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Decompositions of Mixed Graphs with Partial Orientations of the $P_4$.

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Decompositions of Mixed Graphs with Partial Orientations of the $P_4$

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

by

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May 2009

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ABSTRACT

Decompositions of Mixed Graphs with Partial Orientations of the $P_4$

by

Adam Meadows

A decomposition $\mathcal{D}$ of a graph $H$ by a graph $G$ is a partition of the edge set of $H$ such that the subgraph induced by the edges in each part of the partition is isomorphic to $G$. A mixed graph on $V$ vertices is an ordered pair $(V, C)$, where $V$ is a set of vertices, $|V| = v$, and $C$ is a set of ordered and unordered pairs, denoted $(x, y)$ and $[x, y]$ respectively, of elements of $V$ [8]. An ordered pair $(x, y) \in C$ is called an arc of $(V, C)$ and an unordered pair $[x, y] \in C$ is called an edge of graph $(V, C)$. A path on $n$ vertices is denoted as $P_n$. A partial orientation on $G$ is obtained by replacing each edge $[x, y] \in E(G)$ with either $(x, y)$, $(y, x)$, or $[x, y]$ in such a way that there are twice as many arcs as edges. The complete mixed graph on $v$ vertices, denoted $M_v$, is the mixed graph $(V, C)$ where for every pair of distinct vertices $v_1, v_2 \in V$, we have $\{(v_1, v_2), (v_2, v_1), [v_1, v_2]\} \subset C$. The goal of this thesis is to establish necessary and sufficient conditions for decomposition of $M_v$ by all possible partial orientations of $P_4$. 
DEDICATION

I would like to dedicate my thesis to my father. My father has been an inspiration to me my whole life. My father has always been there when I needed him and has always been proud of everything I have achieved or ever done. Thank you for making me the man I am today.
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1 INTRODUCTION

1.1 Basic Graph Theory Definitions

A graph $G$ consists of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates the edges with vertices. If a vertex is associated with an edge, it is an endpoint of that edge. A vertex $v$ is incident to an edge $e$ if $v$ is an endpoint of $e$. An arc, $a = (x, y)$, is considered to be a directed edge from $x$ to $y$. For a vertex $v$ in a graph $G$, the degree of $v$ denoted $deg_G(v)$ is the number of edges of $G$ incident with $v$. For a vertex $v$ in graph $G$, the out degree denoted $od(v)$ of $v$ is the number of vertices of $G$ to which $v$ is adjacent, while the in degree of $G$ denoted $id(v)$ of $v$ is the number of vertices of $G$ from which $v$ is adjacent. We will assume that all graphs are simple, i.e., there are no loops (edges whose endpoints are equal) and no multiple edges [25]. All graphs presented here are finite unless otherwise noted. Examples are given in Figure 1.

![Simple Graph with No Arcs, with Arcs, and with Arcs and Edges.](image)

Figure 1: Simple Graph with No Arcs, with Arcs, and with Arcs and Edges.

The order of $G$, denoted $n(G) = |V(G)|$, is the number of vertices in $G$. The size of $G$, denoted $e(G) = |E(G)|$, is the number of edges in $G$. The maximum degree of $G$, denoted $\Delta(G) = \max\{deg_G(v)\}$, is the largest number of edges incident to a vertex $v$ in $G$. The minimum degree of $G$, denoted as $\delta(G) = \min\{deg_G(v)\}$, is the smallest number of edges incident to a vertex $v$ in $G$ [25].
A path on \( n \) vertices is denoted \( P_n \). The cycle on \( n \) vertices is denoted \( C_n \). A bipartite graph is a graph whose vertices can be divided into two disjoint sets \( V_1 \) and \( V_2 \) such that every edge connects a vertex in \( V_1 \) to one in \( V_2 \); that is, \( V_1 \) and \( V_2 \) are independent sets. Equivalently, a bipartite graph is a graph that does not contain any cycles of odd length. A complete bipartite graph \( G : (V_1 \cup V_2, E) \) is a bipartite graph such that \( v_1v_2 \in E(G) \) if and only if \( v_1 \in V_1 \) and \( v_2 \in V_2 \). The complete bipartite graph with partitions \( |V_1| = m \) and \( |V_2| = n \) is denoted \( K_{m,n} \). The wheel graph, denoted \( W_n \), is a graph with \( n + 1 \) vertices, formed by connecting a single vertex to all the vertices of an \( n - \text{cycle} \). The complete graph is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph on \( n \) vertices is denoted by \( K_n \) [8]. Examples of these graphs are given in Figure 2.

![Special Graphs on Six Vertices](image)

Figure 2: Special Graphs on Six Vertices

A directed graph or digraph, denoted \( D \), consists of a vertex set \( V(D) \), and a set of arcs \( A(D) \). If a directed path leads from \( x \) to \( y \), then \( y \) is said to be a successor of \( x \) and reachable from \( x \), and \( x \) is said to be a predecessor of \( y \). An arc, \( a = (x, y) \), is considered to be directed from \( x \) to \( y \). The arc \( (y, x) \) is the arc \( (x, y) \) inverted.
A mixed graph on $V$ vertices is an ordered pair $(V, C)$, where $V$ is a set of vertices, $|V| = v$, and $C$ is a set of ordered and unordered pairs, denoted $(x, y)$ and $[x, y]$ respectively, of elements of $V$ [8]. An ordered pair $(x, y) \in C$ is called an arc of $(V, C)$ and an unordered pair $[x, y] \in C$ is called an edge of graph $(V, C)$. Figure 3 illustrates a mixed graph on four vertices.

![Figure 3: Mixed Complete Graph on Four Vertices, $M_4$](image)

An isomorphism between two graphs $G$ and $H$ is a bijection $f : V(G) \rightarrow V(H)$ such that $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. Two graphs are isomorphic if such a bijections exists.

1.2 Decompositions

A decomposition $\mathcal{D}$ of a digraph $H$ is a partition of the arc set of $H$. The graph $H$ is called the host graph for the decomposition. For each $P$ of the partition, the subgraph of $H$ induced by $P$ is called a block of the partition. We will be concerned with the case that all blocks are isomorphic to a single block prototype $G$. In this case we say that $\mathcal{D}$ is a $G$-decomposition of $H$ and that $G$ is a divisor of $H$. This
situation is denoted $G|H$. Analogous definitions exist for decompositions of directed and mixed graphs. Figure 4 illustrates a $P_3$-decomposition of $Q_3$.

![Figure 4: A $P_3$-Decomposition of $H$](image)

If $G$ is a simple graph then the mixed graph $M(G)$ has $V(M(G)) = V(G)$, and $C = \{(x, y), [x, y], [y, x]\} \subseteq C$ if and only if $xy \in E(G)$. For convenience, $M(K_n) = M_n$. Figure 5 illustrates a graph and its associated mixed graph.

![Figure 5: $C_4$ and $M(C_4)$](image)

A partial orientation of $G$ is obtained by replacing each edge $[x, y] \in E(G)$ with either $(x, y)$, $(y, x)$, or $[x, y]$. We restrict this to partial orientations in which there are twice as many arcs as edges. Since the mixed complete graph has twice as many arcs as edges, $e(G) \equiv 0 \pmod{3}$. The converse of a directed (mixed) graph $G$, denoted $G^c$, is obtained from $G$ by reversing the orientation on all arcs, i.e., $(x, y) \in G^c$ if and only if $(y, x) \in G$. 

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The goal of this thesis is to establish necessary and sufficient conditions for the decomposition of $M_v$ by all possible partial orientations of $P_4$.

![Partial Orientations of $P_4$](image)

**Figure 6: Partial Orientations of $P_4$**

To aid in this, we give the following proposition.

**Proposition 1.1**

i. If $G|H$ and $H|K$, then $G|K$.

ii. $G|H$ if and only if $G^c|H^c$.

iii. $G|H$ if and only if $M(G)|M(H)$.

**Proof.**

i. Suppose $G|H$ and $H|K$. If $G|H$, then there is a decomposition $\mathcal{D}_1$ of a graph $H$ into copies of $G$. Similarly, since $H|K$, there exists a decomposition $\mathcal{D}_2$ of $K$ into copies of $H$. For each $H$-block in $\mathcal{D}_2$, we replace it with edge disjoint copies of $G$ via $\mathcal{D}_1$. This gives the required $G$-decomposition of $K$.

ii. Suppose $G|H$. If $G|H$, then there is a decomposition $\mathcal{D}$ of a digraph $H$ is a partition of the arc set of $H$. Then take any $G$-block $B$ such that $(x, y) \in B$. Note that $(x, y) \in A(H)$. Then for each $(x, y) \in A(H)$, there exists and $G^c$-block $B^c$ such that $(y, x) \in A(B^c)$. Hence, this gives the required $G^c$-decomposition
of \( H^c \). Conversely, if \( G^c|H^c \) then there is a decomposition \( \mathcal{D} \) of a graph \( H^c \) into copies of \( G^c \). Then for each \((x, y) \in A(H)\), there exists and \( G^c \)-block, \( B^c \), such that \((y, x) \in A(B^c)\). Since \((x, y) \in A(H)\), there exists a \( G \)-block \( B \) such that \((x, y) \in A(B)\). This gives the required \( G \)-decomposition of \( H \).

iii. If \( G|H \), then there is a decomposition \( \mathcal{D} \) of a graph \( H \) into copies of \( G \). Then for each \( xy \in E(H) \), there exists a \( G \)-block \( B \) such that \( xy \in B \). For every 

\[
\{(x, y), (y, x), [x, y]\} \subset C(M(H)),
\]

there exists \( M(G) \)-block \( B \) such that \( \{(x, y), (y, x), [x, y]\} \in C(B) \). This means we have a \( \mathcal{D} \) of a graph \( M(H) \) into copies of \( M(G) \). This then gives the required \( M(G) \)-decomposition of \( M(H) \). Conversely, if \( M(G)|M(H) \) then there is a decomposition \( \mathcal{D} \) of a graph \( M(H) \) into copies of \( M(G) \). For every 

\[
\{(x, y), (y, x), [x, y]\} \subset C(M(H)),
\]

there exists \( M(G) \)-block \( B \) such that \( \{(x, y), (y, x), [x, y]\} \in M(H) \). Hence for each \( xy \in E(M(H)) \) there exists a \( M(G) \)-block \( B \) such that; \( xy \in B \). This means we have a decomposition \( \mathcal{D} \) of a graph \( H \) into copies of \( G \). Hence this gives the required \( G \)-decomposition of \( H \).

\[\blacksquare\]

1.3 Algebra

The constructions that we will use to prove the sufficient conditions for our results will often be algebraic in nature. To facilitate this we give relevant definitions and
Let $a, d$ be integers. If $d$ divides $a$, then there exists $c \in \mathbb{Z}$ such that $cd = a$, and this situation is denoted $d|a$. If there is no such $c$, we say that $d$ does not divide $a$, and this situation is denoted $d \nmid a$. Let $S \subseteq \mathbb{Z}^+$. Then the greatest common divisor of $S$, denoted $gcd(S)$, is the largest $d \in \mathbb{Z}^+$ such that $d|a$ for all $a \in S$ [2, 10, 13].

Define $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n : 0 < x \leq n/2\}$. Define $|x|_n = \min\{|x|, |n - x|\}$. Let $n \in \mathbb{Z}^+$ and $S \subseteq \mathbb{Z}_n^*$ be given. The circulant graph $C_n(S)$ is the undirected graph with vertex set $V = \mathbb{Z}_n$ and edge set: $E = \{xy : x, y \in \mathbb{Z}_n \text{ and } |x - y|_n \in S\}$[3]. Figure 7 illustrates an example of a circulant graph.

![Figure 7: Circulant - $C_6(1, 2)$](image)

Let $\mathbb{Z}_n = \{0, ..., n - 1\}$. If $a \in \mathbb{Z}_n$ and $gcd(a, n) = 1$, then for every $x \in \mathbb{Z}_n$ there exists a $y \in \mathbb{Z}_n$ such that $ya = x$. Hence $a$ generates $\mathbb{Z}_n$. In general, if $gcd(a, n) = d$, then $a$ will generate $d$ disjoint cycles, each of length $n/d$.

Two integers $v$ and $a$ are said to be congruent modulo $n$, if their difference $a - b$ is an integer multiple of $n$. An equivalent definition is that both numbers have the same remainder when divided by $n$. If this is the case, it is expressed as $v \equiv a \pmod{n}$.
2 REVIEW OF LITERATURE

This chapter introduces the motivation behind the study of triple systems and mixed triple systems.

2.1 Triple Systems

Our problem originates from the study of triple systems. This originated with Kirkman’s schoolgirl problem. Kirkman’s schoolgirl problem is a problem in combinatorics solved by Thomas Kirkman in 1847 as Query VI in “The Lady’s and Gentleman’s Diary” [17].

The problem states: “Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.”

The problem can be generalized to $n$ girls, where $n$ is an odd multiple of three. The $n$ girls need to be walking in triplets for $(n - 1)/2$ days, with the requirement that no pair of girls walk in the same row twice. A complete solution to the general case was given by D. K. Ray-Chaudhuri and R. M. Wilson in 1969 [23].

We can show that Kirkman’s Schoolgirl problem is equivalent to the decomposition of $K_{15}$ into copies of $K_3$. Let the fifteen girls be represented by the vertices of $K_{15}$. As any two girls can walk in a row, there is an edge between any two vertices. The “three abreast rows” mean $K_3$-blocks. Further, “no two shall walk abreast” means that the edges of these triangles are disjoint [17]. The solution to this generalization is a Steiner Triple System: $S(2, 3, 6t + 3)$ with parallelism (that is, one in which each
of the $6t + 3$ elements occurs exactly once in each block of $3 - \text{element}$ sets). A solution to Kirkman’s Schoolgirl Problem is given in Table 1.

<table>
<thead>
<tr>
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<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thur</th>
<th>Fri</th>
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<td>13,14,2</td>
<td>15,1,4</td>
<td>6,7,10</td>
</tr>
</tbody>
</table>

Next, this problem was worked on by Jacob Steiner [24], who was unaware at the time of the earlier work by Kirkman. In combinatorial mathematics, a Steiner system is a type of block design. A Steiner system with parameters $l, m, n$, written $STS(l, m, n)$, is an $n$-element set $S$ together with a set of $m - \text{element}$ subsets of $S$ (called blocks) with the property that each $l - \text{element}$ subset of $S$ is contained in exactly one block. A Steiner system with parameters $\ell, m, n$ is often called simply $STS(\ell, m, n)$. An $STS(2, 3, n)$ is called a Steiner triple system, and its blocks are called triples. The number of triples is $n(n - 1)/6$.

A Steiner triple system of order $v$, $STS(v)$, is a decomposition of the complete graph on $v$ vertices, $K_v$, into $3 - \text{cycles}$.

Steiner proved that $v \equiv 1, 3 \pmod{6}$ is necessary [24] and Reiss [21] showed that this was also sufficient. Reiss and Steiner were both unaware that the problem had been posed and solved by Kirkman in 1847 [17].

A Mendelsohn triple system, denoted $MTS(v)$, is a decomposition of $D_v$ into
copies of $A$, where $A$ is the directed graph with $V(D_v) = \{a, b, c\}$,

$$A(A) = \{(a, b), (b, c), (c, a)\}$$

This graph is also called a 3 - circuit. A Mendelsohn triple system of order $v$ is equivalent to an $A$-decomposition of $D_v$ and exists if and only if $v \equiv 0, 1 \pmod{3}, v \neq 6$.

Directed triple system denoted $DTS(v)$ is decomposition of $D_v$ into copies of $B$ where $V(B) = \{a, b, c\}$, $A(B) = \{(a, b), (c, b), (c, a)\}$. A Directed triple system of order $v$ is equivalent to an $B$-decomposition of $D_v$ and exists if and only if $v \equiv 0, 1 \pmod{3}$. A Directed Triple System exists if and only if $v = 0, 1 \pmod{3}$ [8]. Figure 8 illustrates an example of these graphs.

Triple systems were further expanded in Hartman and Mendelsohn’s “Last of the Triple systems”. Figure 9 illustrates the thirteen connected digraphs on three vertices by Hartman and Mendelsohn.

Inspired by Hartman and Mendelsohn, Robert Gardner [8] gave necessary and sufficient conditions for the existence of new triple systems which are given by presenting decompositions of the complete mixed graph into partial orientations of $K_3$. We will denote $T_i$ by the ordered triple $(a, b, c)_i$. Figure 10 gives examples of these
graphs. This is decomposition is called a $T_i$-triple system of order $v$ where $i = 1, 2$ or 3 [8]. The necessary and sufficient conditions are stated in Proposition 2.1.

**Proposition 2.1** [8]

1. A $T_1$-triple system of order $v$ exists if and only if $v \equiv 1 \pmod{2}$.

2. A $T_2$-triple system of order $v$ exists if and only if $v \equiv 1 \pmod{2}$.

3. A $T_3$-triple system of order $v$ exists if and only if $v \equiv 1 \pmod{2}$, $v \notin \{3, 5\}$.

A decomposition of $M_v$ into copies of $T_i$ is a $T_i$ Mixed Triple System of order $v$ [8].
2.2 Ringel’s Conjecture

It has been conjectured by Ringel that any tree $T$ size $m$ will decompose $K_{2m+1}$[?]. Ringel’s Conjecture was the primary motivation for Rosa to introduce valuations. Alexander Rosa developed several valuations on graphs. The most influential are $\beta$ - valuations [22]. These valuations were popularized by Golomb under the name graceful labelings [9].

A graceful labeling of a graph with $n$ vertices and $e$ edges is a labeling of its vertices with distinct integers between 0 and $e$ inclusive, such that each edge is uniquely identified by the positive, or absolute difference, between its endpoints. A graceful labeling will induce a cyclic decomposition of the complete graph [9]. Figure 11 illustrates a graph with a graceful labeling.

![Figure 11: A Gracefully Labeled Graph](image)

An outline of results for graceful labelings and related labelings are given in Gallian’s [7] “Dynamic Survey of Graph Labeling.” This is summarized in Proposition 2.2.

**Proposition 2.2** The following graphs are known to have a graceful labeling:

(i) Trees with at most four endpoints [15, 16, 22, 26].
(ii) Trees in which, at any two vertices $u$ and $v$, there is a $P_k$ with $u$ and $v$ as its endpoints, where $k \leq 5$ [14, 26].

(iii) Trees of order at most 27 [1].

(iv) Complete bipartite graphs [9, 22].

(v) Cycles of length $n$ where $n \equiv 0 \pmod{4}$ or $n \equiv 3 \pmod{4}$ [18].

(vi) $W_n$ (i.e., Wheel Graphs) [6][5].

(vii) The $n$-dimensional hypercube, $Q_n$ [19, 20].

**Proposition 2.3** [3] If $n \geq 2q + 1$, then a graceful graph of size $q$ will decompose $C_n(1, \ldots, q)$.

We will use these results for our constructions in the next chapter.
3 RESULTS

In this chapter we will give necessary and sufficient conditions for the decomposition of $M_v$ using all possible partial orientations of the $P_4$.

As $P_4^5$ is the converse of $P_4^1$, $P_4^6$ is the converse of $P_4^2$, and $P_4^7$ is the converse of $P_4^3$, we only need to find necessary and sufficient conditions of partial orientations of $P_4^1$, $P_4^2$, $P_4^3$, $P_4^4$ by Proposition 1.1. Our notation will be as follows:

i. $[a, b, c, d]_1$ denotes a $P_4^1$-block with \{(b, a), (c, b), [c, d]\}.

ii. $[a, b, c, d]_2$ denotes a $P_4^2$-block with \{(b, a), (b, c), [c, d]\}.

iii. $[a, b, c, d]_3$ denotes a $P_4^3$-block with \{(b, a), (c, d), [b, c]\}.

iv. $[a, b, c, d]_4$ denotes a $P_4^4$-block with \{(b, a), (d, c), [b, c]\}.

Theorem 3.1 i. There exists a $P_4^1$-decomposition of $M(C_n)$ if and only if $n \geq 4$ and $n \equiv 0 \pmod{2}$.

ii. There exists a $P_4^2$-decomposition of $M(C_n)$ if and only if $n \geq 4$.

iii. There exists a $P_4^3$-decomposition of $M(C_n)$ if and only if $n \geq 4$. 

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iv. There exists a $P_4^4$-decomposition of $M(C_n)$ if and only if $n \geq 4$ and $n \equiv 0 \pmod{4}$.

**Proof.** Note $n \geq 4$ is necessary in all cases [4].

i. Take the set of blocks $[2i, 2i + 1, 2i + 2, 2i + 3]_1$ and $[2i, 2i - 1, 2i - 2, 2i - 3]_1$ where $i = 0, 1, 2, ..., n - 1$. This will give $P_4^1$-decomposition for $M(C_n)$, $n \geq 4$ and $n \equiv 0 \pmod{4}$. Suppose $n$ is odd and there exists a $P_4^1$-decomposition of $C_{2n+1}$. Without loss of generality, suppose we start with the block $[0, 1, 2, 3]_1$. We then need the block $[2, 3, 4, 5]_1$ which is forced. Since the arc $(2, 3)$ cannot be in another block, we must use the set of blocks of form $[2i, 2i + 1, 2i + 2, 2i + 3]_1$. However, this gives a contradiction since $i = n$ and as $[0, 1, 2, 3]_1$ and $[2n, 0, 1, 2]_1$ share the arc $(1, 2)$. Thus, there exists a $P_4^1$-decomposition of $M(C_n)$ if and only if $n \geq 4$ and $n \equiv 0 \pmod{2}$.

ii. Take the set of blocks $[i, i + 1, i + 2, i + 3]_2$ where $i = 0, 1, 2, ..., n - 1$. This will yield the $P_4^2$-decomposition of $M(C_n)$, for $n \geq 4$.

iii. Take the set of blocks $[i, i + 1, i + 2, i + 3]_3$ where $i = 0, 1, 2, ..., n - 1$. This will yield the $P_4^3$-decomposition of $M(C_n)$, for $n \geq 4$.

iv. Without loss of generality take the blocks $[0, 1, 2, 3]_4$ and $[4, 3, 2, 1]_4$. This forces the set of blocks $[5, 4, 3, 2]_4$ and $[3, 4, 5, 6]_4$. These in turn force the set of blocks $[4i, 4i + 1, 4i + 2, 4i + 3]_4$, $[4i + 4, 4i + 3, 4i + 2, 4i + 1]_4$, $[4i + 5, 4i + 4, 4i + 3, 4i + 2]_4$, and $[4i + 3, 4i + 4, 4i + 5, 4i + 6]_4$. Thus, if $n \equiv 0 \pmod{4}$ this gives the required $P_4^1$-decomposition for $C_n$, where $n = 4t$, $i = 1, ..., t$ and $n \geq 4$. If
$n \equiv 1 \pmod{4}$, $n = 4t + 1$, we notice that the set of blocks $[0, 1, 2, 3]_4$ and $[n-2, n-1, 0, 1]_4$ share the arc $(0, 1)$. If $n \equiv 2 \pmod{4}$ we will have the set of blocks $[0, 1, 2, 3]_4$, $[n-3, n-2, n-1, 0]_4$, and $[n-1, n-2, n-3, n-4]_4$. Now we are missing the directed arcs $(1, 2), (0, n-1), (n-2, n-1)$ and the undirected edges $[0, 1], [n-1, 0]$. This then forces the block $[n-2, n-1, 0, 1]_4$, but the arc $(0, 1)$ is shared. When $n \equiv 3 \pmod{4}$, $4i + 3 = n$ and $i = (n-3)/4$ we remove the block $[0, 1, 2, 3]_4$. Now we have the set of blocks $[4, 3, 2, 1]_4$, $[2, 1, 0, n-1]_4$ which share the arc $(2, 1)$. Thus, there exists a $P^4_4$-decomposition of $C_n$ if and only if $n \equiv 0 \pmod{4}$, $n \geq 4$.

\begin{quote}
\textbf{Theorem 3.2} \hspace{1em} i. There exists a $P^1_4$-decomposition of $M(W_n)$ if and only if $n \geq 3$.

\hspace{1em} ii. There does not exist a $P^2_4$-decomposition of $M(W_n)$ for any $n$.

\hspace{1em} iii. There exists a $P^3_4$-decomposition of $M(W_n)$ if and only if $n \geq 3$.

\hspace{1em} iv. There exists a $P^4_4$-decomposition of $M(W_n)$ if and only if $n \geq 3$.
\end{quote}

\textit{Proof.} Note that $n \geq 3$ is necessary in all cases [4]. Also note that $c$ denotes the center of the wheel and the vertices of $n$-cycle are denoted by elements of $\mathbb{Z}_n$.

i. Take the set of blocks $[i, i+1, c, i+2]_1$ and $[c, i+2, i+1, i]_1$ for $i = 0, 1, 2..., n-1$.

This will yield a $P^4_4$-decomposition of $M(W_n)$.

ii. Note that $P^2_4$ has only one vertex with positive out degree and this vertex “b” has out degree two, and the vertices on the cycle of $W_n$ have odd out degree. It
follows that a necessary condition to the existence of a $P_4^2$-decomposition of $H$ is for every vertex in $H$ to have an even out degree. Hence there does not exist a $P_4^2$-decomposition of $M(W_n)$.

iii. Take the set of blocks $[i, i + 1, i + 2, c]_3$ and $[i, c, i + 1, i + 2]_3$ for $i = 0, 1, 2, ..., n - 1$. This will yield the $P_4^3$-decomposition of $M(W_n)$ for $n \geq 3$.

iv. Take the set of blocks $[c, i + 2, i + 1, i]_4$ and $[i, c, i + 1, i + 2]_4$ for $i = 0, 1, 2, ..., n - 1$. This will yield the $P_4^4$-decomposition of $M(W_n)$ for $n \geq 3$.

\[\text{Theorem 3.3} \quad \text{i. There exists a } P_4^1\text{-decomposition of } M(C_n(1, 2)) \text{ if and only if } n \geq 4.\]

\[\text{ii. There exists a } P_4^2\text{-decomposition of } M(C_n(1, 2)) \text{ if and only if } n \geq 5.\]

\[\text{iii. There exists a } P_4^3\text{-decomposition of } M(C_n(1, 2)) \text{ if and only if } n \geq 4.\]

\[\text{iv. There exists a } P_4^4\text{-decomposition of } M(C_n(1, 2)) \text{ if and only if } n \geq 4.\]

\[\text{Proof.} \quad \text{Note that in all cases } C_4(1, 2) \text{ is isomorphic to } W_3. \text{ Hence a decomposition of } C_4(1, 2) \text{ exists in all cases except for } P_4^2. \text{ Further, for } n \geq 4 \text{ is necessary in all cases [4]. As such, it suffices to give constructions for } n \geq 5.\]

\[\text{i. Take the set of blocks } [i + 2, i + 3, i + 1, i]_1 \text{ and } [i + 1, i + 3, i + 2, i]_1, \text{ } i = 0, 1, 2, ..., n - 1. \text{ This will give the } P_4^1\text{-decomposition of } M(C_n(1, 2)).\]

\[\text{ii. Take the set of blocks } [i + 1, i, i + 2, i + 3]_2 \text{ and } [i, i + 2, i + 1, i + 3]_2, \text{ } i = 0, 1, 2, ..., n - 1. \text{ This will give the } P_4^2\text{-decomposition of } M(C_n(1, 2)).\]
iii. Take the set of blocks \([i + 3, i + 1, i + 2, i]_3\) and \([i + 4, i + 3, i + 2, i]_3\), \(i = 0, 1, 2, ..., n - 1\). This will give the \(P^3_4\)-decomposition of \(M(C_n(1, 2))\).

iv. Take the blocks \([i, i + 2, i + 3, i + 1]_4\) and \([i + 2, i + 3, i + 1, i]_4\), \(i = 0, 1, 2, ..., n - 1\). This will give the \(P^4_4\)-decomposition of \(M(C_n(1, 2))\).

\[\text{Theorem 3.4} \]

i. There exists a \(P^1_4\)-decomposition of \(M(C_n(1, 2, 3))\) if and only if \(n \geq 6\).

ii. There exists a \(P^2_4\)-decomposition of \(M(C_n(1, 2, 3))\) if and only if \(n \geq 7\).

iii. There exists a \(P^3_4\)-decomposition of \(M(C_n(1, 2, 3))\) if and only if \(n \geq 6\).

iv. There exists a \(P^4_4\)-decomposition of \(M(C_n(1, 2, 3))\) if and only if \(n \geq 6\).

\[\text{Proof.}\] Note that \(n \geq 6\) is necessary in all cases [4].

i. Take the set of blocks \([i + 2, i + 3, i + 1, i]_1\) and \([i + 3, i + 5, i + 2, i]_1\), and \([i + 1, i + 4, i + 3, i]_1\), \(i = 0, 1, 2, ..., n - 1\). This will give the \(P^1_4\)-decomposition of \(M(C_n(1, 2, 3))\).

ii. Note that when \(n = 6\) we have \(C_6(1, 2, 3)\) is isomorphic to \(K_6\). Note that every vertex in \(C_6(1, 2)\) has five out degree arcs, this leaves only one vertex with a positive degree namely “b” which gives a contradiction by an in degree out degree argument from previous work. Hence a necessary condition for all vertices to have an even out degree. In all other cases, take the set of blocks
\[ [i+6, i+3, i+1, i]_2, [i+6, i+5, i+2, i]_2, \text{ and } [i+6, i+4, i+3, i]_2, i = 0, 1, 2, ..., n-1 \]

for \( n \geq 7 \). This will give the \( P^2_4 \)-decomposition of \( M(C_n(1, 2, 3)) \).

iii. Take the set of blocks \([i, i+2, i+1, i+3]_3, [i, i+3, i+1, i+4]_3, \text{ and } [i+2, i+3, i, i+1]_3, i = 0, 1, 2, ..., n-1 \). This will give the \( P^3_4 \)-decomposition of \( M(C_n(1, 2, 3)) \).

iv. Take the blocks \([i+1, i+3, i+2, i]_4, [i+2, i+5, i+3, i]_4, \text{ and } [i+3, i+4, i+1, i]_4, i = 0, 1, 2, ..., n-1 \). This will give the \( P^4_4 \)-decomposition of \( M(C_n(1, 2, 3)) \).

\[ \square \]

**Theorem 3.5** There exists a \( P^i_4 \)-decomposition for \( M(C_n(1, 2, 3, 4)) \) for every \( n \geq 9 \).

**Proof.** Note \( P^i_4 | M(C_4) \) by Theorem 3.1, further \( C_4 \) is graceful [22]. Thus by [3] we have a \( C_4 \)-decomposition of \( C_n(1, 2, 3, 4) \) for every \( n \geq 9 \). Thus, by Proposition 1.1 there is a \( P^i_4 \)-decomposition of \( M(C_n(1, 2, 3, 4)) \) for \( n \geq 9 \).

\[ \square \]

**Lemma 3.6** Let \( K = \{W_p : n \geq 5, p = 0, p \geq 3 \} \). Then there exists a \( K \)-decomposition of \( K(3k + 4p + 1, k) \), for all \( k \).

**Proof.** Treat \( k \) vertices as fixed points. The center of the wheel requires \( k \) pairs differences from \( \mathbb{Z}_{2k+4p+1} \). \( \mathbb{Z}_{2k+4p+1} \) has differences \( \{1, 2, ..., 2p, 2p+1, ..., 2p+k\} \). Use differences \( 2p+1, ..., 2p+k \) to generate wheels. This leaves \( C_{2k+4p+1}(1, ..., 2p) \). Note \( W_p \) has a graceful labeling [5] and \( e(W_p) = 2p \), a graceful graph size \( q \) will decompose \( C_n(1, ..., q) \), where \( n \geq 2q + 1 \) [3].

\[ \square \]
Theorem 3.7 There exists $P_4^1$-decomposition of $M_v$ for every $v \geq 4$.

Proof. Note that $v \geq 4$ is necessary. Let $v \equiv 0 \pmod{4}$. Then $v = 4p + 4$. Use a single vertex as a single fixed point and we have $M(K(4p+3,1))$. Use $2p+1$ differences to generate the outside cycle of the wheel. We know that $W_p|C_{4p+3}(1,\ldots,2p)$ for $p = 0$, $p \geq 3$. Thus, we must decompose the remaining cases $p = 1$ and $p = 2$. For $p = 1$, we have $M(C_7(1,2))$ which by Theorem 3.3 can be decomposed by $P_4^1$. When $p = 2$, we have $M(C_{11}(1,2,3,4))$ which by Theorem 3.5 $M(C_{11}(1,2,3,4))$ can be decomposed by $P_4^1$.

Next, let $v \equiv 1 \pmod{4}$, say $v = 4p + 1$. Note that $K_{4p+1}$ is isomorphic to $C_{4p+1}(1,\ldots,2p)$. Since $W_p|C_{4p+1}(1,\ldots,2p)$ for $p = 0$ and $p \geq 3$, we are left with $p = 1$ and $p = 2$. When $p = 1$, we have $M(C_5(1,2))$ which can be decomposed by $P_4^1$ by Theorem 3.3. For $p = 2$, we have $M(C_9(1,2,3,4))$ has a $P_4^1$-decomposition by Theorem 3.5.

For $v = 6$, take the blocks $[2j + 2, 2j + 1, 2j + 3]_1$, $[i + 2, i + 4, i + 1, i]_1$, $[i + 3, i + 1, i + 2, i]_1$, $i = 0, 1, \ldots, n - 1$ and $j = 0, 1, 2$. This we will give the $P_4^1$-decomposition for $M_6$. For $v = 7$, we have $K_7 = C_7(1,2,3)$ which by Theorem 3.4 can be decomposed by $P_4^1$.

Next, let $v \equiv 2 \pmod{4}$, $v = 4p + 22$. Treat seven vertices as fixed points. We know $W_p|C_{4p+15}(1,\ldots,2p)$, $p = 0$, $p \geq 3$. This leaves the remaining cases $p = 1$ and $p = 2$. When $p = 1$, we have $C_{19}(1,2)$ which can be decomposed by Theorem 3.3. For $p = 2$, we have $M(C_{23}(1,2,3,4))$ and by Theorem 3.5, $M(C_{23}(1,2,3,4))$ can be decomposed by $P_4^1$. Now we are left with $v = 6, 10, 14, 18$. By above there is a decomposition for $v = 6$. When $v = 10$, $K_{10} = W_9 \cup C_9(1,2,3)$. By Theorem
3.2 and Theorem 3.4, we can decompose $M(W_9)$ and $M(C_9(1, 2, 3))$ respectively. Hence there is a $P_4^1$-decomposition for $M_{10}$. When $v = 14$, we have $K_{14} = W_{13} \cup C_{13}(1, 2) \cup C_{13}(3, 4, 5)$. Then take the set of blocks $[i + 4, i + 7, i + 3, i], [i + 2, i + 7, i + 4, i], [i + 7, i, i + 6, i + 1]_1$, and $[i = 0, 1, 2, ..., n - 1$ yields required decomposition of $C_{13}(3, 4, 5)$. By Theorem 3.2 and Theorem 3.3, we can decompose $M(W_{13})$ and $M(C_{13}(1, 2))$. Hence, there is a $P_4^1$-decomposition of $M_{14}$. When $v = 18$, we have $K_{18} = W_{17} \cup C_{17}(1, 2, 3, 4) \cup C_{17}(5, 6, 7)$. We know by Theorem 3.2 and Theorem 3.5, we can decompose $M(W_{17})$ and $M(C_{17}(1, 2, 3, 4))$. Taking the set of blocks $[i + 6, i + 11, i + 5, i], [i + 7, i + 13, i + 6, i], [i + 5, i + 12, i + 7, i]_1$, and $[i = 0, 1, 2, ..., n - 1$ which will give the $P_4^1$-decomposition of $M(C_{17}(5, 6, 7))$.

Finally, let $v \equiv 3 \pmod{4}$, say $v = 4p + 19$. Treat six vertices as fixed points which leaves $4p + 13$. We know $W_p|C_{4p+13}(1, ..., 2p)$ when $p = 0$, $p \geq 3$. When $p = 1$, we have $M(C_{23}(1, 2))$ which can be decomposed by Theorem 3.3. When $p = 2$, we have $M(C_{27}(1, 2, 3, 4))$ and by Theorem 3.5, $M(C_{27}(1, 2, 3, 4))$ has a $P_4^1$-decomposition. We are now left with $p = 7, 11, 15$. From previous work, we know the decomposition exists when $v = 7$. When $v = 11$, we have $K_{11} = C_{11}(1, 2, 3) \cup C_{11}(4, 5)$. The set of blocks $[i + 5, i + 9, i + 4, i], [i + 4, i + 9, i + 5, i]_1$, and $[i = 0, 1, 2, ..., n - 1$ will yield decomposition of $C_{11}(4, 5)$. By Theorem 3.2 and Theorem 3.4 we can decompose $M(W_{13})$ and $C_{13}(1, 2)$. Hence there exists a $P_4^1$-decomposition of $M_{11}$. When $v = 15$, we have $K_{15} = C_{15}(1, 2, 3, 4) \cup C_{15}(5, 6, 7)$. By Theorem 3.5, we can decompose $M(C_{15}(1, 2, 3, 4))$. Take the set of blocks $[i + 6, i + 11, i + 5, i], [i + 7, i + 13, i + 6, i], [i + 5, i + 12, i + 7, i]_1$, and $[i = 0, 1, 2, ..., n - 1$ will yield $P_4^1$-decomposition of $M(C_{15}(5, 6, 7))$.  

■
Theorem 3.8 There exists $P_4^2$-decomposition of $M_v$ if and only if $v \equiv 1 \pmod{2}$, $v \geq 5$.

Proof. If $v \equiv 0 \pmod{2}$, then the directed part of $P_4^2$ will not decompose $D_v$. If $v \leq 3$, there will not be enough vertices [11]. It is known that $C_n|K_v$ [12]. If $v = 2n + 1$. Thus, $M(C_n)|M(K_v)$ and by transitivity, $P_4^2|M_v$ when $v \geq 9$, $v \equiv 1 \pmod{2}$. Thus, the only cases needed to be considered are and $v = 5, 7$. When $v = 5$ we will have $K_5 = C_5 \cup C_5$. When $v = 7$, we will have $K_7 = C_7 \cup C_7 \cup C_7$. 

Theorem 3.9 There exists a $P_4^3$-decomposition of $M_v$ for every $v \geq 4$.

Proof. Note that $v \geq 4$ is necessary. Let $v \equiv 0 \pmod{4}$. Then $v = 4p + 4$. Use a single vertex as a single fixed point and we have $M(K(4p+3,1))$. Use the differences $2p + 1$ to generate the outside cycle of the wheel. We know that $W_p|C_{4p+3}(1,...,2p)$ for $p = 0$, $p \geq 3$. Thus, we must decompose the remaining cases $p = 1$ and $p = 2$. For $p = 1$, we have $M(C_7(1,2))$ which by Theorem 3.3 can be decomposed by $P_4^3$. When $p = 2$, we have $M(C_{11}(1,2,3,4))$ which by Theorem 3.5 $C_{11}(1,2,3,4)$ can be decomposed by $P_4^3$.

Next, let $v \equiv 1 \pmod{4}$, say $v = 4p + 1$. Note that $K_{4p+1}$ is isomorphic to $C_{4p+1}(1,...,2p)$. Since $W_p|C_{4p+1}(1,...,2p)$ for $p = 0$ and $p \geq 3$, we are left with $p = 1$ and $p = 2$. When $p = 1$, we have $M(C_5(1,2))$ which can be decomposed by $P_4^3$ by Theorem 3.3. For $p = 2$, we have $M(C_9(1,2,3,4))$ has a $P_4^3$-decomposition by Theorem 3.5.

For $v = 6$, take the blocks $[2j + 2, 2j + 1, 2j, 2j + 3]_3$, $[i + 2, i + 4, i + 1, i]_3$, $[i + 3, i + 1, i + 2, i]_3$, $i = 0, 1, ..., n - 1$ and $j = 0, 1, 2$. This we will give the $P_4^3$.
decomposition for $M_6$. For $v = 7$, we have $K_7 = C_7(1, 2, 3)$ which by Theorem 3.4 can be decomposed by $P_4^3$.

Next, let $v \equiv 2 \pmod{4}$, $v = 4p + 22$. Treat seven vertices as fixed points. We know $W_p|C_{4p+15}(1, ..., 2p)$, $p = 0$, $p \geq 3$. This leaves the remaining cases $p = 1$ and $p = 2$. When $p = 1$, we have $C_{19}(1, 2)$ which can be decomposed by Theorem 3.3. For $p = 2$, we have $C_{23}(1, 2, 3, 4)$ and by Theorem 3.5, $M(C_{23}(1, 2, 3, 4))$ can be decomposed by $P_4^3$. Now we are left with $v = 6, 10, 14, 18$. By above there is a decomposition for $v = 6$. When $v = 10$, $K_{10} = W_9 \cup C_9(1, 2, 3)$. By Theorem 3.2 and Theorem 3.4, we can decompose $M(W_9)$ and $M(C_9(1, 2, 3))$ respectively. Hence there is a $P_4^3$-decomposition for $M_{10}$. When $v = 14$, we have $K_{14} = W_{13} \cup C_{13}(1, 2) \cup C_{13}(3, 4, 5)$. Taking the set of blocks $[i+3, i+7, i+4, i]_3$, $[i+4, i+9, i+5, i]_3$, and $[i+5, i+8, i+3, i]_3$, $i = 0, 1, 2, ..., n - 1$ yields required decomposition of $M(C_{13}(3, 4, 5))$. By Theorem 3.2 and Theorem 3.3, we can decompose $M(W_{13})$ and $M(C_{13}(1, 2))$. Hence we get the $P_4^3$-decomposition for $M_{14}$. When $v = 18$, we have $K_{18} = W_{17} \cup C_{17}(1, 2, 3, 4) \cup C_{17}(5, 6, 7)$. We know by Theorem 3.2 and Theorem 3.5, $M(W_{17})$, and $M(C_{17}(1, 2, 3, 4))$ can be decomposed by $P_4^3$. Take the set of blocks $[i+6, i+11, i+5, i]_3$, $[i+7, i+13, i+6, i]_3$, and $[i+5, i+12, i+7, i]_3$, $i = 0, 1, 2, ..., n - 1$ we have the $P_4^3$-decomposition $M(C_{17}(5, 6, 7))$.

Finally, let $v \equiv 3 \pmod{4}$, say $v = 4p + 19$. Treat six vertices as fixed points and leaves $4p + 13$. We know $W_p|C_{4p+13}(1, ..., 2p)$ when $p = 0$, $p \geq 3$. When $p = 1$, we have $M(C_{23}(1, 2))$ which can be decomposed by Theorem 3.3. When $p = 2$, we have $M(C_{27}(1, 2, 3, 4))$ and by Theorem 3.5, $M(C_{27}(1, 2, 3, 4))$ has a $P_4^3$-decomposition. We are now left with $p = 7, 11, 15$. From previous work, we know the decomposition exists when $v = 7$. When $v = 11$, we have $K_{11} = C_{11}(1, 2, 3) \cup C_{11}(4, 5)$. The set of blocks
[i, i + 5, i + 1, i + 6]_3, and [i + 1, i + 5, i, i + 4]_3, i = 0, 1, 2, ..., n − 1 will yield the $P_3^3$-decomposition for $M_{11}$. When $v = 15$, we have $K_{15} = C_{15}(1, 2, 3, 4) \cup C_{15}(5, 6, 7)$. By Theorem 3.5, we can decompose $M(C_{15}(1, 2, 3, 4))$ for $P_3^3$. The set of blocks $[i + 6, i + 11, i + 5, i]_3$, $[i + 7, i + 13, i + 6, i]_3$, and $[i + 5, i + 12, i + 7, i]_3$ will yield the $P_3^3$-decomposition of $M(C_{15}(5, 6, 7))$.

**Theorem 3.10** There exists a $P_4^4$-decomposition of $M_v$ for every $v \geq 4$.

**Proof.** Note that $v \geq 4$ is necessary. Let $v \equiv 0 \pmod{4}$. Then $v = 4p + 4$. Use a single vertex as a single fixed point and we have $M(K(4p+3, 1))$. Use $2p+1$ differences to generate the outside cycle of the wheel. We know that $W_p|C_{4p+3}(1, ..., 2p)$ for $p = 0$, $p \geq 3$. Thus, we must decompose the remaining cases $p = 1$ and $p = 2$. For $p = 1$, we have $M(C_7(1, 2))$ which by Theorem 3.3 can be decomposed by $P_4^4$. When $p = 2$, we have $M(C_{11}(1, 2, 3, 4))$ which by Theorem 3.5 $M(C_{11}(1, 2, 3, 4))$ can be decomposed by $P_4^4$.

Next, let $v \equiv 1 \pmod{4}$, say $v = 4p + 1$. Note that $K_{4p+1}$ is isomorphic to $C_{4p+1}(1, ..., 2p)$. Since $W_p|C_{4p+1}(1, ..., 2p)$ for $p = 0$ and $p \geq 3$, we are left with $p = 1$ and $p = 2$. When $p = 1$, we have $M(C_5(1, 2))$ which can be decomposed by $P_4^4$ by Theorem 3.3. For $p = 2$, we have $M(C_9(1, 2, 3, 4))$ has a $P_4^4$-decomposition by Theorem 3.5.

For $v = 6$, take the blocks $[2j + 2, 2j + 1, 2j, 2j + 3]_4$, $[i + 2, i + 4, i + 1, i]_4$, $[i + 3, i + 1, i + 2, i]_4$, $i = 0, 1, ..., n - 1$ and $j = 0, 1, 2$. This we will give the $P_4^4$-decomposition for $M_6$. For $v = 7$, we have $K_7 = C_7(1, 2, 3)$ which by Theorem 3.4 can be decomposed by $P_4^4$. 

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Next, let \( v \equiv 2 \mod 4 \), \( v = 4p + 22 \). Treat seven vertices as fixed points. We know \( W_p|C_{4p+15}(1, \ldots, 2p) \), \( p = 0, p \geq 3 \). This leaves the remaining cases \( p = 1 \) and \( p = 2 \). When \( p = 1 \), we have \( C_{19}(1,2) \) which can be decomposed by Theorem 3.3. For \( p = 2 \), we have \( M(C_{23}(1,2,3,4)) \) and by Theorem 3.5, \( M(C_{23}(1,2,3,4)) \) can be decomposed by \( P_4^4 \). Now we are left with \( v = 6, 10, 14, 18 \). By above there is a decomposition for \( v = 6 \). When \( v = 10 \), \( K_{10} = W_9 \cup C_9(1,2,3) \). By Theorem 3.2 and Theorem 3.4, we can decompose \( M(W_9) \) and \( M(C_9(1,2,3)) \) respectively. Hence there is a \( P_4^4 \)-decomposition for \( M_{10} \). When \( v = 14 \), we have \( K_{14} = W_{13} \cup C_{13}(1,2) \cup C_{13}(3,4,5) \). Taking the set of blocks \([i + 3, i + 7, i + 4, i]_4\), \([i + 4, i + 9, i + 5, i]_4\), and \([i + 5, i + 12, i + 7, i]_4\), \( i = 0, 1, 2, \ldots, n - 1 \) yields required decomposition of \( C_{13}(3,4,5) \). By Theorem 3.2 and Theorem 3.3, we can decompose \( M(W_{13}) \) and \( M(C_{13}(1,2)) \). Hence there is a \( P_4^4 \)-decomposition of \( M_{14} \). When \( v = 18 \), we have \( K_{18} = W_{17} \cup C_{17}(1,2,3,4) \cup C_{17}(5,6,7) \). We know by Theorem 3.2 and Theorem 3.5, how to decompose \( M(W_{17}) \) and \( M(C_{17}(1,2,3,4)) \). Take the set of blocks \([i + 6, i + 11, i + 5, i]_4\), \([i + 7, i + 13, i + 6, i]_4\), and \([i + 5, i + 12, i + 7, i]_4\), \( i = 0, 1, 2, \ldots, n - 1 \) will yield the \( P_4^4 \)-decomposition of \( M(C_{17}(5,6,7)) \).

Finally, let \( v \equiv 3 \mod 4 \), say \( v = 4p + 19 \). Treat six vertices as fixed points and leaves \( 4p + 13 \). We know \( W_p|C_{4p+13}(1, \ldots, 2p) \) when \( p = 0, p \geq 3 \). When \( p = 1 \), we have \( M(C_{23}(1,2)) \) which can be decomposed by Theorem 3.3. When \( p = 2 \), we have \( C_{27}(1,2,3,4) \) and by Theorem 3.5, \( M(C_{27}(1,2,3,4)) \) has a \( P_4^4 \)-decomposition. We are now left with \( p = 7, 11, 15 \). From previous work, we know the decomposition exists when \( v = 7 \). When \( v = 11 \), we have \( K_{11} = C_{11}(1,2,3) \cup C_{11}(4,5) \). The set of blocks \([i, i + 5, i + 1, i + 6]_4\), \([i + 1, i + 5, i, i + 4]_4\), \( i = 0, 1, 2, \ldots, n - 1 \) yields a \( P_4^4 \)-decomposition.
of $M(C_{11}(4,5))$. When $v = 15$ we can decompose $C_{15}(1, 2, 3, 4)$ by Theorem 3.5. Take the set of blocks $[i+6, i+11, i+5, i]_4$, $[i+7, i+13, i+6, i]_4$, and $[i+5, i+12, i+7, i]_4$, $i = 0, 1, 2, ..., n - 1$ will yield $P_4^1$-decomposition of $M(15(5,6,7))$.

**Theorem 3.11** There exists a $P_4^5$-decomposition of $M_v$ for every $v \geq 4$.

*Proof.* As $P_4^5$ is the converse of $P_4^1$ and by Proposition 1.1 the same proof follows from Theorem 3.7.

**Theorem 3.12** There exists a $P_4^6$-decomposition of $M_v$ for every $v \geq 4$.

*Proof.* As $P_4^6$ is the converse of $P_4^6$ and by Proposition 1.1 the same proof follows from Theorem 3.9.

**Theorem 3.13** There exists a $P_4^7$-decomposition of $M_v$ for every $v \geq 4$.

*Proof.* As $P_4^7$ is the converse of $P_4^3$ and by Proposition 1.1 the same proof follows from Theorem 3.10.
Within this paper, we have established the necessary and sufficient conditions for decompositions of $M_v$ and other mixed graphs using partial orientations of $P_4$. There are still several related problems using the partial orientations of the $P_4$. One open problem to look at packings and coverings. A packing is a set of $\{G_1, ..., G_k\}$-blocks where $G_i \subset V(H)$, $\cup_{i=1} E(G_i) \subset E(H)$, and $E(G_i) \cap E(G_j) = \emptyset$. The leave is $L = E(H) \setminus \cup_{i=1} E(G_i)$. A maximal packing of simple graph $H$ with isomorphic copies of graph $G$ is a set $\{G_1, G_2, ..., G_n\}$ where $G_i \cong G$ and $V(G_i) \subset V(H)$ for all $i, j$, $E(G_i) \cap E(G_i) = \emptyset$.

A minimal covering of simple graph $H$ with isomorphic copies of graph $G$ is a set $\{G_1, G_2, ..., G_n\}$ where $G_i \cong G$ and $V(G_i) \subset V(H)$, for every $i$, $H \subseteq \cup_{i=1}^{n} G_i$ is minimal. The graph $P$ is called the padding of the covering.

Another open relation problem is looking at $\lambda M_n$-designs. A $\lambda M_n$-design is defined as a partition of the edges of $\lambda M_n$ into subgraphs called $G$-blocks each of which is isomorphic to $G$ where $G = (V(G), E(G))$. An open problem to consider is taking
the partial orientations of $C_6$, corona graphs, and the graphs in Figure 13. Also there is hybrid decompositions where there are two or more prototypes $G_1$ and $G_2$. Let $K = \{G_1, ..., G_k\}$. There is a $K$-decomposition of $H$ in there are $n_i$ copies of $G_i$. Finally, an open problem to consider is looking at partial orientations of the graphs in Figure 13.


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