Cost Effective Domination in Graphs

Tabitha Lynn McCoy

East Tennessee State University

Follow this and additional works at: http://dc.etsu.edu/etd

Recommended Citation

This Thesis - Open Access is brought to you for free and open access by Digital Commons @ East Tennessee State University. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of Digital Commons @ East Tennessee State University. For more information, please contact dcadmin@etsu.edu.
Cost Effective Domination in Graphs

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

by

Tabitha McCoy

December 2012

Teresa W. Haynes, Ph.D., Chair

Robert A. Beeler, Ph.D.

Debra Knisley, Ph.D.

Keywords: cost effective domination, cost effective domination number
ABSTRACT

Cost Effective Domination in Graphs

by

Tabitha McCoy

A set $S$ of vertices in a graph $G = (V,E)$ is a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. A vertex $v$ in a dominating set $S$ is said to be cost effective if it is adjacent to at least as many vertices in $V \setminus S$ as it is in $S$. A dominating set $S$ is cost effective if every vertex in $S$ is cost effective. The minimum cardinality of a cost effective dominating set of $G$ is the cost effective domination number of $G$. In addition to some preliminary results for general graphs, we give lower and upper bounds on the cost effective domination number of trees in terms of their domination number and characterize the trees that achieve the upper bound. We show that every value of the cost effective domination number between these bounds is realizable.
DEDICATION

I’m dedicating this thesis to my grandfather, Mack Tuggle. He was a true inspiration to me and all those around him of how to live a faithful life in God. He taught me to seek God in all I do, and to never go a single day without praying. It is also he who taught me how to love the ones in my life. I am blessed to have had such a precious man to call my papaw.
ACKNOWLEDGMENTS

First and foremost, I want to thank Dr. Teresa Haynes for all of her encouragement and support throughout this process. The love she has for Graph Theory radiates from her, and without her this thesis would not have been possible. She believed in me when I did not believe in myself, and for that I will always be grateful. I also would like to thank Dr. Stephen Hedetniemi for all of his insight and feedback; it truly was invaluable. I’m also thankful to my committee members, Dr. Robert Beeler and Dr. Debra Knisley, for helping me along the way and during the revision process.

I also can’t go without thanking one of my previous teachers, Mrs. Lisa Henley, for instilling the love of math in me. Through her enthusiasm for math, she inspired me to follow my heart and pursue a major in mathematics. Also, to my fellow graduate students that have grown to become my friends instead of just classmates, thank you for making this journey such a positive one.

I also want to thank my wonderful family for all of their love and support throughout my entire life, and especially during my college career. They are my biggest cheerleaders and have always been there for me. If it was not for them, I would not be the woman I am today. And to my boyfriend Ryan, you are the best. Thank you for being you.
CONTENTS

ABSTRACT ................................................................. 2
DEDICATION ............................................................. 4
ACKNOWLEDGMENTS .................................................... 5
LIST OF FIGURES .......................................................... 7
1 INTRODUCTION .......................................................... 8
  1.1 Introduction to Graph Theory ...................................... 8
  1.2 Cost Effective Domination ......................................... 10
2 BACKGROUND AND TERMINOLOGY ................................. 14
  2.1 Unfriendly Partitions ............................................... 14
  2.2 Differentials in Graphs .......................................... 16
3 PRELIMINARY RESULTS ............................................... 18
4 MAIN RESULTS .......................................................... 23
5 CONCLUDING REMARKS ............................................... 31
BIBLIOGRAPHY ............................................................ 32
VITA ................................................................. 35
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Graphs $K_4, C_5, P_3 \Box P_4$ and $K_{2,3}$</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>Graphs with $\gamma(G_1) = 3$ and $\gamma(G_2) = 4$</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>Graph with $\gamma_c(G) = 7$ and $\gamma_{vce}(G) = 8$</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>Graph with $\gamma_c(G) = 8$ and $\gamma_{vce}(G) = 10$</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>Graphs that do not have cost effective $\gamma$-sets</td>
<td>19</td>
</tr>
<tr>
<td>6</td>
<td>Trees $T_1$ and $T_3$</td>
<td>22</td>
</tr>
<tr>
<td>7</td>
<td>Tree $T$ that achieves the upper bound of Theorem 4.1</td>
<td>27</td>
</tr>
<tr>
<td>8</td>
<td>Tree $T$ with $\gamma(T) = a = 6$ and $\gamma_c(T) = b = 8$</td>
<td>30</td>
</tr>
</tbody>
</table>
1 INTRODUCTION

1.1 Introduction to Graph Theory

A graph $G$ is a nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of 2-element subsets of $V(G)$ called edges. To indicate that a graph $G$ has vertex set $V(G)$ and edge set $E(G)$, we write $G = (V, E)$. We consider simple, finite graphs, that is, graphs with no loops or multiple edges. Each edge $\{u, v\}$ of $G$ is typically denoted by $uv$ or $vu$, and $u$ and $v$ are called adjacent vertices. Two adjacent vertices are called neighbors of each other. The degree of a vertex $v$ in a graph $G$ is the number of vertices in $G$ adjacent to $v$. A vertex $v$ is said to be even or odd, according to whether its degree in $G$ is even or odd. Also, two edges are called adjacent edges if $uv$ and $vw$ are distinct edges in $G$. The vertex $u$ and the edge $uv$ are said to be incident to each other.

The number of vertices in a graph $G$ is the order of $G$, and the number of edges is the size of $G$. We let $|V(G)| = n$ and $|E(G)| = m$. A graph of order 1 is called a trivial graph, and a graph of order 2 or more is called a nontrivial graph. A graph of size 0 is called an empty graph. A nonempty graph has one or more edges. The complete graph of order $n$, denoted $K_n$, is the graph for which every two distinct vertices are adjacent. Thus, $K_n$ has size $n(n - 1)/2$. The path on $n \geq 1$ vertices, denoted $P_n$, is a graph of order $n$ and size $n - 1$. The length of a path is the number of edges it contains. A graph $G$ is connected if for every pair of vertices in $V(G)$, there exists a path between them. The cycle on $n$ vertices, denoted $C_n$, is a closed path, $P_n$, and has order $n$ and size $n$. The length of a cycle is the number of edges
it contains. An acyclic graph has no cycles. A tree is a connected acyclic graph. A graph $G$ is bipartite if $V(G)$ can be partitioned into two independent sets. A complete bipartite graph is a bipartite graph with partitions $V_1$ and $V_2$ such that every vertex in $V_1$ is adjacent to every vertex in $V_2$. If $|V_1| = s$ and $|V_2| = t$, then the complete bipartite graph is denoted $K_{s,t}$ and has order $s + t$ and size $st$. We note that trees are bipartite. A graph $H$ is a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph $G$ is regular if the all vertices of $G$ have the same degree, say $r$. Such graphs are called $r$-regular. A 3-regular graph is also called a cubic graph. The cartesian product of two graphs $G_1$ and $G_2$, commonly denoted by $G_1 \square G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$ and two distinct vertices $(u,v)$ and $(x,y)$ of $G_1 \square G_2$ are adjacent if either

$$(1) \ u = x \text{ and } vy \in E(G_2) \text{ or } (2) \ v = y \text{ and } ux \in E(G_1).$$

Figure 1 gives examples of the graphs $K_4$, $C_5$, $P_3 \square P_4$ and $K_{2,3}$.

For a graph $G = (V,E)$, the open neighborhood of a vertex $u \in V$ is the set $N(u) = \{v \mid uv \in E\}$, and the closed neighborhood of $u$ is the set $N[u] = N(u) \cup \{u\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{u \in S} N(u)$, and the closed neighborhood of a set $S$ is the set $N[S] = N(S) \cup S$. A set $S$ of vertices is independent if no two vertices in $S$ are adjacent and is a dominating set if $N[S] = V$, that is, every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ of a graph $G$ equals the minimum cardinality of a dominating set in $G$ (see Figure 2 for examples where the darkened vertices represent $\gamma(G)$-sets), while the upper domination number $\Gamma(G)$ equals the maximum cardinality of a minimal dominating
set in $G$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. For more details on domination, the reader is referred to *Fundamentals of Domination in Graphs* by Haynes, Hedetniemi, and Henning [13]. The *vertex independence number* $\beta_0(G)$ equals the maximum cardinality of an independent set in $G$, while the *independent domination number* $i(G)$ equals the minimum cardinality of a maximal independent set in $G$. The following inequalities are well-known in domination theory.

**Proposition 1.1** For any graph $G$, $\gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G)$.

We go into more depth with inequality chain from Proposition 1.1 in the Preliminary Results section of this thesis.

1.2 Cost Effective Domination

Motivated by the studies of unfriendly partitions and satisfactory partitions (for example, see [1, 2, 7, 8, 9, 19, 20]), cost effective domination was introduced in [10].
A vertex $v$ in a set $S$ is said to be *cost effective* if it is adjacent to at least as many vertices in $V \setminus S$ as it is in $S$, and $v$ is *very cost effective* if it is adjacent to more vertices in $V \setminus S$ than to vertices in $S$. A set $S$ is *(very) cost effective* if every vertex in $S$ is (very) cost effective. A set $S$ is a *(very) cost effective dominating set* if $S$ is both (very) cost effective and a dominating set.

**Definition 1.2** The *cost effective domination number* $\gamma_{ce}(G)$ of a graph $G$ equals the minimum cardinality of a cost effective dominating set in $G$. The *upper cost effective domination number* $\Gamma_{ce}(G)$ equals the maximum cardinality of a minimal dominating set that is cost effective in $G$. A cost effective dominating set of $G$ with cardinality $\gamma_{ce}(G)$ is called a $\gamma_{ce}(G)$-set. The *very cost effective domination number* $\gamma_{vce}(G)$ and the *upper very cost effective domination number* $\Gamma_{vce}(G)$ are defined similarly.

For examples, consider the graphs $G$ in Figures 3(a) and 4(a) where the darkened vertices represent $\gamma_{ce}(G)$-sets and Figures 3(b) and 4(b) where the darkened vertices represent $\gamma_{vce}(G)$-sets.
Figure 3: Graph with $\gamma_{ce}(G) = 7$ and $\gamma_{vce}(G) = 8$.

Figure 4: Graph with $\gamma_{ce}(G) = 8$ and $\gamma_{vce}(G) = 10$.

It should be pointed out that while the property of being a dominating set is superhereditary, that is, every superset of a dominating set is also a dominating set, the property of being a cost effective dominating set is not superhereditary. This explains why the definition of the upper cost effective domination number does not include the word “minimal” as it does in the definition of the upper domination number. Without the word minimal in the definition of $\Gamma(G)$, the value of $\Gamma(G)$ would equal $n = |V|$ for all graphs.

In terms of application, we assume that maintaining edges in a network has an associated cost, and thus they should be used effectively. We assume that an edge between a vertex in a set $S$ and a vertex in $V \setminus S$ is being used effectively, while an edge between two vertices in $S$ is not necessarily being used cost effectively. Thus, a vertex is considered to be cost effective if at least as many edges incident to it are
being used cost effectively as not being used cost effectively.

Another way of viewing the application is to consider a company, where the set $S$ represents the employees and $V \setminus S$ represents the customers. Certainly the company would want to have only employees that add to its profits. Suppose the company offers a service to both its employees and its customers. Let the edges inside $S$ represent services between employees (internal costs) and let edges between $S$ and $V \setminus S$ represent income from paying customers. If the company allows employees to use the services it offers for free or at a discounted price, then to ensure that each employee $v \in S$ is profitable for the company it would be necessary for $v$ to have at least as many neighbors in $V \setminus S$ as in $S$, that is, $S$ needs to be a cost effective set. In this thesis, we study bounds on the cost effective domination number of graphs.
2 BACKGROUND AND TERMINOLOGY

2.1 Unfriendly Partitions

Cost effective domination is derived from the study of unfriendly partitions of graphs, as follows. Let $C$ be a two-coloring of the vertices of a graph $G$, $C : V \rightarrow \{Red, Blue\}$. For every vertex $u \in V$, define $B(u) = \{v \in N(u), C(v) = Blue\}$ and $R(u) = \{v \in N(u), C(v) = Red\}$. Similarly, define $B(V) = \{v \in V, C(v) = Blue\}$ and $R(V) = \{v \in V, C(v) = Red\}$. A two-coloring produces a bipartition of $V$, $\pi = \{B(V), R(V)\}$. Given such a bipartition $\pi$, we say that an edge $uv \in E$ is bicolored if $C(u) \neq C(v)$. A bipartition $\pi$ is called an unfriendly partition if every vertex $u \in B(V)$ has at least as many neighbors in $R(V)$ as it does in $B(V)$, and every vertex $v \in R(V)$ has at least as many neighbors in $B(V)$ as it does in $R(V)$. That is, if $C(u) = Blue$, then $|B(u)| \leq |R(u)|$, and if $C(u) = Red$, then $|R(u)| \leq |B(u)|$. These types of partitions were defined and studied by Borodin and Koshtochka [3], Aharoni, Milner and Prikry [1], and Shelah and Milner [20], who called these unfriendly partitions. They observed the following, a simple proof of which we provide here.

**Theorem 2.1** [10] Every finite connected graph $G$ of order $n \geq 2$ has an unfriendly partition.

**Proof.** Let $\pi = \{B(V), R(V)\}$ be any bipartition of $V(G)$ having the property that the number of bicolored edges is a maximum. Assume to the contrary that $\pi$ is not an unfriendly partition. Then there must exist a vertex, say $v \in R(V)$, without loss of generality, having more Red neighbors than Blue neighbors. In this case, moving
v to $B(V)$ will increase the number of bicolored edges, contradicting the assumption that $\pi$ has a maximum number of bicolored edges. □

Unfriendly partitions have shown up indirectly in several other lines of research. In [4, 5] the concept of $\alpha$-domination in graphs is defined and studied. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called an $\alpha$-dominating set if for every vertex $v \in V \setminus S$, $|N(v) \cap S|/|N[v]| \geq \alpha$, where $0 \leq \alpha < 1$. In the case where $\alpha \geq 1/2$, every vertex in $V \setminus S$ meets the unfriendly condition in that it has at least as many neighbors in $S$ as it has in $V \setminus S$. However, no unfriendly condition is imposed on the vertices in $S$.

Similarly, in [6, 11, 12, 14, 16, 18] global offensive alliances in graphs are defined and studied. A set $S \subseteq V$ of vertices is called a global offensive alliance if for every vertex $v \in V \setminus S$, $|N(v) \cap S| \geq |N[v] \cap (V \setminus S)|$. As with $\alpha$-domination, if $S$ is a global offensive alliance, then every vertex $v \in V \setminus S$ satisfies the unfriendly condition, in that it has at least as many neighbors in $S$ as it has in $V \setminus S$ if you count the vertex $v$ as one of its own neighbors. But no unfriendly condition is imposed on the vertices in $S$.

A partition that is in some sense dual to an unfriendly partition is a bipartition $\pi = \{B(V), R(V)\}$ called a satisfactory partition such that every vertex $u \in B(V)$ has at least as many neighbors in $B(V)$ as it does in $R(V)$, and every vertex $u \in R(V)$ has at least as many neighbors in $R(V)$ as it has in $B(V)$. That is, if $C(u) = Blue$, then $|B(u)| \geq |R(u)|$, and if $C(u) = Red$, then $|R(u)| \geq |B(u)|$. Satisfactory partitions have been studied in [7, 8, 9] and [19]. However, unlike unfriendly partitions, not every graph has a satisfactory partition. In fact, it is an NP-complete problem to
determine if an arbitrary graph has a satisfactory partition [2].

2.2 Differentials in Graphs

The related concept of differentials in graphs was studied in [17], where the following game was considered for any arbitrary graph $G = (V,E)$. Assume you are allowed to buy as many tokens as you like, say $k$ tokens, at the cost of $1$ each. You then place your tokens on some subset $k$ vertices of $V$. For each vertex of $G$ which is adjacent to a vertex with a token on it, but has no token on itself, you receive $1$. Note that you do not receive any credit for the vertices on which you place a token. Your objective it to maximize your profit, that is, the total value received minus the cost of the tokens bought. $B(X)$ is defined as the set of vertices in $V \setminus X$ that have a neighbor in a set $X$. Based on this game, the differential of a set $X$ is defined to be $\partial(X) = |B(X)| - |X|$, and the differential of a graph to equal the max\{\partial(X)\} for any subset $X$ of $V$.

In [17], it was shown that for any graph $G$,

$$n - 2\gamma(G) \leq \partial(G) \leq n - \gamma(G) - 1,$$

and

$$\Delta(G) - 1 \leq \partial(G).$$

The following realizability result was also given.

**Theorem 2.2** [17] For any triple $(a,b,c)$ of positive integers such that $a \leq b \leq c$ and $c - 2a \leq b \leq c - a - 1$, there exists a tree $T$ having order $n = c, \gamma(T) = a$, and $\partial(T) = b$. 
A subdivision of an edge $uv$ is obtained by removing edge $uv$, adding a new vertex $w$, and adding the new edges $uw$ and $wv$. A wounded spider is the graph formed by subdividing at most $t - 1$ of the edges of a star $K_{1,t}$ for $t \geq 0$. The following gives a characterization of trees that achieve the upper bound for $\partial(T)$, while the characterization of the trees $T$ for which $\partial(T) = n - 2\gamma(T)$ is still being determined.

**Theorem 2.3** [17] A tree $T$ has $\partial(T) = n - \gamma(T) - 1$ if and only if $T$ is a nontrivial wounded spider.

Also in [17], the trees having $\partial(T) = \Delta(T) - 1$ are characterized. For a rooted tree $T$, let $T_u$ denote the subtree of $T$ induced by $u$ and its descendents.

A family $\mathcal{T}$ of trees is defined in [17] as follows. A tree $T$ is in $\mathcal{T}$ if $T$ is a tree rooted at a vertex $v$ of maximum degree $\Delta(T)$ and one of the following properties holds:

1. $v$ is adjacent to exactly one leaf $x$ and for each $u \in N(v) \setminus \{x\}$, $T_u \in \{P_2, P_3\}$, where $u$ is an endvertex of $T_u$, or

2. There exist two vertices $x, y \in N(v)$ such that $T_x \in \{P_1, P_2\}$ and $T_y \in \{P_1, P_2\}$. And, for each $u \in N(v) \setminus \{x, y\}$, the subtree $T_u \in \{P_1, P_2, P_3, P_4, P_5\}$ where $u$ is the center of $T_u$ or $u$ is a leaf of $T_u = P_3$.

We conclude this section with the following theorem.

**Theorem 2.4** [17] A tree $T$ has $\partial(T) = \Delta(T) - 1$ if and only if $T \in \mathcal{T}$.  

17
3 PRELIMINARY RESULTS

This section will begin with some preliminary results that build to the main results of this thesis.

Observation 3.1 Every independent dominating set $S$ in an isolate-free graph $G$ is a very cost effective dominating set.

Corollary 3.2 For any isolate-free graph $G$,

$$\gamma(G) \leq \gamma_{ce}(G) \leq \gamma_{vce}(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_{vce}(G) \leq \Gamma_{ce}(G) \leq \Gamma(G).$$

It is known [13] that $\beta_0(G) = \Gamma(G)$ for bipartite graphs so, from Corollary 3.2, we have that $\beta_0(G) = \Gamma_{vce}(G) = \Gamma_{ce}(G) = \Gamma(G)$ for bipartite graphs. On the other hand, in this section we will see that all combinations of the inequalities in the chain $\gamma(G) \leq \gamma_{ce}(G) \leq \gamma_{vce}(G) \leq i(G)$ are possible, even when restricted to trees. We also give necessary conditions for a graph $G$ to have $\gamma(G) = \gamma_{ce}(G)$, and for a graph $G$ to have $\gamma_{ce}(G) = \gamma_{vce}(G)$. In Section 4, we show that $\gamma_{ce}(T) \leq 2\gamma(T) - 3$ for trees $T$ with $\gamma(T) \geq 3$, and characterize the trees achieving this bound. Then we show that, for trees $T$, every value of the cost effective domination number between $\gamma(T)$ and $2\gamma(T) - 3$ is realizable.

We first give some additional terminology. For a graph $G$ and a subset $S \subseteq V$, we denote the subgraph induced by $S$ as $G[S] = (S, E \cap (S \times S))$. An $S$-external private neighbor of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to $v$ but to no other vertex of $S$. The set of all $S$-external private neighbors of $v \in S$ is called the $S$-external private neighbor set of $v$ and is denoted $epn(v, S)$. A vertex of degree one
is called a leaf (or endvertex), and its neighbor is a support vertex. The double star $S_{r,s}$ is the tree with exactly two adjacent non-leaf vertices, one of which is adjacent to $r$ leaves and the other to $s$ leaves. The corona of graphs $G$ and $H$, denoted $G \circ H$, is the graph formed from one copy of $G$ and $|V(G)|$ copies of $H$, where the $i^{th}$ vertex in $V(G)$ is adjacent to every vertex in the $i^{th}$ copy of $H$.

The inequalities in Corollary 3.2 raise the following interesting questions: Which graphs have a cost effective $\gamma$-set, that is, for which graphs $G$, is $\gamma(G) = \gamma_{ce}(G)$? For which graphs $G$ is $\gamma_{ce}(G) = \gamma_{vce}(G)$?

Note that if $G$ is a cycle or a path $P_k$ for $k \geq 5$, then $\gamma(G) = \gamma_{ce}(G) = \gamma_{vce}(G) = i(G)$. The graphs in Figure 5(a) and 5(b), where the darkened vertices represent $\gamma_{ce}(G)$-sets, have $\gamma_{ce}(G) > \gamma(G)$.

![Figure 5](image)

Figure 5: Graphs that do not have cost effective $\gamma$-sets.

**Observation 3.3** [15] Let $S$ be a cost effective set of $G$. If every vertex in $S$ has odd degree, then $S$ is a very cost effective set of $G$.

**Corollary 3.4** If $G$ has a $\gamma_{ce}(G)$-set that consists of only odd vertices, then $\gamma_{ce}(G) = \gamma_{vce}(G)$.

**Corollary 3.5** If every vertex of $G$ has odd degree, then $\gamma_{ce}(G) = \gamma_{vce}(G)$. 
Note that in particular, \( \gamma_{ce}(G) = \gamma_{vce}(G) \) for cubic graphs.

**Theorem 3.6** If \( G \) has maximum degree \( \Delta(G) \leq 4 \), then \( \gamma(G) = \gamma_{ce}(G) \).

**Proof.** Among all \( \gamma(G) \)-sets, select \( S \) to be one with the minimum number of edges in \( G[S] \). If \( S \) is cost effective, we are finished. Hence, assume to the contrary that there exists a vertex, say \( x \), that is not cost effective. Therefore, \( |N(x) \cap S| > |N(x) \cap (V \setminus S)| \). Thus, \( x \) has at least one neighbor in \( S \). By the minimality of \( S \), \( x \) has at least one external private neighbor, say \( x' \), with respect to \( S \). But since \( \Delta(G) \leq 4 \) and \( |N(x) \cap S| > |N(x) \cap (V \setminus S)| \), it follows that \( N(x) \cap (V \setminus S) = \{x'\} \). But then \( S' = (S \setminus \{x\}) \cup \{x'\} \) is a \( \gamma(G) \)-set with fewer edges in \( G[S'] \) than in \( G[S] \), contradicting our choice of \( S \). Hence, \( S \) is cost effective. \( \square \)

Notice that the tree \( T \) in Figure 5(b) has maximum degree \( \Delta(T) = 5 \) and \( \gamma(T) < \gamma_{ce}(T) \), and thus, the bound \( \Delta(G) \leq 4 \) in Theorem 3.6 is best possible.

From Theorem 3.6, we have the following.

**Corollary 3.7** If \( G \) is a grid graph \( P_m \square P_n \), a cylinder \( C_m \square P_n \), or a torus \( C_m \square C_n \), then \( \gamma(G) = \gamma_{ce}(G) \).

From Observation 3.3 and Theorem 3.6, we have the following:

**Corollary 3.8** If \( G \) is a cubic graph, then \( \gamma(G) = \gamma_{ce}(G) = \gamma_{vce}(G) \).

**Theorem 3.9** If \( \gamma(G) \leq 3 \), then \( \gamma(G) = \gamma_{ce}(G) \).

**Proof.** Clearly, if \( \gamma(G) = 1 \), then \( \gamma(G) = \gamma_{ce}(G) \), so assume that \( 2 \leq \gamma(G) \leq 3 \). Among all \( \gamma(G) \)-sets, select \( S \) to be one with the minimum number of edges in \( G[S] \).
If $S$ is cost effective, then we are finished. Thus, assume that $G$ is not cost effective. Then there exists a vertex $x \in S$, such that $|N(x) \cap S| > |N(x) \cap (V \setminus S)|$. Hence, $x$ has at least one neighbor in $S$. By the minimality of $S$, $x$ has at least one external private neighbor, say $x'$. Hence, $|N(x) \cap S| \geq 2$, implying that $|S| = 3$ and $x$ has two neighbors in $S$ and $|N(x) \cap (V \setminus S)| = 1$, that is, $N(x) \cap (V \setminus S) = \{x'\}$. But then $(S \setminus \{x\}) \cup \{x'\}$ is a $\gamma(G)$-set with fewer edges in its induced subgraph than in $G[S]$, contradicting our choice of $S$. Hence, $S$ is cost effective. □

Notice that the tree $T$ in Figure 5(b) has $\gamma(T) = 4$, but $\gamma_{ce}(T) = 5$, so the bound $\gamma(G) \leq 3$ in Theorem 3.9 is best possible. We conclude this section by showing that all eight combinations of the inequalities $\gamma(G) \leq \gamma_{ce}(G) \leq \gamma_{vce}(G) \leq i(G)$ from Corollary 3.2 are possible, even when restricted to trees. For this purpose, let $K^x_{1,3}$ be the star with center $x$ and leaves $x_1, x_2$, and $x_3$. Let $T^i_j$ be the corona $K^x_{1,3} \circ K_j$.

For the following, $T_i$ satisfies inequality $i$.

1. $\gamma(T) < \gamma_{ce}(T) < \gamma_{vce}(T) < i(T)$.

Let $T_1$ be the tree obtained from $T^2_x \cup T^2_y$ by adding a new leaf vertex adjacent to $x$ and an edge between $x_1$ and $y_1$. See Figure 6(a).

2. $\gamma(T) < \gamma_{ce}(T) < \gamma_{vce}(T) = i(T)$.

Let $T_2$ be the tree obtained from $T^2_x \cup T^2_y$ by adding the edge $xy$.

3. $\gamma(T) < \gamma_{ce}(T) = \gamma_{vce}(T) < i(T)$.

Let $T_3$ be the tree obtained from $T^2_x \cup T^2_y$ by adding the edge $x_1y_1$ and removing the two leaves adjacent to $x_1$. See Figure 6(b).
4. $\gamma(T) < \gamma_{ce}(T) = \gamma_{vce}(T) = i(T)$.

Let $T_4$ be the tree $T^2_x$.

5. $\gamma(T) = \gamma_{ce}(T) < \gamma_{vce}(T) < i(T)$.

Let $T_5$ be the corona $P_6 \circ K_2$.

6. $\gamma(T) = \gamma_{ce}(T) < \gamma_{vce}(T) = i(T)$.

Let $T_6$ be the tree $T^3_x$.

7. $\gamma(T) = \gamma_{ce}(T) = \gamma_{vce}(T) < i(T)$.

Let $T_7$ be the double star $S_{r,s}$ where $2 \leq r \leq s$.

8. $\gamma(T) = \gamma_{ce}(T) = \gamma_{vce}(T) = i(T)$.

Let $T_8$ be the corona $T' \circ K_1$ of any tree $T'$.

Figure 6: Trees $T_1$ and $T_3$. 
In this section, we determine an upper bound on the cost effective domination number of trees and characterize the trees obtaining this bound. We also show that every value of $\gamma_{ce}(T)$ between the upper and lower bounds of Theorem 4.1 is realizable.

**Theorem 4.1** If $T$ is a tree with $\gamma(T) \geq 3$, then $\gamma(T) \leq \gamma_{ce}(T) \leq 2\gamma(T) - 3$, and these bounds are sharp.

**Proof.** The lower bound is direct from Corollary 3.2. Let $S$ be a $\gamma(T)$-set. If $S$ is cost effective, then we are finished. Thus, assume that $S$ is not cost effective and let $U = \{u_1, u_2, ..., u_k\}$ be the vertices of $S$ that are not cost effective with respect to $S$. Let $s_i = |N(u_i) \cap S|$ and $o_i = |N(u_i) \cap (V \setminus S)|$, for $1 \leq i \leq k$. Thus for each $u_i \in U$, $s_i \geq o_i + 1$. Let $U' \subseteq V \setminus S$ be the vertices in $V \setminus S$ whose only neighbors in $S$ are in $U$. Note that since each $u_i$ is not cost effective, $u_i$ has a neighbor in $S$, that is, $s_i \geq 1$. Hence, the minimality of $S$ implies that $u_i$ has at least one external private neighbor with respect to $S$ in $U'$. Thus, $|U'| \geq \sum_{i=1}^{k} |epn(u_i, S)| \geq k$.

We first prove a claim:

**Claim A** $\sum_{i=1}^{k} s_i \leq \gamma(T) + k - 2$.

**Proof.** We establish the bound on the degree sum in $T[S]$ by considering the possible edges of $T[S]$ incident to a vertex in $U$. If both endvertices of an edge are in $U$, then we say the edge is a Type-1 edge, while if one endvertex is in $U$ and the other is in $S \setminus U$, we say the edge is of Type-2. Thus, each Type-1 edge adds 2 to the degree sum in $T[S]$, and each Type-2 edge adds 1. Let $t_i$ be the number of Type-$i$ edges.
Note that if a pair of vertices in $U$ are connected by a path in $T[U]$, then they have no common neighbor in $S \setminus U$, for otherwise a cycle is formed. Let $T[U]$ have $c$ components. Since $T$ is a tree, $t_1 = k - c$, and there are at most $c - 1$ pairs of vertices in $U$ having a common neighbor in $S \setminus U$. By the Pigeonhole Principle, there are at least $t_2 - |S \setminus U|$ pairs of vertices in $U$ having a common neighbor in $S \setminus U$. Thus, $t_2 - |S \setminus U| \leq c - 1$.

Hence, $\sum_{i=1}^{k} s_i = 2t_1 + t_2 \leq 2(k - c) + |S \setminus U| + c - 1 = 2k - 2c + \gamma(T) - k + c - 1 = \gamma(T) + k - c - 1 \leq \gamma(T) + k - 2$. □

Since $s_i \geq o_i + 1$ for each $i, 1 \leq i \leq k$, by Claim A, we have $\sum_{i=1}^{k} o_i \leq \sum_{i=1}^{k} (s_i - 1) \leq \gamma(T) + k - 2 - k = \gamma(T) - 2$. Hence, $|U'| \leq \gamma(T) - 2$.

Next, we give an algorithm to recursively build a cost effective dominating set $S_k$ from a $\gamma(T)$-set $S$. As before, let $U = \{u_1, u_2, \ldots, u_k\}$ be the subset of vertices in $S$ that are not cost effective, and let $U'$ be the set of vertices in $V \setminus S$ whose only neighbors in $S$ are in $U$.

\begin{verbatim}
begin
let $S_0 = S$. 
for $i = 1$ to $k$ do 
  if $u_i$ is cost effective in $S_{i-1}$ 
    then let $S_i = S_{i-1}$ 
  else if epn($u_i, S_{i-1}$) = $\emptyset$ 
    then let $S_i = S_{i-1} \setminus \{u_i\}$ 
  else let $S_i = (S_{i-1} \setminus \{u_i\}) \cup$ epn($u_i, S_{i-1}$) 
for 
end
\end{verbatim}

24
We next prove that the algorithm produces a cost effective dominating set with cardinality at most $2\gamma(T) - 3$.

**Claim B**  The algorithm terminates with a cost effective dominating set, namely $S_k$, and $|S_k| \leq 2\gamma(T) - 3$.

**Proof.** By definition the set $S = S_0$ is a dominating set and the vertices of $S \setminus \{u_1, u_2, ..., u_k\}$ are cost effective in $S$. We define the loop invariant: for $1 \leq i \leq k$, the set $S_i$ is a dominating set and all of the vertices in $S_i \setminus \{u_{i+1}, ..., u_k\}$ are cost effective in $S_i$.

To see that $S_i$ is a dominating set, we note that $S_{i-1}$ is a dominating set, so if $u_i$ is cost effective and $S_i = S_{i-1}$, clearly, $S_i$ is a dominating set. If $u_i$ is not cost effective in the set $S_{i-1}$, then $u_i$ has at least one neighbor in $S_{i-1}$, implying that $u_i$ is dominated by $S_i$. Moreover, the external private neighbors of $u_i$ with respect to $S_{i-1}$ are added to form $S_i$, so $S_i$ is a dominating set.

To see that the set $S_i \setminus \{u_{i+1}, ..., u_k\}$ is cost effective, note if $u_i$ is not cost effective in $S_{i-1}$, then $S_i = S_{i-1} \cup \text{epn}(u_i, S_{i-1})$. Let $X = \text{epn}(u_i, S_{i-1})$. Since $T$ is a tree and each vertex in $X$ is adjacent to $u_i$, $X$ is an independent set. Moreover, since each vertex $x \in X$ is a private neighbor of $u_i$, $x$ has no neighbors in $S_{i-1} \setminus \{u_i\}$. In other words, $X$ is independent in $T[S_i]$ and so the vertices of $X$ are cost effective with respect to $S_i$. Hence, the vertices that are not cost effective in $S_i$ are the at the most the ones that are not cost effective in $S_{i-1} \setminus \{u_i\}$. On iteration $k$, the algorithm terminates with the cost effective dominating set $S_k$. 


It remains to be shown that $|S_k| \leq 2\gamma(T) - 3$. To do this we count the maximum possible vertices being added to form the set $S_k$. Since $U'$ consists of the vertices whose only neighbors in $S$ are in $U$, we have that $\text{epn}(u_i, S) \subseteq U'$ for $1 \leq i \leq k$.

Consider the construction of set $S_k$. At iteration $i$, if $u_i$ is cost effective in $S_{i-1}$, then we let $S_i = S_{i-1}$. Since $u_i \in U$, it is not cost effective in $S$ so we have $|\text{epn}(u_i, S)| \geq 1$. Hence, for our counting purposes, letting $S_i = S_{i-1}$ is essentially the same as removing $u_i$ and replacing it with a vertex from $\text{epn}(u_i, S) \subseteq U'$.

If $u_i$ is not cost effective in $S_{i-1}$, then we remove $u_i$ and add the set $\text{epn}(u_i, S_{i-1})$ to form $S_i$. To show that at most $|U'|$ vertices are added to $S$ to form $S_k$, it suffices to show that $\text{epn}(u_i, S_{i-1}) \subseteq U'$. To see this, suppose to the contrary that $x \in \text{epn}(u_i, S_{i-1})$ and $x \notin U'$. By the definition of $U'$, it follows that $x$ has a neighbor in $S \setminus U$. Since $S \setminus U \subseteq S_{i-1}$, $x$ has a neighbor in $S_{i-1} \setminus U$. But $u_i \in U$, contradicting that $x \in \text{epn}(u_i, S_{i-1})$. Hence, $\text{epn}(u_i, S_{i-1}) \subseteq U'$, and so we may conclude that every vertex added to form $S_k$ is in the set $U'$.

It follows that to form $S_k$ from our original set $S$, we add at most $|U'|$ vertices, while for the purposes of our count, we “remove” $|U| = k$ vertices. Since $|U'| \leq \gamma(T) - 2$, we have $|S_k| \leq |S| - |U| + |U'| \leq \gamma(T) - k + \gamma(T) - 2 = 2\gamma(T) - k - 2 \leq 2\gamma(T) - 3$ for $k \geq 1$. □

By Claim B, $\gamma_c(T) \leq |S_k| \leq 2\gamma(T) - 3$, as desired. We conclude this proof by showing the bounds are sharp. The corona $T \circ K_1$ of any tree $T$ achieves the lower bound. Let $T$ be the corona $K_{1,t} \circ K_{t-1}$. Then $\gamma(T) = t + 1$ and $\gamma_c(T) = 2t - 1 = 2\gamma(T) - 3$, obtaining the upper bound. □
Figure 7: Tree $T$ that achieves the upper bound of Theorem 4.1.

Note that the upper bound on the cost effective domination number of Theorem 4.1 does not hold for the very cost effective domination number of trees. For a counterexample, consider the tree $T = K_{1,t} \circ K_t$ for which $\gamma(T) = \gamma_{ce}(T) = t + 1$ and $\gamma_{vce}(T) = 2t > 2t - 2 = 2(t + 1) - 3 = 2\gamma(T) - 3$.

Next we characterize the trees obtaining the upper bound of Theorem 4.1. For this purpose, we define the family $F$ of trees $T_t$, which are obtained from the star $K_{1,t}$, with center $x$ and leaves $x_1, x_2, ..., x_t$ as follows. Add exactly $t - 1$ new vertices adjacent to $x$, and for $1 \leq i \leq t$, add at least $t - 1$ new vertices adjacent to $x_i$. Note that the corona $K_{1,t} \circ K_{t-1} \in F$.

**Theorem 4.2** A tree $T$ with $\gamma(T) \geq 3$ has $\gamma_{ce}(T) = 2\gamma(T) - 3$ if and only if $T \in F$.

**Proof.** Let $T_t \in F$. Then $\gamma(T_t) = t + 1$, while $\gamma_{ce}(T) = t + t - 1 = 2t - 1 = 2\gamma(T_t) - 3$.

Next assume that $\gamma_{ce}(T) = 2\gamma(T) - 3$. Let $S_k$ be a cost effective dominating set of $T$ formed by the algorithm in the proof of Theorem 4.1. Then, $2\gamma(T) - 3 = \gamma_{ce}(T) \leq |S_k| \leq 2\gamma(T) - k - 2 \leq 2\gamma(T) - 3$. Since we have equality throughout, it follows that $2\gamma(T) - 3 = 2\gamma(T) - k - 2$, implying that for the set $S_k$, we have that $k = 1$. Thus, from our algorithm, we deduce that $T$ has a $\gamma(T)$-set $S$ with exactly one vertex, say $u_1$, that is not cost effective in $S$. Furthermore, $S_k = S_1 = (S \setminus \{u_1\}) \cup \text{epn}(u_1, S)$. Since $\gamma_{ce}(T) = 2\gamma(T) - 3 \leq |S_k| = |S| + |\text{epn}(u_1, S)| = \gamma(T) - 1 + |\text{epn}(u_1, S)| \leq 2\gamma(T) - 3$, ...
we have that $|\text{epn}(u_1, S)| = \gamma(T) - 2$. Moreover, since $u_1$ is not cost effective with respect to $S$, $u_1$ has exactly $\gamma(T) - 1$ neighbors in $S$. Since $T$ is a tree, the induced subgraph $T[S]$ is the star $K_{1,\gamma(T)-1}$ with center $u_1$ and every vertex in $V \setminus S$ is a leaf in $T$. To see that $T \in \mathcal{F}$, we need to show that each vertex in $S \setminus \{u_1\}$ has at least $\gamma(T) - 2$ leaf neighbors in $V \setminus S$. Suppose to the contrary that $x \in S \setminus \{u_1\}$ and $x$ has at most $\gamma(T) - 3$ leaf neighbors in $V \setminus S$. Then $(S \setminus \{x\}) \cup \text{epn}(x, S)$ is a cost effective dominating set of $T$ with cardinality $|S \setminus \{x\}| + |\text{epn}(x, S)| \leq \gamma(T) - 1 + \gamma(T) - 3 < 2\gamma(T) - 3 = \gamma_c(T)$, a contradiction. Thus, $u_1$ has exactly $\gamma(T) - 2$ leaf neighbors, and every vertex in $S \setminus \{u_1\}$ has at least $\gamma(T) - 2$ leaf neighbors, and so $T \in \mathcal{F}$. □

We conclude by showing that all values between the lower and upper bounds of Theorem 4.1 are realizable. Let $K^r_{v_i}$ be the star with center $v$ and leaves $v_1, \ldots, v_t$.

**Theorem 4.3** Given positive integers $a$ and $b$ such that $4 \leq a \leq b \leq 2\gamma(T) - 3$, there exists a tree $T$ having $\gamma(T) = a$ and $\gamma_c(T) = b$.

**Proof.** To construct a tree $T$ having $\gamma(T) = a$ and $\gamma_c(T) = b$, we begin with the tree $(K^r_{v_i} \circ K_{a-2}) \cup K_{y_1}^{y_t} \cup K_{b-a+1}$ and add the edge $xy$. Then, $T$ has $a$ support vertices. We show that $\gamma(T) = a$ and $\gamma_c(T) = b$. First note that since the set of support vertices of $T$ is a dominating set, $\gamma(T) \leq a$, and since every leaf or its support must be in any $\gamma(T)$-set, we have $\gamma(T) \geq a$. Hence, $\gamma(T) = a$.

Let $S = \{x, x_1, \ldots, x_{a-2}, y_1, \ldots, y_{b-a+1}\}$. To see that $S$ is a dominating set, note that every vertex in $S$ is dominated by $S$. Assume $v \in V \setminus S$. Then, $v$ is either a leaf adjacent to $x_i$ or $x$, or $v = y$ and is dominated by $y_j$, for some $i, j$. Hence, $S$ is a dominating set. To see that $S$ is cost effective, note that $y_i$ is independent in $T[S]$, so each $y_i, 1 \leq i \leq b - a + 1$, is cost effective with respect to $S$. Moreover,
$|N(x_i) \cap S| = 1$ and $|N(x_i) \cap (V \setminus S)| = a - 2 \geq 2$, so $x_i$, for $1 \leq i \leq a - 2$, is cost effective. Finally, $|N(x) \cap S| = a - 2 < a - 1 = |N(x) \cap (V \setminus S)|$, so $x$ is cost effective. Hence, $S$ is cost effective, and so $\gamma_{ce}(T) \leq |S| = 1 + a - 2 + b - a + 1 = b$.

Now, let $S^*$ be a $\gamma_{ce}(T)$-set. To dominate $T$, each leaf or its support vertex must be in $S^*$. We show that at least one of the support vertices is not in $S^*$. Assume to the contrary that $S^*$ contains all the support vertices of $T$. That is, $\{x, x_1, ..., x_{a-2}, y\} \subseteq S^*$. But then $|N(x) \cap (V \setminus S^*)| = a - 2 < |N(x) \cap S^*| = a - 1$, contradicting that $S^*$ is a cost effective set. Hence, at least one support vertex, say $w$, of $T$ is not in $S^*$, implying that $S^*$ contains the leaves adjacent to $w$. Let $l_w$ be the number of leaves adjacent to $w$. Recall that $T$ has $a$ support vertices, so $a - 1 + l_w \leq |S^*| = \gamma_{ce}(T) \leq b$. Thus, $l_w \leq b - a + 1$. Since $b \leq 2a - 3$, we have that $b - a + 1 \leq 2a - 3 - a + 1 = a - 2$. Now each support vertex of $T$ is adjacent to either $a - 2$ or $b - a + 1$ leaves and $b - a + 1 \leq a - 2$, so we conclude that each support vertex is adjacent to at least $b - a + 1$ leaves. In particular, $l_w \geq b - a + 1$, and so, $l_w = b - a + 1$. Hence, $\gamma_{ce}(T) = |S^*| \geq a - 1 + l_w = a - 1 + b - a + 1 = b$. Therefore, $\gamma_{ce}(T) = b$. □

For an example, consider the tree $T$ in Figure 8(a) where the darkened vertices represent a $\gamma(T)$-set and Figure 8(b) where the darkened vertices represent a $\gamma_{ce}(T)$-set.
Figure 8: Tree $T$ with $\gamma(T) = a = 6$ and $\gamma_{\alpha}(T) = b = 8$. 

(a) 

(b)
5 CONCLUDING REMARKS

We have determined an upper bound on the cost effective domination number of trees and characterized the trees obtaining the bound. We also showed that every value of $\gamma_{ce}(T)$ between the upper and lower bounds is realizable. We conclude with some open problems suggested by this work:

1. Characterize the trees $T$ for which $\gamma(T) = \gamma_{ce}(T)$.

2. Characterize the trees $T$ for which $\gamma_{ce}(T) = \gamma_{vce}(T)$.

3. Characterize the trees $T$ for which $\gamma_{vce}(T) = i(T)$.

4. We have seen that the upper bound of $2\gamma(T) - 3$ on the cost effective domination number of trees does not hold for the very cost effective domination number. Is there a bound on $\gamma_{vce}(T)$ in terms of $\gamma(T)$ for trees $T$?

5. Although $2\gamma(T) - 3$ is an upper bound on the cost effective domination number for trees, we have not been able to prove or disprove that it is a bound for the cost effective domination number of general graphs. Prove or disprove: For any graph $G$, $\gamma_{ce}(G) \leq 2\gamma(G) - 3$.

6. Investigate bounds on the upper parameters $\Gamma_{ce}(G)$ and $\Gamma_{vce}(G)$.  

31
BIBLIOGRAPHY


VITA

TABITHA MCCOY

Education: B.S. Mathematics, University of Virginia’s College at Wise, Wise, Virginia 2011
                      M.S. Mathematics, East Tennessee State University
                              Johnson City, Tennessee 2013

Professional Experience: National Science Foundation GK-12 Fellow
                              East Tennessee State University
                              Johnson City, Tennessee (2012-2013)

Publications: T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi,
                                 T. L. McCoy, and I. Vasylieva,
                              Cost effective domination in graphs.

                              T. W. Haynes, S. T. Hedetniemi, T. L. McCoy,
                              Cost effective domination.
                              Submitted for publication, August 2012.

Honor Societies: Kappa Mu Epsilon
                              Darden Society