The Last of the Mixed Triple Systems

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by

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ABSTRACT

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In this thesis, we consider the decomposition of the complete mixed graph on v vertices denoted M_v, into every possible mixed graph on three vertices which has (like M_v) twice as many arcs as edges. Direct constructions are given in most cases. Decompositions of the λ-fold complete mixed graph λM_v, are also studied.

DEDICATION

I would love to dedicate this piece of work to my entire family the Jums who are always there when I need them. To my cousin's family the Jams for their moral and financial support and for making it possible for me to attain this level of education. Finally, to my beloved fiancé Delphine Sayani for standing by me all this while.

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Finally, my greatest appreciation goes to my thesis advisor and graduate coordinator, Professor Robert Gardner. He has been very available to see this work through. Worthy of note is the fact that Dr. Beeler and Dr. Gardner made available most of the articles used during this research.

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1 INTRODUCTION AND BASIC DEFINITIONS

Combinatorial design theory is an interesting area of study in combinatorial mathematics. The approach of modeling objects as a set of points (or vertices) and the relation between them as arcs and/or edges comes in handy when studying triple systems in general and mixed triple systems in particular. Graphs provide a visible link between theory and applications that makes them ideal for design theory. For a better understanding of this thesis, we start by giving a comprehensive list of definitions.

A *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates each edge with two vertices (not necessarily distinct) called its end points [14]. A directed graph (digraph) D is simply a graph where the edges have been assigned direction.

A *complete graph* on v vertices, K_v, is a graph on v vertices where every vertex is adjacent to every other vertex in the graph. The *complete digraph* on v vertices, D_v, is formed by replacing each edge in K_v with two arcs of opposite orientation. If $a = (x, y)$ is an arc in the digraph D, then a is said to *join* x to y and a is *incident from* x and incident to y, while x is *incident to* a and y is incident from a. We say that x and y are *adjacent* vertices.

A *mixed graph* on v vertices is an ordered pair (V, C) where V is a set of vertices, of order v and C is a set of ordered and unordered pairs denoted (x, y) and $[x, y]$, respectively, of elements of V. An ordered pair $(x, y) \in C$ is called an *arc* of (V, C) and an unordered pair $[x, y] \in C$ is called an *edge* of (V, C).

The *complete mixed graph* on v vertices, denoted M_v, is the mixed graph (V, C)

where for every pair of distinct vertices u, v, we have $\{(u,v),(v,u),[u,v]\} \subset C$. Figure 1 is an example of the complete mixed graph on three vertices (M_3). The *converse* of a mixed graph (V, C), is the mixed graph (V', C') where $C' = \{(u,v),(v,u) \in C\} \cup \{[u,v] \in C\}$.

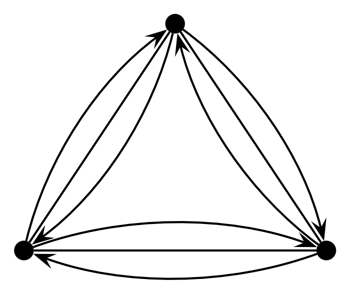

Figure 1: The Complete Mixed Graph on Three Vertices (M_3)

In mixed graphs, the concept of the *out degree*, $od(u)$, of vertex u in M_v refers to the number of vertices of M_v that are adjacent from u. That is, $od(u) = |N_o(u)|$ where the open neighborhood $N_o(u) = \{x \in V(M_v) \mid x \text{ is adjacent from } u\}$. The *in-degree*, $id(u)$, of vertex u in M_v refers to the number of vertices of M_v that are adjacent to u. That is, $id(u) = |N_i(u)|$ where $N_i(u) = \{x \in V(M_v) \mid x \text{ is adjacent to } u\}$. The *degree* of a vertex, $d(u)$, of u is the number of edges incident to u. By the total degree $t(u)$ of vertex u we shall mean the sum: $t(u) = od(u) + id(u) + d(u)$. It thus follows that in M_v the sum of its arcs and edges is always congruent to zero modulo three.

Let K_v (D_v) denote the complete graph (digraph) on v vertices. If G is a graph (digraph) then a *G-decomposition* of K_v (D_v) is a collection $\{G_1, G_2, \ldots G_n\}$ of edge (arc) disjoint subgraphs of K_v (D_v) each of which is isomorphic to G and such that $\bigcup_{i=1}^{n} E(G_i) = E(K_v)$ ($\bigcup_{i=1}^{n} A(G_i) = A(D_v)$), where $E(G)$ ($A(G)$) is the edge (arc) set of G and the $G_i's$ are called the *blocks* of the decomposition.

The decomposition of a graph (digraph) into copies of a graph (digraph) on three vertices is called a *triple system.* A K_3-decomposition of the complete graph on v vertices, K_v is called a *Stiener Triple System* of order v, $STS(v)$. It is well known that $STS(v$ exist if and only if $v \equiv 1$ or 3 (mod 6).

Example: A K_7 decomposition into 3-cycles ($STS(7)$). The following blocks give an $STS(7)$ of the graph in Figure 2: $(0, 1, 3)$, $(1, 2, 4)$, $(2, 3, 5)$, $(3, 4, 6)$, $(4, 5, 0)$, $(5, 6, 1)$, $(6, 0, 2)$.

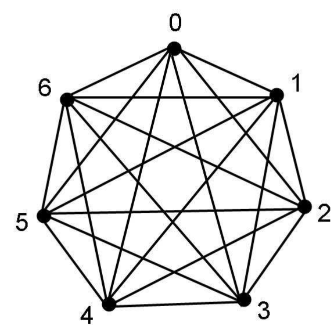

Figure 2: A Complete Graph on Seven Vertices

Generally, whenever a complete graph, digraph, or mixed graph is decomposed into a graph (respectively digraph or mixed graph) on three vertices, the resulting systems is called a *triple system* and the resulting triples are called *blocks.* The complete graph on three vertices K_3 has two orientations namely the 3-circuit and the transitive triple (see Figure 3).

A decomposition D_v into isomorphic copies of the 3-circuit is equivalent to a *Mendelsohn triple system* of order v, denoted $MTS(v)$, and is known to exists if and

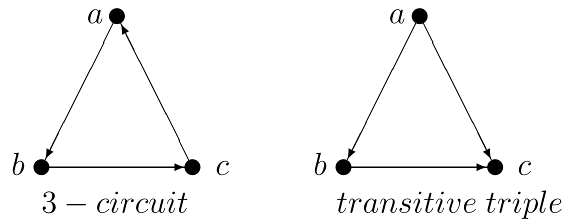

Figure 3: Orientation of a K_3

only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [11]. A decomposition of D_v into isomorphic copies of a *transitive triple*, is equivalent to a *directed triple system,* and exists if and only if $v \equiv 0$ or 1 (mod 3) [10].

The *wheel*, denoted W_n, is the graph obtained from the n-cycle by appending a central universal vertex. We will denote the wheel W_n with center c and cycle $0, a, 2a, ..., (n-1)a$ by $W_n(c : a)$. Note that $\mid V((W_n)) \mid= n+1$ and $\mid E((W_n)) \mid= 2n$. This can be extended to a mixed graph by replacing each edge with an edge, a forward arc, and a backwards arc. We will denote the mixed wheel by with center at c and cycle $0, a, 2a, ..., (n-1)a$ by $W_n^M(c : a)$.

The *circulant*, denoted $C_n(S)$, has vertex set $V(C_n(S)) = \mathbb{Z}_n$ where $\mathbb{Z}_n$ denotes the set of integers modulo n. Two vertices u and v are adjacent if and only if $|u-v|_n \in S$, where $|u - v|_n = min\{(u - v) \pmod n, (v - u) \pmod n\}$. The *mixed circulant* will have an edge, a forward arc, and a backward arc for pair of vertices. edges. Similarly, the complete graph with v vertices and hole of order k, denoted $K(v, k)$, is obtained from K_v by deleting an edge induced subgraph isomorphic to K_k.

Gardner [5] studied the *mixed triple systems* which are the decompositions of the complete mixed graph into each of the partial orientation of a K_3 (see Figure 4).

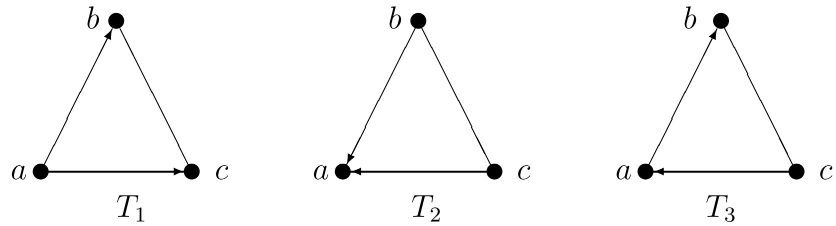

Figure 4: Partial Orientation of a K_3

He showed that a T_1-triple system and T_2-triple system of order v exists if and only if $v \equiv 1 \pmod 2$. He also showed that a T_3-triple system of order v exists if and only if $v \equiv 1 \pmod 2$, $v \notin \{3, 5\}$.

Hartman and Mendelsohn [8] gave necessary and sufficient conditions for the existence of a G-decomposition of λD_v (the complete directed graph on v vertices with each arc taken λ times) for all simple connected digraphs G having three vertices. Their work motivated the research in this thesis where instead of considering the complete digraph, we consider the complete mixed graph on v vertices M_v (with each arc taken λ times) and decompose such a graph into all possible mixed graphs on three vertices with twice as many arcs as edges. There are eighteen such graphs and these are illustrated in Figure 5.

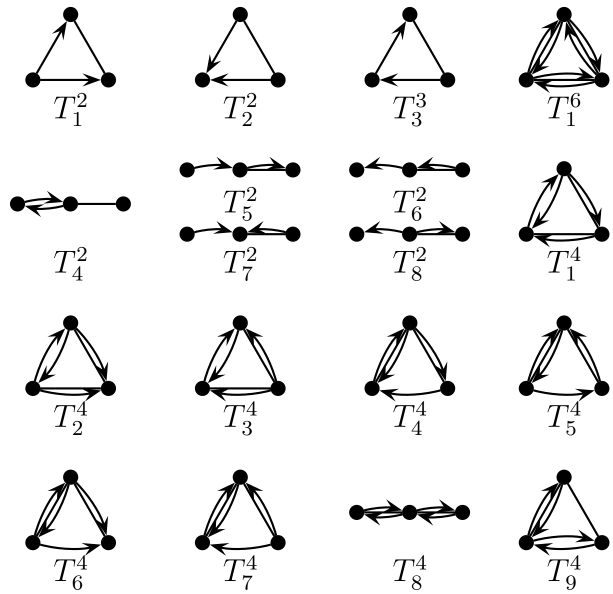

Figure 5: Mixed Triple Systems

We let T_j^i denote the mixed graph on three vertices with twice as many arcs as edges, where i represents the number of arcs, and j a counter. We denote the vertices of T_j^i by the ordered triple $(a, b, c)_j^i$ with arc and edge sets as given in Table 1.

Triple	Ordered Triple	Arc Set	Edge Set
T_1^2	$(a,b,c)_1^2$	$\{(a,b),(a,c)\}$	$\{[b,c]\}$
T_2^2	$(a,b,c)_2^2$	$\{(b,a),(c,a)\}$	$\{[b,c]\}$
T_3^2	$(a,b,c)_3^2$	$\{(a,b),(c,a)\}$	$\{[b,c]\}$
T_4^2	$(a,b,c)_4^2$	$\{(a,b),(b,a)\}$	$\{[c,a]\}$
T_5^2	$(a,b,c)_5^2$	$\{(a,b),(b,c)\}$	$\{[a,c]\}$
T_6^2	$(a,b,c)_6^2$	$\{(b,a),(c,b)\}$	$\{[a,c]\}$
T_7^2	$(a,b,c)_7^2$	$\{(a,b),(c,b)\}$	$\{[a,c]\}$
T_8^2	$(a,b,c)_8^2$	$\{(b,a),(b,c)\}$	$\{[a,c]\}$
T_1^4	$(a,b,c)_1^4$	$\{(a,b),(b,c),(c,a),(a,c)\}$	$\{[a,b],[b,c]\}$
T_2^4	$(a,b,c)_2^4$	$\{(a,b),(c,b),(c,a),(a,c)\}$	$\{[a,b],[b,c]\}$
T_3^4	$(a,b,c)_3^4$	$\{(b,a),(b,c),(c,a),(a,c)\}$	$\{[a,b],[b,c]\}$
T_4^4	$(a,b,c)_4^4$	$\{(a,b),(b,c),(c,a),(c,a)\}$	$\{[a,b],[a,c]\}$
T_5^4	$(a,b,c)_5^4$	$\{(b,a),(c,b),(c,a),(a,c)\}$	$\{[a,b],[a,c]\}$
T_6^4	$(a,b,c)_6^4$	$\{(a,b),(b,c),(c,a),(b,a)\}$	$\{[a,b],[a,c]\}$
T_7^4	$(a,b,c)_7^4$	$\{(b,c),(b,c),(c,a),(a,c)\}$	$\{[a,b],[a,c]\}$
T_8^4	$(a,b,c)_8^4$	$\{(a,b),(b,a),(b,c),(c,b)\}$	$\{[a,b],[b,c]\}$
T_9^4	$(a,b,c)_9^4$	$\{(a,b),(b,a),(c,a),(a,c)\}$	$\{[a,b],[b,c]\}$
T_1^6	$(a,b,c)_1^6$	$\{(a,b),(b,a),(b,c),(c,b),$ $(a,c),(c,a)\}$	$\{[a,b],[b,c],[a,c]\}$

Table 1: Arc and Edge Sets for the Ordered Triples $(a,b,c)_j^i$

This thesis is divided into five chapters. Following this introductory chapter is the second chapter in which we consider the decomposition of the complete mixed graph (M_v). Here we present the results for the decomposition of the complete mixed graph on M_v. The third chapter constitutes the decomposition of the λ-fold complete mixed graph where λ is greater than one. In this chapter we study the decomposition of the λ-fold complete mixed graph λM_v. Necessary and sufficient conditions are given for the existence of a T_j^i-triple systems of order v. A verification of the the methods used is the principal content of the fourth chapter. Here we demonstrate, with the aid of examples, how the difference method and graph labelings are used in our decompositions. Finally in the last chapter, we present our conclusions.

2 DECOMPOSITIONS OF THE COMPLETE MIXED GRAPH

2.1 Introduction

In this chapter, we present the results for the decomposition of the complete mixed graph on v vertices, denoted M_v, into every possible mixed graph on three vertices with twice as many arcs as edges. Decompositions of M_v are defined similarly to decompositions of K_v and D_v. A T_j^i-decomposition of M_v is called a T_i^j-*triple systems*. Here we give direct constructions of the T_j^i-*triple system* of order v when they exist using difference method techniques, except for the cases T_4^4 and T_5^4 where we use the notion of wheels and graph labelings to show the existence of such a decomposition.

Before giving our results, we must remark that the following theorems their proofs is thanks to the herculean effort of Dr. Robert Beeler. They are Lemma 2.6, Theorem 2.7, Theorem 2.8 and Theorem 3.10.

2.2 Results

Theorem 2.1 *[5] A T_1^2-decomposition , a T_2^2-decomposition and a T_3^2-decomposition of M_v exists if and only if $v \equiv 1 \pmod 2$, except when $v = 3$ or $v = 5$ in the case of T_3^2.*

Theorem 2.2 *A T_4^2-decomposition, of M_v exists for all $v \geq 3$.*

Proof. We consider two cases as outlined below.

Case 1. Suppose $v \equiv 1 \pmod 2$, for all $v \geq 3$ say $v = 2k + 1$ and let the vertex set of M_v be $\{0, 1, \cdots, v - 1\}$. Consider the set:

$$\{(3+2i+j, 1+i+j, j)_4^2 | i = 0, 1, 2, \cdots k-2, j = 0, 1, \cdots, 2k\}$$

$$\cup\{(k+1+j, k+j, j)_4^2 | j = 0, 1, \cdots 2k\}$$

where the vertex labels are reduced mudolo v.

Case 2. Suppose $v \equiv 0 \pmod 2$, for all $v \geq 4$ say $v = 2k$ and let the vertex set of M_v be $\{\infty, 0, 1, \cdots, v-2\}$. Take the decomposition of M_{v-1}, where the vertex set of M_{v-1} is $\{0, 1, \cdots, v-2\}$, and union it with the set $\{(1+j, \infty, j)_4^2 | j = 0, 1, \cdots 2k-1\}$
In both cases, the resulting set gives the desired decomposition of T_4^2. ∎

Theorem 2.3 *A T_5^2-decomposition and a T_6^2-decomposition, of M_v exists* $\forall v \geq 3$.

Proof. In order to establish this proof, we consider two cases, v odd and v even.
Case 1. Suppose $v \equiv 1 \pmod 2$, for all $v \geq 3$ say $v = 2k+1$ and let the vertex set of M_v be $\{0, 1, \cdots, v-1\}$. Consider the set:

$$\{(2k+j, 1+i+j, j)_5^2 | i = 0, 1, 2, \cdots k-2, j = 0, 1, \cdots, 2k\}$$

$$\cup\{(k-1+j, k+j, j)_5^2 | j = 0, 1, \cdots 2k\},$$

where the vertex set is reduced mudolo v.

Case 2. Suppose $v \equiv 0 \pmod 2$, for all $v \geq 4$ say $v = 2k$ and let the vertex set of M_v be $\{\infty, 0, 1, \cdots, v-2\}$. Take the decomposition of M_{v-1}, where the vertex set of M_{v-1} is $\{0, 1, \cdots, v-2\}$, and union it with the set $\{(1+j, \infty, j)_5^2 | j = 0, 1, \cdots 2k-1\}$
In both cases, the resulting set gives the desired decomposition of T_5^2.
Since T_6^2 is the converse of T_5^2 the existence of a $T_5^2 - decomposition$ implies the existence of a $T_6^2 - decomposition$ of M_v. ∎

Theorem 2.4 *A T_7^2-decomposition and a T_8^2-decomposition of M_v exists if and only if $v \equiv 1 \pmod 2$*

Proof. For the necessary part of the theorem we observe that if $v \equiv 0 \pmod 2$, then a T_7^2-decomposition of M_v would imply a (using the notation of [8]) T_2-decomposition of the complete digraph D_v. However, such a decomposition of D_v does not exist [8]. Thus v must be odd.

Suppose $v \equiv 1 \pmod 2$ say $v = 2k+1$. Consider the set:

$$\{(2+2i+j, 1+i+j)_7^2 \mid i = 0, 1, 2, \ldots, k-1, j = 0, 1, 2, \ldots, v-1\},$$

where the vertex labels are reduced modulo v. This set is the desired T_7^2-decomposition. Given that T_8^2 is the converse of T_7^2, the existence of a T_7^2 decomposition of M_v implies the existence of a T_8^2 decomposition of M_v. ∎

Theorem 2.5 *Neither T_1^4-decompositions, T_2^4-decompositions, nor T_3^4-decompositions of M_v exist.*

Proof. Suppose one of these decompositions exists. Then there is a block B of the decomposition which contains an edge of the form $[a, b]$ and an arc of the form (a, b). However, it is impossible for the arc (b, a) to be contained in any other block of the decomposition since this would require duplicating either edge $[a, b]$ or arc (a, b). Therefore no such decomposition exists. ∎

Lemma 2.6 *There exists a cyclic W_p-decomposition of $C_n(1, ..., 2p)$.*

Proof. Note that the wheel, W_p, has a graceful labeling [3, 9]. A graceful labeling on a graph of size q will induce a cyclic decomposition of the circulant $C_n(1, ..., q)$, when $n \geq 2q + 1$ [1]. Hence there exists a cyclic W_p-decomposition of $C_n(1, ..., 2p)$. ∎

Theorem 2.7 *Let $k \in \mathbb{N}$, $p \in \mathbb{N} - \{1, 2\}$, and $\mathcal{K} = \{W_j : j \geq 3\}$. There exists a $\mathcal{K}$-decomposition of $K(4p + 3k + 1, k)$.*

Proof. Let $C = \{c_0, ..., c_{k-1}\}$ be the set of mutually non-adjacent vertices in $K(4p + 3k + 1, k)$. Similarly, the set of mutually adjacent vertices in $K(4p + 3k + 1, k)$ can be represented by the elements of $\mathbb{Z}_{4p+2k+1}$. The edges between C and $\mathbb{Z}_{4p+2k+1}$ can be partitioned by the set of wheels $W_{n(2p+k-i)}(c_i : 2p + k - i)$ where $n(2p + k - i)$ denotes the length of the orbit of $2p + k - i$ in $\mathbb{Z}_{4p+2k+1}$. We note that $n(2p + k - i) = \frac{4p+2k+1}{gcd(2p+k-i, 4p+2k+1)} \geq 3$ as $2p + k - i < 4p + 2k + 1$ and $4p + 2k + 1$ is odd. Hence each of these wheels is in the set $\mathcal{K}$. This leaves a set of edges isomorphic to $C_{4p+2k+1}(1, 2, ..., 2p)$. There exists a cyclic W_p-decomposition of $C_{4p+2k+1}(1, 2, ..., 2p)$ by Theorem 2.6. Hence there exists a $\mathcal{K}$-decomposition of $K(4p + 3k + 1, k)$. ■

Theorem 2.8 *A T_4^4-decomposition and T_5^4-decomposition of M_v exists if and only if $v \equiv 0 \pmod 4$ or $v \equiv 1 \pmod 4), v \notin \{5, 8, 9, 12\}$.*

Proof. Since the directed part of T_4^4 must decompose D_v, it follows from [8] that $v \equiv 0, 1 \pmod 4$ and $v \notin \{5, 8\}$ are necessary conditions. Further, [8] gives all decompositions of D_9 into the directed part of T_4^4. However, it is easy to verify that none of these extend to a T_4^4-decomposition of M_9. It suffices to construct the remaining cases. Note that the set of blocks $(i, c, i + 1)_4^4$ for $i = 0, ..., n - 1$ will give a T_4^4-decomposition of W_n^M centered at c.

If $v \equiv 1 \pmod 4$, say $v = 4p + 1$ for $p \geq 3$. Since W_p is a graceful graph[3, 9] of size $2p$, it will decompose K_{4p+1} [13]. It follows that there exists a T_4^4-decomposition of M_v for $v \equiv 1 \pmod 4$ and $v \geq 13$.

If $v \equiv 0 \pmod 4$, say $v = 4p+4$ for $p \geq 3$. Let the vertex set of M_v be given by $\{\infty, 0, ..., 4p+2\}$. Note that $K_v = W(\infty : 2p+1) \cup C_{4p+3}(1, ..., 2p)$. Since W_p is a graceful graph of size $2p$, it follows from [1] that it will decompose $C_{4p+3}(1, ..., 2p)$. It follows that there exists a T_4^4-decomposition of M_v for $v \equiv 0 \pmod 4$ and $v \geq 16$. It remains to show that there exists a T_4^4-decomposition of M_{12}. Since T_5^4 is the converse of T_4^4, it follows that the necessary and sufficient conditions for the existence of a T_5^4-decomposition of M_v are the same. ■

Theorem 2.8 implies the decomposition of D_v into copies of the underling digraph of T_4^4 (and also of the underlying digraph of T_5^4), both of which are results of Hartman and Mendelsohn [8]. However, the proof of the result of Hartman and Mendelsohn is based on a complicated inductive proof, whereas the proof of Lemma 2.6 is somewhat more direct.

Theorem 2.9 *A T_6^4-decomposition and a T_7^4-decomposition of M_v exists if and only if $v \equiv 1 \pmod 4$.*

Proof. Since M_v has $v(v-1)$ arcs and T_6^4 has four arcs, it follows that $4 \mid v(v-1)$. Also, for each vertex u in T_6^4, $od(u) = 2$ it thus follows that $2 \mid (v-1)$. Hence v must be odd. Therefore $v \equiv 1 \pmod 4$ and the necessary condition follows. Sufficiency is established, by considering the following cases:

If $v = 5$, then consider the set:

$$\{(j, 1+j, 2+j)_6^4 \mid j = 0, 1, 2, 3, 4\}.$$

If $v = 9$, then consider the set:

$\{(0,2,1)_6^4, (2,1,5)_6^4, (1,5,8)_6^4, (5,8,0)_6^4, (8,0,4)_6^4, (0,4,7)_6^4, (4,7,1)_6^4, (7,1,3)_6^4, (1,3,6)_6^4,$

$(3,6,0)_6^4, (6,0,2)_6^4, (2,3,4)_6^4, (3,4,5)_6^4, (4,5,6)_6^4, (5,6,7)_6^4, (6,7,8)_6^4, (7,8,2)_6^4, (8,2,3)_6^4\}$.

If $v = 13$, then consider the set:

$$\{(j, 1+j, 2+j)_6^4, (j, 3+j, 6+j)_6^4, (j, 4+j, 8+j)_6^4 \mid j = 0, 1, 2, \ldots, 12\}.$$

If $v = 17$, then consider the set:

$$\{(j, 1+j, 2+j)_6^4, (j, 3+j, 6+j)_6^4, (j, 4+j, 8+j)_6^4, (j, 5+j, 10+j)_6^4 \mid j = 0, 1, 2, \ldots, 16\}.$$

If $v = 29$, then consider the set:

$$\{(j, 1+j, 2+j)_6^4, (j, 4+j, 8+j)_6^4, (j, 5+j, 10+j)_6^4, (j, 6+j, 12+j)_6^4, (j, 7+j, 14+j)_6^4,$$

$$(j, 9+j, 18+j)_6^4, (j, 13+j, 26+j)_6^4 \mid j = 0, 1, 2, \ldots, 28\}.$$

If $v = 33$, then consider the set:

$$\{(j, 2+j, 4+j)_6^4, (j, 3+j, 6+j)_6^4, (j, 10+j, 20+j)_6^4, (j, 12+j, 24+j)_6^4, (j, 14+j, 28+j)_6^4,$$

$$(j, 1+j, 8+j)_6^4, (j, 7+j, 22+j)_6^4, (j, 15+j, 16+j)_6^4 \mid j = 0, 1, 2, \ldots, 32\}.$$

If $v = 49$, then consider the set:

$$\{(j, 2+j, 4+j)_6^4, (j, 6+j, 12+j)_6^4, (j, 7+j, 14+j)_6^4, (j, 8+j, 16+j)_6^4, (j, 17+j, 34+j)_6^4,$$

$$(j, 18+j, 36+j)_6^4, (j, 22+j, 44+j)_6^4, (j, 23+j, 46+j)_6^4, (j, 24+j, 48+j)_6^4, (j, 10+j, 21+j)_6^4,$$

$$(j, 9+j, 19+j)_6^4, (j, 11+j, 20+j)_6^4 \mid j = 0, 1, 2, \ldots, 48\}.$$

If $v = 57$, then consider the set,

$$\{(j, 1+j, 2+j)_6^4, (j, 4+j, 8+j)_6^4, (j, 5+j, 10+j)_6^4, (j, 6+j, 12+j)_6^4, (j, 7+j, 14+j)_6^4,$$

$(j,16{+}j,32{+}j)_6^4,(j,17{+}j,34{+}j)_6^4,(j,21{+}j,42{+}j)_6^4,(j,22{+}j,44{+}j)_6^4,(j,24{+}j,48{+}j)_6^4,$

$(j,27{+}j,54{+}j)_6^4,(j,18{+}j,38{+}j)_6^4,(j,11{+}j,29{+}j)_6^4,(j,20{+}j,31{+}j)_6^4 \mid j=0,1,2,\ldots,56\}.$

If $v=93$, then consider the set:

$\{(j,6{+}j,12{+}j)_6^4,(j,8{+}j,16{+}j)_6^4,(j,11{+}j,22{+}j)_6^4,(j,15{+}j,30{+}j)_6^4,(j,18{+}j,36{+}j)_6^4,$

$(j,20{+}j,40{+}j)_6^4,(j,21{+}j,42{+}j)_6^4,(j,23{+}j,46{+}j)_6^4,(j,24{+}j,48{+}j)_6^4,(j,26{+}j,52{+}j)_6^4,$

$(j,27{+}j,54{+}j)_6^4,(j,28{+}j,56{+}j)_6^4,(j,29{+}j,58{+}j)_6^4,(j,34{+}j,68{+}j)_6^4,(j,38{+}j,76{+}j)_6^4,$

$(j,43{+}j,86{+}j)_6^4,(j,44{+}j,88{+}j)_6^4,(j,1{+}j,32{+}j)_6^4,(j,2{+}j,3{+}j)_6^4,(j,31{+}j,33{+}j)_6^4,$

$$(j,4+j,13+j)_6^4,(j,9+j,19+j)_6^4,(j,10+j,14+j)_6^4 \mid j=0,1,2,\ldots,92\}.$$

In each case, reduce the vertex labels modulo v, and the set gives the desired decomposition. If $v \equiv 1 \pmod 4$, then D_v (and hence M_v) can be decomposed into arc-disjoint copies of elements of $\{D_5,D_9,D_{13},D_{17},D_{29},D_{33},D_{49},D_{57},D_{93}\}$ (and hence arc-disjoint and edge-disjoint copies of the corresponding complete mixed graphs) (Theorem 2.1 of [8]; also see [12, 15]). Therefore, for $v \equiv 1 \pmod 4$, there exists a decomposition of M_v into arc-disjoint and edge-disjoint copies of elements of

$$\{M_5,M_9,M_{13},M_{17},M_{29},M_{33},M_{49},M_{93}\}$$

and, from the above constructions, it follows that there is a T_6^4-decomposition of M_v. ■

We note that the existence of a T_6^4 decomposition of M_v implies the existence of a T_7^4 decomposition of M_v, since they are converses of each other. The proof of Theorem 2.9 follows along the same basic lines as the proof by Hartman and Mendelsohn [8] for the result concerning the corresponding underlying digraph of T_6^4. However, our proof is a bit more direct in the sense that the specific blocks are given for decompositions when $v \in \{5, 9, 13, 17, 29, 33, 49, 57, 93\}$.

Theorem 2.10 *A T_8^4-decomposition of M_v exists if and only if $v \equiv 0 \pmod 4$ or $v \equiv 1 \pmod 4$.*

Proof. Such a decomposition is equivalent to a P_3-decomposition of K_v. It is well-known that such a decomposition exists if and only if $v \equiv 0 \pmod 4$ or $v \equiv 1 \pmod 4$ (see [2]).

Theorem 2.11 *A T_9^4-decomposition of M_v exists if and only if $v \equiv 0$ or $1 \pmod 4$.*

Proof. If $v \equiv 2$ or $3 \pmod 4$, then a T_9^4-decomposition of M_v would imply a (using the notation of [8]) P_3-decomposition of the complete digraph D_v. However, such a decomposition of D_v does not exist [8].

Case 1. Suppose $v \equiv 1 \pmod 4, \forall\, v \geq 5$ say $v = 4k+1$ and let the vertex set of M_v be $\{0, 1, \cdots, v-1\}$. Consider the set:

$\{(j, 2i+j, 2k-2i+j)_9^4 | i = 0, 1, 2, \ldots\ k, j = 0, 1, \cdots, v-1\}$ where the vertex labels are reduced modulo v.

Case 2. Suppose $v \equiv \pmod 4$ for all $v \geq 4$, say $v = 4k$. Associate the arc differences $2k-1$, $2k$, with the fixed point ∞ with the edge difference $2k-1$. Consider the set $\{(j, 2i+j, 2k-2i+j)_8^4 | i = 0, 1, 2, \cdots k-1, j = 0, 1, \cdots\} \cup \{(j, 2k-1+j, \infty)_8^4 | j = 0, 1, \cdots v\}$ where the vertex labels are reduced modulo $v-1$. In both cases, the given sets give the desired decomposition.

Theorem 2.12 *[2] A T_1^6-decomposition of M_v exists if and only if $v \equiv 0 \pmod 6$ or $v \equiv 1 \pmod 6$.*

The following table gives a summary of the results in Chapter 2.

Triple	**Values of v**	**Justification**
T_1^2	$v \equiv 1 \pmod 2$	Theorem 2.1
T_2^2	$v \equiv 1 \pmod 2$	Theorem 2.1
T_3^2	$v \equiv 1 \pmod 2$, $v \neq 3$, $v \neq 5$	Theorem 2.1
T_4^2	$v \geq 3$	Theorem 2.2
T_5^2	$v \geq 3$	Theorem 2.3
T_6^2	$v \geq 3$	Theorem 2.3
T_7^2	$v \equiv 1 \pmod 2$	Theorem 2.4
T_8^2	$v \equiv 1 \pmod 2$	Theorem 2.4
T_1^4	no v	Theorem 2.5
T_2^4	no v	Theorem 2.5
T_3^4	no v	Theorem 2.5
T_4^4	$v \equiv 0$ or $1 \pmod 4$, $v \notin \{5, 8, 9\}$	Theorem 2.8
T_5^4	$v \equiv 0$ or $1 \pmod 4$, $v \notin \{5, 8, 9\}$	Theorem 2.8
T_6^4	$v \equiv 1 \pmod 4$	Theorem 2.9
T_7^4	$v \equiv 1 \pmod 4$	Theorem 2.9
T_8^4	$v \equiv 0$ or $1 \pmod 4$	Theorem 2.10
T_9^4	$v \equiv 0$ or $1 \pmod 4$	Theorem 2.11
T_1^6	$v \equiv 0$ or $1 \pmod 6$	Theorem 2.12

Table 2: Summary of Results in Chapter 2

3 DECOMPOSITION OF THE λ-FOLD COMPLETE MIXED GRAPH

In this Chapter, we consider the case $\lambda > 1$. Here we study decompositions of the complete mixed graph, λM_v. We give T_i^j-*triple systems* whenever it is possible. The same notation as that in chapter 2 is adopted here.

Lemma 3.1 *A T_1^2-decomposition and a T_2^2-decomposition of $2M_v$ exists for all $v \equiv 0$ $(mod\ 2)$.*

Proof. For $v \equiv 0 \pmod 4$, let $v = 4k$ and suppose the vertex set of $2M_v$ is $\{0, 1, \ldots, v-1\}$. Consider the set:

$$\{(j, k+j, 3k+j)_1^2\} \mid j = 0, 1, 2, \ldots, v-1\} \cup \{(j, k-i+j, k+1+i+j)_1^2,$$

$$(j, 3k-1-i+j, 3k+i+j)_1^2 \mid i = 0, 1, 2, \ldots, k-1, j = 0 \ldots 4k-1\}$$

$$\cup\{(j, 3k-1-i+j, 3k+1+i+j)_1^2, (j, k-1-i+j, k+1+i+j)_1^2$$

$$\mid i = 0, 1, 2, \ldots, k-2, j = 0 \ldots 4k-1\},$$

where vertex labels are reduced modulo v. This set is the desired decomposition.

For $v \equiv 2 \pmod 4$, let $v = 4k+2$ and suppose the vertex set of $2M_v$ is $\{\infty, 0, 1, \ldots, v-2\}$. Consider the set:

$$\{(\infty, j, 2k+j)_1^2, (j, \infty, k+1+j)_1^2, (j, \infty, 3k+1+j)_1^2 \mid j = 0, 1, 2, \ldots, 4k\}$$

$$\cup\{(j, k-i+j, k+1+i+j)_1^2, (j, 3k-i+j, 3k+i+j)_1^2, (j, k-i+j, k+2+i+j)_1^2$$

$$\mid i = 0, 1, 2, \ldots, k-1, j = 0, 1, 2, \ldots, 4k\} \cup (j, 3k-i+j, 3k+2+i+k)_1^2$$

$$| \; i = 0, 1, 2, \ldots, k-2, j = 0, 1, 2, \ldots, 4k\}$$

where vertex labels are reduced modulo $v-1$. This set is the desired decomposition. ∎

Theorem 3.2 *A T_1^2-decomposition and a T_2^2-decomposition of λM_v exists if and only if* (1) $\lambda \equiv 0 \pmod 2$ *and* $v \geq 3$, *or* (2) $\lambda \equiv 1 \pmod 2$ *and* $v \equiv 1 \pmod 2$.

Proof. For $\lambda \equiv 1 \pmod 2$ and $v \equiv 1 \pmod 2$, a T_1^2-decomposition of M_v exists by Lemma 2.1, and hence a T_1^2-decomposition of λM_v exists. For $\lambda \equiv 0 \pmod 2$ and $v \equiv 0 \pmod 2$, a T_1^2-decomposition of $2M_v$ exists by Lemma 3.1, and hence a T_1^2-decomposition of λM_v exists.

If $\lambda \equiv 1 \pmod 2$, then the total degree of each vertex of M_v is $3\lambda(v-1)$. Since each vertex of T_1^2 is of total degree 2, thus a necessary condition for a T_1^2-decomposition of λM_v is that $v \equiv 1 \pmod 2$. When both v and λ are odd, a T_1^2-decomposition of M_v exists, and hence a T_1^2-decomposition of λM_v exists. ∎

Lemma 3.3 *A T_3^2-decomposition of $2M_v$ exists for all $v \equiv 0 \pmod 2$.*

Proof. For $v \equiv 0 \pmod 8$, let $v = 8k$ and suppose the vertex set of $2M_v$ is $\{0, 1, 2, \ldots, v-1\}$. Consider the multi-set:

$$\{(j, 1+i+j, 8k-1-i+j)_3^2 \mid i = 0, 1, 2, \ldots, 2k-1, j = 0, 1, 2, \ldots, 8k-1\}$$

$$\cup\{(2k+1+i+j, 4k+2+2i+j, j)_3^2 \mid i = 0, 1, 2, \ldots, 2k-2, j = 0, 1, 2, \ldots, 8k-1\}$$

$$\cup 2 \times \{(4k+2i+j, 1+4i+j, j)_3^2, (6k+2i+j, 4k+1+4i+j, j)_3^2 \mid i = 0, 1, 2, \ldots, k-1,$$

$$j = 0, 1, 2, \ldots, 8k-1\}$$

where the vertex labels are reduced modulo v. This multi-set is the desired decomposition.

For $v \equiv 2 \pmod 4$, let $v = 4k+2$ and suppose the vertex set of $2M_v$ is $\{\infty, 0, 1, 2, \ldots, v-2\}$. Consider the multi-set:

$$(\infty, j, 2+j)_3^2, (1+j, \infty, j)_3^2, (j, 1+j, \infty)_3^2) \mid j = 0, 1, 2, \ldots, 4k\}$$

$$\cup\{(2+i+j, 4+2i+j, j)_3^2 \mid i = 0, 1, 2, \ldots, k-2, j = 0, 1, 2, \ldots, 4k\}$$

$$\cup\{(k+1+i+j, 2k+2+2i+j, j)_3^2, (2k+1+i+j, 1+2i, j)_3^2, (3k+1+i+j, 2k+1+2i+j, j)_3^2 \mid$$

$$i = 0, 1, 2, \ldots, k-1, j = 0, 1, 2, \ldots, 4k\}$$

where the vertex labels are reduced modulo $v-1$. This multi-set is the desired decomposition.

For $v \equiv 4 \pmod 8$, let $v = 8k+4$ and suppose the vertex set of $2M_v$ is $\{0, 1, 2, \ldots, v-1\}$. Consider the multi-set:

$$\{(j, 1+i+j, 8k+3-i+j)_3^2 \mid i = 0, 1, 2, \ldots, 2k, j = 0, 1, 2, \ldots, 8k+3\}$$

$$\cup\{(2k+2+i+j, 4k+4+2i+j, j)_3^2 \mid i = 0, 1, 2, \ldots, 2k-1, j = 0, 1, 2, \ldots, 8k+3\}$$

$$\cup\{2 \times \{(4k+2+2i+j, 1+4i+j, j)_3^2 \mid i = 0, 1, 2, \ldots, k, j = 0, 1, 2, \ldots, 8k+3\}$$

$$\cup 2 \times \{(6k+4+2i+j, 4k+5+4i+j, j)_3^2 \mid$$

$$i = 0, 1, 2, \ldots, k-1, j = 0, 1, 2, \ldots, 8k+3\}$$

where the vertex labels are reduced modulo v. This multi-set is the desired decomposition. ∎

Theorem 3.4 *A T_3^2-decomposition of λM_v exists if and only if* (1) $\lambda \equiv 0 \pmod 2$ *and* $v \geq 3$, *or* (2) $\lambda \equiv 1 \pmod 2$ *and* $v \equiv 1 \pmod 2$.

Proof. The necessary conditions follow as in Theorem 3.2. For $\lambda \geq 1$ and $v \equiv 1 \pmod 2$, a T_3^2-decomposition of M_v exists by Lemma 2.1, and hence a T_3^2-decomposition of λM_v exists. For $\lambda \equiv 0 \pmod 2$ and $v \equiv 0 \pmod 2$, a T_3^2-decomposition of $2M_v$ exists by Lemma 3.3, and hence a T_3^2-decomposition of λM_v exists. ■

Theorem 3.5 *A T_4^2-decomposition, a T_5^2-decomposition, and a T_6^2-decomposition of λM_v exists $\forall v \geq 3$.*

Proof. Since such decompositions exist for $\lambda = 1$ by Theorem 2.3, the result follows trivially. ■

Theorem 3.6 *A T_7^2-decomposition and a T_8^2-decomposition of λM_v exists $\forall v \geq 3$.*

Proof. Suppose $v \equiv 0 \pmod 2$, say $v = 2k$, and $\lambda = 2$. Consider the set of blocks

$$\{(j, i+j, 2k-1+j)_7^2 \mid i = 1, 2, \ldots, 2k-2, j = 0 \ldots 2k-1\}$$

$$\cup\{(j, 2k-1+j, 2k-2+j)_7^2 \mid j = 0 \ldots 2k-1\}.$$

These set of blocks, form a T_7^2-decomposition of $2M_v$ and hence a T_7^2-decomposition of λM_v exists for v even. Suppose $\lambda \equiv 1 \pmod 2$. Then $v \equiv 1 \pmod 2$, and a T_7^2-decomposition of λM_v exists by Theorem 2.8. Since T_8^2 is the converse of T_7^2, the results follow. ■

Theorem 3.7 *A T_1^4-decomposition of λM_v exists for $\lambda \equiv 0 \pmod 4$ and for all v.*

Proof. Suppose $\lambda \equiv 0 \pmod 4$. First let $\lambda = 4$ and v be odd. That is $v = 2k+1$, for all $k \geq 1$. Consider the collection of blocks

$$\{(j, i+1+j, 2k-i+j)_1^4 \mid i = 0, 1, \ldots k-1, j = 0, 1, \ldots 2k\}$$

$$\cup\{(j, 2k-i+j, i+1+j)_1^4 \mid i = 0, 1, \ldots k-1, j = 0, 1, \ldots 2k\}$$

Next let $\lambda = 4$ and v be even. That is $v = 2k$, $k \geq 1$. Consider the collection of blocks

$$\{(j, k-1-i+j, k+i+j)_1^4 \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 2k-1\}$$

$$\cup\{(j, k+1+i+j, k-2-i+j)_1^4 \mid i = 0, 1, \ldots, k-3, j = 0, 1, \ldots, 2k-1\}$$

$$\cup\{(\infty, 0, k)_1^4, (0, \infty, k-1)_1^4, (0, 2k-2, \infty)_1^4\}$$

In both cases, the set of blocks gives a T_1^4-decompositions of $4M_v$, hence a T_1^4-decomposition of λM_v for all $\lambda \equiv 0 \pmod 4$ and $v \in \mathbf{N}$. ■

Conjecture 3.8 *A T_1^4-decomposition of λM_v exists if and only if $\lambda(v-1) \equiv 0$ (mod 4).*

We now present a partial proof for our conjecture. If the proof is completed we will get neccessary and sufficient conditions for the existence of a T_1^4-triple system.

Partial Proof. The arc and edge sets of T_1^4 are $A(T_1^4) = \{(a, b), (b, c), (a, c), (c, a)\}$ and $E(T_1^4) = \{[a, b], [b, c]\}$ respectively, as given in Table 1. Now if λ is odd, then we now show that there is no T_1^4-decomposition of λM_v. Suppose, to the contrary, there is a T_1^4-decomposition of λM_v. Then there is a collection of blocks $B_1, B_2, \ldots, B_\lambda$ where each B_i contains edge $[a, b]$. Since λ is odd, the blocks $B_1, B_2, \ldots, B_\lambda$ contains either an odd number of copies of arc (a, b) and an even number of copies of arc (b, a) or an even number of copies of arc (a, b) and an odd number of copies of arc (b, a). The remaining blocks in the decomposition each either contain both arcs (a, c) and (c, a) or contain neither arc (a, c) nor arc (c, a). But then in the total collection of blocks, there is either an even number of copies of arc (a, c) and an odd number of copies of arc (c, a) or an odd number of copies of arc (a, c) and an even number of copies of arcs (c, a). Thus a contradiction. Therefore, λ must be even.

In M_v, every vertex is in $\lambda(v-1)$ edges and $2\lambda(v-1)$ arcs. For any vertex u in T_1^4, we have that $t(u) = 4$. So it is necessary that $4 \mid 3\lambda(v-1)$. It thus follows that $\lambda(v-1) \equiv 0 \pmod 4$. This establishes the necessary condition of the conjecture. For the conjecture to be true, we must consider two cases.

Case 1. $\lambda \equiv 0 \pmod 4$. First let $\lambda = 4$ and v be odd. This is handled in Theorem 3.7.

Case 2. $\lambda \equiv 0 \pmod 2$. Since $\lambda(v-1) \equiv 0 \pmod 4$, it follows that v is odd. Now let $\lambda = 2$ and $v = 8k+1$, for $k \geq 1$. Consider the set of blocks

$$\{(j, 3k-i+j, 1+i+j)_1^4 \mid i = 0, 1, \ldots k-1, j = 0, 1, \ldots 8k\}$$

$$\{(j, 8k-i+j, 3k+1+i+j)_1^4 \mid i = 0, 1, \ldots k-1, j = 0, 1, \ldots 8k\}$$

$$\{(j, 2k-i+j, 5k+1+i+j)_1^4 \mid i = 0, 1, \ldots 2k-1, j = 0, 1, \ldots 8k\}$$

This set of blocks gives the desired decomposition. The remaining cases to consider are $v \equiv 3 \pmod 8$, $v \equiv 5 \pmod 8$ and $v \equiv 7 \pmod 8$, which we are still working on.

Theorem 3.9 *A T_2^4-decomposition and a T_3^4-decomposition of λM_v exists if and only if*

$$\lambda \equiv 2 \pmod 4, \textit{ and } v \equiv 1 \pmod 2$$

$$\lambda \equiv 0 \pmod 4, \forall v \geq 3.$$

Proof. The necessary condition follow as in Theorem 3.7. For sufficiency, we consider the following cases.

Case 1. $v \equiv 1 \pmod 2$, say $v = 2k+1$, and $\lambda = 2$. Consider the set of blocks

$$\{(2k-1-2i, 2k-i, 0)_2^4 \mid i = 0, 1, 2, \ldots, k-1\}.$$

This is a collection of base blocks for a cyclic T_2^4-decomposition of $2M_v$ and hence a T_2^4-decomposition of λM_v for $\lambda \equiv 2 \pmod 4$.

Case 2. $v \equiv 0 \pmod 2$, say $v = 2k$, and $\lambda = 4$. Consider the set of blocks

$$\{(2k-3-2i+j, 2k-2-i+j, j)_2^4 \mid i = 0, 1, 2, \ldots, k-3, j = 0 \ldots 2k-1\}$$

$$\cup\{(1+j, k+j, j)_2^4, (1, \infty, 0)_2^4, (1, k, \infty)_2^4, (0, k, \infty)_2^4, j = 0 \ldots 2k-1\}.$$

This set of blocks forms a T_2^4-decomposition of $4M_v$. Thus for $\lambda \equiv 0 \pmod 4$ and v even, we get the desired results. Since T_3^4 is the converse of T_2^4 it follows that the existence of a T_2^4-*triple system* implies the existence of a T_3^4-*triple system* ■

Theorem 3.10 *A T_4^4-decomposition and a T_5^4-decomposition of λM_v exists if and only if*

$$\lambda \equiv 1 \pmod 2, v \equiv 0, 1 \pmod 4$$

$$\lambda \equiv 0 \pmod 2, v \geq 4$$

$$\lambda \equiv 0 \pmod 4, v \geq 3$$

$$(\lambda, v) \neq (1, 5), (1, 8), (1, 9), (1, 12).$$

Proof. It suffices to give constructions for the cases not covered in Lemma 2.6.

When $v = 5$ and $\lambda = 2$, take the set of blocks $(i, i+1, i+2)_4^4$ and $(i+2, i, i+1)_4^4$ for $i = 0, 1, 2, 3, 4$.

For $v = 5$ and $\lambda = 3$, take the set of blocks $(i, i+1, i+3)_4^4$, $(i+2, i, i+1)_4^4$ and

$(i+2,i,i+1)_4^4$ for $i=0,1,2,3,4$.

When $v=7$ and $\lambda=2$, take the set of blocks $(i,\infty,i+3)_4^4$ and $(i+2,\infty,i+1)_4^4$, $(2j,2j+1,2j+2)_4^4$, $(2j,2j+1,2j+2)_4^4$ for $i=0,1,2,3,5$ and $j=0,1,2$. All computations are done modulo 6.

When $v=8$, note that $2M_8=2W_7^M(\infty:3)\cup\{(i,i+1,i+2)_4^4,(i+2,i,i+1)_4^4 : i=0,1,2,3,4,5,6\}$.

For $v=8$ and $\lambda=3$, note that $3M_8=3W_7\cup\{(i,i+1,i+2)_4^4,(i+2,i,i+1)_4^4 : i=0,1,2,3,4,5,6\}$.

When $v=9$ and $\lambda=2$, take the set of blocks $(i,i+2,i+4)_4^4$, $(i+1,i+3,i)_4^4$, $(i+3,i+2,i)_4^4$ and $(i,i+3,i+4)_4^4$ for $i=0,1,2,\ldots 8$.

For $v=9$ and $\lambda=3$, take the set of blocks $(i,i+2,i+4)_4^4$,$(i,i+2,i+4)_4^4$, $(i,i+4,i+1)_4^4$,$(i,i+4,i+1)_4^4$, $(i,i+2,i+3)_4^4$ and $(i+3,i+2,i)_4^4$ for $i=0,1,2,\ldots,8$.

For $v=10$ and $\lambda=2$, take the set of blocks $(i,\infty,i+3)_4^4$, $(i,\infty,i+4)_4^4$ and $W_3(i;i+1,i+2,i+4)$ for $i=0,1,\ldots 8$ and all computations are done modulo 9.

For $v=11$ and $\lambda=2$, use the wheels $W_5^M(i,i+2,i+6,i+3,i+5,i+1)$ for $i=0,\ldots,10$.

For $v=12$ and $\lambda=2$, take $2W+11(\infty:2)$ along with the set of blocks $(i,i+2,i+4)_4^4$, $(i+1,i,i+3)_4^4$, $(i+3,i,i+2)_4^4$ for $i=0,\ldots,10$.

For $v=12$ and $\lambda=3$, take $2W+11(\infty:2)$ along with the set of blocks $(i,i+2,i+4)_4^4$, $(i+4,i+2,i)_4^4$, $(i,i+4,i+1)_4^4$, $(i+1,i+4,i)_4^4$, $(i,i+2,i+3)_4^4$ and $(i+3,i+2,i)_4^4$ for $i=0,\ldots,10$.

When $v=14$ and $\lambda=2$ take the set of blocks $(i,c,i+2)_4^4$, $(i,c,i+4)_4^4$, and the wheel

$W_5^M(i : i+1, i+2, i+8, i+3, i+10)$ for $i = 0, ..., 12$.

When $v = 15$ and $\lambda = 2$, take the set of blocks $(i+5, i, i+11)_4^4$, $(i+11, i, i+5)_4^4$, $(i+5, i, i+12)_4^4$, $(i+12, i, i+5)_4^4$, $W_3^M(i : i+1, i+2, i+4)$ for $i = 0, ..., 14$.

For $v = 18$ and $\lambda = 2$, take the set of blocks $(i, c, i+5)_4^4$, $(i, c, i+7)_4^4$, $(i+3, i, i+12)_4^4$, $(i+12, i, i+3)_4^4$, $(i+4, i, i+10)_4^4$, $(i+10, i, i+4)_4^4$, $W_3^M(i : i+1, i+2, i+4)$ for $i = 0, ..., 16$.

For $v = 19$ and $\lambda = 2$, take the wheels $2W_{15}(\infty_1; 7)$, $2W_{15}(\infty_2; 6)$, $2W_{15}(\infty_3; 5)$, $2W_{15}(\infty_4; 4)$, and $W_{15}(\infty_4; 3)$ along with $2M_4$. The remaining edges are partitioned by $W_3(i; i+1, i+2, i+4)$.

When $v = 23$ and $\lambda = 2$, take the set of blocks $(i+3, i, i+22)_4^4$, $(i+22, i, i+3)_4^4$, $(i+8, i, i+18)_5^4$, $(i+18, i, i+8)_4^4$, $(i+7, i, i+21)_4^4$, $(i+21, i, i+7)_4^4$, $(i+6, i, i+17)_4^4$, $(i+17, i, i+6)_4^4$, $W_3^M(i : i+1, i+3, i+8)$, for $i = 0, ..., 22$.

For $v = 26$ and $\lambda = 2$, take the blocks $(i, \infty, i+12)_4^4$, $(i+12, \infty, i)_4^4$, $(i, \infty, i+11)_4^4$, and $(i+11, \infty, i)_4^4$.

The remaining edges are isomorphic to $2M(C_{25}(1, \ldots, 10))$. Since $C_{25}(1, \ldots, 10)$ has a W_5 decomposition, by above the result follows.

For $v = 27$ and $\lambda = 2$, take the set of blocks $(i+1, i, i+9)_4^4$, $(i+9, i, i+1)_4^4$, $(i+2, i, i+12)_4^4$, $(i+12, i, i+2)_4^4$, $(i+2, i, i+16)_4^4$, $(i+16, i, i+2)_4^4$, $(i+4, i, i+11)_4^4$, $(i+11, i, i+4)_4^4$, $(i+1, i, i+23)_4^4$, $(i+23, i, i+1)_4^4$, $W_3^M(i : i+3, i+6, i+12)$, for $i = 0, ..., 26$.

For $v = 30$ and $\lambda = 2$, take the blocks $(i, \infty, i+14)_4^4$, $(i, \infty, i+13)_4^4$, $(i+14, \infty, i)_4^4$, and $(i+13, \infty, i)_4^4$.

The remaining edges are isomorphic to $2M(C_{29}(1, \ldots, 12))$. Since $C_{29}(1, \ldots, 12)$ has a W_6 decomposition, by above the result follows.

For $v = 35$ and $\lambda = 2$, take the wheels $W_{25}(\infty_j : j = 1, \ldots, 12 - j)$ twice each for $j = 0, \ldots, 9$ and a copy of $2M_{10}$. The remaining edges are partitioned by the blocks $(i, i+1, i+2)_4^4$, $(i, i+1, i+2)_4^4$ for $i = 0, \ldots, 24$ and all computations are done modulo 25.

For $v = 35$ and $\lambda = 2$, take the wheels $W_{29}(\infty_1; 1)$, $W_{29}(\infty_1; 1)$, $W_{29}(\infty_2; 5)$, $W_{29}(\infty_i; i+4)$ for $i = 3, \ldots, 11$ twice each along with $2M_{10}$. The remaining edges are partitioned by the blocks $(i, i+3, i+6)_4^4$, $(i+6, i+3, i)_4^4$, $(i, i+2, i+4)_4^4$, $(i+4, i, i+2)_4^4$ for $i = 0, \ldots, 28$ and all computations are done modulo 29.

Let $\lambda = 2$ and $v \equiv 2 \pmod 4$, say $v = 4 + 22$ for $p = 0$ or $p \geq 3$. Note that $M_v = M(4p + 22, 7)$. As the decomposition of $2M_7$ is given above, we need only give the decomposition of $M(4p + 22, 7)$. Note that $K(4p + 22, 7) \cong \cup_{i=1}^{7} W_{4p+15}(\infty_i; 2p + i) \cup C_{4p+15}(1, \ldots, 2p)$. Since $C_{4p+15}(1, \ldots, 2p)$ has a W_p-decomposition by above, the result follows. Let $\lambda = 2$ and $v \equiv 3 \pmod 4$, say $v = 4 + 31$ for $p = 0$ or $p \geq 3$. Note that $M_v = M(4p + 31, 10) \cup M_{10}$. As the decomposition of $2M_{10}$ is given above, we need only give the decomposition of $M(4p + 31, 10)$. Note that $K(4p + 31, 10) \cong \cup_{i=1}^{10} W_{4p+21}(\infty_i; 2p + i) \cup C_{4p+21}(1, \ldots, 2p)$. Since $C_{4p+21}(1, \ldots, 2p)$ has a W_p-decomposition by above, the result follows. Since T_5^4 is the converse of T_4^4, it follows that it has the same necessary and sufficient conditions. ■

Theorem 3.11 *T_6^4-decomposition and a T_7^4-decomposition of λM_v exists if and only if*

$$\lambda \equiv 1 \pmod 2 \text{ and } v \equiv 1 \pmod 4$$

$$\lambda \equiv 2 \pmod 4 \text{ and } v \equiv 1 \mod 2$$

$$\lambda \equiv 0 \pmod 4 \text{ and } v \geq 3.$$

Proof. For the necessary condition see [8]. For sufficiency, we first consider the following cases.

Case 1. $\lambda = 2$ and $v = 3$. Consider the set of blocks $\{(0, 1, 2)_6^4, (1, 2, 0)_6^4, (2, 0, 1)_6^4)\}$ these blocks form a T_6^4-decomposition of $2M_3$.

It is easy to see that there is a K_3-decomposition of $2K_4$ and so there is a D_3-decomposition of $2D_4$, and hence there is a $2D_3$ decomposition of $4D_4$ and a $2M_3$-decomposition of $4M_4$. Combining this final observation with the fact that there is a T_6^4-decomposition of $2M_3$, we see that there is a T_6^4-decomposition of $4M_4$.

Case 2. $\lambda = 4$ and $v = 6$. Consider the set of blocks

$$\{(9, 2, 3)_6^4, (0, 1, 3)_6^4, (0, 5, 1)_6^4, (0, 2, 4)_6^4, (0, 2, 1)_6^4\}$$

these blocks form a T_6^4-decomposition of $4M_6$.

Case 3. $\lambda = 4$ and $v = 8$. Consider the set of blocks

$$\{(0, 3, 7)_6^4, (0, 4, 7)_6^4, (0, 2, 4)_6^4, (0, 5, 2)_6^4, (0, 1, 3)_6^4, (0, 2, 3)_6^4, (0, 1, 2)_6^4\}.$$

This set of blocks along with the images of these blocks under the permutation $(0, 1, 2, 3, 4, 5, 6, 7)$ forms a T_6^4-decomposition of $4M_8$. For $v \equiv 1 \pmod 4$ and λ odd see Theorem 2.9.

Next, suppose $v \equiv 3 \pmod 4$. Note that D_v (and hence M_v) can be decomposed into arc-disjoint copies of $\{D_3, D_5\}$(and hence arc-disjoint and edge disjoint copies of the corresponding complete mixed graphs)(Theorem 2.2 of [8]). Hence, there is a $\{2M_3, 2M_5\}$-decomposition of $2M_v$. Since there is a T_6^4-decomposition of M_5, it follows that there is a T_6^4-decomposition of $2M_5$. From case 1 and the previous statement, it follows that there is a T_6^4 decomposition of λM_v for $\lambda \equiv 2 \pmod 4$ and v odd.

Finally, Theorem 1.5 of [8] states that there exists a $\{D_k \mid k = 3,4,5,6,8\}$-decomposition of D_v for all $v \geq 3$. It thus follows that there is a $\{M_k \mid k = 3,4,5,6,8\}$-decomposition of M_v for all $v \geq 3$. Since there exist a T_6^4-decomposition of $2M_3$, M_5 and $2M_4$, it follows that there is a T_6^4-decomposition of $4M_3$, $4M_4$ and $4M_5$. From case 1 and case 2 above we get a T_6^4-decomposition of $4M_6$ and $4M_8$. It follows that there is a T_6^4-decomposition of M_v for $\lambda \equiv 0 \pmod 4$ and $v \equiv 0 \pmod 2$. ■

The proof of theorem 3.11 follows along the same basic lines as the proof as [8] for the result corresponding to the underlying digraph of T_6^4. However, our proof is a bit more direct in the sense that specific blocks are given when $v \in \{3,4,5,6,8,9,13,17,29,33,49,57,93\}$.

Theorem 3.12 *T_8^4-decomposition of λM_v exists if and only if $\lambda v(v-1) \equiv 0 \pmod 4$*

Proof. Such a decomposition is equivalent to a 2-path decomposition of K_v. It is well known that such a decomposition exists if and only if $\lambda v(v-1) \equiv 0 \pmod 4$. (see [2])

Theorem 3.13 *T_9^4-decomposition of λM_v exists if and only if*

$$v \equiv 0 \text{ or } 1 \pmod 4,$$

$$v \equiv 2 \text{ or } 3 \pmod 4 \text{ and } \lambda \equiv 0 \pmod 2.$$

Proof. Since λM_v has $\lambda v(v-1)$ arcs and T_9^4 has 4 arcs, the necessary conditions follow. Since for $v \equiv 0$ or $1 \pmod 4$ there exists a T_9^4-decomposition of M_v, then there exists a T_9^4 decomposition of λM_v for all λ. Suppose $v \equiv 3 \pmod 4$, say $v = 4k+3$, and $\lambda = 2$. Consider the collection of blocks

$$\{(0, i, 4k+3-i)_9^4 \mid i = 1, 2, \ldots, 2k+1\}.$$

This is a collection of base blocks for a cyclic T_9^4-decomposition of $2M_v$.

Suppose $v \equiv 2 \pmod 4$, say $v = 4k+2$, and $\lambda = 2$. Consider the collection of blocks

$$\{(0, 2k+1+i, 4k-1)_9^4 \mid i = 1, 2, \ldots, k-1\}$$

$$\cup\{(0, 1+i, 2k-1-i)_9^4 \mid i = 1, 2, \ldots, k-2\}$$

$$\cup\{(0, k, \infty)_9^4, (0, 2k, \infty)_9^4\}.$$

This is a collection of base blocks for a cyclic T_9^4-decomposition of $2M_v$. These blocks along with the images under the permutation $(\infty)(0, 1, 2, \ldots, 4k)$ form a T_9^4-decomposition of $2M_v$. Therefore there exist a T_9^4-decomposition of λM_v for all $\lambda \equiv 0$ mod 2. Since T_{10}^4 is the converse of T_9^4 the result follows. ∎

Theorem 3.14 *[7] A T_1^6-decomposition of M_v exists if and only if*

$$\lambda \equiv 1 \text{ or } 5 \pmod 2 \text{ and } v \equiv 1 \text{ or } 3 \pmod 6$$

$$\lambda \equiv 2 \text{ or } 4 \pmod 6 \text{ and } v \equiv 0 \text{ or } 1 \pmod 3$$

$$\lambda \equiv 3 \pmod 6 \text{ and } v \equiv 1 \pmod 2$$

$$\lambda \equiv 0 \pmod 6 \text{ and } v \geq 3.$$

The following table gives a summary of the results obtained in chapter three. Here we give necessary and sufficient conditions for the the existence of a T_j^i-triple system.

Triple	**Values of λ**	**Values of v**	**Justification**
T_1^2, T_2^2	$\lambda \equiv 0 \pmod 2$	$v \geq 3$	Theorem 3.2
	$\lambda \equiv 1 \pmod 2$	$v \equiv 1 \pmod 2$	
T_3^2	$\lambda \equiv 0 \pmod 2$	$v \geq 3$	Theorem 3.4
	$\lambda \equiv 1 \pmod 2$	$v \equiv 1 \pmod 2$	
T_4^2, T_5^2, T_6^2	all λ	$v \geq 3$	Theorem 3.5
T_7^2, T_8^2	all λ	$v \geq 3$	Theorem 3.6
T_1^4	$\lambda \equiv 0 \pmod 4$	$v \geq 3$	Theorem 3.7
	$\lambda \equiv 2 \pmod 4$	$v \equiv 1 \pmod 2$	Conjecture 3.8
T_2^4, T_3^4	$\lambda \equiv 2 \pmod 4$	$v \equiv 1 \pmod 2$	Theorem 3.9
	$\lambda \equiv 0 \pmod 4$	$v \geq 3$	
	$\lambda \equiv 0 \pmod 4$	$v \geq 3$	
T_4^4, T_5^4	$\lambda \equiv 1 \pmod 4$	$v \equiv 0$ or $1 \pmod 4$	Theorem 3.10
	$\lambda \equiv 0 \pmod 2$	$v \geq 4$	
	$\lambda \equiv 0 \pmod 4$	$v \geq 3$	
T_6^4, T_7^4	$\lambda \equiv 1 \pmod 2$	$v \equiv 1 \pmod 4$	
	$\lambda \equiv 2 \pmod 4$	$v \equiv 1 \pmod 2$	
	$\lambda \equiv 0 \pmod 4$	$v \geq 3$	Theorem 3.11
T_8^4	$\lambda \equiv 0 \pmod 4$	v	Theorem 3.12
	$\lambda \equiv 1$ or $3 \pmod 4$	$v \equiv 0$ or $1 \pmod 4$	
	$\lambda \equiv 2 \pmod 4$	$v \equiv 0$ or $1 \pmod 2$	
T_9^4	all λ	$v \equiv 0$ or $1 \pmod 4$	Theorem 3.13
	$\lambda \equiv 0 \pmod 2$	$v \equiv 2$ or $3 \pmod 4$	
T_1^6	$\lambda \equiv 1$ or $5 \pmod 6$	$v \equiv 1$ or $3 \pmod 6$	Theorem 3.14
	$\lambda \equiv 2$ or $4 \pmod 6$	$v \equiv 0$ or $1 \pmod 3$	
	$\lambda \equiv 3 \pmod 6$	$v \equiv 1 \pmod 2$	
	$\lambda \equiv 0 \pmod 6$	$v \geq 3$	

Table 3: Summary of Results in Chapter 3

4 VERIFICATION OF RESULTS AND SOME EXAMPLES

4.1 Difference Method

Here we briefly describe what difference methods and labelings are all about. We then go ahead to illustrate with the use of concrete examples how we use the difference method to carry out our decompositions. We also, by way of an example, show how we use the notion of graph labelings to demonstrate the existence of some of our T_j^i-triple system.

The difference method is a method used for decomposing a complete graph using distances (differences) between vertices. We define difference as follows. Suppose we have a complete graph on v vertices. We begin by labeling the vertices 0 through $v-1$. An arc (a, b) has associated difference $(b-a) \pmod{v}$ and for any edge $[a, b]$ the associated edge difference is given by $\min\{(b-a) \pmod{v}, (a-b) \pmod{v}\}$. The complete mixed graph M_v has $N = v-1$ arc differences and $\lfloor (v-1)/2 \rfloor$ edge differences.

Let H be a graph and let $\gamma = \{G_1, G_2, \ldots, G_n\}$ be a G-decomposition of H. An *automorphism* of this decomposition is a *permutation* of the vertex $V(H)$ which fixes the set γ. That is, if G_i is a block and π a permutation, then $\pi(G_i)$ also forms a block.

We illustrate the difference method by carrying out a T_9^4 decomposition of M_v, the complete mixed graph on v vertices with $v \equiv 1 \pmod{4}$

Example 4.1 *A T_9^4-triple system of order 17.*

Suppose we have the complete mixed graph M_{17}. We observe that $17 \equiv 1 \pmod{4}$, and therefore by Theorem 2.11, there exist a T_9^4-decomposition of M_{17}. The associated arc and edge differences are $\{1,2,3,4,\dots 16\}$ and $\{1,2,3,4,\dots 8\}$ respectively.

Next we denote T_9^4 by the order triple $(a,b,c)_9^4$. Then we have the arc differences $(b-a) \pmod{17}$, $(a-b) \pmod{17}$, $(a-c) \pmod{17}$ and $(c-a) \pmod{17}$, and edge differences $\min\{(b-a) \pmod{17}, (a-b) \pmod{17}\}$ and $\min\{(b-c) \pmod{17}, (c-b) \pmod{17}\}$. Now label the vertices in M_{17} from 0 through 16. Consider the base blocks $(0,2,7)_9^4$, $(0,4,5)_9^4$, $(0,6,3)_9^4$, $(0,8,1)_9^4$. Table 4 gives the associated arc and edge differences for each base block and Table 5 gives a T_9^4-tiple system of order 17.

Base Block	**Arc Differences**	**Edge Differences**
$(0,2,7)_9^4$	$2,15,7,10$	$2,5$
$(0,4,5)_9^4$	$4,13,5,12$	$4,1$
$(0,6,3)_9^4$	$6,11,3,12$	$6,3$
$(0,8,1)_9^4$	$8,9,1,16$	$8,7$

Table 4: Base Blocks and Differences for $v = 17$

Base Block	**Generated Blocks**
$(0,2,7)_9^4$	$(1,3,8)_9^4, (2,4,9)_9^4, (3,5,10)_9^4, (4,6,11)_9^4, (5,7,12)_9^4, (6,8,13)_9^4,$ $(7,9,14)_9^4, (8,10,15)_9^4, (9,11,16)_9^4, (10,12,17)_9^4, (11,13,1)_9^4,$ $(12,14,2)_9^4, (13,15,3)_9^4, (14,16,4)_9^4, (15,0,5)_9^4, (16,1,6)_9^4.$
$(0,4,5)_9^4$	$(1,5,6)_9^4, (2,6,7)_9^4, (3,7,8)_9^4, (4,8,9)_9^4, (5,9,10)_9^4, (6,10,11)_9^4,$ $(7,11,12)_9^4, (8,12,13)_9^4, (9,13,14)_9^4, (10,14,15)_9^4, (11,15,16)_9^4,$ $(12,16,0)_9^4, (13,0,1)_9^4, (14,1,2)_9^4, (15,2,3)_9^4, (16,3,4)_9^4.$
$(0,6,3)_9^4$	$(1,7,4)_9^4, (2,8,5)_9^4, (3,9,6)_9^4, (4,10,7)_9^4, (5,11,8)_9^4, (6,12,9)_9^4,$ $(7,13,10)_9^4, (8,14,11)_9^4, (9,15,12)_9^4, (10,16,13)_9^4, (11,0,14)_9^4,$ $(12,1,15)_9^4, (13,2,16)_9^4, (14,3,0)_9^4, (15,4,1)_9^4, (16,5,3)_9^4.$
$(0,8,1)_9^4$	$(1,9,2)_9^4, (2,10,3)_9^4, (3,11,4)_9^4, (4,12,5)_9^4, (5,13,6)_9^4, (6,14,7)_9^4,$ $(7,15,8)_9^4, (8,16,9)_9^4, (9,0,10)_9^4, (10,1,11)_9^4, (11,2,12)_9^4,$ $(12,3,13)_9^4, (13,4,14)_9^4, (14,5,15)_9^4, (15,6,16)_9^4, (16,7,0)_9^4.$

Table 5: A T_9^4-Triple System of Order 17

We observe that each arc and edge difference is used exactly once. In Table 5, we get a total of 68 blocks, each of which contains two edges and four arcs. If we multiply 68 by 4 we get 272 arcs and 68 by 2 gives us 136 edges. Thus we get all the arc and edge differences.

For a generalization we let $v = 4k + 1$. The associated arc and edge differences are $\{1, 2, 3 \ldots 4k - 1\}$ and $\{1, 2, 3, \ldots (4k - 1)/2\}$ respectively.

Next we consider the base blocks $(0, 2, 2k - 1)_9^4$, $(0, 4, 2k - 3)_9^4$, $(0, 6, 2k - 5)_9^4$, ..., $(0, 2k - 4, 5)_9^4$, $(0, 2k - 2, 3)_9^4$, $(0, 2k, 1)_9^4$ given in general form by $(0, 2i, 2k - 2i + 1)_9^4$, for $i = 1 \ldots k$. The associated arc and edge differences are given in Table 6.

Base Block	**Arc Differences**	**Edge Differences**
$(0, 2, 2k - 1)_9^4$	$2, 2k - 1, 2k + 2, 4k - 1$	$2k - 3, 2$
$(0, 4, 2k - 3)_9^4$	$4, 2k - 3, 2k + 4, 4k - 3$	$2k - 7, 4$
$(0, 6, 2k - 5)_9^4$	$6, 2k - 5, 4k - 5, 4k - 5$	$2k - 4, 6$
$\vdots$	$\vdots$	$\vdots$
$(0, 2k - 4, 5)_9^4$	$2k - 4, 5, 4k - 4, 2k + 5$	$2k - 9, 2k - 4$
$(0, 2k - 2, 3)_9^4$	$2k - 2, 3, 4k - 2, 2k + 3$	$2k - 5, 2k - 2$
$(0,\ 2k,\ 1)_9^4$	$2k, 1, 4k, 2k + 1$	$2k - 1, 2k$

Table 6: Base Blocks and Differences for $v = 4k + 1$

We note all the arc and edge differences are included exactly once. These base blocks are then permuted as follows $\{(j, 2i + j, 2k - 2i + 1 - j)_9^4 \mid i = 1 \ldots k, j = 0, \ldots 4k\}$. This gives a T_9^4-triple system of order $4k + 1$. The same technique is used for the case $\lambda > 1$ except that here the edge and arc differences are taken λ times.

4.2 Wheels and Graceful Labelings

A *graceful labeling* on a graph G with q edges is an injective mapping f from $V(G)$ to $\{0, ..., q\}$ such that the edge labels defined by $f'(uv) = |f(u) - f(v)|$ satisfy $f'(E) = \{1, ..., q\}$ [6, 13]. Graceful labelings were introduced by Rosa under the name β-valuations [13]. These were popularized by Golomb under the name *graceful labelings* [6]. The interested reader is referred to [4] for a list of graceful and related results. All wheels are graceful (see [9]). For an illustration, consider the wheel $W_4 = C_4 + K_1$. Construct a labeling f on W_4 as follows: f assigns the vertex of K_1 the number 0 and f assigns the vertices of the cycle C_4 consecutively the numbers, 8,1,5,2. In order to show that the numbering f is graceful, we verify that for every number $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ there is exactly one edge number i. With these labels we have the following:

$$f'([0, 1]) = 1, f'([0, 2]) = 2, f'([0, 5]) = 5, f'([0, 8]) = 8.$$

$$f'([2, 5]) = 3, f'([5, 1]) = 4, f'([2, 8]) = 6, f'([1, 8]) = 7.$$

Observe that $f'(E) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. We thus see that W_4 is graceful. We now use the notion of graceful labelings to illustrate the existence of a T_4^4-triple system.

Denote T_4^4 by the ordered triple $(a, b, c)_4^4$, with arcs $(a, b), (b, c), (a, c)$ and (c, a), and edges $[a, b], [c, a]$. Next, let's denote by W_n^M the mixed wheel with a cycle of length n and center c.

Lemma 4.2 *There is a T_4^4 -decomposition of the complete mixed wheel W_n^M for $n \geq$ 2.*

Proof. Consider the collection $\{(i, c, 1 + i)_4^4 : i \in \{0, 1, \ldots n - 1\}\}$, where c is fixed. This collection gives the desired decomposition. ∎

Example 4.3 *A T_4^4-decomposition of W_3^M is given by* $\{(0, c, 1)_4^4, (1, c, 2)_4^4, (2, c, 0)_4^4\}$.

Let $v = 17$, then we observe that $v \equiv 1 \pmod 4$. Since W_3 is a graceful graph of size 8, it will decompose the complete graph on 17 vertices K_{17} [13]. Thus, W_3^M will decompose M_{17}. It follows that there is a T_4^4-decomposition of M_{17}. This notion is extended in a similar manner to show the existence of a $T_4^4 - triple\ system$ and a $T_5^4 - triple\ system$ of order v.

5 CONCLUSION

In this thesis, we studied the decomposition of the complete mixed graph M_v into every possible graph on three vertices with twice as many arcs as edges. In most of the cases we gave direct constructions for our decompositions except for the cases T_4^4 and T_5^4 where we used the notion of labeling to show the existence of such a decomposition. We also studied the decomposition of the λ-fold complete mixed graph λM_v. Here we gave constructions to show the existence of a T_j^i-triple system of order v. We showed that there exists a T_j^i-decomposition of λM_v for all T_j^i except for the case T_1^4 when $\lambda = 2$ and v odd where we got a partial results. We are also left with the cases T_4^4 and T_5^4 when $\lambda = 1,\ v = 12$ and $\lambda = 2, v = 6$ to consider.

We must remark here that for every odd value of v we have considered so far, there exists a T_1^4-decomposition of $2M_v$. We will not end this work without noting that some of the proofs presented in this work, are more elegant versions for the proofs of some of the results in [8] for the corresponding underlying digraphs.

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