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Nested (2,r)-regular graphs and their network properties.

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Nested $(2, r)$-Regular Graphs and Their Network Properties

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ABSTRACT

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by

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A graph G is a (t, r)-regular graph if every collection of t independent vertices is collectively adjacent to exactly r vertices. If a graph G is $(2, r)$-regular where $p, s,$ and m are positive integers, and $m \geq 2$, then when n is sufficiently large, then G is isomorphic to $G = K_s + mK_p$, where $2(p-1)+s = r$. A nested $(2, r)$-regular graph is constructed by replacing selected cliques with a $(2, r)$-regular graph and joining the vertices of the peripheral cliques. For example, in a nested 's' graph when $n = s + mp$, we obtain $n = s_1 + m_1p_1 + mp$. The nested 's' graph is now of the form $G_s = K_{s_1} + m_1K_{p_1} + mK_p$. We examine the network properties such as the average path length, clustering coefficient, and the spectrum of these nested graphs.

DEDICATION

I would like to dedicate this to my uncle, Dan Reese Caudell, and aunt, Rachel Brooks Caudell. I love you and miss you both. If it wasn't for your support, love, and devotion to God, your family and friends, I would never would have the drive to make it this far.

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1 INTRODUCTION

Graph Theory is an area of mathematics that has grown considerably since the introduction of the internet. Graphs can be used to model relationships. Such relationships could be a connection between individuals like as in a friendship or between to web sites that share a common theme or link. In order to obtain a better understanding, we can use the techniques in Graph Theory to analyze the behavior of these graphs or networks.

In this thesis, we will first give an overview of previous work on networks and their properties, then implement these ideas on a particular family of graphs. We will also develop a deterministic algorithm to generate a network and study the changes in its properties.

Frequently, networks are changing and growing; their structure is dynamic. Research by several parties, including Strogatz and Watts [14] to Albert and Barabási [1], have shown that networks such as the power grid of the western United States [14] and short-term memory are small world networks [5]. These small-world networks are highly clustered yet have a low average path length. An example of a highly clustered network would be a social networking site. We will formally define these properties in the next section and investigate, in detail, this family of graphs in the next chapter.

1.1 Small World Graphs

The concept of small-world describes networks that despite being large in size, in most cases, there is a relatively short path between any two vertices or nodes. The distance is determined by the number of edges along the shortest path between two nodes. The most popular appearance is the "six degrees of separation" concept that was studied by the psychologist Stanley Milgram [2]. The idea behind the six degrees of separation is that every person in the world can be linked to another by at most six individuals.

In 1967, Stanley Milgram was given funding of $680 by the Laboratory of Social Relations at Harvard to study this small-world phenomenon [11]. He needed to develop a method that would allow him to record the number of acquaintances that linked two individuals. Milgram established two different starting points: the first in Wichita, Kansas, and the second in Omaha, Nebraska. The target for the Kansas Project was a wife of a divinity school student in Cambridge, Massachusetts, and the target for the Nebraska project was a stockbroker that lived in Boston. During the Nebraska project, 160 letters were sent to residents in Omaha to participate in the experiment. The package that was sent included instructions of the experiment, a roster to keep track of who got the letter, and tracer cards that were to be sent back to Milgram for feedback on his study. The instructions stated that the individual must forward the package to a friend or acquaintance of whom he or she were on a first name basis. When the study was concluded the chains varied in length from two to ten intermediate steps, and out of the 160 packages, 44 made the complete journey to the man in Boston, while 126 dropped out. Milgram stated that the median of those chains was five intermediates [11].

The notion of small-world networks was introduced in the 1998 paper, *Collective dynamics of 'small-world' networks*, by Strogatz and Watts [14]. In this paper, they started with a regular ring lattice graph and began to rewire the graph. A regular ring

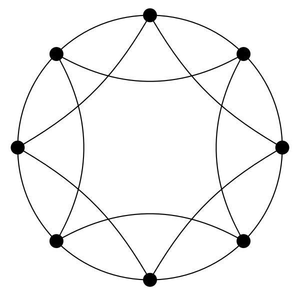

Figure 1: Regular Ring Lattice

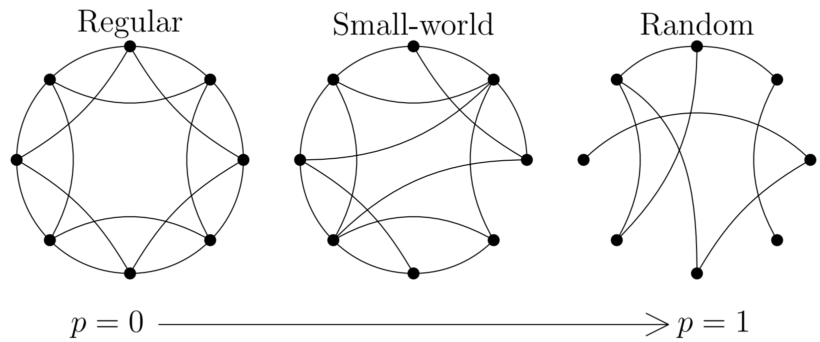

Figure 2: 'Rewiring' of the Ring Lattice

lattice is a graph with n nodes and m edges where every node in the ring is adjacent to its first additional k neighbors and every node is of the same degree. Figure 1 is an example of a regular lattice graph when $k = 2$.

Watts and Strogatz gave each edge a random probability p and began to rewire the graph. By doing this, they were able to tune the graph from regular, where $p = 0$, and completely random, $p = 1$, [14]. Figure 2 demonstrates the process in which they rewired the ring lattice.

Watts and Strogatz described graphs, or networks, as small world if the graphs are highly clustered like a regular graph, yet have a small average path length [14]. The clustering coefficient is defined as follows. Suppose that a vertex v has k_v neighbors;

then at most $k_v(k_v - 1)/2$ edges can exist between them. Let C_v denote the fraction of these allowable edges that actually exist. Define C as the average of C_v over all v. The clustering coefficient can also be found a number of different ways. A path uvw is said to be closed if the edge uw is present. The clustering coefficient can then be defined as the fraction of closed paths of length two in the network,

$$C = \frac{\text{(number of closed paths of length two)}}{\text{number of paths of length two}}.$$

A different way to define the clustering coefficient is

$$\frac{\text{number of triangles} \times 6}{\text{number of paths of length two}}.$$

The clustering coefficient lies in the range from zero to one [12].The average path length, L, is defined as the number of edges in the shortest path between two vertices, averaged over all pairs of vertices [14]. They noticed that as the graph becomes more and more random, the average path length, a global property, grows whereas the clustering coefficient, a local property, remains relatively unchanged in a broad interval between $0 < p < 1$.

Small world networks have been found to be generic for many large, sparse networks found in nature. A few examples are the collaboration of film actors where an edge represents that two actors have worked on a film together, the neural network of the worm *Caenorhabditis elegans*, the power grid of the western United States [14], and short-term memory uses small world networks between neurons [5]. The research of such networks has lead into a new type of network modeling that deals with properties of small world networks along with a network's degree distribution, namely scale-free networks.

Scale-free networks is a term defined in 1999 by Réka Albert and Albert-László Barabási [1]. A scale free network is a connected graph or network with the property that the number of links originating from a given node exhibits a power law distribution $P(k) \sim ck^{-\gamma}$. Barabási and Albert developed an algorithm on how to grow a graph or network to be scale free. First, networks are expanded continuously with the addition of new nodes, and then these new nodes are attached preferentially to sites that are already well connected [1, 2]. They assumed that the probability $\prod$ that a new node will be connected to node i depends on the degree k_i of node i such that

$$\prod(k_i) = \frac{k_i}{\sum_j k_j}.$$

Watts and Strogatz introduced the concept of clustering coefficient and average path length [14]. It later became apparent these properties alone are not sufficient to characterize small-world networks. Albert and Barabási introduced the scale-free property of small-world networks. In this work we will not investigate the scale-free properties of the nested $(2, r)$-regular graphs that will be defined in Chapter 2, but we gain motivation from the research on this property.

1.2 Pseudofractals and Hierarchical Graphs

Barabási, Erzébet Ravasz, and Tamás Vicsek introduced a deterministic algorithm to construct networks with scale-free properties [3]. The construction of such networks follows a hierarchical rule, where each iteration uses components that are created in previous steps [3]. The construction can be as follows. Let the initial step be a vertex. Then, in the next iteration, add two more vertices and connect them to the initial vertex. You now have constructed a P_3. In the next step, add two more copies of a P_3 and connect the mid-point of the initial P_3 with the outer vertices of the two new P_3's. This construction can be continued indefinitely [13]. Figure 3 is an example of a hierarchical network or graph.

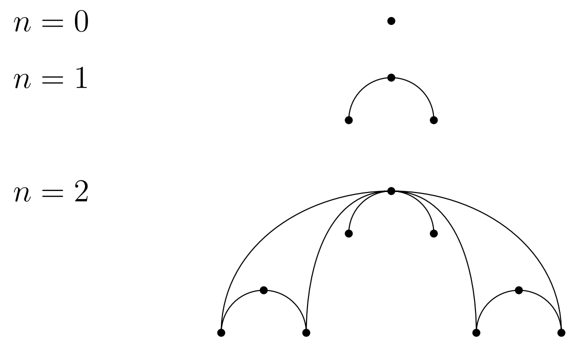

Figure 3: Hierarchical Network

The pseudofractal is another example of a deterministic graph construction that has been proposed by S. N. Dorogovstev, et al. to model the growth of scale-free networks [6]. The graph is constructed in a similar manner to that of the hierarchical graphs. The graph grows at each step by connecting together three copies of the graph in the previous step [13]. Figure 4 gives an example of one such graph.

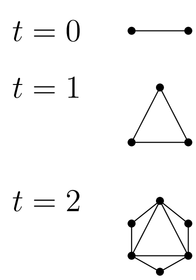

Figure 4: Pseudofractal Graph

1.3 (t, r)-Regular Graphs

As defined in [4], a *graph* G is a finite nonempty set of objects called *vertices* together with a set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$, while the *edge set* is denoted by $E(G)$.

The edge $e = \{u, v\}$ is said to *join* the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent vertices*, while u and e are *incident*, as are v and e. The cardinality of the vertex set of G is called the *order* of G and is commonly denoted by $n(G)$ or n. The cardinality of its edge set is the *size* of G and is often denoted by $m(G)$ or m. The *degree of a vertex* v in a graph G is the number of edges of G incident with v, which is denoted by deg v.

For a connected graph G, we define the *distance*, denoted $d(u, v)$, between two vertices u and v as the minimum of the lengths of the $u - v$ paths of G. The *eccentricity*, denoted $e(v)$, of a vertex v is the number $\max_{u \in V(G)} d(u, v)$. That is, $e(v)$ is the distance between v and a vertex farthest from v. The *diameter* of G, denoted diam G, is the maximum eccentricity among the vertices of G.

A graph G is *regular of degree* r if deg $v = r$ for each vertex v of G. Graphs are called r-regular if all vertices are of the same degree r and a graph is *complete* if every two of its vertices are adjacent. A complete graph is of order n and size m is therefore a regular graph of degree $n - 1$ having $m = n(n-1)/2$. We denote this graph by K_n. The *join* $G = G_1 + G_2$, sometimes denoted $G = G_1 \bigvee G_2$, has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \text{ and } v \in V(G_2)\}$. Two vertices are *independent* if they are not adjacent to one another. Figure 5 is an example demonstrating some of these properties.

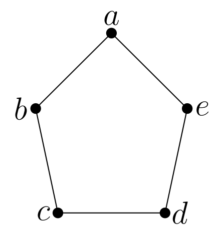

Figure 5: A 2-regular graph where a and d are independent and the diam G is 2.

A graph G is (t, r)-regular if every collection of t independent vertices is collectively adjacent to exactly r vertices. An r-regular graph is a $(1, r)$-regular graph, and in a 1996 paper by R. Faudree and D. Knisley the characterization of $(2, r)$-regular graphs was defined [7].

Theorem 1.1 [7] *Let r, s, and p be nonnegative integers and let G be a $(2, r)$-regular graph of order n. If n is sufficiently large, then G is isomorphic to $K_s + mK_p$ where $2(p-1) + s = r$. There are exactly $\lfloor \frac{r+1}{2} \rfloor$ such graphs.*

Figure 6 is the generalized construction of these graphs.

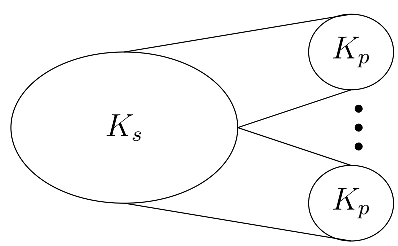

Figure 6: General form of a $(2, r)$-regular graph

Consider the example in Figure 7. Notice that the cardinality of the neighborhood of any two independent vertices is always 20 and hence this graph is a $(2, 20)$-regular graph. The degree distribution of this graph is bimodal. In fact, all $(2, r)$-regular graphs of the form $K_s + mK_p$ have a bimodal degree distribution, where the vertices have either a degree of $s + mp$ or $(p-1) + s$. Two examples of a $(2,3)$-regular graph are given in Figure 8.

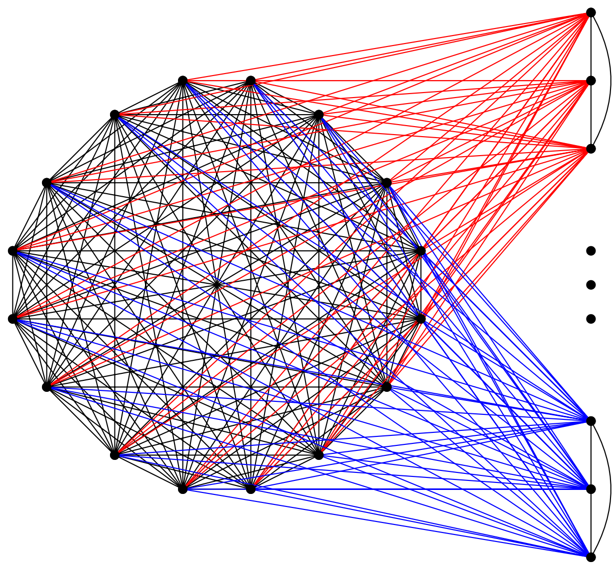

Figure 7: $K_{16} + mK_3$

To construct these graphs, we first need to solve the equation $2(p-1) + s = r$ for some value r. The kernel, denoted $Ker_t(G)$, is the set of vertices that are not in any of the independent sets of t of G. The shell, denoted $Shell_t(G)$, is the set of all vertices excluding the kernel [8]. Here p is the order of the each peripheral clique in the shell, and s is the order of the kernel. The graphs in Figure 7 are examples of $(2,r)$-regular graphs, when $r = 3$.

To find p and s, we first set up the equation:

$$2(p-1) + s = 3$$

We then solve for p and s, and our solutions are as follows: when $p = 1$, $s = 3$ and when $p = 2,\ s = 1$.

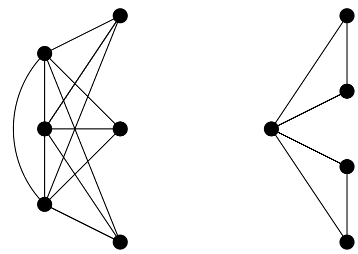

Figure 8: Examples of (2,3)-regular graphs

The graphs in Figure 8 are the only solutions since p, s are elements of the positive integers. As r increases, the number of $(2, r)$-regular graphs also increase for those values. Figure 9 is an example of a $(2, r)$-regular graph when $r = 6$. To construct such a graph, we use our equation defined above.

$$2(p - 1) + s = 6$$

Here we have the solution set $\{s, p, m\} = \{6, 1, 2\}$.

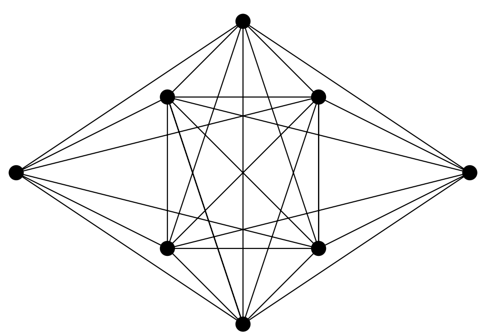

Figure 9: (2,6)-regular graph

In a 2003 paper by Jamison and Johnson, it was found that this characterization does not hold for $r \geq 3$, but the structure of such graphs are very similar to this construction when the order of the graph is sufficiently large [8].

Corollary 1.2 *For $t \geq 3$, $r \geq 1$, and $n \geq N(t,r)$, every (t,r)-regular graph of order n may be constructed as follows: choose integers $a \geq 0$, $p \geq 1$, and $m \geq t$ such that $r = t(p-1)+a$ and $n = mp+a$; if $a = 0$ take $G = mK_p$; otherwise, take a graph H with $n(H) = a$ and $\alpha(H) < t$ and construct G by putting in some of the edges between H and mK_p in $H \bigvee mK_p$ so that in G, for each $k \in \{1,...,\alpha(H)\}$, and each independent set of vertices $S \subseteq V(H)$ with $|S| = k$, no more than $t-k-1$ of the p-cliques of mK_p contain vertices not adjacent to any vertex of S.*

In this paper, Jamison and Johnson found that when $r \geq 3$ that the kernel is almost complete and the shell is mostly joined to the kernel [8].

2 NESTED (2,r)-REGULAR GRAPHS

Recall that the clustering coefficient of a graph ranges from zero to one. In general, a sparse graph is small-world if the clustering coefficient is high and the average path length is low. We find that the $(2, r)$-regular graphs have a high clustering coefficient and short average path length, but they are not sparse.

By motivation from the work on pseudofractals [3], and hierarchical graphs [13], we developed a deterministic construction to reduce the number of edges in $(2, r)$-regular graphs. In our construction, we explored the technique of 'nesting' or replacing specific cliques of the graph with a $(2, r)$-regular graph of the same order as the clique and only joining the vertices of the peripheral cliques. In this section, we will define nested 's', nested 'p', and nested 's, p'.

2.1 Nested 's'

For a nested 's' graph, we replace the center clique, K_s, with a $(2, r)$-regular graph of the form $G_1 = K_{s_1} + m_1 K_{p_1}$. In the formula $n = s + mp$ where $s = s_1 + m_1 p_1$, we get that $n = s_1 + m_1 p_1 + mp$. The nested 's' graph is now of the form $G_s = K_{s_1} + m_1 K_{p_1} + m K_p$. Figure 10 is the general form of the nested 's' graphs.

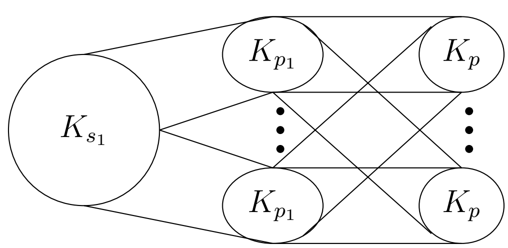

Figure 10: General form of a nested 's' $(2, r)$-regular graph

Figure 11 and 12 are examples of nesting a (2,r)-regular. The graph in Figure 11 is a $(2,10)$-regular graph with 99 edges, and then we nested the graph with a $(2,2)$-regular graph of the form $K_2 + 4K_1$. The nested 's' graph contains 69 edges. By nesting the graph, we reduce the number of edges in the graph.

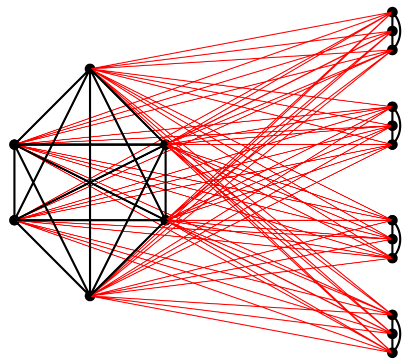

Figure 11: $(2,10)$-regular graph

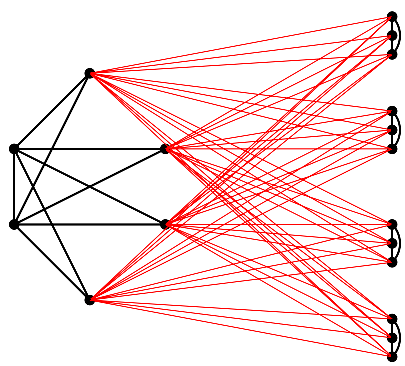

Figure 12: Nested $(2,2)$-regular graph

2.2 Nested 'p'

For a nested 'p' graph, we replace the peripheral cliques, K_p's, with $(2,r)$-regular graphs of the form $G_1 = K_{s_1} + m_1 K_{p_1}$. In the formula $n = s + mp$ where $p = s_1 + m_1 p_1$, we get that $n = s + m(s_1 + m_1 p_1)$. The nested 's' graph is now of the form $G_p = K_{s_1} + m(K_{s_1} + m_1 K_{p_1})$. Figure 13 is the general form of the nested 'p' graphs.

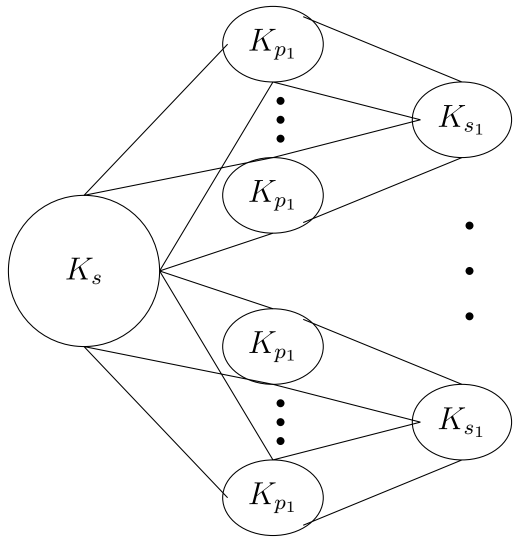

Figure 13: General form of a Nested 'p' $(2, r)$-regular graph

2.3 Nested 's, p'

For a nested 's, p' graph, we replace both the center clique, K_s, and the peripheral cliques, K_p's, with a $(2, r)$-regular graph of the form $G_1 = K_{s_1} + m_1 K_{p_1}$ and $G_2 = K_{s_2} + m_1 K_{p_2}$. In the formula $n = s + mp$ where $s = s_1 + m_1 p_1$ and $p = s_2 + m_2 p_2$, we get that $n = s_1 + m_1 p_1 + m(s_2 + m_2 p_2)$. The nested '$s, p$' graph is now of the form $G_{s,p} = K_{s_1} + m_1 K_{p_1} + m(K_{s_2} + m_2 K_{p_2})$. Figure 14 is the general form of the nested 's, p' graphs.

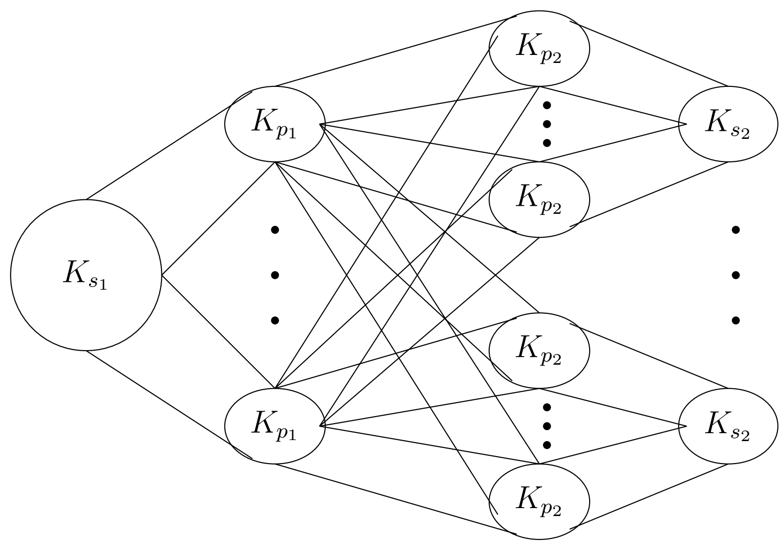

Figure 14: General form of a nested 's, p' $(2, r)$-regular graph

3 NETWORK PROPERTIES

In *Network Properties of (t,r)-regular graphs for small t*, Knisley et al. studied the network properties of $(2, r)$-regular graphs [9].

Theorem 3.1 [9] *Let s, m, and p be nonnegative integers and let $r = 2(p-1) + s$. Let G be a $(2, r)$-regular graph of the form $K_s + mK_p$. If s and p are fixed constants and $m \to \infty$, then $L \to 2$ and $C \to 1$.*

In the proof of this theorem, the generalized formulas for the average path length and clustering coefficients of the $(2, r)$-regular graph were determined. They found the average path length to be

$$L = \frac{\binom{s}{2} + smp + m\binom{p}{2} + 2[\binom{mp}{2} - m\binom{p}{2}]}{\binom{s+mp}{2}}$$

and the clustering coefficient as

$$C = \frac{s\left(\frac{\binom{s-1}{2}+m\binom{p}{2}+mp(s-1)}{\binom{s+mp-1}{2}}\right) + mp}{s + mp}.$$

We will now determine the network properties of the nested $(2, r)$-regular graphs.

3.1 Average Path Length of Nested $(2, r)$-Regular Graphs

Theorem 3.2 *The average path length of a nested 's' graph is*

$$L = 1 + \frac{p^2\binom{m}{2} + p_1^2\binom{m_1}{2} + s_1mp}{\binom{n}{2}}.$$

Proof: Since the diameter of the graph is 2, the only possible paths are those of length 1 and length 2. The total number of paths of length one are the total number of edges in the graph, therefore there are $P_1 = \binom{s_1}{2} + m_1\binom{p_1}{2} + m\binom{p}{2} + s_1m_1p_1 + m_1p_1mp$. By definition of average path length, we found that the total number of paths of length 2 are $P_2 = \binom{n}{2} - P_1$. Average path length, for graphs of diameter 2, is defined as follows

$$L = \frac{P_1 + 2P_2}{\binom{n}{2}}.$$

By substituting in our P_1 and P_2, we obtain the following

$$\begin{aligned} L &= \frac{\binom{s_1}{2} + m_1\binom{p_1}{2} + m\binom{p}{2} + s_1m_1p_1 + m_1p_1mp}{\binom{n}{2}} \\ &+ \frac{2(\binom{n}{2} - \binom{s_1}{2} + m_1\binom{p_1}{2} + m\binom{p}{2} + s_1m_1p_1 + m_1p_1mp)}{\binom{n}{2}}. \end{aligned}$$

Which reduces to

$$L = \frac{\binom{n}{2} + p^2\binom{m}{2} + p_1^2\binom{m_1}{2} + s_1mp}{\binom{n}{2}}.$$

Thus, the average path length of a nested 's' is

$$L = 1 + \frac{p^2\binom{m}{2} + p_1^2\binom{m_1}{2} + s_1mp}{\binom{n}{2}}. \blacksquare$$

Theorem 3.3 *The average path length of a nested 'p' graph is*

$$L = 1 + \frac{\binom{m}{2}(4s_1m_1p_1 + 3s_1^2) + p_1^2\binom{mm_1}{2} + sms_1}{\binom{n}{2}}.$$

Proof: The diameter of a nested 'p' graph is 4, and the only possible paths are of length 1, 2, 3, and 4. The total number of paths of length 1 are $P_1 = \binom{s}{2} + m\binom{s_1}{2} + mm_1\binom{p_1}{2} + ms_1m_1p_1 + smm_1p_1$, paths of length 2 $P_2 = sms_1 + p_1^2\binom{mm_1}{2}$, paths of length 3 $P_3 = m^2s_1m_1p_1 - ms_1m_1p_1$ and paths of length 4 $P_4 = s_1^2\binom{m}{2}$. The average path length, for graphs of diameter 4, is defined as follows

$$L = \frac{P_1 + 2P_2 + 3P_3 + 4P_4}{\binom{n}{2}}.$$

By substituting and simplifying, we obtain the following

$$L = \frac{\binom{n}{2} + \binom{m}{2}(4s_1m_1p_1 + 3s_1^2) + p_1^2\binom{mm_1}{2} + sms_1}{\binom{n}{2}}.$$

Therefore, the average path length of a nested 'p' graph is

$$L = 1 + \frac{\binom{m}{2}(4s_1m_1p_1 + 3s_1^2) + p_1^2\binom{mm_1}{2} + sms_1}{\binom{n}{2}}. \blacksquare$$

Theorem 3.4 *The average path length of a nested 's, p' graph is*

$$L = 1 + \frac{\binom{m}{2}(4s_2m_2p_2 + 3s_2^2) + p_2^2\binom{mm_2}{2} + p_1^2\binom{m_1}{2} + m(s_1m_2p_2 + m_1p_1s_2 + s_1s_2)}{\binom{n}{2}}.$$

Proof: The diameter of a nested 's, p' graph is 4, and the only possible paths are of length 1, 2, 3, and 4. The total number of paths of length 1 are $P_1 = \binom{s_1}{2} + m_1\binom{p_1}{2} + mm_2\binom{p_2}{2} + m\binom{s_2}{2} + ms_2m_2p_2 + mm_1p_1m_2p_2 + s_1m_1p_1$, paths of length 2 $P_2 = p_1^2\binom{m_1}{2} + p_2^2\binom{mm_2}{2} + ms_1m_2p_2 + mm_1p_1s_2$, paths of length 3 $P_3 = ms_2s_1 + m(m-1)s_2m_2p_2$ and paths of length 4 $P_4 = s_2^2\binom{m}{2}$. The average path length, for graphs of diameter 4, is defined as follows

$$L = \frac{P_1 + 2P_2 + 3P_3 + 4P_4}{\binom{n}{2}}.$$

By substituting and simplifying, we obtain the following

$$\frac{\binom{n}{2} + \binom{m}{2}(4s_2m_2p_2 + 3s_2^2) + p_2^2\binom{mm_2}{2} + p_1^2\binom{m_1}{2} + m(s_1m_2p_2 + m_1p_1s_2 + s_1s_2)}{\binom{n}{2}}.$$

Therefore, the average path length of a nested 's, p' graph is

$$L = 1 + \frac{\binom{m}{2}(4s_2m_2p_2 + 3s_2^2) + p_2^2\binom{mm_2}{2} + p_1^2\binom{m_1}{2} + m(s_1m_2p_2 + m_1p_1s_2 + s_1s_2)}{\binom{n}{2}}. \blacksquare$$

3.2 Clustering Coefficient of Nested $(2, r)$-Regular Graphs

Theorem 3.5 *The clustering coefficient of a nested 's' graph is*

$$C = \frac{s_1(C_U) + m_1p_1(C_V) + mp(C_W)}{s_1 + m_1p_1 + mp}.$$

Proof: By the structure of the nested 's' graph, there are only three distinct values for the clustering coefficients of the vertices. Let $\mathcal{C}_u$ denote the set of vertices whose clustering coefficient is $C_U = \frac{m_1\binom{p_1+s_1-1}{2}-(m_1-1)\binom{s_1-1}{2}}{\binom{s_1-1+m_1p_1}{2}}$, $\mathcal{C}_v$ denote the set of vertices whose clustering coefficient is $C_V = \frac{m\binom{p_1-1+p}{2}-m\binom{p_1-1}{2}+\binom{p_1-1+s_1}{2}}{\binom{p_1-1+s_1+mp}{2}}$, and $\mathcal{C}_w$ denote the set of vertices whose clustering coefficient is $C_W = \frac{m_1\binom{p-1+p_1}{2}-(m_1-1)\binom{p-1}{2}}{\binom{p-1+m_1p_1}{2}}$. The clustering coefficient of the graph is defined as the sum of the clustering coefficients of the graph divided by the total number of vertices in the graph. Therefore, the clustering coefficient of a nested 's' graph is

$$C = \frac{s_1(C_U) + m_1p_1(C_V) + mp(C_W)}{s_1 + m_1p_1 + mp}. \blacksquare$$

Theorem 3.6 *The clustering coefficient of a nested 'p' graph is*

$$C = \frac{s(C_U) + mm_1p_1(C_V) + ms_1(C_W)}{s + ms_1 + mm_1p_1}.$$

Proof: By the structure of the nested 'p' graph, there are only three distinct values for the clustering coefficients of the vertices. Let $\mathcal{C}_u$ denote the set of vertices whose clustering coefficient is $C_U = \frac{mm_1\binom{s-1p_1}{2}-(mm_1-1)\binom{s-1}{2}}{\binom{s-1+mm_1p_1}{2}}$, $\mathcal{C}_v$ denote the set of vertices whose clustering coefficient is $C_V = \frac{\binom{p_1-1+s}{2}+\binom{p_1-1+s_1}{2}-\binom{p-1}{2}}{\binom{p_1-1+s+s_1}{2}}$, and $\mathcal{C}_w$ denote the set of vertices whose clustering coefficient is $C_W = \frac{m_1\binom{s_1-1+p_1}{2}-(m_1-1)\binom{s_1-1}{2}}{\binom{s_1-1+m_1p_1}{2}}$. The clustering coefficient of the graph is defined as the sum of the clustering coefficients of the graph divided by the total number of vertices in the graph. Therefore, the clustering coefficient of a nested 'p' graph is

$$C = \frac{s(C_U) + mm_1p_1(C_V) + ms_1(C_W)}{s + ms_1 + mm_1p_1}. \blacksquare$$

Theorem 3.7 *The clustering coefficient of a nested 's, p' graph is*

$$C = \frac{s_1(C_U) + m_1p_1(C_V) + mm_2p_2(C_W) + ms_2(C_X)}{s_1 + m_1p_1 + ms_2 + mm_2p_2}.$$

Proof: By the structure of the nested 's, p' graph, there are only four distinct values for the clustering coefficients of the vertices. Let $\mathcal{C}_u$ denote the set of vertices whose clustering coefficient is

$$C_U = \frac{m_1\binom{s_1-1+p_1}{2} - (m_1-1)\binom{s_1-1}{2}}{\binom{s_1-1+m_1p_1}{2}},$$

$\mathcal{C}_v$ denote the set of vertices whose clustering coefficient is

$$C_V = \frac{\binom{p_1-1+s_1}{2} + mm_2\binom{p_1-1+p_2}{2} - mm_2\binom{p_1-1}{2}}{\binom{p_1-1+s_1+mm_2p_2}{2}},$$

$\mathcal{C}_w$ denote the set of vertices whose clustering coefficient is

$$C_W = \frac{\binom{p_2-1+s_2}{2} + m_1\binom{p_2-1+p_1}{2} - m_1\binom{p_2-1}{2}}{\binom{p_2-1+s_2+m_1p_1}{2}},$$

and $\mathcal{C}_x$ denote the set of vertices whose clustering coefficient is

$$C_X = \frac{m_2\binom{s_2-1+p_2}{2} - (m_2-1)\binom{s_2-1}{2}}{\binom{s_2-1+m_2p_2}{2}}.$$

The clustering coefficient of the graph is defined as the sum of the clustering coefficients of the graph divided by the total number of vertices in the graph. Therefore, the clustering coefficient of a nested 's, p' graph is

$$C = \frac{s_1(C_U) + m_1p_1(C_V) + mm_2p_2(C_W) + ms_2(C_X)}{s_1 + m_1p_1 + ms_2 + mm_2p_2}. \blacksquare$$

4 SPECTRAL PROPERTIES OF $(2, r)$-REGULAR GRAPHS AND NESTED 's' GRAPHS

The Laplacian Matrix is defined as

$$L = D - A$$

where L is the obtained Laplacian matrix, D is the diagonal matrix whose diagonal consists of the degrees of the vertices, and A is the adjacency matrix. In the adjacency matrix, if two vertices are adjacent to each other the element in the corresponding matrix is a one, otherwise it is a zero [4]. The Laplacian of a $(2, r)$-regular graph, of the form $K_s + mK_p$, is of the form

$$L = \left[\begin{array}{c|cccc} C & & -\mathbf{1}^T & & \\ \hline & P & 0 & \cdots & 0 \\ -\mathbf{1} & 0 & P & \ddots & \vdots \\ & \vdots & \ddots & \ddots & 0 \\ & 0 & \cdots & 0 & P \end{array}\right].$$

In the matrix above, $-\mathbf{1}$ represents the $mp \times s$ matrix in which every entry is -1, and the matrices C and P are of the form

$$C = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{bmatrix}, \quad P = \begin{bmatrix} q & -1 & \cdots & -1 \\ -1 & q & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & q \end{bmatrix},$$

where $n = s + mp$ and $q = p + s - 1$. We then studied various $(2, r)$-regular graphs and their eigenvalues. We observed that there was an overall pattern with the values, and searched for a generalized form of each of the distinct values in terms of n, s, m, and p. In Table 1, we have shown the eigenvalues of five $(2, r)$-regular graphs.

Table 1: Eigenvalues of $(2, r)$-regular graphs

EigVal	$K_{10} + 4K_3$	$K_5 + 7K_2$	$K_{11} + 3K_6$	$K_5 + 5K_5$	$K_2 + 11K_9$
λ_1	0	0	0	0	0
λ_2	10(3)	5(6)	11(2)	5(4)	2(10)
λ_3	13(8)	7(7)	17(15)	10(20)	11(88)
λ_4	22(10)	19(5)	29(11)	30(5)	101(2)

We observed that the eigenvalues of the Laplacian L are

- $\lambda_1 = 0$ with multiplicity 1
- $\lambda_2 = s$ with multiplicity $m - 1$
- $\lambda_3 = p + s$ with multiplicity $m(p - 1)$
- $\lambda_4 = n$ with multiplicity s.

The Laplacian of a nested 's' graph is of the form

$$L = \left[\begin{array}{c|cccc|cccc} C_1 & & -\mathbf{1}^T & & & & \mathbf{0}^T & & \\ \hline & P & 0 & \cdots & 0 & & & & \\ -\mathbf{1} & 0 & P & \ddots & \vdots & & -\mathbf{1}_*^T & & \\ & \vdots & \ddots & \ddots & 0 & & & & \\ & 0 & \cdots & 0 & P & & & & \\ \hline & & & & & Q & 0 & \cdots & 0 \\ \mathbf{0} & & -\mathbf{1}_* & & & 0 & Q & \ddots & \vdots \\ & & & & & \vdots & \ddots & \ddots & 0 \\ & & & & & 0 & \cdots & 0 & Q \end{array}\right]$$

where $-\mathbf{1}$ represents the $s_1 \times m_1 p_1$ matrix and $-\mathbf{1}_*$ represents the $mp \times m_1 p_1$ matrix in which every entry is -1. The matrices C_1, P, and Q are of the forms

$$C_1 = \begin{bmatrix} s_1 - 1 + m_1 p_1 & -1 & \cdots & -1 \\ -1 & s_1 - 1 + m_1 p_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & s_1 - 1 + m_1 p_1 \end{bmatrix},$$

$$P = \begin{bmatrix} q_* & -1 & \cdots & -1 \\ -1 & q_* & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & q_* \end{bmatrix}, \text{ and } Q = \begin{bmatrix} y & -1 & \cdots & -1 \\ -1 & y & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & y \end{bmatrix}$$

respectively, where $q_* = s_1 + (p_1 - 1) + mp$ and $y = m_1 p_1 + p - 1$.

We attempted to find an overall pattern with the eigenvalues of our nested 's' graphs. Tables 3 and 4 are examples of the eigenvalues of various nested 's' graphs. Here we have fixed all values except for the s_1.

Table 2: Nested s graph where $m_1 = 4$, $p_1 = 3$, $m = 6$ and $p = 1$

EigVal	$s_1 = 1$	$s_1 = 2$	$s_1 = 3$	$s_1 = 4$	$s_1 = 5$	$s_1 = 6$	$s_1 = 7$
λ_1	0	0	0	0	0	0	0
λ_2	7 (3)	8 (3)	9 (3)	10 (3)	11 (3)	12 (3)	13 (3)
λ_3	10 (8)	11 (8)	12 (8)	13 (8)	14 (8)	15 (8)	16 (8)
λ_4	12 (6)	12 (6)	12 (6)	12 (6)	12 (6)	12 (6)	12 (6)
λ_5		14 (1)	15 (2)	16 (3)	17 (4)	18 (5)	19 (6)
λ_6	19	20	21	22	23	24	25

Table 3: Nested s graph where $m_1 = 4$, $p_1 = 3$, $m = 7$ and $p = 1$

EigVal	$s_1 = 1$	$s_1 = 2$	$s_1 = 3$	$s_1 = 4$	$s_1 = 5$	$s_1 = 6$	$s_1 = 7$
λ_1	0	0	0	0	0	0	0
λ_2	8 (3)	9 (3)	10 (3)	11 (3)	12 (3)	13 (3)	14 (3)
λ_3	11 (8)	12 (8)	13 (8)	14 (8)	15 (8)	16 (8)	17 (8)
λ_4	12 (7)	12 (7)	12 (7)	12 (7)	12 (7)	12 (7)	12 (7)
λ_5		14 (1)	15 (2)	16 (3)	17 (4)	18 (5)	19 (6)
λ_6	20	21	22	23	24	25	26

5 CONCLUSION

At the beginning of this thesis we investigated the research on networks or graphs, including their network properties. We observed that there is a growing interest in the study of developing methods that can be used to grow a network in a deterministic fashion. We began to develop our own deterministic method to reduce the number of links or edges in a preexisting network and investigate its new network properties.

Our primary goal of this thesis was to develop a deterministic method to reduce the number of edges in a $(2, r)$-regular graph using a nested graph approach. We successfully achieved a new method in reducing edges in this particular family of graphs. We were also able to define the clustering coefficient and average path length of three distinct nested graphs.

We then investigated the Laplacian Matrices of the $(2, r)$-regular graphs and the nested 's' graphs. We were successful in showing the generalized forms of the matrices of these graphs. In addition, we found the general form of the eigenvalues of the $(2, r)$-regular graphs. In future work, one should determine the general form of the eigenvalues of the nested 's' graphs. We have opened the doors for further research into these nested graphs. Some open problems which came out of this research are:

- Determine the Laplacian of the two remaining nested graphs.
- What is the general form of the set of eigenvalues of each of the nested graphs?
- What values do the average path length and clustering coefficient of each nested graph approach as the graph grows?

We have introduced a novel method to construct a network with high connectivity properties. These graphs will be of interest to those working in the field.

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