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Liar's Domination in Grid Graphs

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ABSTRACT

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As introduced by Slater in 2008, liar's domination provides a way of modeling protection devices where one may be faulty. Assume each vertex of a graph G is the possible location for an intruder such as a thief. A protection device at a vertex v is assumed to be able to detect the intruder at any vertex in its closed neighborhood $N[v]$ and identify at which vertex in $N[v]$ the intruder is located. A liar's dominating set can identify an intruder's location even when any one device in the neighborhood of the intruder vertex can misidentify any vertex in its closed neighborhood as the intruder location or fail to report an intruder in its closed neighborhood. In this thesis, we present the liar's domination number for the grid graphs $P_2 \Box P_\infty$, $P_2 \Box P_c$, $P_3 \Box P_\infty$, and give bounds for other grid graphs.

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1 BACKGROUND

Our main objective in this thesis is to investigate liar's domination in grid graphs. It is useful to our discussion of liar's domination to first understand the fundamentals of graph theory.

1.1 Basic Graph Theory Definitions

As defined in Haynes, Hedetniemi, and Slater [3], a graph $G = (V, E)$ consists of a nonempty set V, or $V(G)$, and a collection E, or $E(G)$, of unordered pairs $\{uv\}$ for $u, v \in V$. We call each element in V a *vertex* and each element in E an *edge*. If any two vertices have an edge between them, we say these vertices are *adjacent*. The number of vertices, or the cardinality of V, is called the *order* of G and is denoted $|V|$, and $|E|$ is called the *size* of G.

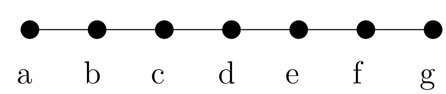

Figure 1: The path P_7

For example the graph in Figure 1, namely, the path P_7 has vertex set $V = \{a, b, c, d, e, f, g\}$, and the order of P_7 is $|V| = 7$. The edge set of P_7 is $E = \{ab, bc, cd, de, ef, fg\}$, and the size of P_7 is $|E| = 6$.

The *open neighborhood* $N(v)$ of the vertex v consists of the set of vertices adjacent to v, that is, $N(v) = \{u \in V |\ uv \in E\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup v$. In the graph in Figure 1, $N(a) = \{b\}$ and $N[a] = \{a, b\}$. For a set $S \subseteq V$, the *open neighborhood* $N(S)$ is defined to be $\cup_{v \in S} N(v)$, and the *closed neighborhood* of S is $N[S] = N(S) \cup S$. The *degree* of a vertex v is the number of edges incident with v, or $|N(v)|$, and is denoted $deg(v)$. The *minimum* and *maximum* degrees of vertices in $V(G)$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. In the graph in Figure 1, $deg(a) = 1$, $\delta(P_7) = 1$, and $\Delta(P_7) = 2$.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N[S] = V(G)$, that is, a set S is a dominating set if every element in $V \setminus S$ is adjacent to an element in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $S \subseteq V(G)$ is a *double dominating set* of G if every element in $V \setminus S$ is adjacent to at least two elements in S and every element in S is adjacent to another element in S. The *double domination number* is the minimum cardinality of a double dominating set of G and is denoted $\gamma_{\times 2}(G)$. In general, as defined in Harary and Haynes [2], a set $S \subseteq V(G)$ is a k-tuple dominating set if $|N[v] \cap S| \geq k$ for every $v \in V(G)$, and the minimum cardinality of a k-tuple dominating set of G is denoted $\gamma_{\times k}(G)$.

1.2 Liar's Domination

As introduced by Slater in 2008, a graph $G = (V, E)$ may be used to model a building, network, or computer system with each vertex in $V(G)$ representing an area in the building, hub in a computer network, or processor in a computer system. The edges in $E(G)$ could represent connections such as hallways in a building, adjacent hubs in a network, or adjacent processors in a system. Each vertex in the graph is a possible location for a thief, saboteur, fire in a facility, or fault in a computer network, henceforth reffered to as an intruder. Protection devices are placed at certain vertices, or locations, to protect them from intruders. A protection device at vertex v can detect intruders in adjacent areas and at v itself. Thus, if a protection device is placed at vertex v, then that protection device can detect an intruder in $N[v]$. A protection device at v serves two purposes: to correctly identify the intruder vertex in $N[v]$ and to correctly report the intruder location. We assume that each protection device at v is able to detect an intruder in $N[v]$, specify the location in $N[v]$ at which the intruder is located, and correctly report the intruder location.

To have some fault-tolerance in the system, at most one protection device is allowed to "lie", or misreport the vertex in its closed neighborhood at which the intruder is located. When there is an intruder in the closed neighborhood of a protection device, the device can misreport in two ways: it can report an incorrect vertex in its closed neighborhood as the intruder vertex, or it can fail to report any vertex in its closed neighborhood as the intruder vertex. It is also assumed that only detection devices that are in the closed neighborhood of the intruder vertex can report, so there can be no "false alarms".

As defined in Slater [7], a dominating set $S \subseteq V(G)$ is a *liar's dominating set* if for any vertex $v \in V(G)$ if all or all but one of the vertices in $N[v] \cap S$ report v as the intruder location, and at most one vertex w in $N[v] \cap S$ either reports a vertex $x \in N[w]$ or fails to report any vertex, then the vertex v can be correctly identified as the intruder vertex. In other words, if an intruder is at any vertex v, then the protection devices outside of $N[v]$ are assumed to not report any intruder, one vertex $w \in N[v] \cap S$ can report nothing or any vertex in $N[w]$ as the intruder vertex, every other element of $N[v] \cap S$ will correctly report vertex v as the intruder location, and

v will be correctly identified as the intruder vertex. The minimum cardinality of a liar's dominating set for graph G is called the *liar's domination number* and is denoted $\gamma_{LR}(G)$.

In order to detect an intruder in any graph, a dominating set is needed. But since any one device can fail to detect the intruder, a double dominating set is required. Let G be the graph in Figure 2.

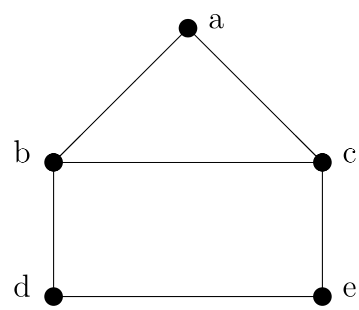

Figure 2: House graph

Since no two vertices can double dominate this graph, we need at least 3 vertices in any double dominating set. Let $S = \{b, c, d\}$ and note that S is a double dominating set of G. We will check to see if S is also a liar's dominating set. Let us assume an intruder is at vertex a. The device at d will not report anything, because there are no false alarms. Let us say the device at c correctly identifies vertex a as the intruder vertex and correctly reports it. The device at b, however, can "lie" and report vertex c as the intruder vertex. Since we cannot determine the location of the intruder with this set, it is not a liar's dominating set. Moreover, no double dominating set of cardinality 3 is a liar's dominating set, so $\gamma_{LR}(G) \geq 4$. If we let $S = \{a, b, c, d\}$ and check all possible intruder locations and liars, we can determine that this is a liar's dominating set, and thus, $\gamma_{LR}(G) = 4$.

1.3 Grid Graphs

Let P_n denote the path on n vertices. As defined in Klobucar [5], the cartesian product of two graph G and H, denoted $G \Box H$, is a graph with the vertex set $V(G) \times V(H)$ and $((uv), (wx)) \in E(G \Box H)$ if and only if either $u = w$ and $vx \in E(H)$, or $uv \in E(G)$ and $v = x$. In other words, we replace every vertex in G with a copy of H, and then corresponding vertices in the different copies of H are made adjacent whenever the original vertices in G are adjacent.

A *grid graph* is created by taking the cartesian product between two paths, P_r and P_c. For example, the cartesian product between $G = P_2$ and $H = P_7$ is $P_2 \Box P_7$, which is shown in Figure 3.

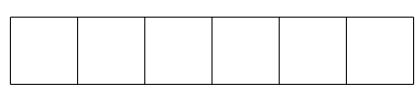

Figure 3: The cartesian product of P_2 and P_7

2 LIAR'S DOMINATION

In this section, we survey known results in liar's domination.

2.1 Important Properties

In general, we follow the notation and terminology of [3]. We will employ the following useful results from from Slater [7].

Theorem 1 [7] *If $S \subseteq V(G)$ is an LDS of G, then each component of the induced subgraph $G[S]$ contains at least three vertices.*

Since every liar's dominating set of G must double dominate G and every triple dominating set of G is a liar's dominating set, we have the following result.

Theorem 2 [7] *For every connected graph of order $n \geq 3$, we have $\gamma_{\times 2}(G) \leq \gamma_{LR}(G)$, and if G has minimum degree $\delta(G) \geq 2$, then $\gamma_{\times 2}(G) \leq \gamma_{LR}(G) \leq \gamma_{\times 3}(G)$.*

Theorem 3 [7] *A vertex set $S \subseteq V(G)$ is a LDS if and only if (1) S double dominates every $v \in V(G)$ and (2) for every pair of u, v of distinct vertices we have $|(N[u] \cup N[v]) \cap S| \geq 3$.*

2.2 Previous Work

Slater [7] introduced liar's domination in 2008. Since then there has been some progress in finding the liar's domination number of different graphs. While most results will not help us with finding the liar's domination number of grid graphs, we do have some useful results from Slater [7] which serve as a starting point. Slater [7] states and proves the following results.

The following theorems give a lower bound on γ_{LR} for all graphs G.

Theorem 4 [7] *If a graph G of order $n = |V(G)|$ has maximum degree $\Delta(G) = r$ (in particular, if G is regular of degree r), then $\gamma_{LR}(G) \geq \frac{6}{3r+2}n$.*

Theorem 5 [7] *For a graph G of order $n = |V(G)|$ and size $m = |E(G)|$, we have $\gamma_{LR}(G) \geq \frac{3}{4}(2n - m)$.*

The next result determines the liar's domination number for any path P_n.

Theorem 6 [7] *For a path P_n of order n, $\gamma_{LR}(P_n) = \left\lceil \frac{3}{4}(n+1) \right\rceil$, and $\gamma_{LR}(P_n) = \frac{3}{4}(n+1)$ if an only if $n = 4k + 3$.*

3 LIAR'S DOMINATION IN GRID GRAPHS

We have seen that the liar's domination number has been determined for some graph families. We study liar's domination on grid graphs. We determine the liar's domination number for $P_2\Box P_c$, give bounds for larger grid graphs, and determine the percentage of vertices in a $\gamma_{LR}(G)$-set for $G \in \{P_2\Box P_\infty, P_3\Box P_\infty\}$. To aid in our discussion about the cardinality of a liar's dominating set in an *infinite* graph, we need to define the liar's domination number in terms of a percentage. The parameter $\gamma_{LR}\%$ is defined by $\gamma_{LR}\% = \min\{\limsup |V(G_k) \cap S|/|V(G_k)|\}$, where G_k is the induced subgraph $P_r\Box P_k$ of $P_r\Box P_\infty$ for $r \geq 2$.

We will denote the vertices of a $P_r\Box P_c$ grid by $v_{i,j}$ for $1 \leq i \leq r$ and $1 \leq j \leq c$. We refer to the set of vertices $R_i = \{v_{i,1}, v_{i,2}, ..., v_{i,c}\}$ as the *ith* row; and the set of vertices $C_j = \{v_{1,j}, v_{2,j}, ..., v_{r,j}\}$ as the *jth* column of $P_r\Box P_c$. For a set S of vertices, we say that a column C_j is *S-empty* if $C_j \cap S = \emptyset$; and we say that it is *S-full* if all the vertices of C_j are in S, that is, if $|C_j \cap S| = r$.

3.1 Ladders

We refer to the grid graph $P_2\Box P_c$ and the infinite grid $P_2\Box P_\infty$ as *ladders*. We will prove the following theorems on the liar's domination number of ladders.

Theorem 7 *For the infinite ladder $P_2\Box P_\infty$, $\gamma_{LR}\%(P_2\Box P_\infty) = \frac{7}{12}$.*

Theorem 8 *For the finite ladder $P_2\Box P_c$, where $c \geq 2$,*

$$\gamma_{LR}(P_2\Box P_c) = \begin{cases} 7\lfloor \frac{c}{6} \rfloor + k + 1 & \text{if } c \neq 4; \\ 6 & \text{if } c = 4. \end{cases}$$

A subgraph induced by $k \geq 2$ consecutive columns of the $P_2 \Box P_\infty$ or the $P_2 \Box P_c$ is called a *k-block*. We first prove four lemmas necessary to the proofs of Theorems 1 and 2.

In these lemmas, we let B be an arbitrary k-block of a ladder. Abusing notation for the ease of discussion in the proofs of these lemmas, we refer to the columns of this arbitrary block B as $C_1, C_2, ..., C_k$ and the vertex $v_{i,j}$ is in row i and column j, for $i \in \{1, 2\}$ and $1 \leq j \leq k$.

Lemma 9 *Let $G = P_2 \Box P_c$ for $c \geq 2$, S be a $\gamma_{LR}(G)$-set, and B be an arbitrarty k-block of G. Then $|V(B) \cap S| \geq k$; and if B is preceded or succeeded by an S-empty column, then $|V(B) \cap S| \geq k + 1$.*

Proof. If no column of B is S-empty, then clearly $|V(B) \cap S| \geq k$. Moreover, if B is preceded (respectively, succeeded) by an S-empty column, then C_1 (respectively, C_k) is S-full. Since no column is S-empty, we have $|V(B) \cap S| \geq k + 1$.

Hence, assume that at least one column of B is S-empty. We proceed by induction on k. If $k = 2$ and C_1 is S-empty, then, by Theorem 3, C_2 is S-full. By symmetry, if C_2 is S-empty, then C_1 is S-full. Thus, $|V(B) \cap S| \geq 2 = k$. If B is preceded (respectively, succeeded) by an S-empty column, then Theorem 1 implies that at least one vertex from C_2 (respectively, C_1) is in S. In this case, $|V(B) \cap S| \geq 3 = k + 1$, as desired.

Let $k = 3$. If C_1 or C_3 is S-empty, then C_2 is S-full. Theorem 1 implies that at least one vertex from $C_1 \cup C_3$ is in S. Hence, $|V(B) \cap S| \geq 3 = k$. If C_2 is S-empty, then both C_1 and C_3 are S-full, and so $|V(B) \cap S| \geq 4 = k + 1$. Suppose B is preceded (respectively, succeeded) by an S-empty column. Then C_1 (respectively, C_3) is S-full and by Theorem 1, C_2 has at least one vertex in S. Then at least one additional vertex is in S from $C_2 \cup C_3$ (respectively, $C_1 \cup C_2$) in order to dominate C_3 (respectively, C_1). Hence, $|V(B) \cap S| \geq 4 = k + 1$.

Thus the base case holds for $k \in \{2, 3\}$. Assume the result is true for all k', where $2 \leq k' < k$. Let $k \geq 4$, and consider the block B' obtained by removing the last two columns of B. Since $k \geq 4$, B has $k - 2 \geq 2$ columns. By our inductive hypothesis, $|V(B') \cap S| \geq k - 2$, and if B, and hence, B', is preceded by an S-empty column, then $|V(B') \cap S| \geq k - 1$. By our inductive hypothesis, at least two additional vertices are in S from the final two columns of B, so $|V(B) \cap S| \geq k - 2 + 2 = k$, and if B is preceded by an S-empty column, $|V(B) \cap S| \geq k - 1 + 2 = k + 1$. Finally, suppose B is succeeded by an S-empty column. Then by our inductive hypothesis, $|(C_{k-1} \cup C_k) \cap S| \geq 3$, implying that $|V(B) \cap S| \geq |V(B') \cap S| + 3 = k - 2 + 3 = k + 1$, as desired. $\square$

Lemma 10 *Let $G \in \{P_2 \square P_c, P_2 \square P_\infty\}$, B be an arbitrary 6-block in G, and S be a $\gamma_{LR}(G)$-set. If $|V(B) \cap S| = 6$, then the first and last columns of B are S-empty.*

Proof. Let $G \in \{P_2 \square P_c, P_2 \square P_\infty\}$, and let S be a $\gamma_{LR}(G)$-set. Let B be an arbitrary 6-block of G. By Lemma 9, $|V(B) \cap S| \geq 6$. Assume that $|V(B) \cap S| = 6$. To show that C_1 and C_4 are S-empty, we prove a series of claims.

Claim 1 *At least one column of B is S-empty.*

Proof. Assume that no column of B is S-empty. Since $|V(B) \cap S| = 6$, it follows that each column of B has exactly one vertex in S. Without loss of generality, we may assume that $v_{1,1} \in S$.

If $v_{1,2} \in S$, then to double dominate $v_{2,2}$, we have $v_{2,3} \in S$. Since each column contributes exactly one vertex to S, Theorem 1 implies that $v_{2,4}$ and $v_{2,5}$ are in S. But then $v_{1,4}$ is not double dominated, a contradiction.

Thus, $v_{2,2} \in S$. Again, Theorem 1 implies that $v_{2,3}$ and $v_{2,4}$ are in S. But then $v_{1,3}$ is not double dominated by S, a contradiction. ($\Box$).

By Claim 1, we may assume that at least one column, say C_i, of B is S-empty.

Claim 2 *If column C_i is S-empty, then $i \notin \{2, 3, 4, 5\}$.*

Proof. By symmetry, it suffices to show that the result holds for $i \in \{2, 3\}$. If $i = 3$, then C_2 and C_4 are S-full to double dominate the vertices of C_3. Moreover, Theorem 1 implies that at least one vertex from each of C_1 and C_5 is in S. But then at least one more vertex from $C_5 \cup C_6$ is in S to dominate the vertices of C_6, contradicting that $|V(B) \cap S| = 6$. Thus, we may assume that C_3 is not S-empty. If C_2 is S-empty, then C_1 and C_3 are S-full to double dominate C_2, and by Theorem 1, at least one vertex of C_4 is in S. Without loss of generality, let $v_{1,4} \in S$. Now, at least two vertices from $\{v_{2,4}, v_{1,5}, v_{2,5}, v_{2,6}\}$ are in S to double dominate the vertex $v_{2,5}$, and so $|V(B) \cap S| \geq 7$, again a contradiction. ($\Box$)

Hence, if $|V(B) \cap S| = 6$ and C_i is S-empty, then $i \in \{1, 6\}$. Thus, we may assume that S contains at least one vertex from each of columns 2 through 5.

Claim 3 *Both C_1 and C_6 are S-empty.*

Proof. From previous claims, at least one of C_1 and C_6 is S-empty. Without loss of generality, assume that C_1 is S-empty. Then C_2 is S-full in order to dominate C_1. By Claim 2, S contains at least one vertex from each of columns 3, 4, and 5. Suppose for the purpose of a contradiction that $C_6 \cap S \neq \emptyset$. Then each of the columns 2 through 6 have exactly 1 vertex in S. Without loss of generality, we may assume that $v_{1,6} \in S$. Then $v_{2,6} \notin S$. If $v_{1,5} \in S$, then to double dominate $v_{2,5}$, we have $v_{2,4} \in S$. Theorem 1 implies that $v_{2,3} \in S$. But then $|(N[v_{1,4}] \cup N[v_{2,5}]) \cap S| = 2$, contradicting Theorem 3. Hence, $v_{1,5} \notin S$, so $v_{2,5} \in S$. Since each of C_3, C_4 and C_5 have exactly one vertex in S, Theorem 1 implies that $v_{2,3} \in S$ and $v_{2,4} \in S$. But then $v_{1,4}$ is not double dominated by S, a contradiction. Thus, both C_1 and C_6 are S-empty. (□)

Our result follows directly from Claims 1, 2, and 3. □

Lemma 11 *Let $G \in \{P_2 \Box P_c, P_2 \Box P_\infty\}$, B be an arbitrary 6-block in G, and S be a $\gamma_{LR}(G)$-set. If B is immediately preceded and succeeded by an S-empty column, then $|V(B) \cap S| \geq 9$.*

Proof. Suppose B is preceded and succeeded by S-empty columns. Since S double dominates these S-empty columns, both C_1 and C_6 are full, and Theorem 1 implies that S contains at least one vertex from each of C_2 and C_5. Without loss of generality, assume that $v_{1,2} \in S$.

If C_3 is S-empty, then C_2 and C_4 are S-full, implying that $|V(B) \cap S| \geq 9$. Hence, we may assume that C_3 has at least one vertex in S, and by symmetry, C_4 has at least one vertex in S. If any C_i, for $2 \leq i \leq 5$, is S-full, then we are finished. Hence, assume that $|V(C_i) \cap S| = 1$ for $2 \leq i \leq 5$.

If $v_{1,3} \in S$, then to double dominate $v_{2,3}$, we have $v_{2,4} \in S$. Further, Theorem 1 implies that $v_{2,5} \in S$. But then $|(N[v_{2,3}] \cup N[v_{1,4}]) \cap S| = 2$, a contradiction to Theorem 3.

If $v_{2,3} \in S$, then Theorem 1 implies that $v_{2,4}$ and $v_{2,5}$ are in S. But then $v_{1,4}$ is not double dominated by S, again a contradiction. Hence, $|V(B) \cap S| \geq 9$. $\square$

Let $G \in \{P_2 \square P_c, P_2 \square P_\infty\}$ and S be a $\gamma_{LR}(G)$-set. It simplifies our discussion if we are able to discuss 6-blocks of G by their "position" with respect to each other. Again, being loose with notation, if we begin with an arbitrary 6-block of G, we count the 6 columns immediately preceding B as its *predecessor block* and the 6 columns immediately following B as its *successor block*. We say that B is consecutive with its predecessor and successor blocks. If B' and B'' are two 6-blocks of G such that $B', B_1, B_2, ..., B_k, B''$ is a sequence of consecutive 6-blocks, then we say that blocks $B_1, B_2, ..., B_k$ *separate* blocks B' and B''. We have seen by Lemma 9 that for any 6-block B, at least 6 vertices of B are in S. We call a 6-block having exactly six vertices in S, a *good block*.

Lemma 12 *Let $G \in \{P_2 \square P_c, P_2 \square P_\infty\}$ and S be a $\gamma_{LR}(G)$-set. If B' and B'' are good blocks separated by blocks $B_1, B_2, ..., B_k$ in G, then either $|V(B_i) \cap S| \geq 9$ for some i, $1 \leq i \leq k$, or $|V(B_i) \cap S| = 8 = |V(B_j) \cap S|$ for some integers i and j, $1 \leq i \neq j \leq k$,*

Proof. Suppose B' and B'' are good blocks separated by blocks $B_1, B_2, ..., B_k$ in G. It follows from Lemma 10 that every block has at least 6 vertices in S; and if B is a good block, then it begins and ends with S-empty columns. Since S must at least double dominate G, it follows that no two good blocks of G are consecutive. Hence, $k \geq 1$. We may assume that $|V(B_i) \cap S| \geq 7$ for otherwise, one of the B_i is a good block, and we let $B'' = B_i$. By Lemma 11, our result holds if $k = 1$, so we may assume that $k \geq 2$. We prove a claim.

Claim 1 If B_i is preceded by an S-empty column and $|V(B_i) \cap S| = 7$, then the last column of B_i is S-empty.

Proof. Suppose B_i is preceded by an S-empty column and $|V(B_i) \cap S| = 7$. Lemma 11 implies that B_i is not succeeded by an S-empty column. Label the columns of B_i as $C_1, C_2, ..., C_6$. Then C_1 is S-full, and Theorem 1 implies that at least one vertex of C_2 is in S, say $v_{1,2} \in S$. To double dominate the vertices of $C_5 \cup C_6$, S contains at least two vertices from $C_5 \cup C_6$. Now to double dominate $v_{2,3}$, at least two vertices from $N[v_{2,3}]$ are in S. Since only 7 vertices from B_i are in S, it follows that exactly two vertices from $C_5 \cup C_6$ are in S and exactly two vertices from $N[v_{2,3}]$ are in S. Hence, $v_{1,4} \notin S$.

If C_5 is S-full, then C_6 is S-empty, and the claim is proven. If C_6 is S full, then C_5 is S-empty. But then $v_{1,5}$ is not double dominated by S, a contradiction. Hence, we may assume that exactly one vertex from C_5 and exactly one vertex from C_6 is in S.

If $v_{1,5} \in S$, then since $v_{1,4} \notin S$, Theorem 1 implies that $v_{1,6} \in S$. To double dominate $v_{2,5}$, we have $v_{2,4} \in S$. But then no matter which vertex from $N[v_{2,3}] - \{v_{2,4}\}$ is in S, Theorem 1 is violated. Hence, $v_{1,5} \notin S$.

If $v_{2,5} \in S$, then $v_{1,6} \in S$ to double dominate $v_{1,5}$. Then Theorem 1 implies that $v_{2,3}$ and $v_{2,4}$ are in S to be in a component with $v_{2,5}$. Now $v_{1,4}$ is not double dominated, a contradiction. Hence, C_6 is empty. (□)

We now return to the proof of our lemma. By Claim 1, if every B_i has exactly 7 vertices in S, then B_k ends in an S-empty column. By Lemma 12, B'' begins in an S-empty column. Thus, we have two consecutive S-empty columns, a contradiction.

Hence, at least one B_i has at least 8 vertices in S.

Beginning with B_1, let B_i be the first 6-block that has at least 8 vertices in S. Then, by symmetry, beginning at B_k moving to the left, let B_j be the first block having at least 8 vertices in S. Hence, by Claim 1, B_{i-1} ends in an empty column and B_{j+1} begins in an empty column. If $i = j$, then by Lemma 11, $|V(B_i) \cap S| \geq 9$. Hence, $i \neq j$, and so $|V(B_i) \cap S| \geq 8$ and $V(B_j) \cap S| \geq 8$. $\square$

Recall the statement of Theorem 1.

Theorem 1 For the infinite ladder $P_2 \square P_\infty$, $\gamma_{LR}\%(P_2 \square P_\infty) = \frac{7}{12}$.

Proof. For the infinite ladder $G = P_2 \square P_\infty$, we show that the percentage of vertices in a $\gamma_{LR}(G)$-set is $\frac{7}{12}$. We first establish an upper bound by noting that the set S of darkened vertices shown by the pattern in Figure 4 is a liar's dominating set of G. In this pattern, the block of columns labeled a through f has exactly six darkened vertices, columns 1 through 6 have exactly seven darkened vertices, and there are exactly seven darkened vertices in columns 7 through 12. Repeating the pattern established in columns 7 through 12 infinitely to the right and the pattern in columns 1 through 6 infinitely to the left yields 7/12 of the vertices in both directions. Hence, $\gamma_{LR}\%(G) \leq \frac{7}{12}$.

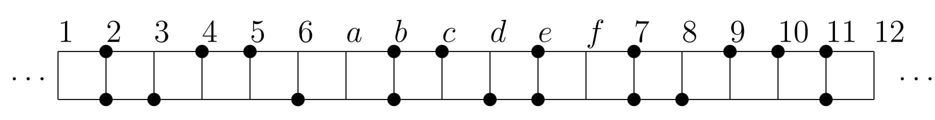

Figure 4: Pattern for $\gamma_{LR}\%(P_2 \square P_\infty) \leq \frac{7}{12}$

To prove the lower bound, let S be a $\gamma_{LR}(G)$-set. By Lemma 9, every 6-block of G has at least 6 vertices in S. If there is a finite number of good blocks, then clearly

$\gamma_{LR}\%(P_2\Box P_\infty) \geq \frac{7}{12}$. If there are two or more good blocks, then by Lemma 12, between every pair of good blocks there exists a block B_i having $|V(B_i) \cap S| \geq 9$ for some i, or two blocks B_i and B_j for some integers i and j, $i \neq j$ such that $|V(B_i) \cap S| = 8 = |V(B_j) \cap S|$. Hence, if there are an infinite number of good blocks, $\gamma_{LR}\%(P_2\Box P_\infty) \geq \frac{7}{12}$. In both cases the lower bound holds, so $\gamma_{LR}\%(P_2\Box P_\infty) = \frac{7}{12}$.
$\Box$

We now turn our attention to the finite ladder.

Observation 13 *For the finite ladder $P_2\Box P_c$, any $\gamma_{LR}(P_2\Box P_c)$-set S contains at least 3 vertices from the first two columns and at least 3 vertices from the last two columns, that is, $|(C_1 \cup C_2) \cap S| \geq 3$ and $|(C_{c-1} \cup C_c) \cap S| \geq 3$.*

Definition 14 We call the subgraph formed from $P_2\Box P_c$ by removing the first two and last two columns an *internal ladder.* Thus, the internal ladder of $P_2\Box P_c$ is the subgraph $P_2\Box P_{c-4}$ beginning with column 3 and ending with column $c-2$ in the original $P_2\Box P_c$.

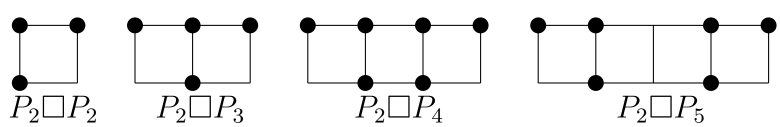

Figure 5: $\gamma_{LR}(P_2\Box P_c)$-sets for $c \leq 5$

Theorem 2 *For the finite graph $P_2\Box P_c$ where $c \geq 2$,*

$$\gamma_{LR}(P_2\Box P_c) = \begin{cases} 7\lfloor \frac{c}{6} \rfloor + k + 1 & \text{if } c \neq 4; \\ 6 & \text{if } c = 4. \end{cases}$$

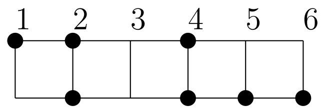

Figure 6: The block pattern

where k is the remainder of $c/6$.

Proof. We note that the darkened vertices illustrated in Figure 5 for the ladders $P_2\Box P_c$ for $c \in \{2,3,4,5\}$ are liar's dominating set for $P_2\Box P_c$. By Observation 13, the sets are $\gamma_{LR}(P_2\Box P_c)$-sets for $2 \le c \le 5$, and our result holds. Hence, we may assume that $c \ge 6$.

We first establish the upper bound. Using the block pattern illustrated in Figure 6, let S_b be the set of darkened vertices formed by repeating this pattern starting at column 1 and continuing on $\lfloor \frac{c}{6} \rfloor$ consecutive blocks of $P_2\Box P_c$. For $c \equiv 0 \pmod 6$, let $S = S_b \cup \{v_{1,c}\}$. For $c \equiv 1 \pmod 6$, let $S = S_b \cup \{v_{1,c}, v_{2,c}\}$. For $c \equiv 2 \pmod 6$, let $S = S_b \cup \{v_{1,c-1}, v_{2,c-1}, v_{1,c}\}$. For $c \equiv 3 \pmod 6$, let $S = S_b \cup \{v_{1,c-2}, v_{1,c-1}, v_{1,c}, v_{2,c-1}\}$. For $c \equiv 4 \pmod 6$, let $S = S_b \cup \{v_{1,c-3}, v_{2,c-3}, v_{1,c-1}, v_{2,c-1}, v_{1,c}\}$. For $c \equiv 5 \pmod 6$, let $S = S_b \cup \{v_{1,c-4}, v_{1,c-3}, v_{1,c-1}, v_{1,c}, v_{2,c-3}, v_{2,c-1}\}$. In each case, it is straightforward to check that S is an liar's dominating set of $P_2\Box P_c$, and hence, $\gamma_{LR}(G_{2,c}) \le 7\left\lfloor \frac{c}{6} \right\rfloor + k + 1$, where k is the remainder of $c/6$.

To establish the necessary lower bound, let S be a $\gamma_{LR}(P_2\Box P_c)$-set. By Observation 13, there are at least 6 vertices in S from $C_1 \cup C_2 \cup C_{c-1} \cup C_c$. We may assume that C_2 and C_{c-1} are S-full vertices since they dominate at least as many vertices in $P_2\Box P_c$ as the vertices in columns C_1 and C_c do. We consider the internal ladder having $c-4$ columns, where the vertices in the first and last columns, namely, C_3

and C_{c-2} in $P_2 \Box P_c$, are dominated exactly once by $S \cap (C_1 \cup C_2 \cup C_{c-1} \cup C_c)$.

We begin with the first column of the internal ladder and group the columns into consecutive 6-blocks, say $B_1, B_2, ..., B_j$, where $c - 4 = 6j + d$ and d is the remainder of $(c-4)/6$. Thus, there are $j = \lfloor (c-4)/6 \rfloor$ 6-blocks with d extra columns in the internal ladder. By Lemma 9, $|V(B_i) \cap S| \geq 6$ for $1 \leq i \leq j$.

Assume that there are at least two good blocks in $B_1, B_2, ..., B_j$, say B_a and B_b, where $a < b$. Then by Lemma 12, $|V(B_i) \cap S| \geq 9$ for some $a < i < b$ or $|V(B_i) \cap S| = 8$ and $|V(B_p) \cap S| = 8$ for some $a < i \neq p < b$. This is the case between any pair of good blocks. Hence, if there are at least two good blocks or no good blocks, $|\bigcup^j{}_{i=1} V(B_i) \cap S| \geq 7j$. Moreover, if there is exactly one good block, $|\bigcup^j{}_{i=1} V(B_i) \cap S| \geq 7(j-1) + 6$. From Lemma 10 and the proof of Lemma 12, we deduce that if $|\bigcup^j{}_{i=1} V(B_i) \cap S| = 7(j-1) + 6 = 7j - 1$, then exactly one of the B_i's is a good block and that B_j ends in an S-empty column. Thus, to count the minimum number of vertices in S, we add $|\bigcup^j{}_{i=1} V(B_i) \cap S|$ plus 6 for the vertices from the first two and last two columns of G plus the number of vertices in S from the remaining d columns of the internal ladder.

We consider six cases based on d.

Case 1: $d = 0$. Then $c \equiv 4 \pmod{6}$, so $k = 4$. It follows that $|S| \geq 7(j-1) + 6 + 6 = 7j + 5 = 7\left\lfloor \frac{c-4}{6} \right\rfloor + 5 = 7\left\lfloor \frac{c}{6} \right\rfloor + k + 1$.

Case 2: $d = 1$. Then $c \equiv 5 \pmod{6}$, so $k = 5$. If there are no good blocks in $\{B_1, B_2, ..., B_j\}$, then $|S| \geq 7j + 6 = 7\lfloor \frac{c-4}{6} \rfloor + 6 = 7\lfloor \frac{c}{6} \rfloor + k + 1$, as desired. If there is a good block, then B_j ends in an S-empty column, that is, C_{c-3} is an S-empty column, implying that C_{c-2} is S-full. Hence, $|S| \geq 7(j-1) + 6 + 6 + 2 = 7\lfloor \frac{c-4}{6} \rfloor - 7 + 14 =$

$7\lfloor \frac{c}{6} \rfloor + k + 1$.

Case 3: $d = 2$. Then $c \equiv 0 \pmod 6$, so $k = 0$. By Lemma 9, $|S \cap (C_{c-2} \cup C_{c-3})| \geq 2$. If there is no good block in $\{B_1, B_2, ..., B_j\}$, then $|S| \geq 7\left\lfloor \frac{c-4}{6} \right\rfloor + 6 + 2 = 7\left\lfloor \frac{c}{6} \right\rfloor + 1$. If there is a good block, then since B_j ends in an S-empty column, by Lemma 9 $|S \cap (C_{c-2} \cup C_{c-3})| \geq 3$. Hence, $|S| \geq 7(j-1) + 6 + 6 + 3 = 7\left\lfloor \frac{c-4}{6} \right\rfloor + 8 = 7\left\lfloor \frac{c}{6} \right\rfloor + 1$.

Case 4: $d = 3$. Then $c \equiv 1 (\bmod 6)$, so $k = 1$. By Lemma 9, $|S \cap (C_{c-4} \cup C_{c-3} \cup C_{c-2})| \geq 3$. Moreover, if there is a good block in $B_1, B_2, ..., B_j$, then B_j ends in an empty column, and so by Lemma 9, $|S \cap (C_{c-4} \cup C_{c-3} \cup C_{c-2})| \geq 4$. Hence, either $|S| \geq 7j + 6 + 3 = 7\left\lfloor \frac{c-4}{6} \right\rfloor + 9$ or $|S| \geq 7(j-1) + 6 + 6 + 4 = 7\left\lfloor \frac{c-4}{6} \right\rfloor + 9$. In both cases, $|S| \geq 7\left\lfloor \frac{c}{6} \right\rfloor + k + 1$.

Case 5: $d = 4$. Then $c \equiv 2 (\bmod 6)$, so $k = 2$. By Lemma 9, $|S \cap (C_{c-5} \cup C_{c-4} \cup C_{c-3} \cup C_{c-2})| \geq 4$, and using a similar argument as in previous cases, if there is a good block, then $|S \cap (C_{c-5} \cup C_{c-4} \cup C_{c-3} \cup C_{c-2})| \geq 5$. If there is no good block, then $|S| \geq 7\lfloor \frac{c-4}{6} \rfloor + 6 + 4 = 7\lfloor \frac{c}{6} \rfloor + k + 1$. If there is a good block, $|S| \geq 7(j-1) + 6 + 6 + 5 = 7\lfloor \frac{c}{6} \rfloor + k + 1$.

Case 6: $d = 5$. Then $c \equiv 3 \pmod 6$, so $k = 3$. By Lemma 9, $|S \cap (C_{c-6} \cup C_{c-5} \cup C_{c-4} \cup C_{c-3} \cup C_{c-2})| \geq 5$, and using a similar argument as in previous cases, if there is a good block, then $|S \cap (C_{c-6} \cup C_{c-5} \cup C_{c-4} \cup C_{c-3} \cup C_{c-2})| \geq 6$. If there is no good block, then $|S| \geq 7\left\lfloor \frac{c-4}{6} \right\rfloor + 6 + 5 = 7\left\lfloor \frac{c}{6} \right\rfloor + k + 1$. If there is a good block, $|S| \geq 7(j-1) + 6 + 6 + 6 = 7\left\lfloor \frac{c-4}{6} \right\rfloor + 11 = 7\left\lfloor \frac{c}{6} \right\rfloor + k + 1$.

Thus, in every case, $\gamma_{LR}(P_2 \square P_c) \geq 7\left\lfloor \frac{c}{6} \right\rfloor + k + 1$, and so for $2 \leq c \neq 4$, we have $\gamma_{LR}(P_2 \square P_c) = 7\left\lfloor \frac{c}{6} \right\rfloor + k + 1$. $\square$

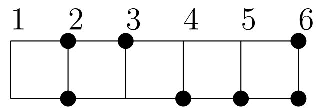

Figure 7: The internal ladder block pattern

3.2 $P_3 \Box P_\infty$

We now shift our focus to the $P_3 \Box P_\infty$ grid. Similar to the ladder, we define a *4-block* as four consecutive columns of the $P_3 \Box P_\infty$. Our goal is to determine $\gamma_{LR}\%(G)$ for $P_3 \Box P_\infty$. We desire to prove the following theorem.

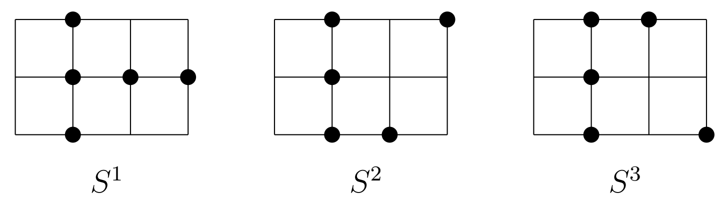

Figure 8: Good block configurations

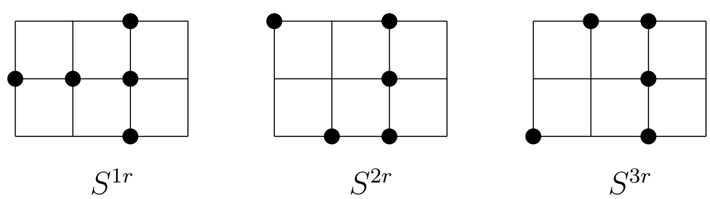

Figure 9: Reflected good block configurations

Theorem 15 *For the infinite graph $P_3 \Box P_\infty$, $\gamma_{LR}\%(P_3 \Box P_\infty) = \frac{1}{2}$.*

Let B be an arbitrary 4-block, and let S be a $\gamma_{LR}(P_3 \Box P_\infty)$-set.

Lemma 16 *Let $G = P_3 \Box P_\infty$, B be any 4-block in G, and S be a γ_{LR}-set. Then $|S \cap V(B)| \geq 5$ and if $|S \cap V(B)| = 5$, then the first or last column of B is S-empty.*

If there exists a 4-block, B, such that $|S \cap V(B)| = 5$, then we call B a good block. Every good block configuration is illustrated by one of the patterns in Figures 8 and 9.

Proof. Let $G = P_3 \Box P_\infty$, B be any 4-block in G, and S be a γ_{LR}-set. Label the columns of B as C_1, C_2, C_3, and C_4. Observe that in every block B, C_2 and C_3 must be at least double dominated by the vertices of $V(B) \cap S$ with C_1 and C_4 at least dominated. If $|(C_2 \cup C_3) \cap S| \leq 1$, then C_2 or C_3 is not double dominated. Hence, $|(C_2 \cup C_3) \cap S| \geq 2$. If $|(C_2 \cup C_3) \cap S| \geq 6$, we are finished. Hence, assume that $2 \leq |(C_2 \cup C_3) \cap S| \leq 5$, implying that at most one of C_2 and C_3 is S-full. If $|(C_2 \cup C_3) \cap S| = 5$, then at least one vertex of $C_1 \cup C_4$ is not dominated by the vertices of $C_2 \cup C_3$ implying that S contains a vertex from $C_1 \cup C_4$ and so $|V(B) \cap S| \geq 6$.

Suppose $|(C_2 \cup C_3) \cap S| = 2$. If $|C_2 \cap S| = 2$, then C_3 is not double dominated. Similarly, both vertices of S are not in C_3. Hence, $|C_2 \cap S| = 1$ and $|C_3 \cap S| = 1$. If $v_{1,2}$ and $v_{1,3}$, or $v_{3,2}$ and $v_{3,3}$ are in S, then C_3 is not double dominated is not dominated by S. If $v_{2,2} \in S$ and $v_{2,3} \in S$, then $\{v_{1,1}, v_{3,1}, v_{1,4}, v_{3,4}\} \subseteq S$ to double dominate the vertices of $C_2 \cup C_3$. Hence, $|V(B) \cap S| \geq 6$. If $v_{1,2} \in S$ and $v_{2,3} \in S$, then C_2 is not double dominated. Similarly, if any of the pairs $v_{1,3}$ and $v_{2,2}$, $v_{3,2}$ and $v_{2,3}$, and $v_{2,2}$ and $v_{3,3}$ is in S, then S does not double dominate $C_2 \cup C_3$. If $v_{1,2} \in S$ and $v_{3,3} \in S$, then C_1 and C_4 are S-full and $|V(B) \cap S| \geq 8$.

Let $|C_2 \cup C_3| = 3$. If C_2 is S-full, then C_3 is S-empty implying that C_4 is S-full. Hence, $|V(B) \cap S| \geq 6$. Thus, C_2 is not S-full. Similarly, C_3 is not S-full. Without loss of generality, we may assume that exactly two vertices from C_2 and one vertex from C_3 are in S. We consider two cases depending on the vertices of C_2. Assume the vertices of $C_2 \cap S$ are adjacent. Without loss of generality, we may assume that $v_{1,2} \in S$ and $v_{2,2} \in S$. Then, if $v_{1,3} \in S$, then $v_{3,3}$ is not double dominated by S, a contradiction. If $v_{2,3} \in S$, then to double dominate $\{v_{3,2}, v_{3,3}, v_{3,1}, v_{1,4}\}$, S requires at least three vertices from $C_1 \cup C_4$. Thus, $|V(B) \cap S| \geq 6$. If $v_{3,3} \in S$, then Theorem 1 implies that $v_{4,4} \in S$ and at least one of $v_{1,1}$ and $v_{2,1}$ is in S. Moreover, to double dominate $v_{1,3}$, it follows that $v_{1,4} \in S$. Thus, $|V(B) \cap S| \geq 6$.

If the vertices in $C_2 \cap S$ are not adjacent, then it must be that $v_{2,2} \notin S$. Hence, $v_{1,2} \in S$ and $v_{3,2} \in S$. If $v_{1,3} \in S$, then at least two vertices from C_4 are in S to double dominate $\{v_{2,3}, v_{3,3}\}$, and at least one vertex from C_1 is in S to double dominate $v_{2,1}$. Thus, $|V(B) \cap S| \geq 6$. Similarly, the result holds in $v_{3,3} \in S$. If $v_{2,3} \in S$, then Theorem 1 implies that $v_{2,4} \in S$, $v_{1,1} \in S$, and $v_{3,1} \in S$. Hence, $|V(B) \cap S| \geq 6$.

Thus, we may assume that $|(C_2 \cup C_4) \cap S| = 4$. If $|(C_1 \cup C_4) \cap S| \geq 2$, then we are finished. Hence, let $|(C_1 \cup C_4) \cap S| = 1$, that is $|V(B) \cap S| = 5$. It follows that one of C_1 and C_4 is S-empty. Without loss of generality, assume that C_1 is S-empty. Then C_2 is S-full. Now, $|C_3 \cap S| = 1$ and $|C_4 \cap S| = 1$. To double dominate $C_3 \cup C_4$, each vertex of $C_3 \cup C_4$ must be dominated by $|(C_3 \cup C_4) \cap S|$. Hence, either $\{v_{1,2}, v_{3,4}\} \subseteq S$, $\{v_{1,4}, v_{3,3}\} \subseteq S$, or $\{v_{2,3}, v_{2,4}\} \subseteq S$. By symmetry, the same holds for C_4 is S-empty. Since in all other cases, $|V(B) \cap S| \geq 6$, this establishes the block patterns (shown in Figures 8 and 9) necessary for a *good* 4-block. By symmetry, either C_1 or C_4 is S-empty and C_2 or C_3, respectively, is S-full. $\square$

Lemma 17 *If C_1 and C_4 are S-full in any 4-block B, then $|V(B) \cap S| \geq 8$.*

Proof. Let B be an arbitrary 4-block, and let C_1 and C_4 be S-full. Then by Theorem 3, we must at least double dominate C_2 and C_4, so $|(C_2 \cup C_3) \cap S| \geq 2$. Thus,

$|V(B) \cap S| \geq 8$. $\square$

Lemma 18 *If a 4-block B_i is preceded and succeeded by good blocks, then $|V(B_i) \cap S| \geq 8$.*

Proof. Label the columns in B_i as 1 through 4. Let S^1 be the first configuration of vertices in Figure 8. Notice that the first block in Figure 9 is a reflection of S^1. Let the reflection of S^1 be S^{1r}. Let the second configuration of vertices in Figure 8 be labeled S^2 and its reflection labeled S^{2r}. Let the third configuration of vertices in Figure 8 be labeled S^3 and its reflection labeled S^{3r}. Let S be a γ_{LR}-set. We now consider cases based on good block configurations in B_{i-1} and B_{i+1}.

Case 1: Let B_{i-1} have any configuration in Figure 9 or S^2 or S^3 from Figure 8. If B_{i-1} has a configuration from Figure 9, then Theorem 3 implies that C_1 is S-full. If B_{i-1} has S^2 or S^3, without loss of generality, say S^2, then Theorem 1 implies that $v_{1,1} \in S$. Moreover, to double dominate the vertices of the last column of B_{i-1}, $v_{2,1}$ and $v_{3,1}$ are in S. Thus, in any case, C_1 is S-full.

If B_i is succeeded by any configuration in Figure 9 or S^{2r} or S^{3r} in Figure 8, then a similar argument to above shows that C_4 is S-full. Then to double dominate the vertices of $C_2 \cup C_3$ at least two vertices from $C_2 \cup C_3$ are in S, implying that $|V(B_i) \cap S| \geq 8$ as desired.

Assume that B_i is succeeded by S^{1r}. Then $v_{1,4} \in S$ and $v_{2,4} \in S$ to double dominate the vertices of B_{i+1}. If $v_{2,4} \in S$, then as before S contains 2 vertices from $C_2 \cup C_3$ and we are finished.

Hence, assume that $v_{2,4} \notin S$. Then Theorem 1 implies that $v_{1,3}$ and $v_{3,3}$ are in S. Moreover, either $v_{2,3} \in S$ or both $v_{1,2}$ and $v_{3,2}$ are in S. In either case, $|V(B) \cap S| \geq 8$.

Case 2: Let B_{i-1} have configuration S^1. Using a symmetric argument to above, we are finished for all cases except when B_{i+1} has configuration S^{1r}. Then to double dominate the vertices of B_{i-1} and B_{i+1}, we have that $v_{1,1}, v_{3,1}, v_{1,4}$, and $v_{3,4}$ are in S. By Theorem 1, either $v_{2,1} \in S$ or both $v_{1,2}$ and $v_{3,2}$ are in S. Similarly, either $v_{2,4} \in S$ or both $v_{1,3}$ and $v_{3,3}$ are in S. If $\{v_{1,2}, v_{3,2}, v_{1,3}, v_{3,3}\} \subseteq S$, then $|V(B_i) \cap S| \geq 8$, and we are finished.

Hence, without loss of generality, we may assume that $v_{2,1} \in S$, that is, C_1 is S-full. If $v_{2,4} \in S$, then to double dominate $C_2 \cup C_3$, at least two additional vertices from $C_2 \cup C_3$ are in S, implying that $|V(B_i) \cap S| \geq 8$.

Thus, $v_{2,4} \notin S$, and so $v_{1,3}$ and $v_{3,3}$ are in S. Theorem 1 implies that at least one additional vertex from $C_2 \cup C_3$ is in S. Again, $|V(B_i) \cap S| \geq 8$. $\square$

Lemma 19 *If B_i is preceded by a good block and $|V(B_i) \cap S| = 6$, then the last column in B_i is S-empty.*

Proof. Label the columns in B_i as C_1 through C_4, and suppose B_i is preceded by a good block and $|V(B_i) \cap S| = 6$. If C_4 is S-empty, we are finished, so assume that $|C_4 \cap S| \geq 1$. If B_{i-1} has any configuration in Figure 9, or S^2, or S^3, of Figure 8 then by Theorem 3, C_1 is S-full. Theorem 18 implies that B_i is not succeeded by a good block. Then there are exactly three vertices from $C_2 \cup C_3 \cup C_4$ in S. But then the vertices of $C_2 \cup C_3$ are not double dominated.

If B_{i-1} has the configuration S^1, then $v_{1,1} \in S$ and $v_{3,1} \in S$ to double dominate the vertices of B_{i-1}. If $v_{2,1} \in S$, then C_1 is S-full and since C_4 is not S-empty, there are exactly two vertices from $C_2 \cup C_3$ in S. This is the same as the previous case where the vertices of $C_2 \cup C_3$ are not double dominated. Hence, $v_{2,1} \notin S$. Theorem 1 implies that $v_{1,2}$ and $v_{3,2}$ are in S. Moreover, either $v_{2,2} \in S$ or both $v_{1,3}$ and $v_{3,3}$ are in S. For the latter, $v_{2,4}$ is double dominated by S, a contradiction. Hence, $v_{2,2} \in S$, and since $|V(B_i) \cap S| = 6$, exactly one vertex of C_4 is in S and C_3 is S-empty. But then at least one vertex of C_3 is not double dominated, a contradiction. Hence, we conclude that C_4 is S-empty. $\square$

Lemma 20 *Let $G = P_3 \Box P_\infty$, and let S be a $\gamma_{LR}(G)$-set. If B' and B'' are good blocks separated by $B_1, B_2, ..., B_k$, then $|V(B_i) \cap S| \geq 8$ for some i, $1 \leq i \leq k$ or $|V(B_i) \cap S| = 7 = |V(B_j) \cap S|$ for some integers i, j, $i \neq j$.*

Proof. Let B' and B'' be good blocks separated by blocks $B_1, B_2, ..., B_k$ in $P_3 \Box P_\infty$. Lemma 16 implies that $k \geq 1$. We assume that $|V(B_i) \cap S| \geq 6$, $1 \leq i \leq k$, otherwise one of the B_i's is a good block, and we let it be B''. By Lemma 18, our result holds for $k = 1$, so we assume that $k \geq 2$. By Lemma 16, B' ends in an S-empty column or a column with 1 vertex in S, and B'' begins in an S-empty column or a column with 1 vertex in S. Assume for a contradiction that $|V(B_i) \cap S| = 6$ for $1 \leq i \leq k$. By Lemma 19, B_1 ends in an S-empty column and B_k ends in an S-empty column. But then at least one vertex of B'' is not double dominated by S. Hence, at least one B_i has at least 7 vertices in S.

Beginning with B_1, let B_i be the first 4-block that has at least 7 vertices in S. By symmetry, beginning at B_k moving to the left, let B_j be the first block having at least 7 vertices in S. Hence, by Lemma 19, B_{i-1} ends in an empty column and B_{j+1} begins in an empty column. If $i = j$, then by Lemma 17, $|V(B_i) \cap S| \geq 8$ and we are finished. Hence, $i \neq j$, and so $|V(B_i) \cap S| \geq 7$ and $|V(B_j) \cap S| \geq 7$ as desired. $\square$

Theorem 15 *For the infinite graph $P_3 \Box P_\infty$, $\gamma_{LR}\%(P_3 \Box P_\infty) = \frac{1}{2}$.*

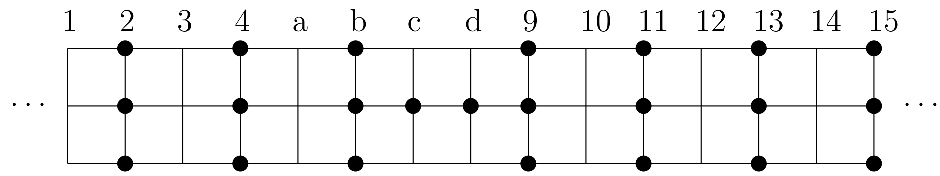

Figure 10: Upper bound for $\gamma_{LR}(P_3\Box P_\infty)$

Proof. For the infinite graph $G = P_3\Box P_\infty$, we show that the percentage of vertices in a $\gamma_{LR}(P_3\Box P_\infty)$-set is $\frac{1}{2}$. We establish the upper bound by noting the set S of darkened vertices shown by the pattern in Figure 10 is a liar's dominating set of $P_3\Box P_\infty$. We have that in every block, B, there are at least 6 vertices in S, except for possibly one block, B_i, which has 5 vertices is S. Repeating the pattern in columns 12 through 15 infinitely to the right and the pattern in columns 1 through 4 infinitely to the left yields $\frac{1}{2}$ of the vertices in both directions. Thus, $\gamma_{LR}\%(P_3\Box P_\infty) \leq \frac{1}{2}$.

To prove the lower bound, let S be a $\gamma_{LR}(G)$-set. By Lemma 16, every 4-block of G has at least 5 vertices in S. If there is at most one good block, then clearly $\gamma_{LR}\%(P_3\Box P_\infty) \geq \frac{1}{2}$. If there are two or more good blocks, then by Lemma 20, between every pair of good blocks there exists a block $|V(B_i) \cap S| \geq 8$ for some i or $1 \leq i \leq k$ or $|V(B_i) \cap S| = 7 = |V(B_j) \cap S|$ for some integers i, j, $i \neq j$. Hence, if there is more than one good block, the average number in S is at least $\frac{1}{2}$. Thus, $\gamma_{LR}\%(P_3\Box P_\infty) = \frac{1}{2}$. $\Box$

3.3 Bounds on Other Grids

Next consider the finite grid graph $P_3\Box P_c$.

Lemma 21 *In the finite graph $P_3\Box P_c$, $|(C_1 \cup C_2) \cap S| \geq 3$ and $|(C_c \cup C_{c-1}) \cap S| \geq 3$, where c is number of columns in $P_3\Box P_c$.*

Proof. Since every liar's dominating set must be a double dominating set, it follows that any liar's dominating set of $P_3 \square P_c$ must have at least 3 vertices from the first two columns in order to double dominate the C_1. Similarly, every liar's dominating set requires 3 vertices from the last two columns. $\square$

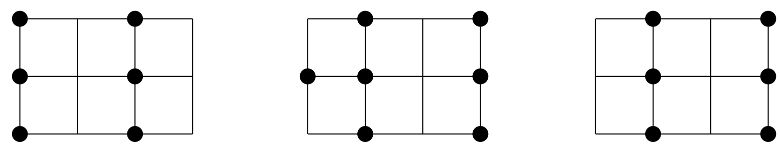

Figure 11: Patterns for finite upper bound

Conjecture 22 *For the finite grid $P_3 \square P_c$, $\gamma_{LR}(P_3 \square P_c) = \lceil \frac{n}{2} \rceil + 1$.*

Proposition 23 *For the finite grid $P_3 \square P_c$, $\gamma_{LR}(P_3 \square P_c) \leq \lceil \frac{n}{2} \rceil + 1$.*

Proof. Let $B \equiv P_3 \square P_4$ be the first 4-block pattern shown in Figure 11. Let S_b be the set of darkened vertices formed by repeating the first block pattern $\lfloor \frac{c}{4} \rfloor$ times on $P_3 \square P_c$.

Let T_b be the set of darkened vertices formed by beginning with the second block pattern in Figure 11 and repeating the third block pattern shown in Figure 11, starting with the first column to the right of C_4, $\lfloor \frac{c-4}{4} \rfloor$ times on $P_3 \square P_c$.

For $c \equiv 0(\text{mod}4)$, let $S = S_b \cup \{v_{2,c}\}$. For $c \equiv 1(\text{mod}4)$, let $S = T_b \cup \{v_{2,c}\}$. For $c \equiv 2(\text{mod}4)$, let $S = S_b \cup \{v_{1,c-1}, v_{2,c-1}, v_{3,c-1}, v_{2,c}\}$. For $c \equiv 3(\text{mod}4)$, let $S = T_b \cup \{v_{1,c-1}, v_{2,c-1}, v_{3,c-1}, v_{2,c}\}$. In each case, S is a liar's dominating set of $P_3 \square P_c$, and hence, $\gamma_{LR}(P_3 \square P_c) \leq \lceil \frac{n}{2} \rceil + 1$. $\square$

It remains an open problem to prove that $\gamma_{LR}(P_3 \square P_c) \geq \lceil \frac{n}{2} \rceil + 1$.

We attain an upper bound on grid graphs of the form $P_n \square P_\infty$, for $n \geq 4$.

Let S be any liar's dominating set. Let every other column of $P_n \square P_\infty$ be in S. We can easily check that S is a liar's dominating set and thus, $\gamma_{LR}\%(P_n \square P_\infty) \leq \frac{1}{2}$. Thus, we have the following proposition.

Proposition 24 *For $G = P_n \square P_\infty$, where $n \geq 4$, $\gamma_{LR}\%(G) \leq \frac{1}{2}$.*

It is an open problem to prove that $\gamma_{LR}\%(P_n \square P_\infty) \geq \frac{1}{2}$, for $n \geq 4$.

Conjecture 25 *For $G = P_n \square P_\infty$, where $n \geq 4$, $\gamma_{LR}\%(G) = \frac{1}{2}$.*

For the infinite grid $\mathbb{Z} \square \mathbb{Z}$, we give an upper bound for the percentage of vertices in a $\gamma_{LR}(\mathbb{Z} \square \mathbb{Z})$-set. Let S be the tiling showing in Figure 12. It is easy to check that the percentage of vertices in S is $\frac{9}{20}$ if we use this tiling on the entire grid. It remains an open problem to show that the percentage of vertices in S is at least $\frac{9}{20}$.

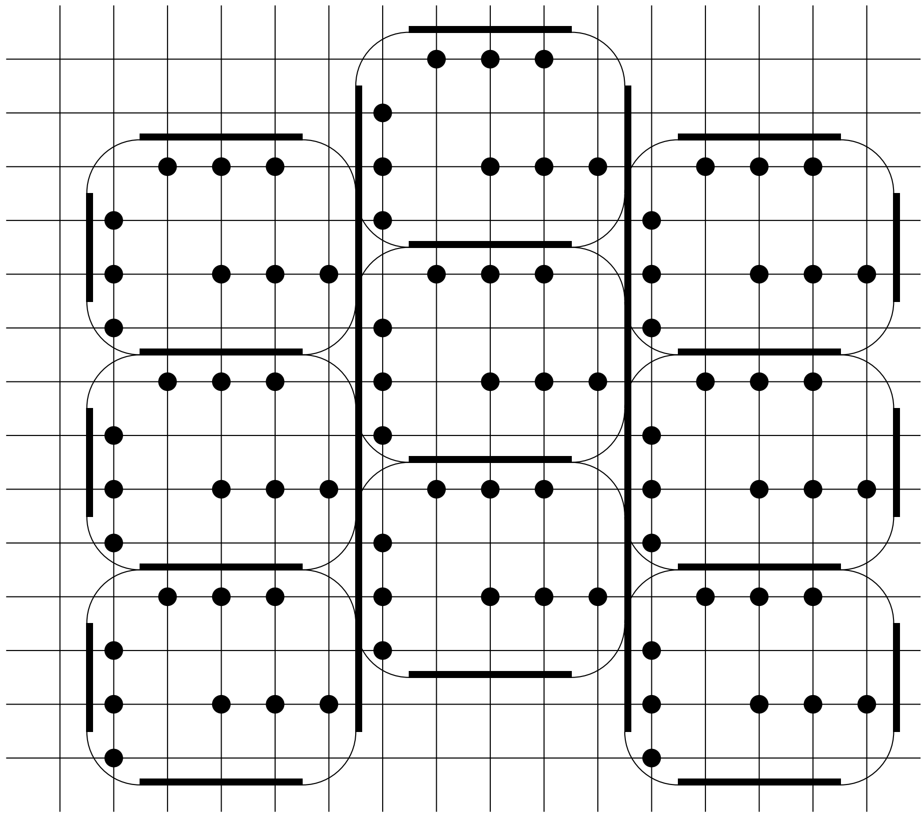

Figure 12: An upper bound for γ_{LR} for the infinite grid graph

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