The Last of the Mixed Triple Systems

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ABSTRACT

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In this thesis, we consider the decomposition of the complete mixed graph on $v$ vertices denoted $M_v$, into every possible mixed graph on three vertices which has (like $M_v$) twice as many arcs as edges. Direct constructions are given in most cases. Decompositions of the $\lambda$-fold complete mixed graph $\lambda M_v$, are also studied.
DEDICATION

I would love to dedicate this piece of work to my entire family the Jums who are always there when I need them. To my cousin’s family the Jams for their moral and financial support and for making it possible for me to attain this level of education. Finally, to my beloved fiancé Delphine Sayani for standing by me all this while.
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1 INTRODUCTION AND BASIC DEFINITIONS

Combinatorial design theory is an interesting area of study in combinatorial mathematics. The approach of modeling objects as a set of points (or vertices) and the relation between them as arcs and/or edges comes in handy when studying triple systems in general and mixed triple systems in particular. Graphs provide a visible link between theory and applications that makes them ideal for design theory. For a better understanding of this thesis, we start by giving a comprehensive list of definitions.

A graph $G$ is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates each edge with two vertices (not necessarily distinct) called its end points [14]. A directed graph (digraph) $D$ is simply a graph where the edges have been assigned direction.

A complete graph on $v$ vertices, $K_v$, is a graph on $v$ vertices where every vertex is adjacent to every other vertex in the graph. The complete digraph on $v$ vertices, $D_v$, is formed by replacing each edge in $K_v$ with two arcs of opposite orientation. If $a = (x, y)$ is an arc in the digraph $D$, then $a$ is said to join $x$ to $y$ and $a$ is incident from $x$ and incident to $y$, while $x$ is incident to $a$ and $y$ is incident from $a$. We say that $x$ and $y$ are adjacent vertices.

A mixed graph on $v$ vertices is an ordered pair $(V, C)$ where $V$ is a set of vertices, of order $v$ and $C$ is a set of ordered and unordered pairs denoted $(x, y)$ and $[x, y]$, respectively, of elements of $V$. An ordered pair $(x, y) \in C$ is called an arc of $(V, C)$ and an unordered pair $[x, y] \in C$ is called an edge of $(V, C)$.

The complete mixed graph on $v$ vertices, denoted $M_v$, is the mixed graph $(V, C)$
where for every pair of distinct vertices $u, v$, we have $\{(u, v), (v, u), [u, v]\} \subset C$. Figure 1 is an example of the complete mixed graph on three vertices ($M_3$). The converse of a mixed graph $(V, C)$, is the mixed graph $(V', C')$ where $C' = \{(u, v), (v, u) \in C\} \cup \{[u, v] \in C\}$.

![Figure 1: The Complete Mixed Graph on Three Vertices ($M_3$)](image)

In mixed graphs, the concept of the out degree, $od(u)$, of vertex $u$ in $M_v$ refers to the number of vertices of $M_v$ that are adjacent from $u$. That is, $od(u) = |N_o(u)|$ where the open neighborhood $N_o(u) = \{x \in V(M_v) \mid x \text{ is adjacent from } u\}$. The in-degree, $id(u)$, of vertex $u$ in $M_v$ refers to the number of vertices of $M_v$ that are adjacent to $u$. That is, $id(u) = |N_i(u)|$ where $N_i(u) = \{x \in V(M_v) \mid x \text{ is adjacent to } u\}$. The degree of a vertex, $d(u)$, of $u$ is the number of edges incident to $u$. By the total degree $t(u)$ of vertex $u$ we shall mean the sum: $t(u) = od(u) + id(u) + d(u)$. It thus follows that in $M_v$ the sum of its arcs and edges is always congruent to zero modulo three.

Let $K_v$ ($D_v$) denote the complete graph (digraph) on $v$ vertices. If $G$ is a graph (digraph) then a $G$-decomposition of $K_v$ ($D_v$) is a collection $\{G_1, G_2, \ldots G_n\}$ of edge (arc) disjoint subgraphs of $K_v$ ($D_v$) each of which is isomorphic to $G$ and such that $\bigcup_{i=1}^{n} E(G_i) = E(K_v)$ ($\bigcup_{i=1}^{n} A(G_i) = A(D_v)$), where $E(G)$ ($A(G)$) is the edge (arc)
set of $G$ and the $G'_i$s are called the *blocks* of the decomposition.

The decomposition of a graph (digraph) into copies of a graph (digraph) on three vertices is called a *triple system*. A $K_3$-decomposition of the complete graph on $v$ vertices, $K_v$, is called a *Stiener Triple System* of order $v$, $STS(v)$. It is well known that $STS(v)$ exist if and only if $v \equiv 1$ or $3 \pmod{6}$.

Example: A $K_7$ decomposition into 3-cycles ($STS(7)$). The following blocks give an $STS(7)$ of the graph in Figure 2: $(0, 1, 3), (1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 0), (5, 6, 1), (6, 0, 2)$.

![Figure 2: A Complete Graph on Seven Vertices](image)

Generally, whenever a complete graph, digraph, or mixed graph is decomposed into a graph (respectively digraph or mixed graph) on three vertices, the resulting systems is called a *triple system* and the resulting triples are called *blocks*. The complete graph on three vertices $K_3$ has two orientations namely the 3-circuit and the transitive triple (see Figure 3).

A decomposition $D_v$ into isomorphic copies of the 3-circuit is equivalent to a *Mendelsohn triple system* of order $v$, denoted $MTS(v)$, and is known to exists if and
only if $v \equiv 0 \text{ or } 1 \pmod{3}$, $v \neq 6$ [11]. A decomposition of $D_v$ into isomorphic copies of a *transitive triple*, is equivalent to a *directed triple system*, and exists if and only if $v \equiv 0 \text{ or } 1 \pmod{3}$ [10].

The *wheel*, denoted $W_n$, is the graph obtained from the $n$-cycle by appending a central universal vertex. We will denote the wheel $W_n$ with center $c$ and cycle $0, a, 2a, ..., (n-1)a$ by $W_n(c:a)$. Note that $|V((W_n))| = n + 1$ and $|E((W_n))| = 2n$. This can be extended to a mixed graph by replacing each edge with an edge, a forward arc, and a backwards arc. We will denote the mixed wheel by with center at $c$ and cycle $0, a, 2a, ..., (n-1)a$ by $W^M_n(c:a)$.

The *circulant*, denoted $C_n(S)$, has vertex set $V(C_n(S)) = \mathbb{Z}_n$ where $\mathbb{Z}_n$ denotes the set of integers modulo $n$. Two vertices $u$ and $v$ are adjacent if and only if $|u - v|_n \in S$, where $|u - v|_n = \min\{(u - v) \pmod{n}, (v - u) \pmod{n}\}$. The *mixed circulant* will have an edge, a forward arc, and a backward arc for pair of vertices. edges. Similarly, the complete graph with $v$ vertices and hole of order $k$, denoted $K(v,k)$, is obtained from $K_v$ by deleting an edge induced subgraph isomorphic to $K_k$.

Gardner [5] studied the *mixed triple systems* which are the decompositions of the complete mixed graph into each of the partial orientation of a $K_3$ (see Figure 4).
He showed that a $T_1$-triple system and $T_2$-triple system of order $v$ exists if and only if $v \equiv 1 \pmod{2}$. He also showed that a $T_3$-triple system of order $v$ exists if and only if $v \equiv 1 \pmod{2}$, $v \notin \{3, 5\}$.

Hartman and Mendelsohn [8] gave necessary and sufficient conditions for the existence of a $G$-decomposition of $\lambda D_v$ (the complete directed graph on $v$ vertices with each arc taken $\lambda$ times) for all simple connected digraphs $G$ having three vertices. Their work motivated the research in this thesis where instead of considering the complete digraph, we consider the complete mixed graph on $v$ vertices $M_v$ (with each arc taken $\lambda$ times) and decompose such a graph into all possible mixed graphs on three vertices with twice as many arcs as edges. There are eighteen such graphs and these are illustrated in Figure 5.
We let $T^i_j$ denote the mixed graph on three vertices with twice as many arcs as edges, where $i$ represents the number of arcs, and $j$ a counter. We denote the vertices of $T^i_j$ by the ordered triple $(a, b, c)^i_j$ with arc and edge sets as given in Table 1.

![Figure 5: Mixed Triple Systems](image)
This thesis is divided into five chapters. Following this introductory chapter is the second chapter in which we consider the decomposition of the complete mixed graph \( (M_v) \). Here we present the results for the decomposition of the complete mixed graph on \( M_v \). The third chapter constitutes the decomposition of the \( \lambda \)-fold complete mixed graph where \( \lambda \) is greater than one. In this chapter we study the decomposition of the \( \lambda \)-fold complete mixed graph \( \lambda M_v \). Necessary and sufficient conditions are given for the existence of a \( T^4_j \)-triple systems of order \( v \). A verification of the the methods used is the principal content of the fourth chapter. Here we demonstrate, with the aid of examples, how the difference method and graph labelings are used in our decompositions. Finally in the last chapter, we present our conclusions.
2 DECOMPOSITIONS OF THE COMPLETE MIXED GRAPH

2.1 Introduction

In this chapter, we present the results for the decomposition of the complete mixed graph on \( v \) vertices, denoted \( M_v \), into every possible mixed graph on three vertices with twice as many arcs as edges. Decompositions of \( M_v \) are defined similarly to decompositions of \( K_v \) and \( D_v \). A \( T^i_j \)-decomposition of \( M_v \) is called a \( T^i_j \)-triple system. Here we give direct constructions of the \( T^i_j \)-triple system of order \( v \) when they exist using difference method techniques, except for the cases \( T^4_4 \) and \( T^4_5 \) where we use the notion of wheels and graph labelings to show the existence of such a decomposition.

Before giving our results, we must remark that the following theorems their proofs is thanks to the herculean effort of Dr. Robert Beeler. They are Lemma 2.6, Theorem 2.7, Theorem 2.8 and Theorem 3.10.

2.2 Results

**Theorem 2.1** [5] A \( T^2_2 \)-decomposition, a \( T^2_3 \)-decomposition and a \( T^2_3 \)-decomposition of \( M_v \) exists if and only if \( v \equiv 1 \) (mod 2), except when \( v = 3 \) or \( v = 5 \) in the case of \( T^2_3 \).

**Theorem 2.2** A \( T^2_4 \)-decomposition, of \( M_v \) exists for all \( v \geq 3 \).

**Proof.** We consider two cases as outlined below.

**Case 1.** Suppose \( v \equiv 1 \) (mod 2), for all \( v \geq 3 \) say \( v = 2k + 1 \) and let the vertex set of \( M_v \) be \( \{0, 1, \ldots, v - 1\} \). Consider the set:
\{(3 + 2i + j, 1 + i + j, j)_{2}^{2}|i = 0, 1, 2, \cdots k - 2, j = 0, 1, \cdots, 2k\}
\cup\{(k + 1 + j, k + j, j)_{2}^{2}|j = 0, 1, \cdots, 2k\}

where the vertex labels are reduced modulo \(v\).

**Case 2.** Suppose \(v \equiv 0 \pmod{2}\), for all \(v \geq 4\) say \(v = 2k\) and let the vertex set of \(M_{v}\) be \(\{\infty, 0, 1, \cdots, v - 2\}\). Take the decomposition of \(M_{v-1}\), where the vertex set of \(M_{v-1}\) is \(\{0, 1, \cdots, v - 2\}\), and union it with the set \(\{(1 + j, \infty, j)_{2}^{2}|j = 0, 1, \cdots, 2k - 1\}\)

In both cases, the resulting set gives the desired decomposition of \(T_{4}^{2}\).

**Theorem 2.3** A \(T_{5}^{2}\)-decomposition and a \(T_{6}^{2}\)-decomposition, of \(M_{v}\) exists \(\forall v \geq 3\).

**Proof.** In order to establish this proof, we consider two cases, \(v\) odd and \(v\) even.

**Case 1.** Suppose \(v \equiv 1 \pmod{2}\), for all \(v \geq 3\) say \(v = 2k + 1\) and let the vertex set of \(M_{v}\) be \(\{0, 1, \cdots, v - 1\}\). Consider the set:

\n\{(2k + j, 1 + i + j, j)_{2}^{3}|i = 0, 1, 2, \cdots k - 2, j = 0, 1, \cdots, 2k\}
\cup\{(k - 1 + j, k + j, j)_{2}^{3}|j = 0, 1, \cdots, 2k\},

where the vertex set is reduced modulo \(v\).

**Case 2.** Suppose \(v \equiv 0 \pmod{2}\), for all \(v \geq 4\) say \(v = 2k\) and let the vertex set of \(M_{v}\) be \(\{\infty, 0, 1, \cdots, v - 2\}\). Take the decomposition of \(M_{v-1}\), where the vertex set of \(M_{v-1}\) is \(\{0, 1, \cdots, v - 2\}\), and union it with the set \(\{(1 + j, \infty, j)_{2}^{2}|j = 0, 1, \cdots, 2k - 1\}\)

In both cases, the resulting set gives the desired decomposition of \(T_{5}^{2}\).

Since \(T_{6}^{2}\) is the converse of \(T_{5}^{2}\) the existence of a \(T_{5}^{2} - \text{decomposition} \) implies the existence of a \(T_{6}^{2} - \text{decomposition} \) of \(M_{v}\).\n
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Theorem 2.4 A $T^2_7$-decomposition and a $T^2_8$-decomposition of $M_v$ exists if and only if $v \equiv 1 \pmod{2}$

Proof. For the necessary part of the theorem we observe that if $v \equiv 0 \pmod{2}$, then a $T^2_7$-decomposition of $M_v$ would imply a (using the notation of [8]) $T_2$-decomposition of the complete digraph $D_v$. However, such a decomposition of $D_v$ does not exist [8]. Thus $v$ must be odd.

Suppose $v \equiv 1 \pmod{2}$ say $v = 2k + 1$. Consider the set:

\[
\{(2 + 2i + j, 1 + i + j)^2_7 \mid i = 0, 1, 2, \ldots, k - 1, j = 0, 1, 2, \ldots, v - 1\},
\]

where the vertex labels are reduced modulo $v$. This set is the desired $T^2_7$-decomposition. Given that $T^2_8$ is the converse of $T^2_7$, the existence of a $T^2_7$ decomposition of $M_v$ implies the existence of a $T^2_8$ decomposition of $M_v$.

Theorem 2.5 Neither $T^4_1$-decompositions, $T^4_2$-decompositions, nor $T^4_3$-decompositions of $M_v$ exist.

Proof. Suppose one of these decompositions exists. Then there is a block $B$ of the decomposition which contains an edge of the form $[a, b]$ and an arc of the form $(a, b)$. However, it is impossible for the arc $(b, a)$ to be contained in any other block of the decomposition since this would require duplicating either edge $[a, b]$ or arc $(a, b)$. Therefore no such decomposition exists.

Lemma 2.6 There exists a cyclic $W_p$-decomposition of $C_n(1, \ldots, 2p)$.

Proof. Note that the wheel, $W_p$, has a graceful labeling [3, 9]. A graceful labeling on a graph of size $q$ will induce a cyclic decomposition of the circulant $C_n(1, \ldots, q)$, when $n \geq 2q + 1$ [1]. Hence there exists a cyclic $W_p$-decomposition of $C_n(1, \ldots, 2p)$.
**Theorem 2.7** Let $k \in \mathbb{N}$, $p \in \mathbb{N} - \{1, 2\}$, and $\mathcal{K} = \{W_j : j \geq 3\}$. There exists a $\mathcal{K}$-decomposition of $K(4p + 3k + 1, k)$.

**Proof.** Let $C = \{c_0, \ldots, c_{k-1}\}$ be the set of mutually non-adjacent vertices in $K(4p + 3k + 1, k)$. Similarly, the set of mutually adjacent vertices in $K(4p + 3k + 1, k)$ can be represented by the elements of $\mathbb{Z}_{4p+2k+1}$. The edges between $C$ and $\mathbb{Z}_{4p+2k+1}$ can be partitioned by the set of wheels $W_{n(2p+k-i)}(c_i : 2p + k - i)$ where $n(2p + k - i)$ denotes the length of the orbit of $2p + k - i$ in $\mathbb{Z}_{4p+2k+1}$. We note that $n(2p + k - i) = \frac{4p+2k+1}{\gcd(2p+k-i, 4p+2k+1)} \geq 3$ as $2p + k - i < 4p + 2k + 1$ and $4p + 2k + 1$ is odd. Hence each of these wheels is in the set $\mathcal{K}$. This leaves a set of edges isomorphic to $C_{4p+2k+1}(1, 2, \ldots, 2p)$. There exists a cyclic $W_p$-decomposition of $C_{4p+2k+1}(1, 2, \ldots, 2p)$ by Theorem 2.6. Hence there exists a $\mathcal{K}$-decomposition of $K(4p + 3k + 1, k)$.

**Theorem 2.8** A $T^4_4$-decomposition and $T^4_5$-decomposition of $M_v$ exists if and only if $v \equiv 0 \pmod{4}$ or $v \equiv 1 \pmod{4}$, $v \notin \{5, 8, 9, 12\}$.

**Proof.** Since the directed part of $T^4_4$ must decompose $D_v$, it follows from [8] that $v \equiv 0, 1 \pmod{4}$ and $v \notin \{5, 8\}$ are necessary conditions. Further, [8] gives all decompositions of $D_9$ into the directed part of $T^4_4$. However, it is easy to verify that none of these extend to a $T^4_4$-decomposition of $M_9$. It suffices to construct the remaining cases. Note that the set of blocks $(i, c, i + 1)^4_4$ for $i = 0, \ldots, n - 1$ will give a $T^4_4$-decomposition of $W^M_n$ centered at $c$.

If $v \equiv 1 \pmod{4}$, say $v = 4p + 1$ for $p \geq 3$. Since $W_p$ is a graceful graph [3, 9] of size $2p$, it will decompose $K_{4p+1}$ [13]. It follows that there exists a $T^4_4$-decomposition of $M_v$ for $v \equiv 1 \pmod{4}$ and $v \geq 13$. 

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If \( v \equiv 0 \) (mod 4), say \( v = 4p + 4 \) for \( p \geq 3 \). Let the vertex set of \( M_v \) be given by \( \{\infty, 0, ..., 4p + 2\} \). Note that \( K_v = W(\infty : 2p + 1) \cup C_{4p+3}(1, ..., 2p) \). Since \( W_p \) is a graceful graph of size \( 2p \), it follows from [1] that it will decompose \( C_{4p+3}(1, ..., 2p) \).

It follows that there exists a \( T_4^4 \)-decomposition of \( M_v \) for \( v \equiv 0 \) (mod 4) and \( v \geq 16 \). It remains to show that there exists a \( T_4^4 \)-decomposition of \( M_{12} \). Since \( T_5^4 \) is the converse of \( T_4^4 \), it follows that the necessary and sufficient conditions for the existence of a \( T_5^4 \)-decomposition of \( M_v \) are the same.

Theorem 2.8 implies the decomposition of \( D_v \) into copies of the underlying digraph of \( T_4^4 \) (and also of the underlying digraph of \( T_5^4 \)), both of which are results of Hartman and Mendelsohn [8]. However, the proof of the result of Hartman and Mendelsohn is based on a complicated inductive proof, whereas the proof of Lemma 2.6 is somewhat more direct.

**Theorem 2.9** A \( T_6^4 \)-decomposition and a \( T_7^4 \)-decomposition of \( M_v \) exists if and only if \( v \equiv 1 \) (mod 4).

**Proof.** Since \( M_v \) has \( v(v - 1) \) arcs and \( T_6^4 \) has four arcs, it follows that \( 4 \mid v(v - 1) \).

Also, for each vertex \( u \) in \( T_6^4 \), \( od(u) = 2 \) it thus follows that \( 2 \mid (v - 1) \). Hence \( v \) must be odd. Therefore \( v \equiv 1 \) (mod 4) and the necessary condition follows. Sufficiency is established, by considering the following cases:

If \( v = 5 \), then consider the set:

\[
\{ (j, 1 + j, 2 + j)_6^4 \mid j = 0, 1, 2, 3, 4 \}.
\]

If \( v = 9 \), then consider the set:

\[
\{ (0, 2, 1)_6^4, (2, 1, 5)_6^4, (1, 5, 8)_6^4, (5, 8, 0)_6^4, (8, 0, 4)_6^4, (0, 4, 7)_6^4, (4, 7, 1)_6^4, (7, 1, 3)_6^4, (1, 3, 6)_6^4, \}
\]
If $v = 13$, then consider the set:

$$\{(j, 1 + j, 2 + j)^4_6, (j, 3 + j, 6 + j)^4_6, (j, 4 + j, 8 + j)^4_6 \mid j = 0, 1, 2, \ldots, 12\}.$$ 

If $v = 17$, then consider the set:

$$\{(j, 1 + j, 2 + j)^4_6, (j, 3 + j, 6 + j)^4_6, (j, 4 + j, 8 + j)^4_6, (j, 5 + j, 10 + j)^4_6 \mid j = 0, 1, 2, \ldots, 16\}.$$ 

If $v = 29$, then consider the set:

$$\{(j, 1 + j, 2 + j)^4_6, (j, 4 + j, 8 + j)^4_6, (j, 5 + j, 10 + j)^4_6, (j, 6 + j, 12 + j)^4_6, (j, 7 + j, 14 + j)^4_6,$$

$$(j, 9 + j, 18 + j)^4_6, (j, 13 + j, 26 + j)^4_6 \mid j = 0, 1, 2, \ldots, 28\}.$$ 

If $v = 33$, then consider the set:

$$\{(j, 2 + j, 4 + j)^4_6, (j, 3 + j, 6 + j)^4_6, (j, 10 + j, 20 + j)^4_6, (j, 12 + j, 24 + j)^4_6, (j, 14 + j, 28 + j)^4_6,$$

$$(j, 1 + j, 8 + j)^4_6, (j, 7 + j, 22 + j)^4_6, (j, 15 + j, 16 + j)^4_6 \mid j = 0, 1, 2, \ldots, 32\}.$$ 

If $v = 49$, then consider the set:

$$\{(j, 2 + j, 4 + j)^4_6, (j, 6 + j, 12 + j)^4_6, (j, 7 + j, 14 + j)^4_6, (j, 8 + j, 16 + j)^4_6, (j, 17 + j, 34 + j)^4_6,$$

$$(j, 18 + j, 36 + j)^4_6, (j, 22 + j, 44 + j)^4_6, (j, 23 + j, 46 + j)^4_6, (j, 24 + j, 48 + j)^4_6, (j, 10 + j, 21 + j)^4_6,$$

$$(j, 9 + j, 19 + j)^4_6, (j, 11 + j, 20 + j)^4_6 \mid j = 0, 1, 2, \ldots, 48\}.$$ 

If $v = 57$, then consider the set:

$$\{(j, 1 + j, 2 + j)^4_6, (j, 4 + j, 8 + j)^4_6, (j, 5 + j, 10 + j)^4_6, (j, 6 + j, 12 + j)^4_6, (j, 7 + j, 14 + j)^4_6,$$

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If \( v = 93 \), then consider the set:

\[
\{(j, 6 + j, 12 + j)_6^4, (j, 8 + j, 16 + j)_6^4, (j, 11 + j, 22 + j)_6^4, (j, 15 + j, 30 + j)_6^4, (j, 18 + j, 36 + j)_6^4, \\
(j, 20 + j, 40 + j)_6^4, (j, 21 + j, 42 + j)_6^4, (j, 23 + j, 46 + j)_6^4, (j, 24 + j, 48 + j)_6^4, (j, 26 + j, 52 + j)_6^4, \\
(j, 27 + j, 54 + j)_6^4, (j, 28 + j, 56 + j)_6^4, (j, 29 + j, 58 + j)_6^4, (j, 34 + j, 68 + j)_6^4, (j, 38 + j, 76 + j)_6^4, \\
(j, 43 + j, 86 + j)_6^4, (j, 44 + j, 88 + j)_6^4, (j, 1 + j, 32 + j)_6^4, (j, 2 + j, 3 + j)_6^4, (j, 31 + j, 33 + j)_6^4, \\
(j, 4 + j, 13 + j)_6^4, (j, 9 + j, 19 + j)_6^4, (j, 10 + j, 14 + j)_6^4 | j = 0, 1, 2, \ldots, 92\}.
\]

In each case, reduce the vertex labels modulo \( v \), and the set gives the desired decomposition. If \( v \equiv 1 \pmod{4} \), then \( D_v \) (and hence \( M_v \)) can be decomposed into arc-disjoint copies of elements of \( \{D_5, D_9, D_{13}, D_{17}, D_{29}, D_{33}, D_{49}, D_{57}, D_{93}\} \) (and hence arc-disjoint and edge-disjoint copies of the corresponding complete mixed graphs) (Theorem 2.1 of [8]; also see [12, 15]). Therefore, for \( v \equiv 1 \pmod{4} \), there exists a decomposition of \( M_v \) into arc-disjoint and edge-disjoint copies of elements of

\[
\{M_5, M_9, M_{13}, M_{17}, M_{29}, M_{33}, M_{49}, M_{93}\}
\]

and, from the above constructions, it follows that there is a \( T_6^4 \)-decomposition of \( M_v \). ■

We note that the existence of a \( T_6^4 \)-decomposition of \( M_v \) implies the existence of a \( T_7^4 \)-decomposition of \( M_v \), since they are converses of each other. The proof of Theorem 2.9 follows along the same basic lines as the proof by Hartman and Mendelsohn [8] for the result concerning the corresponding underlying digraph of \( T_6^4 \). However, our proof
is a bit more direct in the sense that the specific blocks are given for decompositions when \( v \in \{5, 9, 13, 17, 29, 33, 49, 57, 93\} \).

**Theorem 2.10** A \( T_8^4 \)-decomposition of \( M_v \) exists if and only if \( v \equiv 0 \) (mod 4) or \( v \equiv 1 \) (mod 4).

**Proof.** Such a decomposition is equivalent to a \( P_3 \)-decomposition of \( K_v \). It is well-known that such a decomposition exists if and only if \( v \equiv 0 \) (mod 4) or \( v \equiv 1 \) (mod 4) (see [2]).

**Theorem 2.11** A \( T_9^4 \)-decomposition of \( M_v \) exists if and only if \( v \equiv 0 \) or 1 (mod 4).

**Proof.** If \( v \equiv 2 \) or 3 (mod 4), then a \( T_9^4 \)-decomposition of \( M_v \) would imply a (using the notation of [8]) \( P_3 \)-decomposition of the complete digraph \( D_v \). However, such a decomposition of \( D_v \) does not exist [8].

**Case 1.** Suppose \( v \equiv 1 \) (mod 4), \( \forall v \geq 5 \) say \( v = 4k + 1 \) and let the vertex set of \( M_v \) be \( \{0, 1, \cdots, v - 1\} \). Consider the set:
\[
\{(j, 2i + j, 2k - 2i + j)_{8|} | i = 0, 1, 2, \cdots k, j = 0, 1, \cdots, v - 1\}
\]
where the vertex labels are reduced modulo \( v \).

**Case 2.** Suppose \( v \equiv \) (mod 4) for all \( v \geq 4 \), say \( v = 4k \). Associate the arc differences 2\( k - 1 \), 2\( k \), with the fixed point \( \infty \) with the edge difference 2\( k - 1 \). Consider the set
\[
\{(j, 2i + j, 2k - 2i + j)_{8|} | i = 0, 1, 2, \cdots k - 1, j = 0, 1, \cdots \} \cup \{(j, 2k - 1 + j, \infty)_{8|} | j = 0, 1, \cdots v\}
\]
where the vertex labels are reduced modulo \( v - 1 \). In both cases, the given sets give the desired decomposition.

**Theorem 2.12** [2] A \( T_9^6 \)-decomposition of \( M_v \) exists if and only if \( v \equiv 0 \) (mod 6) or \( v \equiv 1 \) (mod 6).
The following table gives a summary of the results in Chapter 2.

<table>
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<td>$v \equiv 1 \pmod{2}$</td>
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<td>$T^2_3$</td>
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<tr>
<td>$T^4_3$</td>
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<td>$T^4_7$</td>
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Table 2: Summary of Results in Chapter 2
3 DECOMPOSITION OF THE $\lambda$-FOLD COMPLETE MIXED GRAPH

In this Chapter, we consider the case $\lambda > 1$. Here we study decompositions of the complete mixed graph, $\lambda M_v$. We give $T_i^j$-triple systems whenever it is possible. The same notation as that in chapter 2 is adopted here.

Lemma 3.1 A $T_1^2$-decomposition and a $T_2^2$-decomposition of $2M_v$ exists for all $v \equiv 0 \pmod{2}$.

Proof. For $v \equiv 0 \pmod{4}$, let $v = 4k$ and suppose the vertex set of $2M_v$ is \{0, 1, \ldots, v - 1\}. Consider the set:

$$\{(j, k + j, 3k + j)^2_1 \mid j = 0, 1, 2, \ldots, v - 1\} \cup \{(j, k - i + j, k + 1 + i + j)^2_1, (j, k - 1 - i + j, 3k + j)^2_1 \mid i = 0, 1, 2, \ldots, k - 1, j = 0 \ldots 4k - 1\}$$

$$\cup \{(j, 3k - 1 - i + j, 3k + 1 + i + j)^2_1, (j, k - 1 - i + j, k + 1 + i + j)^2_1 \mid i = 0, 1, 2, \ldots, k - 2, j = 0 \ldots 4k - 1\},$$

where vertex labels are reduced modulo $v$. This set is the desired decomposition.

For $v \equiv 2 \pmod{4}$, let $v = 4k + 2$ and suppose the vertex set of $2M_v$ is \{\infty, 0, 1, \ldots, v - 2\}. Consider the set:

$$\{(\infty, j, 2k + j)^2_1, (j, \infty, k + 1 + j)^2_1, (j, \infty, 3k + 1 + j)^2_1 \mid j = 0, 1, 2, \ldots, 4k\}$$

$$\cup \{(j, k - i + j, k + 1 + i + j)^2_1, (j, 3k - i + j, 3k + i + j)^2_1, (j, k - i + j, k + 2 + i + j)^2_1 \mid i = 0, 1, 2, \ldots, k - 1, j = 0, 1, 2, \ldots, 4k\} \cup (j, 3k - i + j, 3k + 2 + i + k)^2_1$$

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\[ i = 0, 1, 2, \ldots, k - 2, j = 0, 1, 2, \ldots, 4k \}

where vertex labels are reduced modulo \( v - 1 \). This set is the desired decomposition. ■

**Theorem 3.2**  A \( T^2_1 \)-decomposition and a \( T^2_2 \)-decomposition of \( \lambda M_v \) exists if and only if (1) \( \lambda \equiv 0 \pmod{2} \) and \( v \geq 3 \), or (2) \( \lambda \equiv 1 \pmod{2} \) and \( v \equiv 1 \pmod{2} \).

**Proof.** For \( \lambda \equiv 1 \pmod{2} \) and \( v \equiv 1 \pmod{2} \), a \( T^2_1 \)-decomposition of \( M_v \) exists by Lemma 2.1, and hence a \( T^2_1 \)-decomposition of \( \lambda M_v \) exists. For \( \lambda \equiv 0 \pmod{2} \) and \( v \equiv 0 \pmod{2} \), a \( T^2_1 \)-decomposition of \( 2M_v \) exists by Lemma 3.1, and hence a \( T^2_1 \)-decomposition of \( \lambda M_v \) exists.

If \( \lambda \equiv 1 \pmod{2} \), then the total degree of each vertex of \( M_v \) is \( 3\lambda(v - 1) \). Since each vertex of \( T^2_1 \) is of total degree 2, thus a necessary condition for a \( T^2_1 \)-decomposition of \( \lambda M_v \) is that \( v \equiv 1 \pmod{2} \). When both \( v \) and \( \lambda \) are odd, a \( T^2_1 \)-decomposition of \( M_v \) exists, and hence a \( T^2_1 \)-decomposition of \( \lambda M_v \) exists. ■

**Lemma 3.3**  A \( T^2_3 \)-decomposition of \( 2M_v \) exists for all \( v \equiv 0 \pmod{2} \).

**Proof.** For \( v \equiv 0 \pmod{8} \), let \( v = 8k \) and suppose the vertex set of \( 2M_v \) is \( \{0, 1, 2, \ldots, v - 1\} \). Consider the multi-set:

\[
\{(j, 1 + i + j, 8k - 1 - i + j)_{3^2} \mid i = 0, 1, 2, \ldots, 2k - 1, j = 0, 1, 2, \ldots, 8k - 1\}
\]

\[
\cup \{(2k + 1 + i + j, 4k + 2 + 2i + j, j)_{3^2} \mid i = 0, 1, 2, \ldots, 2k - 2, j = 0, 1, 2, \ldots, 8k - 1\}
\]

\[
\cup 2 \times \{(4k + 2i + j, 1 + 4i + j, 6k + 2i + j, 4k + 4i + j, j)_{3^2} \mid i = 0, 1, 2, \ldots, k - 1, j = 0, 1, 2, \ldots, 8k - 1\}
\]

where the vertex labels are reduced modulo \( v \). This multi-set is the desired decomposition.

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For \( v \equiv 2 \pmod{4} \), let \( v = 4k + 2 \) and suppose the vertex set of \( 2M_v \) is \( \{\infty, 0, 1, 2, \ldots, v - 2\} \). Consider the multi-set:

\[
(\infty, j, 2 + j)^2_3, (1 + j, \infty, j)^2_3, (j, 1 + j, \infty)^2_3) \mid j = 0, 1, 2, \ldots, 4k
\]

\[
\cup\{(2 + i + j, 4 + 2i + j, j)^2_3 \mid i = 0, 1, 2, \ldots, k - 2, j = 0, 1, 2, \ldots, 4k\}
\]

\[
\cup\{(k+1+i+j, 2k+2+2i+j, j)^2_3, (2k+1+i+j, 1+2i, j)^2_3, (3k+1+i+j, 2k+1+2i+j, j)^2_3 \mid i = 0, 1, 2, \ldots, k - 1, j = 0, 1, 2, \ldots, 4k\}
\]

where the vertex labels are reduced modulo \( v - 1 \). This multi-set is the desired decomposition.

For \( v \equiv 4 \pmod{8} \), let \( v = 8k + 4 \) and suppose the vertex set of \( 2M_v \) is \( \{0, 1, 2, \ldots, v - 1\} \). Consider the multi-set:

\[
\{(j, 1 + i + j, 8k + 3 - i + j)^2_3 \mid i = 0, 1, 2, \ldots, 2k, j = 0, 1, 2, \ldots, 8k + 3\}
\]

\[
\cup\{(2k + 2 + i + j, 4k + 4 + 2i + j, j)^2_3 \mid i = 0, 1, 2, \ldots, 2k - 1, j = 0, 1, 2, \ldots, 8k + 3\}
\]

\[
\cup\{2 \times \{(4k + 2 + 2i + j, 1 + 4i + j, j)^2_3 \mid i = 0, 1, 2, \ldots, k, j = 0, 1, 2, \ldots, 8k + 3\}
\]

\[
\cup\{2 \times \{(6k + 4 + 2i + j, 4k + 5 + 4i + j, j)^2_3 \mid i = 0, 1, 2, \ldots, k - 1, j = 0, 1, 2, \ldots, 8k + 3\}
\]

where the vertex labels are reduced modulo \( v \). This multi-set is the desired decomposition.

\[\blacksquare\]

**Theorem 3.4** A \( T^2_3 \)-decomposition of \( \lambda M_v \) exists if and only if (1) \( \lambda \equiv 0 \pmod{2} \) and \( v \geq 3 \), or (2) \( \lambda \equiv 1 \pmod{2} \) and \( v \equiv 1 \pmod{2} \).
**Proof.** The necessary conditions follow as in Theorem 3.2. For \( \lambda \geq 1 \) and \( v \equiv 1 \) (mod 2), a \( T_3^2 \)-decomposition of \( M_v \) exists by Lemma 2.1, and hence a \( T_3^2 \)-decomposition of \( \lambda M_v \) exists. For \( \lambda \equiv 0 \) (mod 2) and \( v \equiv 0 \) (mod 2), a \( T_3^2 \)-decomposition of \( 2M_v \) exists by Lemma 3.3, and hence a \( T_3^2 \)-decomposition of \( \lambda M_v \) exists.

**Theorem 3.5** A \( T_4^2 \)-decomposition, a \( T_5^2 \)-decomposition, and a \( T_6^2 \)-decomposition of \( \lambda M_v \) exists \( \forall v \geq 3 \).

**Proof.** Since such decompositions exist for \( \lambda = 1 \) by Theorem 2.3, the result follows trivially.

**Theorem 3.6** A \( T_7^2 \)-decomposition and a \( T_8^2 \)-decomposition of \( \lambda M_v \) exists \( \forall v \geq 3 \).

**Proof.** Suppose \( v \equiv 0 \) (mod 2), say \( v = 2k \), and \( \lambda = 2 \). Consider the set of blocks

\[
\{(j, i + j, 2k - 1 + j)^2 \mid i = 1, 2, \ldots, 2k - 2, j = 0 \ldots 2k - 1\}
\]

\[
\cup\{(j, 2k - 1 + j, 2k - 2 + j)^2 \mid j = 0 \ldots 2k - 1\}.
\]

These set of blocks, form a \( T_7^2 \)-decomposition of \( 2M_v \) and hence a \( T_7^2 \)-decomposition of \( \lambda M_v \) exists for \( v \) even. Suppose \( \lambda \equiv 1 \) (mod 2). Then \( v \equiv 1 \) (mod 2), and a \( T_7^2 \)-decomposition of \( \lambda M_v \) exists by Theorem 2.8. Since \( T_8^2 \) is the converse of \( T_7^2 \), the results follow.

**Theorem 3.7** A \( T_1^4 \)-decomposition of \( \lambda M_v \) exists for \( \lambda \equiv 0 \) (mod 4) and for all \( v \).

**Proof.** Suppose \( \lambda \equiv 0 \) (mod 4). First let \( \lambda = 4 \) and \( v \) be odd. That is \( v = 2k + 1 \), for all \( k \geq 1 \). Consider the collection of blocks

\[
\{(j, i + 1 + j, 2k - i + j)^4 \mid i = 0, 1, \ldots k - 1, j = 0, 1, \ldots 2k\}
\]
Next let $\lambda = 4$ and $v$ be even. That is $v = 2k$, $k \geq 1$. Consider the collection of blocks

\[ \bigcup \{(j, k - 1 - i + j, k + i + j)_{4,1}^4 | i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 2k - 1\} \]

\[ \bigcup \{(j, k + 1 + i + j, k - 2 - i + j)_{4,1}^4 | i = 0, 1, \ldots, k - 3, j = 0, 1, \ldots, 2k - 1\} \]

\[ \bigcup \{(\infty, 0, k)_{4,1}^4, (0, \infty, k - 1)_{4,1}^4, (0, 2k - 2, \infty)_{4,1}^4\} \]

In both cases, the set of blocks gives a $T_{4,1}^4$-decompositions of $4M_v$, hence a $T_{4,1}^4$-decomposition of $\lambda M_v$ for all $\lambda \equiv 0 \pmod{4}$ and $v \in \mathbb{N}$.

**Conjecture 3.8** A $T_{4,1}^4$-decomposition of $\lambda M_v$ exists if and only if $\lambda(v - 1) \equiv 0 \pmod{4}$.

We now present a partial proof for our conjecture. If the proof is completed we will get neccessary and sufficient conditions for the existence of a $T_{4,1}^4$-triple system.

**Partial Proof.** The arc and edge sets of $T_{4,1}^4$ are $A(T_{4,1}^4) = \{(a, b), (b, c), (a, c), (c, a)\}$ and $E(T_{4,1}^4) = \{[a, b], [b, c]\}$ respectively, as given in Table 1. Now if $\lambda$ is odd, then we now show that there is no $T_{4,1}^4$-decomposition of $\lambda M_v$. Suppose, to the contrary, there is a $T_{4,1}^4$-decomposition of $\lambda M_v$. Then there is a collection of blocks $B_1, B_2, \ldots, B_\lambda$ where each $B_i$ contains edge $[a, b]$. Since $\lambda$ is odd, the blocks $B_1, B_2, \ldots, B_\lambda$ contains either an odd number of copies of arc $(a, b)$ and an even number of copies of arc $(b, a)$ or an even number of copies of arc $(a, b)$ and an odd number of copies of arc $(b, a)$. The remaining blocks in the decomposition each either contain both arcs $(a, c)$ and $(c, a)$ or contain neither arc $(a, c)$ nor arc $(c, a)$. But then in the total collection of
blocks, there is either an even number of copies of arc \((a, c)\) and an odd number of copies of arc \((c, a)\) or an odd number of copies of arc \((a, c)\) and an even number of copies of arcs \((c, a)\). Thus a contradiction. Therefore, \(\lambda\) must be even.

In \(M_v\), every vertex is in \(\lambda(v - 1)\) edges and \(2\lambda(v - 1)\) arcs. For any vertex \(u\) in \(T_4\), we have that \(t(u) = 4\). So it is necessary that \(4 \mid 3\lambda(v - 1)\). It thus follows that \(\lambda(v - 1) \equiv 0 \pmod{4}\). This establishes the necessary condition of the conjecture. For the conjecture to be true, we must consider two cases.

**Case 1.** \(\lambda \equiv 0 \pmod{4}\). First let \(\lambda = 4\) and \(v\) be odd. This is handled in Theorem 3.7.

**Case 2.** \(\lambda \equiv 0 \pmod{2}\). Since \(\lambda(v - 1) \equiv 0 \pmod{4}\), it follows that \(v\) is odd. Now let \(\lambda = 2\) and \(v = 8k + 1\), for \(k \geq 1\). Consider the set of blocks

\[
\{(j, 3k - i + j, 1 + i + j) | i = 0, 1, \ldots k - 1, j = 0, 1, \ldots 8k\}
\]

\[
\{(j, 8k - i + j, 3k + 1 + i + j) | i = 0, 1, \ldots k - 1, j = 0, 1, \ldots 8k\}
\]

\[
\{(j, 2k - i + j, 5k + 1 + i + j) | i = 0, 1, \ldots 2k-1, j = 0, 1, \ldots 8k\}
\]

This set of blocks gives the desired decomposition. The remaining cases to consider are \(v \equiv 3 \pmod{8}\), \(v \equiv 5 \pmod{8}\) and \(v \equiv 7 \pmod{8}\), which we are still working on.

**Theorem 3.9**  A \(T_2^4\)-decomposition and a \(T_3^4\)-decomposition of \(\lambda M_v\) exists if and only if

\[
\lambda \equiv 2 \pmod{4}, \text{ and } v \equiv 1 \pmod{2}
\]

\[
\lambda \equiv 0 \pmod{4}, \forall v \geq 3.
\]

**Proof.** The necessary condition follow as in Theorem 3.7. For sufficiency, we consider the following cases.
**Case 1.** $v \equiv 1 \pmod{2}$, say $v = 2k + 1$, and $\lambda = 2$. Consider the set of blocks

$$\{(2k - 1 - 2i, 2k - i, 0)^4_2 \mid i = 0, 1, 2, \ldots, k - 1\}.$$ 

This is a collection of base blocks for a cyclic $T^4_2$-decomposition of $2M_v$ and hence a $T^3_2$-decomposition of $\lambda M_v$ for $\lambda \equiv 2 \pmod{4}$.

**Case 2.** $v \equiv 0 \pmod{2}$, say $v = 2k$, and $\lambda = 4$. Consider the set of blocks

$$\{(2k - 3 - 2i + j, 2k - 2 - i + j, j)^4_2 \mid i = 0, 1, 2, \ldots, k - 3, j = 0 \ldots 2k - 1\}$$

$$\cup\{(1 + j, k + j, j)^4_2, (1, \infty, 0)^4_2, (1, k, \infty)^4_2, (0, k, \infty)^4_2, j = 0 \ldots 2k - 1\}.$$ 

This set of blocks forms a $T^4_2$-decomposition of $4M_v$. Thus for $\lambda \equiv 0 \pmod{4}$ and $v$ even, we get the desired results. Since $T^4_3$ is the converse of $T^4_2$ it follows that the existence of a $T^4_2$-triple system implies the existence of a $T^4_3$-triple system.

**Theorem 3.10** A $T^4_1$-decomposition and a $T^4_2$-decomposition of $\lambda M_v$ exists if and only if

$$\lambda \equiv 1 \pmod{2}, v \equiv 0, 1 \pmod{4}$$

$$\lambda \equiv 0 \pmod{2}, v \geq 4$$

$$\lambda \equiv 0 \pmod{4}, v \geq 3$$

$$(\lambda, v) \neq (1, 5), (1, 8), (1, 9), (1, 12).$$

**Proof.** It suffices to give constructions for the cases not covered in Lemma 2.6.

When $v = 5$ and $\lambda = 2$, take the set of blocks $(i, i + 1, i + 2)^4_4$ and $(i + 2, i, i + 1)^4_4$ for $i = 0, 1, 2, 3, 4$.

For $v = 5$ and $\lambda = 3$, take the set of blocks $(i, i + 1, i + 3)^4_4$, $(i + 2, i, i + 1)^4_4$ and
When \( v = 7 \) and \( \lambda = 2 \), take the set of blocks \( (i, \infty, i + 3)_{4}^{4} \) and \( (i + 2, \infty, i + 1)_{4}^{4} \),
\( (2j, 2j + 1, 2j + 2)_{4}^{4} \), \( (2j, 2j + 1, 2j + 2)_{4}^{4} \) for \( i = 0, 1, 2, 3, 5 \) and \( j = 0, 1, 2 \). All computations are done modulo 6.

When \( v = 8 \), note that \( 2M_{8} = 2W_{7}^{M}(\infty : 3) \cup \{(i, i + 1, i + 2)_{4}^{4}, (i + 2, i, i + 1)_{4}^{4} : i = 0, 1, 2, 3, 4, 5, 6\}\).

For \( v = 8 \) and \( \lambda = 3 \), note that \( 3M_{8} = 3W_{7} \cup \{(i, i + 1, i + 2)_{4}^{4}, (i + 2, i, i + 1)_{4}^{4} : i = 0, 1, 2, 3, 4, 5, 6\}\).

When \( v = 9 \) and \( \lambda = 2 \), take the set of blocks \( (i, i + 2, i + 4)_{4}^{4}, (i + 1, i + 3, i)_{4}^{4},
(i + 3, i + 2, i)_{4}^{4} \) and \( (i, i + 3, i + 4)_{4}^{4} \) for \( i = 0, 1, 2, \ldots, 8 \).

For \( v = 9 \) and \( \lambda = 3 \), take the set of blocks \( (i, i + 2, i + 4)_{4}^{4}, (i, i + 2, i + 4)_{4}^{4},
(i, i + 4, i + 1)_{4}^{4}, (i, i + 4, i + 1)_{4}^{4}, (i, i + 2, i + 3)_{4}^{4} \) and \( (i + 3, i + 2, i)_{4}^{4} \) for \( i = 0, 1, 2, \ldots, 8 \).

For \( v = 10 \) and \( \lambda = 2 \), take the set of blocks \( (i, \infty, i + 3)_{4}^{4}, (i, \infty, i + 4)_{4}^{4} \) and
\( W_{3}(i; i + 1, i + 2, i + 4) \) for \( i = 0, 1, \ldots, 8 \) and all computations are done modulo 9.

For \( v = 11 \) and \( \lambda = 2 \), use the wheels \( W_{5}^{M}(i, i + 2, i + 6, i + 3, i + 5, i + 1) \) for
\( i = 0, \ldots, 10 \).

For \( v = 12 \) and \( \lambda = 2 \), take \( 2W + 11(\infty : 2) \) along with the set of blocks \( (i, i + 2, i + 4)_{4}^{4},
(i + 1, i, i + 3)_{4}^{4}, (i + 3, i, i + 2)_{4}^{4} \) for \( i = 0, \ldots, 10 \).

For \( v = 12 \) and \( \lambda = 3 \), take \( 2W + 11(\infty : 2) \) along with the set of blocks \( (i, i + 2, i + 4)_{4}^{4},
(i + 4, i + 2, i)_{4}^{4}, (i, i + 4, i + 1)_{4}^{4}, (i + 1, i + 4, i)_{4}^{4}, (i, i + 2, i + 3)_{4}^{4} \) and \( (i + 3, i + 2, i)_{4}^{4} \)
for \( i = 0, \ldots, 10 \).

When \( v = 14 \) and \( \lambda = 2 \) take the set of blocks \( (i, c, i + 2)_{4}^{4}, (i, c, i + 4)_{4}^{4} \) and the wheel
$W_5^M(i : i + 1, i + 2, i + 8, i + 3, i + 10)$ for $i = 0, ..., 12$.

When $v = 15$ and $\lambda = 2$, take the set of blocks $(i + 5, i, i + 11)_4^4$, $(i + 11, i, i + 5)_4^4$, $(i + 5, i, i + 12)_4^4$, $(i + 12, i, i + 5)_4^4$, $W_3^M(i : i + 1, i + 2, i + 4)$ for $i = 0, ..., 14$.

For $v = 18$ and $\lambda = 2$, take the set of blocks $(i, c, i + 5)_4^4$, $(i, c, i + 7)_4^4$, $(i + 3, i, i + 12)_4^4$, $(i + 12, i, i + 3)_4^4$, $(i + 4, i, i + 10)_4^4$, $(i + 10, i, i + 4)_4^4$, $W_3^M(i : i + 1, i + 2, i + 4)$ for $i = 0, ..., 16$.

For $v = 19$ and $\lambda = 2$, take the wheels $2W_{15}(\infty_1; 7)$, $2W_{15}(\infty_2; 6)$, $2W_{15}(\infty_3; 5)$, $2W_{15}(\infty_4; 4)$, and $W_{15}(\infty_4; 3)$ along with $2M_4$. The remaining edges are partitioned by $W_3(i : i + 1, i + 2, i + 4)$.

When $v = 23$ and $\lambda = 2$, take the set of blocks $(i + 3, i, i + 22)_4^4$, $(i + 22, i, i + 3)_4^4$, $(i + 8, i, i + 18)_5^4$, $(i + 18, i, i + 8)_4^4$, $(i + 7, i, i + 21)_4^4$, $(i + 21, i, i + 7)_4^4$, $(i + 6, i, i + 17)_4^4$, $(i + 17, i, i + 6)_4^4$, $W_3^M(i : i + 1, i + 3, i + 8)$, for $i = 0, ..., 22$.

For $v = 26$ and $\lambda = 2$, take the blocks $(i, \infty, i + 12)_4^4$, $(i + 12, \infty, i)_4^4$, $(i, \infty, i + 11)_4^4$, and $(i + 11, \infty, i)_4^4$.

The remaining edges are isomorphic to $2M(C_{25}(1, \ldots, 10))$. Since $C_{25}(1, \ldots, 10)$ has a $W_5$ decomposition, by above the result follows.

For $v = 27$ and $\lambda = 2$, take the set of blocks $(i + 1, i, i + 9)_4^4$, $(i + 9, i, i + 1)_4^4$, $(i + 2, i, i + 12)_4^4$, $(i + 12, i, i + 2)_4^4$, $(i + 2, i, i + 16)_4^4$, $(i + 16, i, i + 2)_4^4$, $(i + 4, i, i + 11)_4^4$, $(i + 11, i, i + 4)_4^4$, $(i + 23, i, i + 1)_4^4$, $W_3^M(i : i + 3, i + 6, i + 12)$, for $i = 0, ..., 26$.

For $v = 30$ and $\lambda = 2$, take the blocks $(i, \infty, i + 14)_4^4$, $(i, \infty, i + 13)_4^4$, $(i + 14, \infty, i)_4^4$, and $(i + 13, \infty, i)_4^4$.

The remaining edges are isomorphic to $2M(C_{29}(1, \ldots, 12))$. Since $C_{29}(1, \ldots, 12)$ has
a $W_6$ decomposition, by above the result follows.

For $v = 35$ and $\lambda = 2$, take the wheels $W_{25}(\infty_j : j = 1, \ldots, 12 - j)$ twice each for $j = 0, \ldots, 9$ and a copy of $2M_{10}$. The remaining edges are partitioned by the blocks $(i, i + 1, i + 2)_{4}^4$, $(i, i + 1, i + 2)_{4}^4$ for $i = 0, \ldots, 24$ and all computations are done modulo 25.

For $v = 35$ and $\lambda = 2$, take the wheels $W_{29}(\infty_j ; 1)$, $W_{29}(\infty_j ; 1)$, $W_{29}(\infty_i ; 5)$, $W_{29}(\infty_i ; i + 4)$ for $i = 3, \ldots, 11$ twice each along with $2M_{10}$. The remaining edges are partitioned by the blocks $(i, i + 3, i + 6)_{4}^4$, $(i + 6, i + 3, i)_{4}^4$, $(i, i + 2, i + 4)_{4}^4$, $(i + 4, i, i + 2)_{4}^4$ for $i = 0, \ldots, 28$ and all computations are done modulo 29.

Let $\lambda = 2$ and $v \equiv 2 \pmod{4}$, say $v = 4 + 22$ for $p = 0$ or $p \geq 3$. Note that $M_v = M(4p + 22, 7)$. As the decomposition of $2M_7$ is given above, we need only give the decomposition of $M(4p + 22, 7)$. Note that $K(4p + 22, 7) \cong \bigcup_{i=1}^{2p} W_{4p+15}(\infty_i ; 2p + i) \cup C_{4p+15}(1, \ldots, 2p)$. Since $C_{4p+15}(1, \ldots, 2p)$ has a $W_p$-decomposition by above, the result follows. Let $\lambda = 2$ and $v \equiv 3 \pmod{4}$, say $v = 4 + 31$ for $p = 0$ or $p \geq 3$. Note that $M_v = M(4p + 31, 10) \cup M_{10}$. As the decomposition of $2M_{10}$ is given above, we need only give the decomposition of $M(4p + 31, 10)$. Note that $K(4p + 31, 10) \cong \bigcup_{i=1}^{2p} W_{4p+21}(\infty_i ; 2p + i) \cup C_{4p+21}(1, \ldots, 2p)$. Since $C_{4p+21}(1, \ldots, 2p)$ has a $W_p$-decomposition by above, the result follows. Since $T_5^4$ is the converse of $T_4^4$, it follows that it has the same necessary and sufficient conditions.

Theorem 3.11 $T_6^4$-decomposition and a $T_7^4$-decomposition of $\lambda M_v$ exists if and only if

$$\lambda \equiv 1 \pmod{2} \text{ and } v \equiv 1 \pmod{4}$$

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\[ \lambda \equiv 2 \pmod{4} \quad \text{and} \quad v \equiv 1 \pmod{2} \]
\[ \lambda \equiv 0 \pmod{4} \quad \text{and} \quad v \geq 3. \]

**Proof.** For the necessary condition see [8]. For sufficiency, we first consider the following cases.

**Case 1.** \(\lambda = 2\) and \(v = 3\). Consider the set of blocks \(\{ (0,1,2,6)^4, (1,2,0,6)^4, (2,0,1,6)^4 \} \)
these blocks form a \(T_6^4\)-decomposition of \(2M_3\).

It is easy to see that there is a \(K_3\)-decomposition of \(2K_4\) and so there is a \(D_3\)-decomposition of \(2D_4\), and hence there is a \(2D_3\) decomposition of \(4D_4\) and a \(2M_3\)-decomposition of \(4M_4\). Combining this final observation with the fact that there is a \(T_6^4\)-decomposition of \(2M_3\), we see that there is a \(T_6^4\)-decomposition of \(4M_4\).

**Case 2.** \(\lambda = 4\) and \(v = 6\). Consider the set of blocks
\[ \{ (0,2,3,6)^4, (0,1,3,6)^4, (0,5,1,6)^4, (0,2,4,6)^4, (0,2,1,6)^4 \} \]
these blocks form a \(T_6^4\)-decomposition of \(4M_6\).

**Case 3.** \(\lambda = 4\) and \(v = 8\). Consider the set of blocks
\[ \{ (0,3,7,6)^4, (0,4,7,6)^4, (0,2,4,6)^4, (0,5,2,6)^4, (0,1,3,6)^4, (0,2,3,6)^4, (0,1,2,6)^4 \} \]
This set of blocks along with the images of these blocks under the permutation \( (0,1,2,3,4,5,6,7) \) forms a \(T_6^4\)-decomposition of \(4M_8\). For \( v \equiv 1 \pmod{4} \) and \( \lambda \) odd see Theorem 2.9.

Next, suppose \( v \equiv 3 \pmod{4} \). Note that \( D_v \) (and hence \( M_v \)) can be decomposed into arc-disjoint copies of \( \{ D_3, D_5 \} \) (and hence arc-disjoint and edge disjoint copies of the corresponding complete mixed graphs) (Theorem 2.2 of [8]). Hence, there is
a \{2M_3, 2M_5\}-decomposition of 2M_v. Since there is a $T^4_6$-decomposition of $M_5$, it follows that there is a $T^4_6$-decomposition of $2M_5$. From case 1 and the previous statement, it follows that there is a $T^4_6$ decomposition of $\lambda M_v$ for $\lambda \equiv 2 \pmod{4}$ and $v$ odd.

Finally, Theorem 1.5 of [8] states that there exists a \{D_k \mid k = 3, 4, 5, 6, 8\}-decomposition of $D_v$ for all $v \geq 3$. It thus follows that there is a \{M_k \mid k = 3, 4, 5, 6, 8\}-decomposition of $M_v$ for all $v \geq 3$. Since there exist a $T^4_6$-decomposition of $2M_3$, $M_5$ and $2M_4$, it follows that there is a $T^4_6$-decomposition of $4M_3$, $4M_4$ and $4M_5$. From case 1 and case 2 above we get a $T^4_6$-decomposition of $4M_6$ and $4M_8$. It follows that there is a $T^4_6$-decomposition of $M_v$ for $\lambda \equiv 0 \pmod{4}$ and $v \equiv 0 \pmod{2}$.

The proof of theorem 3.11 follows along the same basic lines as the proof as [8] for the result corresponding to the underlying digraph of $T^4_6$. However, our proof is a bit more direct in the sense that specific blocks are given when $v \in \{3, 4, 5, 6, 8, 9, 13, 17, 29, 33, 49, 57, 93\}$.

**Theorem 3.12** $T^4_8$-decomposition of $\lambda M_v$ exists if and only if $\lambda v(v-1) \equiv 0 \pmod{4}$

**Proof.** Such a decomposition is equivalent to a 2-path decomposition of $K_v$. It is well known that such a decomposition exists if and only if $\lambda v(v-1) \equiv 0 \pmod{4}$. (see [2])

**Theorem 3.13** $T^4_9$-decomposition of $\lambda M_v$ exists if and only if

\[ v \equiv 0 \text{ or } 1 \pmod{4}, \]

\[ v \equiv 2 \text{ or } 3 \pmod{4} \text{ and } \lambda \equiv 0 \pmod{2}. \]
Proof. Since $\lambda M_v$ has $\lambda v(v - 1)$ arcs and $T_9^4$ has 4 arcs, the necessary conditions follow. Since for $v \equiv 0$ or 1 (mod 4) there exists a $T_9^4$-decomposition of $M_v$, then there exists a $T_9^4$ decomposition of $\lambda M_v$ for all $\lambda$. Suppose $v \equiv 3$ (mod 4), say $v = 4k + 3$, and $\lambda = 2$. Consider the collection of blocks

$$\{(0, i, 4k + 3 - i)^4_9 \mid i = 1, 2, \ldots, 2k + 1\}.$$  

This is a collection of base blocks for a cyclic $T_9^4$-decomposition of $2M_v$.

Suppose $v \equiv 2$ (mod 4), say $v = 4k + 2$, and $\lambda = 2$. Consider the collection of blocks

$$\{(0, 2k + 1 + i, 4k - 1)^4_9 \mid i = 1, 2, \ldots, k - 1\}$$

$$\cup\{(0, 1 + i, 2k - 1 - i)^4_9 \mid i = 1, 2, \ldots, k - 2\}$$

$$\cup\{(0, k, \infty)^4_9, (0, 2k, \infty)^4_9\}.$$  

This is a collection of base blocks for a cyclic $T_9^4$-decomposition of $2M_v$. These blocks along with the images under the permutation $(\infty)(0, 1, 2, \ldots, 4k)$ form a $T_9^4$-decomposition of $2M_v$. Therefore there exist a $T_9^4$-decomposition of $\lambda M_v$ for all $\lambda \equiv 0$ mod 2. Since $T_9^4$ is the converse of $T_9^4$ the result follows.

\[\square\]

**Theorem 3.14** [7] A $T_9^6$-decomposition of $M_v$ exists if and only if

$$\lambda \equiv 1 \text{ or } 5 \pmod{2} \text{ and } v \equiv 1 \text{ or } 3 \pmod{6}$$

$$\lambda \equiv 2 \text{ or } 4 \pmod{6} \text{ and } v \equiv 0 \text{ or } 1 \pmod{3}$$

$$\lambda \equiv 3 \pmod{6} \text{ and } v \equiv 1 \pmod{2}$$

$$\lambda \equiv 0 \pmod{6} \text{ and } v \geq 3.$$  

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The following table gives a summary of the results obtained in chapter three. Here we give necessary and sufficient conditions for the existence of a $T_j^i$-triple system.

<table>
<thead>
<tr>
<th>Triple</th>
<th>Values of $\lambda$</th>
<th>Values of $v$</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_2^2, T_2^2$</td>
<td>$\lambda \equiv 0 \pmod{2}$</td>
<td>$v \geq 3$</td>
<td>Theorem 3.2</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 1 \pmod{2}$</td>
<td>$v \equiv 1 \pmod{2}$</td>
<td></td>
</tr>
<tr>
<td>$T_3^2$</td>
<td>$\lambda \equiv 0 \pmod{2}$</td>
<td>$v \geq 3$</td>
<td>Theorem 3.4</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 1 \pmod{2}$</td>
<td>$v \equiv 1 \pmod{2}$</td>
<td></td>
</tr>
<tr>
<td>$T_2^4, T_2^2, T_6^2$</td>
<td>all $\lambda$</td>
<td>$v \geq 3$</td>
<td>Theorem 3.5</td>
</tr>
<tr>
<td>$T_2^6, T_6^8$</td>
<td>all $\lambda$</td>
<td>$v \geq 3$</td>
<td>Theorem 3.6</td>
</tr>
<tr>
<td>$T_4^1$</td>
<td>$\lambda \equiv 0 \pmod{4}$</td>
<td>$v \geq 3$</td>
<td>Theorem 3.7, Conjecture 3.8</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 2 \pmod{4}$</td>
<td>$v \equiv 1 \pmod{2}$</td>
<td></td>
</tr>
<tr>
<td>$T_2^4, T_3^4$</td>
<td>$\lambda \equiv 2 \pmod{4}$</td>
<td>$v \equiv 1 \pmod{2}$</td>
<td>Theorem 3.9</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 0 \pmod{4}$</td>
<td>$v \geq 3$</td>
<td></td>
</tr>
<tr>
<td>$T_4^1, T_5^3$</td>
<td>$\lambda \equiv 1 \pmod{4}$</td>
<td>$v \equiv 0 \text{ or } 1 \pmod{4}$</td>
<td>Theorem 3.10</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 0 \pmod{2}$</td>
<td>$v \geq 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 0 \pmod{4}$</td>
<td>$v \geq 3$</td>
<td></td>
</tr>
<tr>
<td>$T_6^4, T_7^4$</td>
<td>$\lambda \equiv 1 \pmod{2}$</td>
<td>$v \equiv 1 \pmod{2}$</td>
<td>Theorem 3.11</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 2 \pmod{4}$</td>
<td>$v \equiv 1 \pmod{2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 0 \pmod{4}$</td>
<td>$v \geq 3$</td>
<td></td>
</tr>
<tr>
<td>$T_8^4$</td>
<td>$\lambda \equiv 0 \pmod{4}$</td>
<td>$v \equiv 0 \text{ or } 1 \pmod{4}$</td>
<td>Theorem 3.12</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 1 \pmod{4}$</td>
<td>$v \equiv 0 \text{ or } 1 \pmod{4}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 2 \pmod{4}$</td>
<td>$v \equiv 0 \text{ or } 1 \pmod{2}$</td>
<td></td>
</tr>
<tr>
<td>$T_9^4$</td>
<td>all $\lambda$</td>
<td>$v \equiv 0 \text{ or } 1 \pmod{4}$</td>
<td>Theorem 3.13</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 0 \pmod{2}$</td>
<td>$v \equiv 2 \text{ or } 3 \pmod{4}$</td>
<td></td>
</tr>
<tr>
<td>$T_1^4$</td>
<td>$\lambda \equiv 1 \pmod{2}$</td>
<td>$v \equiv 1 \pmod{4}$</td>
<td>Theorem 3.14</td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 1 \pmod{5}$</td>
<td>$v \equiv 1 \pmod{3}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 2 \pmod{4}$</td>
<td>$v \equiv 0 \pmod{2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 3 \pmod{6}$</td>
<td>$v \equiv 1 \pmod{2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda \equiv 0 \pmod{6}$</td>
<td>$v \geq 3$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Summary of Results in Chapter 3
4 VERIFICATION OF RESULTS AND SOME EXAMPLES

4.1 Difference Method

Here we briefly describe what difference methods and labelings are all about. We then go ahead to illustrate with the use of concrete examples how we use the difference method to carry out our decompositions. We also, by way of an example, show how we use the notion of graph labelings to demonstrate the existence of some of our $T^4_9$-triple system.

The difference method is a method used for decomposing a complete graph using distances (differences) between vertices. We define difference as follows. Suppose we have a complete graph on $v$ vertices. We begin by labeling the vertices 0 through $v - 1$. An arc $(a, b)$ has associated difference $(b - a) \pmod{v}$ and for any edge $[a, b]$ the associated edge difference is given by $\min\{(b - a) \pmod{v}, (a - b) \pmod{v}\}$. The complete mixed graph $M_v$ has $N = v - 1$ arc differences and $\lfloor (v - 1)/2 \rfloor$ edge differences.

Let $H$ be a graph and let $\gamma = \{G_1, G_2, \ldots, G_n\}$ be a $G$-decomposition of $H$. An automorphism of this decomposition is a permutation of the vertex $V(H)$ which fixes the set $\gamma$. That is, if $G_i$ is a block and $\pi$ a permutation, then $\pi(G_i)$ also forms a block.

We illustrate the difference method by carrying out a $T^4_9$ decomposition of $M_v$, the complete mixed graph on $v$ vertices with $v \equiv 1 \pmod{4}$

**Example 4.1** A $T^4_9$-triple system of order 17.

Suppose we have the complete mixed graph $M_{17}$. We observe that $17 \equiv 1 \pmod{4}$,
and therefore by Theorem 2.11, there exist a $T^4_9$-decomposition of $M_{17}$. The associated arc and edge differences are \{1, 2, 3, 4, \ldots, 16\} and \{1, 2, 3, 4, \ldots, 8\} respectively.

Next we denote $T^4_9$ by the order triple $(a, b, c)^4_9$. Then we have the arc differences $(b - a) \pmod{17}$, $(a - b) \pmod{17}$, $(a - c) \pmod{17}$ and $(c - a) \pmod{17}$, and edge differences $\min\{(b - a) \pmod{17}, (a - b) \pmod{17}\}$ and $\min\{(b - c) \pmod{17}, (c - b) \pmod{17}\}$. Now label the vertices in $M_{17}$ from 0 through 16. Consider the base blocks $(0, 2, 7)^4_9$, $(0, 4, 5)^4_9$, $(0, 6, 3)^4_9$, $(0, 8, 1)^4_9$. Table 4 gives the associated arc and edge differences for each base block and Table 5 gives a $T^4_9$-triple system of order 17.

<table>
<thead>
<tr>
<th>Base Block</th>
<th>Arc Differences</th>
<th>Edge Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 2, 7)^4_9$</td>
<td>2, 15, 7, 10</td>
<td>2, 5</td>
</tr>
<tr>
<td>$(0, 4, 5)^4_9$</td>
<td>4, 13, 5, 12</td>
<td>4, 1</td>
</tr>
<tr>
<td>$(0, 6, 3)^4_9$</td>
<td>6, 11, 3, 12</td>
<td>6, 3</td>
</tr>
<tr>
<td>$(0, 8, 1)^4_9$</td>
<td>8, 9, 1, 16</td>
<td>8, 7</td>
</tr>
</tbody>
</table>

Table 4: Base Blocks and Differences for $v = 17$

<table>
<thead>
<tr>
<th>Base Block</th>
<th>Generated Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 2, 7)^4_9$</td>
<td>$(1, 3, 8)^4_9$, $(2, 4, 9)^4_9$, $(3, 5, 10)^4_9$, $(4, 6, 11)^4_9$, $(5, 7, 12)^4_9$, $(6, 8, 13)^4_9$, $(7, 9, 14)^4_9$, $(8, 10, 15)^4_9$, $(9, 11, 16)^4_9$, $(10, 12, 17)^4_9$, $(11, 13, 1)^4_9$, $(12, 14, 2)^4_9$, $(13, 15, 3)^4_9$, $(14, 16, 4)^4_9$, $(15, 0, 5)^4_9$, $(16, 1, 6)^4_9$.</td>
</tr>
<tr>
<td>$(0, 4, 5)^4_9$</td>
<td>$(1, 5, 6)^4_9$, $(2, 6, 7)^4_9$, $(3, 7, 8)^4_9$, $(4, 8, 9)^4_9$, $(5, 9, 10)^4_9$, $(6, 10, 11)^4_9$, $(7, 11, 12)^4_9$, $(8, 12, 13)^4_9$, $(9, 13, 14)^4_9$, $(10, 14, 15)^4_9$, $(11, 15, 16)^4_9$, $(12, 16, 0)^4_9$, $(13, 0, 1)^4_9$, $(14, 1, 2)^4_9$, $(15, 2, 3)^4_9$, $(16, 3, 4)^4_9$.</td>
</tr>
<tr>
<td>$(0, 6, 3)^4_9$</td>
<td>$(1, 7, 4)^4_9$, $(2, 8, 5)^4_9$, $(3, 9, 6)^4_9$, $(4, 10, 7)^4_9$, $(5, 11, 8)^4_9$, $(6, 12, 9)^4_9$, $(7, 13, 10)^4_9$, $(8, 14, 11)^4_9$, $(9, 15, 12)^4_9$, $(10, 16, 13)^4_9$, $(11, 0, 14)^4_9$, $(12, 1, 15)^4_9$, $(13, 2, 16)^4_9$, $(14, 3, 0)^4_9$, $(15, 4, 1)^4_9$, $(16, 5, 3)^4_9$.</td>
</tr>
<tr>
<td>$(0, 8, 1)^4_9$</td>
<td>$(1, 9, 2)^4_9$, $(2, 10, 3)^4_9$, $(3, 11, 4)^4_9$, $(4, 12, 5)^4_9$, $(5, 13, 6)^4_9$, $(6, 14, 7)^4_9$, $(7, 15, 8)^4_9$, $(8, 16, 9)^4_9$, $(9, 0, 10)^4_9$, $(10, 1, 11)^4_9$, $(11, 2, 12)^4_9$, $(12, 3, 13)^4_9$, $(13, 4, 14)^4_9$, $(14, 5, 15)^4_9$, $(15, 6, 16)^4_9$, $(16, 7, 0)^4_9$.</td>
</tr>
</tbody>
</table>

Table 5: A $T^4_9$-Triple System of Order 17
We observe that each arc and edge difference is used exactly once. In Table 5, we get a total of 68 blocks, each of which contains two edges and four arcs. If we multiply 68 by 4 we get 272 arcs and 68 by 2 gives us 136 edges. Thus we get all the arc and edge differences.

For a generalization we let $v = 4k + 1$. The associated arc and edge differences are $\{1, 2, 3, \ldots, 4k - 1\}$ and $\{1, 2, 3, \ldots, (4k - 1)/2\}$ respectively.

Next we consider the base blocks $(0, 2k - 1)^4_g$, $(0, 4, 2k - 3)^4_g$, $(0, 6, 2k - 5)^4_g$, . . . , $(0, 2k - 4, 5)^4_g$, $(0, 2k - 2, 3)^4_g$, $(0, 2k, 1)^4_g$ given in general form by $(0, 2i, 2k - 2i + 1)^4_g$, for $i = 1 \ldots k$. The associated arc and edge differences are given in Table 6.

<table>
<thead>
<tr>
<th>Base Block</th>
<th>Arc Differences</th>
<th>Edge Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 2k - 1)^4_g$</td>
<td>$2k - 1, 2k + 2$</td>
<td>$4k - 1$</td>
</tr>
<tr>
<td>$(0, 4, 2k - 3)^4_g$</td>
<td>$4k - 3, 2k + 4$</td>
<td>$4k - 3$</td>
</tr>
<tr>
<td>$(0, 6, 2k - 5)^4_g$</td>
<td>$6k - 5, 4k - 5$</td>
<td>$4k - 5$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$(0, 2k - 4, 5)^4_g$</td>
<td>$2k - 4, 5, 4k - 4$</td>
<td>$2k + 5$</td>
</tr>
<tr>
<td>$(0, 2k - 2, 3)^4_g$</td>
<td>$2k - 2, 3, 4k - 2$</td>
<td>$2k + 3$</td>
</tr>
<tr>
<td>$(0, 2k, 1)^4_g$</td>
<td>$2k, 1, 4k, 2k + 1$</td>
<td>$2k - 1, 2k$</td>
</tr>
</tbody>
</table>

Table 6: Base Blocks and Differences for $v = 4k + 1$

We note all the arc and edge differences are included exactly once. These base blocks are then permuted as follows $\{(j, 2i + j, 2k - 2i + 1 - j)^4_g | i = 1 \ldots k, j = 0, \ldots, 4k\}$. This gives a $T^4_g$-triple system of order $4k + 1$. The same technique is used for the case $\lambda > 1$ except that here the edge and arc differences are taken $\lambda$ times.

4.2 Wheels and Graceful Labelings
A graceful labeling on a graph $G$ with $q$ edges is an injective mapping $f$ from $V(G)$ to $\{0, ..., q\}$ such that the edge labels defined by $f'(uv) = |f(u) - f(v)|$ satisfy $f'(E) = \{1, ..., q\}$ [6, 13]. Graceful labelings were introduced by Rosa under the name $\beta$-valuations [13]. These were popularized by Golomb under the name graceful labelings [6]. The interested reader is referred to [4] for a list of graceful and related results. All wheels are graceful (see [9]). For an illustration, consider the wheel $W_4 = C_4 + K_1$. Construct a labeling $f$ on $W_4$ as follows: $f$ assigns the vertex of $K_1$ the number 0 and $f$ assigns the vertices of the cycle $C_4$ consecutively the numbers, 8,1,5,2. In order to show that the numbering $f$ is graceful, we verify that for every number $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ there is exactly one edge number $i$. With these labels we have the following:

\[
    f'([0, 1]) = 1, \quad f'([0, 2]) = 2, \quad f'([0, 5]) = 5, \quad f'([0, 8]) = 8.
\]

\[
    f'([2, 5]) = 3, \quad f'([5, 1]) = 4, \quad f'([2, 8]) = 6, \quad f'([1, 8]) = 7.
\]

Observe that $f'(E) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. We thus see that $W_4$ is graceful. We now use the notion of graceful labelings to illustrate the existence of a $T_4^4$-triple system.

Denote $T_4^4$ by the ordered triple $(a, b, c)^4_4$, with arcs $(a, b), (b, c), (a, c)$ and $(c, a)$, and edges $[a, b], [c, a]$. Next, let’s denote by $W_n^M$ the mixed wheel with a cycle of length $n$ and center $c$.

\textbf{Lemma 4.2} There is a $T_4^4$-decomposition of the complete mixed wheel $W_n^M$ for $n \geq 2$.

\textbf{Proof.} Consider the collection $\{(i, c, 1 + i)^4_4 : i \in \{0, 1, \ldots n - 1\}\}$, where $c$ is fixed. This collection gives the desired decomposition. \hfill \blacksquare

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Example 4.3 A $T^4_4$-decomposition of $W_3^M$ is given by \{$(0, c, 1)^4_4$, $(1, c, 2)^4_4$, $(2, c, 0)^4_4$\}.

Let $v = 17$, then we observe that $v \equiv 1 \pmod{4}$. Since $W_3$ is a graceful graph of size 8, it will decompose the complete graph on 17 vertices $K_{17}$ [13]. Thus, $W_3^M$ will decompose $M_{17}$. It follows that there is a $T^4_4$-decomposition of $M_{17}$. This notion is extended in a similar manner to show the existence of a $T^4_4$ -- triple system and a $T^4_5$ -- triple system of order $v$. 
In this thesis, we studied the decomposition of the complete mixed graph $M_v$ into every possible graph on three vertices with twice as many arcs as edges. In most of the cases we gave direct constructions for our decompositions except for the cases $T_4^1$ and $T_5^3$ where we used the notion of labeling to show the existence of such a decomposition. We also studied the decomposition of the $\lambda$-fold complete mixed graph $\lambda M_v$. Here we gave constructions to show the existence of a $T_j^i$-triple system of order $v$. We showed that there exists a $T_j^i$-decomposition of $\lambda M_v$ for all $T_j^i$ except for the case $T_4^1$ when $\lambda = 2$ and $v$ odd where we got a partial results. We are also left with the cases $T_4^1$ and $T_5^3$ when $\lambda = 1$, $v = 12$ and $\lambda = 2$, $v = 6$ to consider.

We must remark here that for every odd value of $v$ we have considered so far, there exists a $T_4^1$-decomposition of $2M_v$. We will not end this work without noting that some of the proofs presented in this work, are more elegant versions for the proofs of some of the results in [8] for the corresponding underlying digraphs.
BIBLIOGRAPHY


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