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Cyclic, f-Cyclic, and Bicyclic Decompositions of the Complete Graph into the 4-Cycle with a Pendant Edge.

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Cyclic, f-Cyclic, and Bicyclic Decompositions of the Complete Graph into the 4-Cycle with a Pendant Edge

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by

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ABSTRACT

Cyclic, f-Cyclic, and Bicyclic Decompositions of the Complete Graph into the 4-Cycle with a Pendant Edge

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In this paper, we consider decompositions of the complete graph on v vertices into 4-cycles with a pendant edge. In part, we will consider decompositions which admit automorphisms consisting of:

(1) a single cycle of length v,

(2) f fixed points and a cycle of length $v - f$, or

(3) two disjoint cycles.

The purpose of this thesis is to give necessary and sufficient conditions for the existence of cyclic, f-cyclic, and bicyclic Q-decompositions of K_v.

DEDICATION

This thesis is dedicated to my parents, Shelton and Tamara Cantrell, for all their love and support throughout my collegiate career. To my extended family, thank you for all your words of encouragment. A very special thank you to my friends, Shane Bray, Adam Delforge, Brent Farley, Justin Ivory, Jeanne Larson, and Joshua Powers, for two of the best years of my life.

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1 INTRODUCTION

Design theory is a branch of combinatorial mathematics that contains many interesting areas of study and has many applications. Design theory is used in computer science, telecommunications, traffic management, and environmental conservation [5]. A few of the more interesting areas of study within design theory are those of decompositions, packings, and coverings of graphs.

A *graph*, G, consists of two sets: a non-empty set of *vertices*, V, and a set of *edges*, E. There are finite and infinite graphs. In this paper we only consider finite graphs. Two vertices are *adjacent* if they have an edge in common. An edge, $e = \{v, w\}$, is said to be *incident* with vertices v and w. A graph on v vertices in which every vertex is adjacent to every other vertex is a *complete graph* on v vertices and is denoted K_v. The degree of a vertex, v, is defined as the number of edges incident with v [8].

Also of interest are directed graphs. In a *directed graph* (or *digraph*) edges are replaced with *arcs* that are assigned a direction. *Complete directed graphs*, D_v, are similar to complete graphs. In these graphs, each edge is replaced by two arcs of opposite orientation [8].

A *G-decomposition of graph H* is a set of subgraphs, $\gamma = \{G_1, G_2, \ldots, G_n\}$, where $G_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$, and $\displaystyle\bigcup_{i=1}^{n} E(G_i) = E(H)$. A set $\gamma' \subset \gamma$ is a *subsystem* of the G-decomposition of H if $\bigcup_{G \in \gamma'} E(G) = E(H')$ for some subgraph H' of H. The study of graph decompositions is a vibrant area of research [4]. Of relevance to our study are decompositions of K_v. For example, in Figure 1, we have decomposed K_5 into 5–cycles.

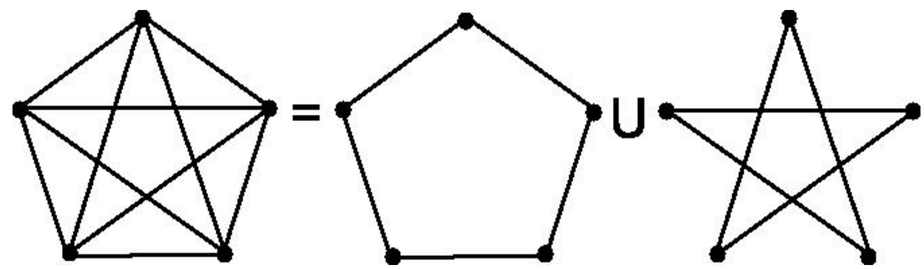

Figure 1: Decomposition of a K_5 into 5–cycles.

The G_i are called *blocks* of the decomposition. In particular, a 3-cycle (C_3) decomposition of K_v exists if and only if $v \equiv 1$ or 3 (mod 6). These were the first decompositions to be studied and are called *Steiner triple systems* of order v, denoted $STS(v)$ [12, 16, 17]. Directed graphs can also be decomposed. Instead of edge sets, $E(G)$, we now have arc sets, $A(G)$. Thus, orientations were given to 3-cycles. The only orientations of a 3-cycle, the 3-circuit and the transitive triple, are shown in Figure 2.

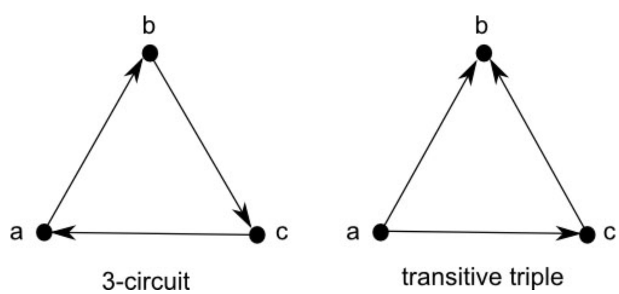

Figure 2: 3-circuit and transitive triple.

The next decompostions studied were *Mendelsohn triple systems* of order v, $MTS(v)$, and *directed triple systems of order v, DTS*(v) [11, 13]. In these decompostions, a D_v

is decomposed into 3-circuits and transitive triples, respectively. A $MTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [13]. A $DTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3) [11].

There are several other notable decompostions of K_v. It is well known that a C_4-decomposition of K_v exists if and only if $v \equiv 1 \pmod 8$[1]. Let L denote the graph with $V(L) = \{a, b, c, d\}$ and $E(L) = \{(a, b), (b, c), (a, c), (a, d)\}$, i.e., the 3-cycle with a pendant edge. An L-decomposition of K_v exists if and only if $v \equiv 0$ or 1 (mod 8) [3]. Let Q denote the graph with $V(Q) = \{a, b, c, d, e\}$ and $E(Q) = \{(a, b), (b, c), (c, d), (a, d), (a, e)\}$, the 4-cycle with a pendant edge. We denote such Q as $[a, b, c, d; e]$, as in Figure 3. A Q-decomposition of K_v exists if and only if $v \equiv 0$ or 1 (mod 5), $v \geq 10$ [2].

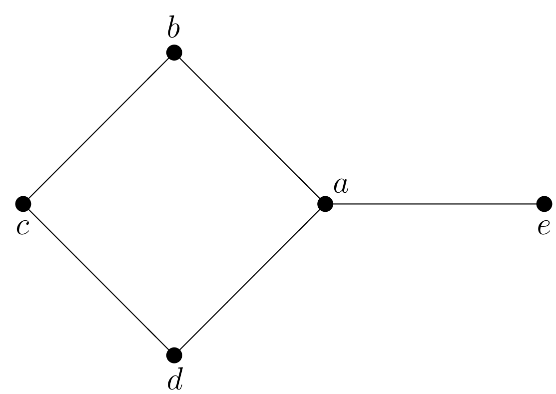

Figure 3: $Q = [a, b, c, d; e]$.

An *automorphism* of a G-decomposition of H is a permutation π of $V(G)$ which fixes the set γ. The *orbit* of a block G_i under π is the set $\{\pi^n(G_i) \mid n \in \mathbb{N}\}$ and the *length* of the orbit of G_i is the cardinality of the orbit of G_i. A set, B, of blocks is a set of *base blocks* under permutation π if the orbits of the blocks of B generate an G-decomposition of H and the orbits of the elements of B are disjoint.

An automorphism is said to be *cyclic* if it consists of a single cycle. A *f-cyclic* automorphism consists of f fixed points and a single cycle. An automorphism is *bicyclic* if it consists of two disjoint cycles. A common method of construction for graph decompositions is the use of difference methods and cyclic permutations. A cyclic C_3-decomposition of K_v exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [15]. It is well known that a cyclic C_4-decomposition of K_v exists if and only if $v \equiv 1$ (mod 8)[1]. A cyclic L-decomposition of K_v exists if and only if $v \equiv 1$ (mod 8) [3, 9].

The f-cyclic automorphism was introduced by Micale and Pennisi in connection with oriented triple systems, which are concerned with decompositions of complete digraphs into orientations of a 3-cylce [14]. When discussing bicyclic automorphisms, we assume that the cycles have lengths M and N where $M \leq N$. A bicyclic C_3-decomposition of K_v exists if and only if:

(i) $v = M + N \equiv 1$ or 3 (mod 6),

(ii) $M \equiv 1$ or 3 (mod 6), $M \neq 9$ ($M > 1$), and $M \mid N$ [6].

A bicyclic L-decomposition of K_v exists if and only if:

(i) $N = 2M$ and $v = M + N \equiv 9$ (mod 24), or

(ii) $m \equiv 1$ (mod 8) and $N = kM$ where $k \equiv 7$ (mod 8) [9].

The purpose of this thesis is to give necessary and sufficient conditions for the existence of cyclic, f-cyclic, and bicyclic Q-decompositions of K_v.

2 CYCLIC AND ROTATIONAL Q-DECOMPOSITIONS

The following result, in conjunction with unpublished work of Dr. Robert Gardner and Gary Coker [7, 10], gives necessary and sufficient conditions for the existence of a cyclic Q-decompositions of K_v.

Theorem 2.1 [7, 10] *A cyclic Q-decomposition of K_v exists if and only if $v \equiv 1$ (mod 10).*

Proof. We consider cyclic Q-decompositions of K_v where $V(K_v) = \{0, 1, 2, \ldots, (v-1)\}$ and where the cyclic permutation is $\pi = (0, 1, 2, \ldots, v-1)$.

Suppose such a system exists for $v \equiv 0$ or 6 (mod 10). By raising π to the $v/2$ power, we see that the edge $(0, v/2)$ is fixed by interchanging the vertices 0 and $v/2$. Since the edge $(0, v/2)$ is in exactly one copy of Q in the decomposition, then this copy of Q must be fixed by $\pi^{v/2}$. However, it is not possible to fix Q with a permutation which interchanges the ends of an edge. Therefore such systems do not exist.

Now suppose that $v \equiv 5$ (mod 10). The length of the orbit of each edge and every block G_i of set γ is v. Therefore the orbits of the G_i create a partition of γ into $|\gamma|/v$ sets. But with $v \equiv 5$ (mod 10), v does not divide $|\gamma|$ and so such a system does not exist.

Suppose $v \equiv 1$ (mod 10), say $v = 10k+1$. If $v = 11$, consider $\{[0, 1, 5, 3; 6]\}$. If $v = 21$, consider $\{[0, 1, 5, 3; 6], [0, 7, 17, 9; 5]\}$. If $v \geq 31$, consider $\{[0, 1, 5, 3; 6], [0, 7, 17, 9; 5]\}$ $\cup\{[0, 5i + 11, 10i + 25, 5i + 13; 5 + 15] \mid i = 0, 1, \ldots, k-3\}$. In each case, a set of base blocks is given for a cyclic Q-decomposition of K_v under π. $\square$

The following figure illustrates a cyclic Q-decompositions of K_{11}. Starting with the block $\{[0, 1, 5, 3; 6]\}$, we obtain K_{11} by rotating the block around the vertices.

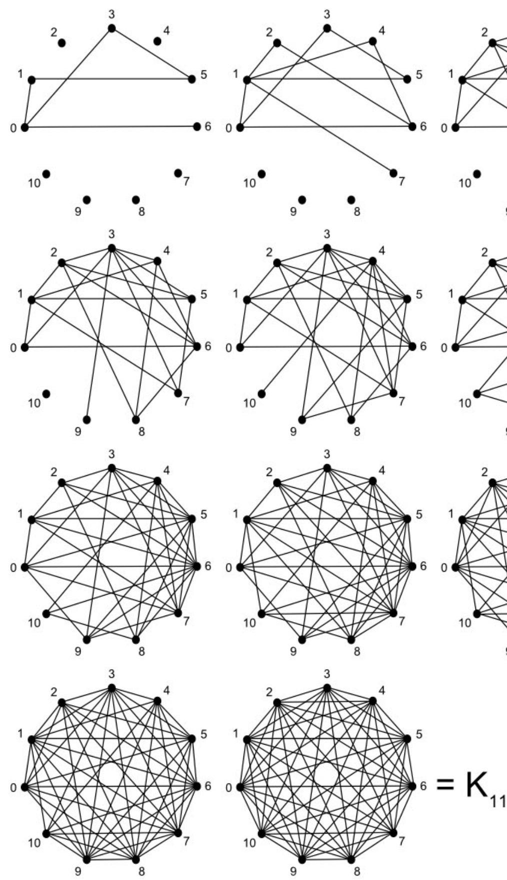

Figure 4: Cyclic Q-decomposition of K_{11}.

A special case of a bicyclic permutation is a permutation consisting of a single fixed point and a single cycle ($M = 1$ and $N = v - 1$ in the notation of Section 1). A graph decomposition admitting such a partition is said to be *rotational* (or 1-*rotational*). The following unpublished theorem, proven by Dr. Robert Gardner [10], classifies rotational Q-decompositions of K_v.

Theorem 2.2 [10] *A rotational Q-decomposition of K_v exists if and only if $v \equiv 0$ (mod 10).*

Proof. In such a system, the length of the orbit of each block is $v - 1$. Therefore the number of edges must be a multiple of $5(v - 1)$. Now $|E(K_v)| = \frac{v(v-1)}{2}$, so it follows that $v \equiv 0$ (mod 10) is necessary. So suppose $v \equiv 0$ (mod 10), say $v = 10k$, $V(K_v) = \{\infty, 0, 1, 2, \ldots, (v-1)\}$ and $\pi = (\infty)(0, 1, 2, \cdots, (v-1))$. Consider the set of blocks:

$$\{[0, 1, 5, 3; \infty]\} \cup \{[0, 5i + 5, 10i + 13, 5i + 7; 5i + 9] \mid i = 0, 1, \ldots, k - 2\}.$$

This is a set of base blocks for a rotational Q-decomposition of K_v under π. $\square$

3 THE f-CYCLIC RESULTS

We now consider a permutation of a Q-decomposition of K_v where the permutation consists of f fixed points and a cycle of length $v-f$.

Lemma 3.1 *The fixed points of a f-cyclic automorphism of a Q-decomposition of K_v form a subsystem. That is, if π is the f-cyclic automorphism, (a,b) is an edge of a block B, and $\pi(a)=a$, $\pi(b)=b$, then each vertex of B is fixed by π.*

Proof. We let the vertex set of K_v be $\{\infty_1,\infty_2\ldots,\infty_f\}\cup\{0,1,\ldots,(v-f-1)\}$ and the f-cyclic permutation be $(\infty_1)(\infty_2)\cdots(\infty_f)(0,1,\ldots(v-f-1))$. Now, edge (a,b) appears in exactly one block of a decomposition. Since (a,b) is in both B and $\pi(B)$, it must be that $B=\pi(B)$. The only way to fix an edge of B without fixing all vertices of B is to fix three vertices of B and interchange the other two. If this is the case, then π must consist of several (at least three) fixed points and a transposition. Assume that the vertices in the transposition are c and d. Edge (c,d) must be in some block, but π fixes edge (c,d) and hence must fix block B'. However, it is impossible to fix B while interchanging the vertices of one of its edges. Therefore π cannot consist of fixed points and a transposition and it must be that π fixes all vertices of B. □

Lemma 3.1 along with the necessary conditions for the existence of an Q-decomposition of K_v implies Lemma 3.2.

Lemma 3.2 *In a f-cyclic Q-decomposition of K_v, it is necessary that $f \equiv 0$ or 1 (mod 5), $f \geq 10$.*

Lemma 3.3 *In a f-cyclic Q-decomposition of K_v, it is necessary that $f \leq (v-1)/9$.*

Proof. Suppose a block of such a decomposition contains edges of the forms (∞_i, a) and (∞_i, b) where $a, b \in \mathbb{Z}_{v-f}$ with $a > b$. Then π^{b-a} maps edge (∞_i, a) to (∞_i, b). Since (∞_i, b) occurs in only one block, π^{b-a} must fix this block. But the only way to fix B without fixing each vertex is to fix three of the vertices of Q and interchange the other two. So π^{b-a} must consist of fixed points and transpositions. However, the pendant edge must be fixed by π^{b-a} and this can occur only if both vertices of the pendant edge are fixed. But this contradicts Lemma 3.1. Therefore no block of an f-cyclic Q-decomposition may include edges of the forms (∞_i, a) and (∞_i, b) where $a, b \in \mathbb{Z}_{v-f}$.

By Lemma 3.1, we see that the admissible blocks of such a decomposition must be of the following forms only: $B_\infty = [\infty_i, \infty_j, \infty_k, \infty_l; \infty_m]$, $B_{C\infty} = [a, b, c, d; \infty_i]$, and $B_C = [a, b, c, d; e]$ where $a, b, c, d \in \mathbb{Z}_{v-f}$. Block B_∞ is fixed by π and all blocks of this form make up a Q-decomposition of K_f. So there are $f(f-1)/10$ such blocks. The length of the orbit of a block of type $B_{C\infty}$ is $v - f$. The orbit of this block contains all edges of the form (∞_i, a) for fixed i and any $a \in \mathbb{Z}_{v-f}$. Therefore, there must be $f(v-f)$ blocks of this form. These blocks contain $4f(v-f)$ edges of the form (a, b) where $a, b \in \mathbb{Z}_{v-f}$. Since K_v has $(v-f)(v-f-1)/2$ such edges, it is necessary that $4f(v-f) \leq (v-f)(v-f-1)/2$, or $f \leq (v-1)/9$. □

Lemma 3.4 *At least one of the following conditions is necessary for the existence of a f-cyclic Q-decomposition of K_v:*

(i) If $v \equiv 0 \pmod{10}$, then $f \equiv 1 \pmod{10}$;

(ii) If $v \equiv 1 \pmod{10}$, then $f \equiv 0 \pmod{10}$;

(iii) If $v \equiv 5 \pmod{10}$, then $f \equiv 6 \pmod{10}$;

(iv) If $v \equiv 6 \pmod{10}$, then $f \equiv 5 \pmod{10}$.

Proof. With the notation of Lemma 3.3, the number of edges of the form (a, b), where $a \in \mathbb{Z}_{v-f}$, which are *not* in blocks of the form $B_{c\infty}$ is

$$\frac{(v-f)(v-f-1)}{2} - 4f(v-f) = (v-f)\left(\frac{v-9f-1)}{2}\right).$$

These edges must be contained in blocks of the form B_c. Since each such block contains five such edges, there must be $(v-f)(v-9f-1)/10$ such blocks. The lengths of the orbit of each B_c is $v-f$, and so there must be $(v-9f-1)/10$ base blocks of the form B_c. Since $v \equiv 0$ or $1 \pmod 5$ and $f \equiv 0$ or $1 \pmod 5$, the conditions on v and f follow. $\Box$

Theorem 3.5 *A f-cyclic Q-decomposition of K_v exists if and only if $f \leq (v-1)/9$ and*

(i) If $v \equiv 0 \pmod{10}$, then $f \equiv 1 \pmod{10}$;

(ii) If $v \equiv 1 \pmod{10}$, then $f \equiv 0 \pmod{10}$;

(iii) If $v \equiv 5 \pmod{10}$, then $f \equiv 6 \pmod{10}$;

(iv) If $v \equiv 6 \pmod{10}$, then $f \equiv 5 \pmod{10}$.

Proof. The necessary conditions follow from Lemmas 3.3 and 3.4. For sufficiency, consider the set:

$$\{[0, 4i + 1, 8i + 5, 4i + 3; \infty_{i+1}] \mid i = 0, 1, 2, \ldots, f - 1\}$$

$$\cup\{[0, 5i+(4f+1), 10i+(8f+5), 5i+(4f+3); 5i+(4f+5)] \mid i = 0, 1, 2, \ldots, (v-9f-11)/10\}.$$

This is a set of base blocks for a f-cyclic Q-decomposition of K_v for the necessary value of v and f. □

4 THE BICYCLIC RESULTS

In this chapter we consider bicyclic Q-decompositions of K_v where the vertex set of K_v is $\{0_1, 1_1, 2_1, \ldots, (M-1)_1, 0_2, 1_2, 2_2, \ldots, (N-1)_2\}$ and the automorphism is $(0_1, 1_1, 2_1, \cdots, (M-1)_1)(0_2, 1_2, 2_2, \cdots, (N-1)_2)$. Therefore, we have the following results.

Lemma 4.1 *In a bicyclic Q-decomposition of K_v, neither M nor N can be even.*

Proof. An argument similar to that used in the proof of Theorem 2.1 can be used to show that in a bicyclic automorphism, neither M nor N can be even (or there is the same uniqueness problem with edge $(0, M/2)$ or edge $(0, N/2)$, respectively).□

Lemma 4.2 *If a bicyclic Q-decomposition of K_v exists where $M < N$, then $M \equiv 1$ $(mod\ 10)$.*

Proof. Suppose a bicyclic Q-decomposition of K_v exists where $M < N$ and let π be the bicyclic automorphism. Assume that there is a block B of the decomposition with vertex set $V(B) = \{v_1, w_1, x_i, y_j, z_k\}$ and edge set satisfying $(v_1, w_1) \subset E(B)$. Then π^M fixes edge (v_1, w_1) and hence must fix B. The only way to fix $Q = [a, b, c, d; e]$ without fixing all of the vertices is to fix the vertices a, c, and e and to interchange vertices b and d. Therefore, such a π satisfies the property that π^M fixes three vertices of B, say v_1, w_1, and x_1, and interchanges the other two vertices, y_2 and z_2. In this case, π^M must consist of M fixed points and $N/2$ transpositions (and so $N = 2M$). However, as seen in Lemma 4.1, N cannot be even. Hence all vertices of B must be fixed by π^M and in fact $V(B) = \{v_1, w_1, x_1, y_1, z_1\}$. That is, if a block of a bicyclic decomposition has one edge with vertices in $\{0_1, 1_1, 2_1, \ldots, (M-1)_1\}$, then all vertices of the block lie in this set. In fact, such blocks form a subsystem of the bicyclic decomposition. If we restrict π to these blocks, we see that they form a cyclic Q-decomposition of K_M and by Theorem 2.1, $M \equiv 1 \pmod{10}$. $\square$

The following lemma, due to the work of Gary Coker [7], gives the neccessary and sufficent conditions for a bicyclic Q-decomposition to exist with cycles of the same length.

Lemma 4.3 [7] *A bicyclic Q-decomposition of K_v admitting an automorphism consisting of two disjoint cycles of the same length exists if and only if $v \equiv 6$ (mod 20), $v \geq 26$.*

Proof. With $M = N$ and $v = 2M$, we have from Lemma 4.1 that a necessary condition is $v \equiv 2 \pmod 4$. Since $v \equiv 0$ or $1 \pmod 5$, it is necessary that $v \equiv 6$ or $10 \pmod{20}$. Now if $v \equiv 10 \pmod{20}$, then $M \equiv 5 \pmod{10}$ and the length of the orbit of each edge and every block $G_i \in \gamma$ is M. Therefore, the orbits of the G_i create a partition of γ into $|\gamma|/M$ sets. But with $v \equiv 10 \pmod{20}$, $M = v/2$ does not divide $|\gamma|$ and so such a system does not exist.

Now suppose $M = v/2 \equiv 3 \pmod{10}$, i.e., $M = 10k + 3$. Consider the set:

$$\{[0_p, 1_p, 5_p, 3_p; 6_p], [0_1, (5k+3)_2, 2_1, 5k+2_2; 0_2], [0_1, (5k+5)_2, 6_1, (5k+4)_2; 5_1],$$

$$[0_2, (5k+7)_1, 10_2, (5k+6)_1; 5_2] \mid p = 1, 2\}$$

$$\cup\{[0_p, (7+5i)_p, (17+10i)_p, (9+5i)_p; (11+5i)_p],$$

$$[0_1, (5k+9+4i)_2, (14+8i)_1, (5k+8+4i)_2; (1+i)_2],$$

$$[0_1, (5k+11+4i)_2, (18+8i)_1, (5k+10+4i)_2; (10k+2-i)_2] \mid p = 1, 2\}.$$

This a set of base blocks for a bicyclic Q-decomposition of K_v as needed. □

Lemma 4.4 *If a bicyclic Q-decomposition of K_v exists with $M < N$, then $M \equiv 1$ $(mod\ 10)$ and $N = kM$ where $k \equiv 9$ $(mod\ 10)$.*

Proof. By Lemma 4.2, $M \equiv 1 \pmod{10}$. Suppose all edges of the form (x_1, y_2) are contained in blocks consisting only of such edges (a possibility since Q is bipartite). Then the blocks with vertices from $\{0_2, 1_2, 2_2, \ldots, (N-1)_2\}$ form a cyclic Q-decomposition of K_N and by Theorem 2.1, $N \equiv 1 \pmod{10}$. But then $v = M+N \equiv 2 \pmod{10}$. Since $v \equiv 0$ or $1 \pmod 5$, this is impossible. Therefore, if a bicyclic Q-decomposition exists with $M < N$, then there must be some block B which contains both edges of the form (x_1, y_2) and (y_2, z_2) (it follows from the proof of Lemma 4.2 that no block can contain both edges of the form (x_1, y_1) and (y_1, z_2)). If we apply π^N to such a block, the edge (y_2, z_2) is fixed. Therefore, the block containing (y_2, z_2) is fixed. As in Lemma 4.2, this can be accomplished by interchanging two of the other vertices of B, but this would require that π^N contains $M/2$ transpositions, a contradiction. Therefore, all vertices of B must be fixed and, in particular, x_1 must be fixed. Therefore, M is a multiple of N: $N = kM$ for some positive integer k.

From Lemma 4.2, we see that every edge of the form (x_1, y_1) is in a block of the form $[a_1, b_1, c_1, d_1; e_1]$. Any edge of the form (x_1, y_2) or the form (x_2, y_2) has an orbit of length N and there are $MN + N(N-1)/2$ such edges. Therefore, any block consisting of such edges also has an orbit of length N and the total number of edges in this orbit is $5N$. This implies that $5N$ divides $MN + N(N-1)/2$, or that

$M+(N-1)/2 = M+(kM-1)/2 \equiv 0 \pmod 5$, from which follows the result $k \equiv 9 \pmod{10}$. $\square$

Lemma 4.5 *A bicyclic Q-decomposition of K_v with $M < N$ exists if and only if $M \equiv 1$ (mod 10) and $N = kM$ where $k \equiv 9$ (mod 10).*

Proof. The case when $M = 1$ follows from Theorem 2.2. For $M > 1$, consider the following collection of blocks:

$$\{[0_2, 4_2, 2_2, 3_2; 0_1]\} \cup \{[0_1, (4+5i)_2, 2_1, (3+5i)_2; (5+5i)_2] \mid i = 0, 1, \ldots, (M-6)/5\}$$

$$\cup\{[0_2, (5+5i)_2, (13+10i)_2, (7+5i)_2; (9+5i)_2] \mid i = 0, 1, \ldots, (N-19)/10\}.$$

These blocks, along with the base blocks of a cyclic Q-decomposition of K_M on vertex set $\{0_1, 1_1, \ldots, (M-1)_1\}$, form a set of base blocks for a bicyclic Q-decomposition of K_v as needed. $\square$

Lemmas 4.2 to 4.5 combine to give necessary and sufficient conditions for a bicyclic Q-decomposition of K_v.

Theorem 4.6 *A bicyclic Q-decomposition of K_v, where the bicyclic automorphism consists of disjoint cycles of lengths M and N where $M \leq N$ exists if and only if*

(i) $M = N \equiv 3$ (mod 10), $M = N \geq 13$, or

(ii) $M \equiv 1$ (mod 10) and $N = kM$ where $k \equiv 9$ (mod 10).

5 THE DIFFERENCE METHOD AND CONCLUSION

In this chapter, we will explore the *difference method* used to obtain the results in the previous chapters. Define a *pure difference of type i* associated with edge (a_i, b_i) as $\min\{\ |a-b| (\mathrm{mod}\ N)\ ,\ |b-a| (\mathrm{mod}\ N)\}$, where N is the length of the cycle. The set of all pure differences is $\{1, 2, 3 \ldots \lfloor N/2 \rfloor\}$. Define a *mixed difference* with associated edge (a_1, b_2) as $(b-a)(\mathrm{mod}\ M)$. The set of all mixed differences is $\{0, 1, 2 \ldots M-1\}$. To ensure that each edge of K_v is present after applying the given permutation, each difference is used exactly once in one of the base blocks. The following two examples illustate the difference method.

Consider the f-cyclic graph where $f = 6$ and $N = 49$, for a total of 55 vertices. In this example, all of the differences are of the pure type 1 variety. The subscript of 1 is omitted since the graph only contains one cycle. The set of all pure type 1 differences is $\{1, 2, 3 \ldots 24\}$. The base blocks described by the proof of Theroem 3.5 can be seen in Figure 5.

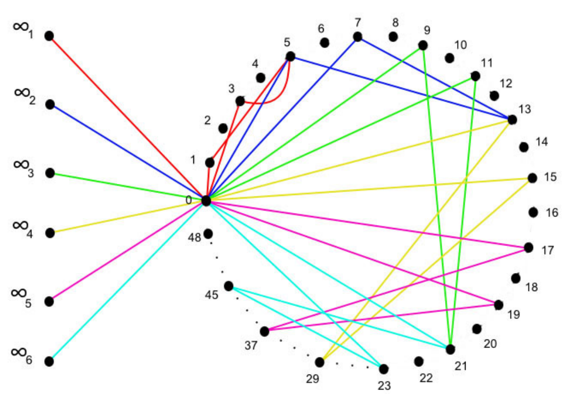

Figure 5: $f = 6$, $N = 49$.

The existence of each edge of K_{55} is ensured since each of the differences is used exactly once in one of these blocks. For example, edge $(7, 13)$ of K_{55} will be present after applying the given permuation since the associated difference of 6 is used in the block $[0, 5, 13, 7; \infty_2]$. A similar argument can be made for each of the differences in the set. Thus, these blocks, along with a Q-decompostion of K_6 on vertex set $\{\infty_1, \infty_2, \ldots \infty_6\}$, form a f-cylic Q-decomposition of K_{55}.

For the second example, consider the bicylic graph where $M = 11$ and $N = 99$, for a total of 110 vertices. In this example, we have both mixed and pure differences. The set of all mixed differences is {0,1,. . . ,10} and the set of all pure type 1 and 2 differences are $\{1, 2, \ldots, 49\}$ and $\{1, 2, 3, 4, 5\}$, respectively. A cyclic Q-decompostion of K_{11} can be seen in Figure 4, thus, we will only consider differences of the mixed and pure type 2 variety. Again, we will show that each difference is used exactly once in one of the blocks. Consider the following table:

Block	Mixed Differences	Pure Type 2 Differences
$[0_2, 4_2, 2_2, 3_2; 0_1]$	0	$1, 2, 3, 4$
$[0_1, 4_2, 2_1, 3_2; 5_2]$	$1, 2, 3, 4, 5$	-
$[0_1, 9_2, 2_1, 8_2; 10_2]$	$6, 7, 8, 9, 10$	-
$[0_2, 5_2, 13_2, 7_2; 9_2]$	-	$5, 6, 7, 8, 9$
$[0_2, 10_2, 23_2, 12_2; 14_2]$	-	$10, 11, 12, 13, 14$
$[0_2, 15_2, 33_2, 17_2; 19_2]$	-	$15, 16, 17, 18, 19$
$[0_2, 20_2, 43_2, 22_2; 24_2]$	-	$20, 21, 22, 23, 24$
$[0_2, 25_2, 53_2, 27_2; 29_2]$	-	$25, 26, 27, 28, 29$
$[0_2, 30_2, 63_2, 32_2; 34_2]$	-	$30, 31, 32, 33, 34$
$[0_2, 35_2, 73_2, 37_2; 39_2]$	-	$35, 36, 37, 38, 39$
$[0_2, 40_2, 83_2, 42_2; 44_2]$	-	$40, 41, 42, 43, 44$
$[0_2, 45_2, 93_2, 47_2; 49_2]$	-	$45, 46, 47, 48, 49$

Table 1: Base blocks and differences for $M = 11$, $N = 99$.

Table 1 lists the base blocks described in the proof of Lemma 4.5 and the associated differences for each block. This table makes it easy to see that each mixed and pure type 2 difference is used exactly once, ensuring that each edge of K_{110} will be present under the given permutation. These blocks combined with a cyclic Q-decomposition of K_{11}, depicted in Figure 4, form a bicyclic Q-decomposition of K_{110}.

In this thesis, the necessary and sufficent conditions for the existence of cyclic, f-cyclic, and bicyclic Q-decompositions of the complete graph on v vertices have been given. The next logical step would be to consider tricylic automorphisms. Some results have been proven concerning tricyclic Stiener Triple Systems [10]. Currently, work is being done related to decompositions, packings, and coverings of K_v using the 6-cycle with a pendant edge [10]. A natural generalization would be to study decompositions of K_v into n-cycles with a pendant edge. Since automorphisms of graph decompositions are widely studied, one direction for future research could include the automorphism question for Q-decompositions of D_v.

BIBLIOGRAPHY

[1] J. C. Bermond, C. Huang, A. Rosa, and D. Sotteau, Balanced Cycle and Curcuit Designs: Even Cases, *Ars Combinatoria* **5** (1978), 293–318.

[2] J. C. Bermond, C. Huang, A. Rosa, and D. Sotteau, Decompositions of Complete Graphs into Isomorphic Subgraphs with Five Vertices, *Ars Combinatoria* **10** (1980), 211–254.

[3] J. C. Bermond and J. Schönheim, G-Decompositions of K_n where G has Four Vertices or Less, *Discrete Mathematics* **19** (1977), 113–120.

[4] J. Bosák, *Decompositions of Graphs*, Boston: Kluwer Academic Publishers, 1990.

[5] F. Buckley and M. Lewinter, A Friendly Introduction to Graph Theory, *Pearson Education, Inc.* (2003), 261–262.

[6] R. Callahan and R. Gardner, Bicyclic Steiner Triple Systems, *Discrete Mathematics* **128** (1994), 35–44.

[7] G. Coker, personal communication.

[8] R. Diestel, Graph Theory, *Springer–Verlag Berlin Heidelberg* (2005), 1–28.

[9] R. Gardner, Bicyclic Decompositions of K_v into Copies of $K_3 \cup \{e\}$, *Utilitas Mathematica* **54** (1998), 51–57.

[10] R. Gardner, personal communication.

[11] S. Hung and N. Mendelshon, Directed Triple Systems, *Journal of Combinatorial Theory Series A,* **14** (1973), 310–318.

[12] T. Kirkman, On a Problem in Combinatorics, *Cambridge and Dublin Mathematics Journal* **2** (1847), 191–204.

[13] N. Mendelshon, A Natural Generalizations of Steiner Triple Systems, *Computers in Number Theory,* A.O. Atkin and B. Birch, Academic Press, London, 1971.

[14] B. Micale and M. Pennisi, The spectrum of d-cyclic oriented triple systems, *Ars Combinatoria* **48** (1998), 219–223.

[15] R. Peltesohn, A Solution to Both of Heffter's Difference Problems (in German), *Compositio Mathematica* **6** (1939), 251–257.

[16] M. Reiss, Über eine Steinersche combinatorsche Aufgabe welche in 45sten Bande dieses Journals, Seite 181, gestellt worden ist, *Journal für die Reine und angewandte Mathematik (Crelles Journal)* **56** (1859), 326-344.

[17] J. Steiner, Combinatorische Aufgabe, *Journal für die Reine und angewandte Mathematik (Crelles Journal)* **45** (1853), 181-182.

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