline{G})$ =6. This graph is known as the *join* of $\overline{K}_n$ and P_2. It is symbolized by $\overline{K}_n + P_2$, and is illustrated in Figure 6 for $n = 4$. □

4.9 Miscellaneous Results

Proposition 33 *Given a graph G of order n, $G\overline{G}$ is Eulerian if, and only if, every vertex in G is of odd degree, and n is odd.*

Proof: Assume that n is odd, and that $deg_G(u)$ is odd for every $u \in V(G)$. Since $deg_G(u) + deg_{\overline{G}}(\overline{u}) = n - 1$ for every $u \in V(G)$, it follows that $deg_{\overline{G}}(\overline{u}) = (n-1) - deg_G(u)$. Hence, $deg_{\overline{G}}(\overline{u})$is odd. Joining corresponding vertices between a copy of G and $\overline{G}$ creates $G\overline{G}$ and all the vertices will have even degree thus by Theorem 4.1 in [3] $G\overline{G}$ is Eulerian. Now, assume that $G\overline{G}$ is Eulerian. By Theorem 4.1 in [3], it follows that $deg_{G\overline{G}}(v)$ is even for every $v \in V(G\overline{G})$. Removing the edges between the corresponding vertices of G and $\overline{G}$ reduces the degree of every vertex in $V(G\overline{G})$ by 1. Hence, $deg_G(u)$ is odd for every $u \in V(G)$. Similarly $deg_{\overline{G}}(\overline{u})$ is odd for every $\overline{u} \in V(\overline{G})$. Since $deg_G(u) + deg_{\overline{G}}(\overline{u}) = n - 1$ for every $u \in V(G)$, it follows that $n - 1$ is even thus n is odd.□

Lemma 34 *If u and v are adjacent vertices in $V(G)$ and u and v do not dominate $V(G)$, then $G\overline{G}$ contains a 5-cycle.*

Proof: Let u and v be any 2 adjacent vertices in $V(G)$. If u and v do not dominate V(G), then there exists a vertex $w \in V(G)$ that is not adjacent in G to u or v. In $\overline{G}$, $\overline{w}$ is adjacent to $\overline{u}$ and $\overline{v}$. Thus the vertices $\{u, v, \overline{u}, \overline{v}, \overline{w}\}$ form a 5-cycle $(u, \overline{u}, \overline{w}, \overline{v}, v)$ in $G\overline{G}$. □

Proposition 35 *The complementary prism $G\overline{G}$ is bipartite, if, and only if, $G \in \{K_1, \overline{K}_1, K_2, \overline{K}_2\}$.*

Proof: If $G \in \{K_1, \overline{K}_1\}$, then $G\overline{G} = K_2$, which is bipartite. If $G \in \{K_2, \overline{K}_2\}$, then $G\overline{G} = P_4$, which is bipartite. Now, assume that $G\overline{G}$ is bipartite. And assume $|V(G)| = n \geq 3$. By Theorem 1.2.18 in [3], it follows that it is necessary for $G\overline{G}$ to have no odd-cycles. Hence, by Lemma 34 it is necessary for every pair of adjacent vertices in G to dominate $V(G)$. Analogously, it is necessary for every pair of adjacent vertices in $\overline{G}$ to dominate $V(\overline{G})$. This means that any two non-adjacent vertices in G cannot have a common neighbor. The graph G must have at least two adjacent vertices since if all the vertices of G are isolated, then $\overline{G}$ is a complete graph of order at least three therefore, $G\overline{G}$ contains an odd cycle and consequently is not bipartite. Hence, assume that there are two adjacent vertices $u, v \in V(G)$. Consider $N(u)$ and $N(v)$. Both $N(u)$ and $N(v)$ must be complete graphs since any two vertices in $N(u)$ or $N(v)$ share a common neighbor hence, must be adjacent. Additionally, $N(u) \cup N(v)$ must be a complete graph since if there exists a vertex $r \in N(u)$ and $j \in N(v)$, j and

r share the vertex u in common (since $u \in N(v)$ and $N(v)$ is complete). Finally since $v \in N(u)$ and $u \in N(v)$, it follows that u and v are both adjacent to every vertex in the graph thus G is complete. Since $|V(G)| \geq 3$, G contains a 3 cycle thus $G\overline{G}$ is not bipartite. Hence, it is necessary for $|V(G)| \leq 2$. If $|V(G)| = 2$, $G\overline{G} = P_4$ If $|V(G)| = 1$, $G\overline{G} = P_2$. Since both P_2 and P_4 are bipartite, we are finished. □

Proposition 36 *For any graph G of order $n \geq 2$,* $\max(\chi(G), \chi(\overline{G})) \leq \chi(G\overline{G}) \leq \max(\Delta(G) + 1, n - \delta(G))$. *These bounds are sharp.*

Proof: Brooks Theorem states that given a connected graph H, if H is not a complete graph or an odd cycle, then $\chi(H) \leq \Delta(H)$. The complementary prism $G\overline{G}$ is connected and since it has $2n$ vertices, it is not an odd cycle. Additionally, since each vertex $v \in V(G)$ dominates exactly one vertex $\overline{v} \in V(\overline{G})$, and since $n \geq 2$, it follows that $G\overline{G}$ is not a complete graph. The above allows us to conclude that $\chi(G\overline{G}) \leq \Delta(G\overline{G})$. But $\Delta(G\overline{G}) = \max(\Delta(G), \Delta(\overline{G})) + 1$. Since $\Delta(\overline{G}) = n - 1 - \delta(G)$, it follows that $\chi(G\overline{G}) \leq \max(\Delta(G) + 1, n - \delta(G))$.

For sharpness, let $G = K_n$, then $\Delta(G) = n - 1$ and $\delta(G) = n - 1$ therefore, $\max(\Delta(G)+1, n-\delta(G)) = \max(n, 1) = n$. Since $\omega(G) = n$, it follows that $\chi(G\overline{G}) \geq n$. Color the copy of G in $G\overline{G}$ with n colors. Color each vertex $\overline{v} \in V(\overline{G})$ with a different color than that utilized to color v. This produces a coloring of $G\overline{G}$ of size n thus $\chi(G\overline{G}) = n = \max(\Delta(G) + 1, n - \delta(G))$.

In order to color $G\overline{G}$, at least $\max(\chi(G), \chi(\overline{G}))$ colors are needed. Therefore, $\max(\chi(G), \chi(\overline{G})) \leq \chi(G\overline{G})$.

For sharpness, let G be a star of order $n \geq 4$. Then it follows that $\chi(G) = 2$

and $\chi(\overline{G}) = n-1$. Hence, $\max(\chi(G), \chi(\overline{G})) = n-1$. Therefore, $\chi(G\overline{G}) \geq n-1$. Note that $\overline{G} = K_1 \cup K_{n-1}$. Color the vertices of the induced K_{n-1} in $G\overline{G}$ with $n-1$ colors. Call them $\{c_1 \ldots c_{n-1}\}$. Color the support vertex $v \in V(G)$ color c_1 and color $\overline{v} \in V(\overline{G})$ color c_2. Now for each leaf vertex $u \in V(G)$, color the leaf vertex with color c_2 unless $\overline{u}$ is already colored c_2. When this occurs, color the vertex c_3. The result is a coloring of $G\overline{G}$ with $n-1$ colors. Therefore, $\chi(G\overline{G}) = \max(\chi(G), \chi(\overline{G})) = n-1$.
$\square$

Proposition 37 *If G is a graph of order $n \geq 10$, then $G\overline{G}$ is not planar.*

Proof: Let G be any graph of order n. Hence, $m(G\overline{G}) = (n^2+n)/2$. By Theorem 11.5 in [3], every graph H of order $b \geq 5$ and size at least $3b-5$ contains either K_5 or a subdivision of K_5. By Kuratowski's Theorem, such a graph is non-planar. In the present case, $|V(G\overline{G})| = 2n$. Solving the equation $(n^2+n)/2 \geq 6n-5$ yields that $n \geq 10$ Therefore, if $n \geq 10$, then $G\overline{G}$ is non-planar. $\square$

Proposition 38 *For any graph G, $\chi(G\overline{G}) \leq \max(\chi(G), \chi(\overline{G})) + \lceil \min(\chi(G), \chi(\overline{G}))/2 \rceil$.*

Proof: Assume, without loss of generality, that $\chi(G) \geq \chi(\overline{G})$. In $G\overline{G}$, properly color $V(G)$ with the $\chi(G)$ colors $\{c_1, \ldots c_{\chi(G)}\}$. Properly color $V(\overline{G})$ with the colors $\{c_1, \ldots c_{\chi(\overline{G})}\}$. Let $\{A_1 \ldots A_{\chi(G)}\}$ represent the subgraphs of G induced by each of the color classes defined by the $\chi(G)$ coloring of $V(G)$. In $\overline{G}$, each of $\{\overline{A}_1 \ldots \overline{A}_{\chi(G)}\}$ forms a complete graph. Since G and $\overline{G}$ have been properly colored, the only possible color conflicts that can occur in $G\overline{G}$, are between a vertex $u \in V(G)$ and $\overline{u} \in V(\overline{G})$. Since each of the $\{\overline{A}_1 \ldots \overline{A}_{\chi(G)}\}$ forms a complete graph, there can be at most one conflicting vertex $u \in A_i$, $1 \leq i \leq \chi(G)$ for each color class of G. We will show the result by iteratively constructing a proper coloring of $G\overline{G}$ of cardinality $\leq \chi(G) + \lceil \chi(\overline{G})/2 \rceil$. Label the vertices of $V(G)$ that conflict with vertices of $V(\overline{G})$ as $\{u_1 \ldots u_r\}$. Note that $1 \leq r \leq \chi(\overline{G})$.

Step 1 Set index $j = 1$. Let S= $\{u_1 \ldots u_r\}$ and T= $\{\overline{u}_1 \ldots \overline{u}_r\}$. If $\mid S \mid \geq 2$, then goto step 2 else goto step 4.

Step 2 Select $u_i \in S$, color u_i with color $c_{\chi(G)+j}$. Color a different vertex $\overline{u}_k \in T$ $k \neq i$ with color $c_{\chi(G)+j}$.

Step 3 Set S $= S \setminus \{u_i, u_k\}$. Set T= $T \setminus \{\overline{u}_i, \overline{u}_k\}$. Set j=j+1.

Step 4 If $\mid S \mid \geq 2$, return to step 2. If $\mid S \mid = 1$ then color the sole vertex of set S color $c_{\chi(G)+j}$ and goto end else set j=j-1 and goto end.

Step 5 end

When the above process is complete, a proper $\chi(G) + j$ coloring of $G\overline{G}$ will be constructed. This process terminates after j iterations, and $j \leq \lceil \chi(\overline{G})/2 \rceil$. Therefore, $\chi(G\overline{G}) \leq \chi(G) + j \leq \chi(G) + \lceil \chi(\overline{G})/2 \rceil$.$\Box$

5 CONCLUSION

Our primary goal in this thesis was to obtain results for restrained domination in complementary prisms similar to those obtained in [8] for domination and total domination in complementary prisms. We successfully achieved this, and were able to show, in addition, that the restrained domination number of a complementary prism is realizable for every integer value between its lower and upper bound. Our results illustrated the contrasts between restrained domination and domination in complementary prisms in such areas as the lower bound on $\gamma_r(G\overline{G})$, seen in Theorem 16.

We subsequently investigated some other selected domination parameters in complementary prisms, such as $distance - k$ domination and $2 - step$ domination, and obtained results on the bounds for each of these. We found that for some domination parameters, such as $2 - step$ domination, the parameter only takes on a few possible values for complementary prisms. A consequence of our efforts was the miscellaneous results we discovered along the way dealing with the chromatic number of a complementary prism, planarity of complementary prisms and the characterization of when a complementary prism is Eulerian or bipartite.

Complementary prisms are a deeply intriguing family of graphs, whose study is just beginning. Many problems are still left to be solved, and new results are left to be discovered. Some unsolved problems and unanswered questions which arose during this research are:

- Characterize the graphs G for which $\gamma_r(G\overline{G}) = \gamma_r(G) + \gamma_r(\overline{G})$.

- Characterize the graphs G for which $\gamma_r(G\overline{G}) = 2\max\{\gamma(G), \gamma(\overline{G})\}$.

- Characterize the graphs G for which $\gamma_r(G\overline{G}) = 4$.

- Are the bounds given in Theorem 30 for $\gamma_{F_3}(G\overline{G})$ sharp? At this point, many examples exist for the lower bound, but an example for the upper bound seems elusive.

- Is $\gamma_{F_5}(G\overline{G})$ realizable throughout the range of its limits?

- Can the upper bound on $\chi(G\overline{G})$ be improved?

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VITA

WYATT J. DESORMEAUX

Education:	B.S. Mathematics and Physics, East Tennessee State University, Johnson City, Tennessee 1993
	M.S. Mathematical Sciences, East Tennessee State University, Johnson City, Tennessee 2008
Professional Experience:	Graduate Teaching Assistant, University of Florida, Gainesville, Florida, 1995–1996
	Graduate Teaching Assistant, East Tennessee State University, Johnson City, Tennessee, 2007-2008
Awards and Honors:	Faculty Award Department of Physics, East Tennessee State University, Johnson City, Tennessee 1993
	Excellence in Teaching Award, East Tennessee State University, Johnson City, Tennessee, 2007-2008
Publications:	W. Desormeaux and T. W. Haynes, "Restrained Domination in Complementary Prisms," submitted.