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Omnisculptures.

Cihan Eroglu
East Tennessee State University

Omnisculptures

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by

Cihan Eroglu

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Anant Godbole, Ph.D., Chair

Teresa Haynes, Ph.D.

Michel Helfgott, Ph.D.

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ABSTRACT

Omnisculptures

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In this thesis we will study conditions for the existence of minimal sized omnipatterns in higher dimensions. We will introduce recent work conducted on one dimensional and two dimensional patterns known as omnisequences and omnimosaics, respectively. These have been studied by Abraham et al [3] and Banks et al [2]. The three dimensional patterns we study are called omnisculptures, and will be the focus of this thesis. A (K, a) omnisequence of length n is a string of letters that contains each of the a^k words of length k over $[A]$=(1,2,...a) as a substring. An omnimosaic $O(n, k, a)$ is an $n \times n$ matrix, with entries from the set A=1,2,...,a, that contains each of the $\{a^{k^2}\}$ $k \times k$ matrices over A as a submatrix. An omnisculpture is an $n \times n \times n$ sculpture (a three dimensional matrix) with entries from set $A = \{1, 2, ..., a\}$ that contains all the a^{k^3} $k \times k \times k$ subsculptures as an embedded submatrix of the larger sculpture. We will show that for given k, the existence of a minimal omnisculpture is guaranteed when

$$\frac{ka^{\frac{k^2}{3}}}{e} \leq n \leq \frac{ka^{\frac{k^2}{3}}}{e}(1 + \epsilon)$$

and $\epsilon = \epsilon_k \to 0$ is a sufficiently small function of k.

DEDICATION

I would like to dedicate this thesis to my son Gabriel Eroglu. He is the single best thing that has happened in my life. Without his smile, and obsession with superheros, my life would not be as joyful and interesting.

CONTENTS

LIST OF FIGURES

A **vertex** is an arbitrary point v from a set $V(G)$.

An **edge** is a line that connects one vertex to other, or it could also connect a vertex to itself. In an abstract sense, an edge is thus a multiset of two vertices.

A **graph** G is a pair consisting of **vertex set** $V(G)$, and an **edge set** $E(G)$, which is a relation that associates with each two vertices (not necessarily distinct) their **endpoints**.

A **bipartite graph** (Fig. 1)is a graph whose set of vertices can be decomposed into two disjoint sets such that no two vertices in the same set are adjacent, i.e. connected by an edge. For a simple graph the adjacency matrix (Fig. 2) (sometimes

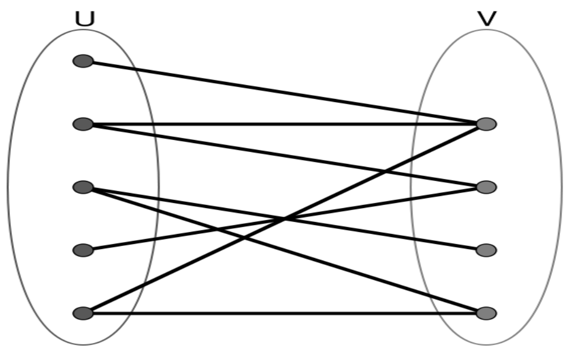
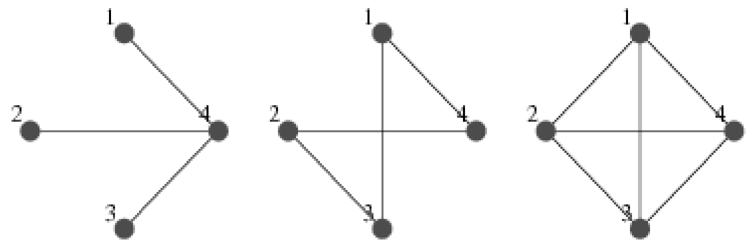


Figure 1: Bipartite Graph

called the connection matrix) is a matrix of rows and columns labeled by graph vertices, with a 1 or 0 in position (v_i, v_j) according to whether v_i and v_j are adjacent or not. For any simple graph with no self-loops, the adjacency matrix must have zeros on the diagonal. Below are some examples of adjacency matrices. Notice that



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure 2: Adjacency Matrix

any adjacency matrix is always symmetric. The adjacency matrix A of a bipartite graph whose parts have r and s vertices has the form

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

where B is an $r \times s$ matrix and O is an all-zero matrix. As you can see, the matrix B uniquely represents the bipartite graphs, and it is commonly called a biadjacency matrix.

A **subgraph** is a graph whose vertices and edges are subsets of the vertices and edges another graph.

An **induced subgraph** (Fig. 3) is a subgraph obtained by the deletion of a vertex or multiple vertices. [7]

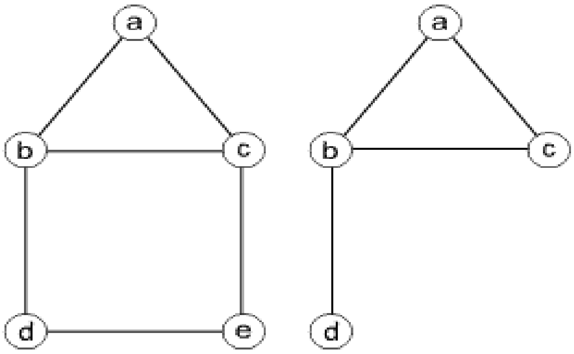


Figure 3: Induced Subgraph

A **universal graph** on n vertices is a graph that contains all k induced subgraphs on vertices (where $k < n$) on some selection of vertices $1 \leq v, < v_z < ... < v_k \leq n$. [4]

2 OMNISEQUENCES

Introduction

Our research is related to finding some embedded patterns in a large set of data. Imagine we are looking for certain words in a text and we start reading this text and we stop when we find the corresponding letters. A good example of this would be the codes that are allegedly embedded in the Bible [11]. It is also similar to a more extreme case where we recently read in an online article by Fox [12] that Russian spies communicated with each other via a special code that was embedded in an image. Our research is very familiar with this example. We are interested in certain patterns in a large set of random data and we will use our mathematical reasoning to find some properties and their relation to the large data set.

Omnisequences or Omnibus Sequences

Imagine we have a 3 letter alphabet a, b, c and assume that a computer generates a random string of letters that only contains these letters. We want this string of the letters to contain all the words of size 2 in this alphabet as a sequence. The shortest string that this computer can generate would be a,b,c,a,b,c. We can pick any letter from the first three letters and then pick the next letter from the next three. We say that the string of a,b,c,a,b,c is two-omnibus over our 3 letter alphabet. You might notice that the string of 6 letters was the shortest string that contains all of the two-letter words; basically we rewrite our alphabet two times back-to-back. Similarly, if we were to find the shortest string that contains every 3-letter words, then we would need to list our alphabet 3 times back-to-back (three-omnibus over our three letter alphabet).

Note that we define a string of letters n as k-omnibus over an alphabet $[a]$=: (1,2,..y) if it contains all of the a^k words as a substring. A minimal k-omnibus aequence can be obtained by writing the alphabet back to back k times. Before we begin getting in to the details of our research, let us begin with some necessary theorems, definitions and corollaries that have previously been proved in this context.

DeBruijn Theorem[3]

For each a and k there exists a cyclic sequence of length a^k *that contains as a substring each k-letter word over [a] :=*$\{1,2,....,a\}$ *precisely once.*

Proof To prove this we will use Eulerian cycles to create the required cyclic sequence. First, let S be the set of all $(k-1)$ length that words are generated by using $\{1,..,a\}$. Let G be a graph such that each vertex is labeled with the elements of S. Note that there are a^{k-1} elements of S and therefore a^{k-1} vertices of G. Next, we need to create an Euler cycle in G with a^k edges. Let $1 \leq i \leq a$ and assume we start with the vertex $(a_1,....,a_{k-1})$ and connect it with directed edges towards the vertex set of $(a_2,...,a_{k-1},1),(a_2,...,a_{k-1},2)....(a_2,...,a_{k-1},a)$ and also let the vertex set $(1,a_1,...,a_{k-1}),(2,a_1,...,a_{k-1}),...,(a,a_1,...,a_{k-1})$ be connected with a directed edges towards $(a_1,....,a_{k-1})$. There are total of a directed outgoing edges from the vertex $(a_1,....,a_{k-1})$ and similarly there are a directed incoming edges. Therefore, each vertex has the same indegree and outdegree and, therefore, G is Eulerian graph and there are $a * a^{k-1} = a^k$ edges of G. Now we start from any vertex in G and follow the Eulerian cycle of any path. We label each edge with the union labeling of two consecutive vertices, for example the edge from the vertex $(a_1, ...a_{k-1})$ to the vertex $(a_2, ...a_{k-1}, a_k)$ would be labeled as $(a_1, ...a_{k-1}, a_k)$. Note that the length of the edge labeling is k and there are a^k edges and therefore we covered all the k-letter words. Once every edge is labeled with a k-letter word, we can pick any edge and follow any Eulerian path and combine all edge labelings by not repeating any consecutive letter. By doing this, we have created a cyclic sequence of length a^k that contains each k-letter words over $[a] := 1, 2, ..., a$.

Example: Let $k = 3$, $a = 2$. Note that the sequence 11100010 is a cyclic sequence that contains all 3-length words that are generated by $[a] := \{0, 1\}$ as a string.

U-trail

The cyclic sequence above is also referred to as an U-cycle. There is also a shortest sequence that contains all k-length letters that does not loop around. Such shortest sequences are called U-trails. Notice that, in the example above, in order to get 011 we need to loop around, and count in the first two letters of the sequence. Instead of looping around we can just repeat the first k-1 letters of the sequence at the end of the sequence and this will ensure that we have all k-length words. Therefore, the length of the U-trial that contains all k-length words would be $a^k + k - 1$.

Coin Flipping Example [3]

Suppose that we decide to flip a coin and we want to know how many times we need to flip in order to have all the 6-length patterns of trials somewhere in the sequence. We know there are $2^6 = 64$ patterns that we will need to have in this sequence of trials. In a perfect world, we know that the shortest length of the sequence would be

$a^k + k - 1 = 2^6 + 6 - 1 = 69$ (this sequence exists by DeBruijn's Theorem [3]) but, in reality, this number is expected to be around $k * a^k * \log(a)$ (this is a deep fact proved in the probability literature [5]).

Omnisequences

Omnisequences are similar to U-trials except that gaps are allowed. A (k, a) omnisequence of length n is a string of letters that contains each of the a^k words of length k over $[a]$=: (1,2,...a). Assume we have a randomly generated string of letters {a,b,c}, and let this string be $babc|abaac|cba|bcbacbabcb....$ As you notice, we have divided this string into parts where these parts contain all the letters of the alphabet (a, b, c) and these are the shortest such sequences. Each of the parts that contains all the letters of the alphabet is also called a **waddle**. We have only selected the first 3 waddles because we can generate every 3-letter alphabet over a, b, c by choosing the first letter from the first waddle and the second and third from the second and third waddles respectively. Since each waddle contains every letter, then we can generate all the 3-length words and therefore we say the string $babc|abaac|cba$ is a (3,3) omnisequence $(k = 3, a = 3)$. Such a string is also called *3-omni* over the alphabet (a, b, c). Note that if we include the next waddle, then we would have a *4-omnisequence* over (a, b, c). Basically, for a string of letters to be k-omni we just need to repeat the alphabet of $[a] = (1, 2, 3, .., h)$ k times.

Waddle Lemma

A sequence S is *k-omni* if and if only there exists a pairwise-disjoint collection P of "completed sets of coupons" (1-omni substrings of S) such that $|P| \geq k$.

Proof: We can easily see from the definition above that this proof is trivial when showing sufficiency. Necessity is also very simple; suppose there exist $m < k$ pairwise disjoint 1-omni substrings of S. Since we can only create every substring of m letters, then a length of k-substring with more than m letters $A = (a_1 a_2 ... a_m t..t)$, where t is a letter after a_m in the mth string is impossible. Then A cannot be a subsequence of the string. This is a contradiction. □

Coupon Collector Problem [9]

The definition of this problem is elementary. Basically, we are collecting "coupons" and once we collect all the coupons in the set we are finished. Collecting baseball cards that are hidden in cereal boxes is a good example of this kind of problem. How many boxes of cereal on average do we need to buy in order to have all n baseball cards? The answer to this question could be shown with some probability and statistical analysis. We see that the expected waiting time for a complete collection is about $n \log n$.

Proof: There are n types of coupons (baseball cards). Let X_i denote the number of trials from i-th success, till the $(i+1)$-th success, where "success" is defined as collection of a new coupon. Clearly, the number of trials performed is

$$X = \sum_{i=0}^{n-1} X_i \text{ Note that } X_i \sim \textit{Geometric} \text{ where } \mathrm{P}(X_i{=}x) = (1-p_i)^{x-1} p_i$$

$x = 1, 2, 3, ...$, where $P_i = \dfrac{n-i}{n}$, so that

$$\begin{aligned}
E(X_i) &= \sum_{x=1}^{\infty} xP(X_i = x) = \sum x(1-p_i)^{x-1}p_i \\
&= p_i \sum x(1-p_i)^{x-1} \\
&= p_i \sum ((1-p_i)^x))' \\
&= p_i(\frac{1-p_i}{p_i})' \\
&= p_i(\frac{p_i(1-p_i)' - (1-p_i)p_i'}{p_i^2})(-1) \\
&= \frac{1}{p_i} \\
\text{Thus, } E(\text{Coupon Collecton Time}) &= \frac{1}{\frac{n}{n}} + \frac{1}{\frac{n-1}{n}} + \frac{1}{\frac{n-2}{n}} + ... + \frac{1}{\frac{n-(n-1)}{n}} \\
&= \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + ... + \frac{n}{1} \\
&= n(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}) \\
&\approx n \int_1^n \frac{dx}{x} \approx n \ln(n)
\end{aligned}$$

It follows that an (a, k) omnisequence takes about $ak \ln(a)$ trials on average. $\square$

3 OMNIMOSAICS

Introduction

Until now, we dealt with one dimensional sequences and now we want to know what happens in the two dimensional world. In other words, is there such a omni pattern in two dimensions? The answer is yes. There exists a collection of data in a plane where two dimensional information is embedded. Imagine a picture that is so full of color combinations so that every picture of certain size is embedded in this large image. Such large pictures are called a omnimosaic. Note that, if we want to find all $k \times k$ matrices over the set A={1,2,...,a}, then we need a large matrix ($n \times n$) that contains all the $k \times k$ matrices. Note that precisely there are a^{k^2} such $k \times k$ matrices. An *omnimosaic* $O(n, k, a)$ is an $n \times n$ matrix, with entries from the set A={1,2,...,a}, that contains, as a submatrix, each of the a^{k^2} $k \times k$ matrices over A. When k, a are fixed, the smallest n for which an $O(n, k, a)$ omnimosaic exists is denoted by $w(k, a)$; for example $w(2, 2) = 4$ since for $a = 2$ there exists a 4×4 matrix as follows

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

We claim that above matrix contains all the 2×2 matrices over the binary digits of {0,1}. Note that we can pick two columns and two rows and the intersection of these columns and rows is a 2×2 matrix and note that with all the possible combinations we can create all of the $2^{2^2} = 16$ matrices of size 2×2. [6]

Now imagine we are trying to find $w(4, 3)$ over $a = \{0, 1, 2\}$. This is not easy but suppose we are trying to find any matrix that contains all of the $a^{k^2} = 3^{4^2}$ submatrices.

How can we construct such a matrix? The answer is through a fairly easy method that has been developed by Banks et al.[2].

Thin Strip Construction by Katie R. Banks [2]

This method is a very efficient method of creating a omnimosaic that contains all $k \times k$ matrices over $(1, ..., a)$. The idea is to first create every k-length word over the set a and then we stack all the k-length letters as a thin strip. For $w(4, 3)$ we need to list all 4-letter words over $a = \{0, 1, 2\}$. There are $3^4 = 81$ such words over $a = \{0, 1, 2\}$ and we start stacking all of these words together. For example

$$a^k = 81\text{-}rows \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ . & & & \\ . & & & \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

Now we repeat the above matrix k times (in this case k=4).

$$a^k = 81\text{-}rows \begin{pmatrix} . & & & \\ . & & & \end{pmatrix}$$

$$a^k = 81\text{-}rows \begin{pmatrix} . & & & \\ . & & & \end{pmatrix}$$

$$a^k = 81\text{-}rows \begin{pmatrix} . & & & \\ . & & & \end{pmatrix}$$

$$a^k = 81\text{-}rows \begin{pmatrix} . & & & \\ . & & & \end{pmatrix}$$

The above step ensures that we can reselect repeated rows. For example, if we were to create a 4×4 matrix with all zeros, then we would pick the first zero row from the first block, then the first zero row from the second block and so forth for the rest.

As we see, there are $k * a^k * k = k^2 * a^k = 4 * 3^4 * 4$ entries. Note that this matrix is not a sequence matrix. Nicholas George Triantafillou[3] created a construction of these matrices by converting the above matrix to a square matrix. The arithmetic is as follows

$$\begin{aligned} k * a^k * k &= k^2 * a^k \\ &= (k * a^{k/2})(k * a^{k/2}) \end{aligned}$$

so that an $n \times n$ matrix can be created with

$$n \approx (k * a^{k/2})$$

Pigeon Hole Principle Applies to Omnimosaics

In the paper *Omnimosaics* by Katie Bank et al[2], it is shown that a $k \times k$ omnimosaic can be embedded in a smaller matrix than $(k * a^{k/2})$x$(k * a^{k/2})$. But if $n < \dfrac{k * a^{k/2}}{e}$, then probability of the array being an omnimosaic is zero. This can be shown as follows:

We want to know the minimum size of an $n \times n$ matrix that contains all $k \times k$ matrices. We select k rows and k columns and consider corresponding $k \times k$ matrix. Basically we are going to look at all the combinations of $\binom{n}{k}$ rows and $\binom{n}{k}$ columns. The following property for $\binom{n}{k}$ can be established by using the binomial expansion and Stirling's approximation($k! \approx \sqrt{2\pi k}(\frac{k}{e})^k$).

$$
\begin{aligned}
\binom{n}{k} &= \frac{n(n-1)...(n-k+1)}{k!} \leq \frac{n^k}{k!} = \\
&= \frac{n^k}{\sqrt{2\pi k}(\frac{k}{e})^k} \leq \frac{n^k}{(\frac{k}{e})^k} = (\frac{ne}{k})^k, i.e. \\
\binom{n}{k} &\leq (\frac{ne}{k})^k, or \\
\binom{n}{k}^2 &\leq (\frac{ne}{k})^{2k}
\end{aligned}
$$

We know that there are a^{k^2} $k \times k$ matrices over a={1,2,...,a} then the following must hold:

$$
\begin{aligned}
(\frac{ne}{k})^{2k} &\geq a^{k^2}, i.e \\
\frac{ne}{k} &\geq a^{\frac{k^2}{2k}} = a^{\frac{k}{2}}, or \\
n &\geq \frac{ka^{\frac{k}{2}}}{e}.
\end{aligned}
$$

This completes the proof.

Above, we established a lower bound of omnimosaics by the pigeon hole principle. Next, we will concentrate on establishing an upper bound on the minimum size of omnimosaics and we will try to lower the upper bound that was created by Banks et al. [2]. This, too, was done in the omnimosaics paper by Banks et al[2], who showed that $w(k,a) \leq \frac{ka^{\frac{k}{2}}}{e}(1+\epsilon)$. We will extend this result to three dimensions.

Recall that the adjacency matrix of a graph is a symmetric matrix. We will start with an example. Note that the adjacency matrix of a 3-universal graph needs to contain all the 3-vertex graphs as induced subgraph. We know there are 8 graphs that exist over 3 vertices. In other words, an edge between two vertex is either present or not ($a = 2$). Since there are 3 vertices, there must be 8 different graphs that could be drawn between 3 vertices. The adjacency matrix of the complete graph K_3 or 3 vertices is

$$\begin{pmatrix} & x_1 & x_2 & x_3 \\ x_1 & 0 & 1 & 1 \\ x_2 & 1 & 0 & 1 \\ x_3 & 1 & 1 & 0 \end{pmatrix}$$

If there exist no edge between x_2 and x_3, the adjacency matrix would be

$$\begin{pmatrix} & x_1 & x_2 & x_3 \\ x_1 & 0 & 1 & 1 \\ x_2 & 1 & 0 & 0 \\ x_3 & 1 & 0 & 0 \end{pmatrix}$$

Similarly remaining 6 graphs would be represented by the following

$$\begin{pmatrix} & x_1 & x_2 & x_3 \\ x_1 & 0 & 1 & 0 \\ x_2 & 1 & 0 & 1 \\ x_3 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} & x_1 & x_2 & x_3 \\ x_1 & 0 & 0 & 1 \\ x_2 & 0 & 0 & 1 \\ x_3 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} & x_1 & x_2 & x_3 \\ x_1 & 0 & 1 & 0 \\ x_2 & 1 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} & x_1 & x_2 & x_3 \\ x_1 & 0 & 0 & 1 \\ x_2 & 0 & 0 & 0 \\ x_3 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now consider the following adjacency matrix of universal graph G that contains all the adjacency matrices above:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{pmatrix} a & b & c & d & e & f \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that we if we pick the c, d, e rows and c, d, e columns then we will pick the first matrix that we mentioned on three vertices. We can pick the empty graph by selecting rows a, b, c and columns, a, b, c. Similarly, all of the other adjacency matrices can be found same way.

Introduction to the Probability Model

We want an $n \times n$ matrix to include all $k \times k$ matrices as a submatrix in order to call this matrix an omnimosaic. Moreover, if one of the $k \times k$ matrices does not exist, then it is not an $n \times n$ omnimosaic. Let us fill in an $n \times n$ array with letters of the alphabet $(1, ..., a)$ randomly and independently. Now, let X be the random variable that counts the number of missing $k \times k$ matrices.

Clearly, $X \geq 1$ implies that the structure is not an omnimosaic. Thus, $P(X \geq 1) < 1$ implies that there exists an $n \times n$ omnimosaic. We know by *Markov's Inequality* that for any $a > 0$ $P(|X| \geq a) \leq \dfrac{E(X)}{a}$, so that choosing $a = 1$ we get $P(X \geq 1) \leq E(X)$. If we can show that $E(X) \to 0$, it will follow that an omnimosaic exists with high probability. Our strategy can be summarized by this paragraph.

By the definition of omnimosaics, there are a total of a^{k^2} $k \times k$ matrices that need to be present as submatrices. Let j represent the jth $k \times k$ matrix. Then,

$$\begin{aligned} E(X) &= E(\sum_{j=1}^{a^{k^2}} I_j) \\ &= \sum E(I_j) \\ &= \sum P(j\text{th } k \times k \text{ matrix is missing}) \\ &= \sum P(j\text{th } k \times k \text{ matrix doesn't occur in any of the} \binom{n}{k}^2 \text{ locations}) \\ &= \sum P_j, \quad \text{say.} \end{aligned} \tag{1}$$

Note that P_j is the probability of the jth $k \times k$ matrix not being present on any of the $\binom{n}{k}^2$ possible locations. In other words,

$$
\begin{aligned}
P_j &= P(\bigcap_{\ell=1}^{L} E_{j,\ell}), \text{ where} \\
L &= \binom{n}{k}^2 \text{ and}
\end{aligned}
$$

$$E_{j,\ell} = P(\text{jth matrix does not occur at location } \ell).$$

Let Z_j = number of occurrences of the$j$th matrix. It follows that

$$
\begin{aligned}
P_j &= P(Z_j = 0), \text{ where} \\
E(Z_j) &= L \times \frac{1}{a^{k^2}} = \frac{\binom{n}{k}^2}{a^{k^2}}. \qquad (2)
\end{aligned}
$$

Applying Suen's Inequality

Suen's inequality (Theorem 2 of [1]) says that

$$P_j \le e^{-\lambda_j + \Delta_j e^{2\delta_j}},$$

where $\lambda_j = E(Z_j) = \frac{\binom{n}{k}^2}{a^{k^2}}$, $\delta_j = \max_{\ell} \sum_{j \sim \ell} P(I_j = 1)$, and $\ell \sim j$ if the ℓth and jth locations share at least one position (at least one row and column.) Thus,

$$\begin{aligned}
\delta_j &= \max_{\ell}\binom{\text{number of}}{\text{intersecting locations}}\frac{1}{a^{k^2}}\\
&= \sum_{r=1}^{k}\sum_{c=1}^{k}\binom{k}{r}\binom{n-k}{k-r}\binom{k}{c}\binom{n-k}{k-c}\frac{1}{a^{k^2}}\\
&\quad (\text{where } r+c<2k)\\
&\le \binom{k}{1}^2\binom{n-1}{k-1}^2\frac{1}{a^{k^2}}, \text{ so}\\
\delta_j &\le k^2\binom{n-1}{k-1}^2\frac{1}{a^{k^2}}.
\end{aligned} \tag{3}$$

The computation of Δ_j is the main component of this proof. We will briefly go over this and note that we will use similar method in more detail when we study omnisculptures.

We have

$$\begin{aligned}
\Delta_j &= \sum_{\ell=1}^{\binom{n}{k}^2}\sum_{\ell\sim j} P(I_j=1, I_\ell=1),\\
&\quad (\text{where } j\sim\ell \text{ if the } j\text{th and } \ell\text{th locations share at least one position})\\
&\le \frac{\binom{n}{k}^2}{a^{2k^2}}\sum_{\substack{r,c=1\\ r+c<2k}}^{k}\binom{k}{r}\binom{k}{c}\binom{n}{k-r}\binom{n}{k-c}a^{rc}\\
&= \frac{\binom{n}{k}^2}{a^{2k^2}}\sum_{\substack{r,c=1\\ r+c<2k}}^{k}\Phi(r,c),\\
&\le \frac{\binom{n}{k}^2}{a^{2k^2}}k^2\max\{\Phi(r,c)\colon\ 1\le r,c\le k; r+c<2k\}
\end{aligned} \tag{4}$$

where $\Phi(r,c)=\binom{k}{r}\binom{k}{c}\binom{n}{k-r}\binom{n}{k-c}a^{rc}$.

Further analysis of $\Phi(r, c)$ by Banks et al. [2] resulted in the following lemmas:

Lemma 1: Given c, $1 \leq c \leq k$, $\Phi(., c)$ is either monotone or unimodal as a function of r.

Lemma 2: $\Phi(1, 1) \geq \Phi(2, 1)$ if $n \geq \frac{k^2 a}{2} + (k - 2)$

Lemma 3: $\Phi(k, k) \geq \Phi(k - 1, k)$ if $n \leq \frac{a^k}{k}$

Lemma 4: $\Phi(k - 1, k) \geq \Phi(1, 1)$

Lemma 5: $\Phi(r, r)$ is first decreasing and then increases as a function of r.

As a result of these lemmas, we note that the maximum of $\Phi(r, c)$ is $\Phi(k - 1, k)$.

Since

$$\Delta_j \leq \frac{\binom{n}{k}^2}{a^{2k^2}} k^2 \max\{\Phi(r, c)\}, \tag{5}$$

it follows that

$$\begin{aligned} \Delta_j &\leq \frac{\binom{n}{k}^2}{a^{2k^2}} k^2 \Phi(k - 1, k) \\ &= \frac{\binom{n}{k}^2}{a^{2k^2}} n k^3 a^{k(k-1)}. \end{aligned} \tag{6}$$

Since $\lambda = \frac{\binom{n}{k}^2}{a^{k^2}}$, (6) yields

$$\Delta_j = \frac{n k^3 \lambda}{a^k}. \tag{7}$$

We see from (3) that

$$\begin{aligned} \delta_j &\leq k^2 \binom{n-1}{k-1}^2 \frac{1}{a^{k^2}} \leq k^2 \binom{n}{k-1}^2 \frac{1}{a^{k^2}} \\ &= \frac{k^4}{(n - k + 1)^2} \lambda \\ &\leq \frac{2k^4}{n^2} \lambda. \end{aligned} \tag{8}$$

Now we can plug (7) and (8) into Suen's inequality to get

$$P_j \leq e^{\left(-\lambda+\lambda\frac{nk^3}{a^k}e^{\frac{4k^4\lambda}{n^2}}\right)}.$$

Note that $e^{\frac{4k^4\lambda}{n^2}} \to 1$ since $\frac{4k^4\lambda}{n^2} \to 0$, so that

$$\begin{aligned} P(\text{Structure is not an Omnimosaic}) &\leq \sum_j P_j \\ &\leq a^{k^2} \times e^{-\lambda+\frac{2\lambda nk^3}{a^k}} \end{aligned} \tag{9}$$

Since we can assume that $n \leq ka^{k/2}$,

$$\begin{aligned} \text{P(not omni)} &\leq a^{k^2} \times e^{(-\lambda+2\lambda(\frac{ka^{k/2}k^3}{a^k}))} \\ &= a^{k^2} \times e^{(-\lambda+2\lambda\frac{k^4}{a^{k/2}})} \end{aligned}$$

Assuming

$$\frac{ka^{k/2}}{e} \leq n \leq \frac{k \times a^{k/2}}{e}(1+\epsilon),$$

plugging the above upper bound into

$$\lambda \leq (\frac{ne}{k})^{2k}\frac{1}{a^{k^2}} \leq k^4,$$

we see that

$$\text{P(not omni)} \leq a^{k^2}e^{(-\lambda+2\frac{k^8}{a^{k/2}})} \to 0, \tag{10}$$

as desired.

4 OMNISCULPTURES

Introduction

We have studied omnipatterns in one dimension and in two dimensions, and now we want to know what happens in three dimensions. First of all, let us explore a three dimensional "data set." Imagine one has k $n \times m$ matrices and we stack them on top of each other to create a three dimensional box matrix which has dimensions of $n \times m \times k$. In other words, imagine that one has k transparent pages, and there are $n \times m$ matrices on each of these k pages. Then, we stack all these pages as a booklet to create a three dimensional data set. It could be hard to visualize this sort of data but we can always break a three dimensional box into two dimensional data by taking each page of paper and placing it on a flat surface. Once we have a three dimensional matrix, how can we find omnipatterns?

An omnisculpture is an $n \times n \times n$ sculpture (a three dimensional matrix) with entries from set $A = \{1, 2, ..., a\}$ that contains all the a^{k^3} $k \times k \times k$ subculptures as an embedded submatrix of the larger sculpture. In other words, a three dimensional $n \times n \times n$ data set is k-omni if it contains all the a^{k^3} $k \times k \times k$ three dimensional data sets as an embedded submatrix where each entry of the data set is from the set $A = \{1, 2, .., a\}$. Three dimensional omnisculptures will be denoted by $\theta(n, k, a)$.

Example of Omnisculpture $\theta(5, 2, 2)$

We need to construct a 5×5 omnisculpture with entries 0 and 1 that contains each $2 \times 2 \times 2$ subsculpture. As we can see, there are $a^{k^3} = 2^{2^3} = 256$ $2 \times 2 \times 2$ subsculptures that need to be embedded. Note that $5 \times 5 \times 5$ is the smallest such sculpture that contains all the $2 \times 2 \times 2$ subscultures. We will show this by utilizing the pigeon hole principle later on.

We will show this sculpture on this paper by listing all 5 of the 5×5 matrices. The first x, y face is the matrix z_1 and so on.

Matrix z_1

$$\left(\begin{array}{cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & 1 & 0 & 0 & 0 & 0 \\ y_2 & 1 & 0 & 1 & 0 & 1 \\ y_3 & 0 & 0 & 0 & 0 & 1 \\ y_4 & 1 & 0 & 1 & 0 & 1 \\ y_5 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

Matrix z_2

$$\left(\begin{array}{cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & 0 & 0 & 1 & 0 & 1 \\ y_2 & 0 & 1 & 1 & 0 & 0 \\ y_3 & 1 & 0 & 0 & 1 & 0 \\ y_4 & 0 & 1 & 1 & 1 & 1 \\ y_5 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Matrix z_3

$$\left(\begin{array}{cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & 0 & 0 & 1 & 0 & 1 \\ y_2 & 1 & 1 & 0 & 0 & 1 \\ y_3 & 0 & 1 & 0 & 1 & 1 \\ y_4 & 0 & 1 & 1 & 0 & 0 \\ y_5 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

Matrix z_4

$$\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline y_1 & 1 & 0 & 1 & 0 & 1 \\ y_2 & 0 & 1 & 1 & 1 & 1 \\ y_3 & 1 & 1 & 1 & 1 & 0 \\ y_4 & 1 & 0 & 0 & 0 & 0 \\ y_5 & 1 & 1 & 0 & 1 & 0 \end{array}$$

Matrix z_5

$$\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline y_1 & 1 & 0 & 0 & 0 & 0 \\ y_2 & 1 & 1 & 1 & 0 & 1 \\ y_3 & 0 & 0 & 1 & 1 & 1 \\ y_4 & 0 & 1 & 0 & 1 & 1 \\ y_5 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Note that a $2 \times 2 \times 2$ three-dimensional matrix with all entries zero is embedded on z_1 and z_5 matrices with the following x, y coordinates.

$$\begin{array}{c|cc} & x_2 & x_4 \\ \hline y_1 & 0 & 0 \\ & & \\ y_5 & 0 & 0 \end{array}$$

Similiarly, we can find a $2 \times 2 \times 2$ three-dimensional matrix with all entries one on z_1 and z_2 matrices with the following x, y coordinates,

$$\begin{array}{c|cc} & x_3 & x_5 \\ \hline & & \\ & & \\ y_4 & 1 & 1 \\ y_5 & 1 & 1 \end{array}$$

As a final example we can find a $2 \times 2 \times 2$ three-dimensional matrix with a front face

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

and back face of

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

by picking z_1 and z_4 with the following x, y coordinates,

$$\begin{array}{c|cc} & x_2 & x_5 \\ \hline y_2 & 0 & 1 \\ y_5 & 0 & 1 \end{array}$$

and

$$\begin{array}{c|cc} & x_2 & x_5 \\ \hline y_2 & 1 & 1 \\ y_5 & 1 & 0 \end{array}$$

Block Design Construction

We can do a simple construction of the desired k-omnisculpture by listing all the a^{k^3} embedded subsculptures and then building a cube containing all these subsculptures. For example, in order to form a 2-omnisculpture with entries 0 and 1, we will need to form all 256 $2 \times 2 \times 2$ subsculptures and then imagine building a cube with all of these subsculptures. So, this large cube will have $2 \times 2 \times 2 \times 2^{2^3}$ entries in total. Then we would have $n \times n \times n = n^3 = 2^3 \times 2^{2^3}$ total entries. We can find n by simply taking the cube root of the above equation, yielding $n = 2^{\frac{3}{3}} \times 2^{\frac{2^3}{3}}$ where $n = 2 \times 2^{\frac{8}{3}}$. Thus, a 2-omnisculpture can be constructed by a block design when $n \approx 13$.

In general, we will need all a^{k^3} $k \times k \times k$ subsculptures over $A = \{1, 2, .., a\}$. Then we can construct an $n \times n \times n$ block design omnisculpture where

$$n \times n \times n = (k \times k \times k) \times a^{k^3},$$

or

$$n = k \times a^{\frac{k^3}{3}}.$$

Actually, we can do a block design similar to the thin strip construction. This does better, but we skip the details. We claim that the above crude construction that gave $n = k \times a^{\frac{k^3}{3}}$ is very large and we will show that for given k, the existence of a minimal omnisculpture is guaranteed when

$$\frac{ka^{\frac{k^2}{3}}}{e} \leq n \leq \frac{ka^{\frac{k^2}{3}}}{e}(1+\epsilon)$$

First, we will show the right hand side of the inequality above, that is $\frac{ka^{\frac{k^2}{3}}}{e} \leq n$. We can show this in a similar way as with omnimosaics.

Applying the Pigeon Hole Principle

We claim that if $n \leq \dfrac{ka^{\frac{k^2}{3}}}{e}$, where all the entries are from $A = \{1, 2, ..a\}$, then the probability of the $n \times n \times n$ array being a k-omnisculpture is zero. In other words, not all of the $k \times k \times k$ subsculptures can possibly be embedded in the $n \times n \times n$ block. This can be shown as follows:

We are interested to know the minimum size of $n \times n \times n$ sculpture that contains all $k \times k \times k$ subsculptures. Since $n \times n \times n$ constitutes three dimensional data, we will select k rows, k columns and k faces (in $\binom{n}{k}^3$ ways). Recall Stirling's approximation ($k! \approx \sqrt{2\pi k}(\frac{k}{e})^k$). [8] Using this we get

$$\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!} \leq \frac{n^k}{k!} =$$

$$= \frac{n^k}{\sqrt{2\pi k}(\frac{k}{e})^k} \leq \frac{n^k}{(\frac{k}{e})^k} = (\frac{ne}{k})^k,$$

and thus

$$\binom{n}{k}^3 \leq (\frac{ne}{k})^{3k}$$

Since there are total of a^{k^3} $k \times k \times k$ sculptures over $\{1, 2, ..., a\}$, the following must hold:

$$(\frac{ne}{k})^{3k} \geq a^{k^3},$$

i.e

$$\frac{ne}{k} \geq a^{\frac{k^3}{3k}} = a^{\frac{k^2}{3}},$$

or

$$n \geq \frac{ka^{\frac{k^2}{3}}}{e}.$$

This completes the proof. $\square$

Note that the example of the omnisculpture $\theta(5,2,2)$ that we computed in the beginning of the omnisculpture section cannot be generated when $n < 5$. Since n is arbitrarily small in this case, we do not need to appeal to Stirling's Approximation. But by the pigeon hole principle, we must have

$$\binom{n}{k}^3 \geq a^{k^3},$$

where $a, k = 2$, or

$$\binom{n}{2}^3 \geq 256,$$

i.e., $n \geq 5$.

Probability Models for Omnisculptures

We want an $n \times n \times n$ sculpture data set to include all $k \times k \times k$ sculptures as a subsculpture in order to call this sculpture an omnisculpture. Moreover, if one of the $k \times k \times k$ subsculptures does not exist, then this is not an $n \times n \times n$ omnisculpture. Let us fill in an $n \times n \times n$ data space with letters of the alphabet $(1, ..., a)$ randomly and independently. Now, let X be the random variable that counts the number of missing $k \times k \times k$ subsculptures. Then X ≥ 1 implies that the structure is not an omnisculpture. Thus, $P(X \geq 1) < 1$ implies that there exists an $n \times n \times n$ omnisculpture. We know by Markov's Inequality [5] that $P(|X| \geq a) \leq \dfrac{E(X)}{a}$ for each $a \geq 0$, so choosing $a = 1$ yields $P(X \geq 1) \leq E(X)$. If we can show that $E(X) \to 0$, then it will follow that an omnisculpture exists with high probability. Since there are a total of a^{k^3} $k \times k \times k$ sculptures that need to be present as subsculptures, we let j represent the jth $k \times k \times k$ sculpture. Then,

$$
\begin{aligned}
E(X) &= E(\sum_{j=1}^{a^{k^3}} I_j) \\
&= \sum E(I_j) \\
&= \sum P(j\text{th } k \times k \times k \text{ sculpture is missing}) \\
&= \sum P(j\text{th } k \times k \times k \text{ sculpture isn't present in any of the} \binom{n}{k}^3 \text{possible locations}) \\
&= \sum P_j, \qquad (1)
\end{aligned}
$$

where P_j is probability of the jth $k \times k \times k$ subsculpture not being present in any location of the $\binom{n}{k}^3$ possible ones. In other words,

$$
\begin{aligned}
P_j &= P(\bigcap_{\ell=1}^{L} E_{j,\ell}), \text{ where} \\
L &= \binom{n}{k}^3, \text{ and} \\
E_{j,\ell} &= P(j\text{th sculpture does not occur at location } \ell).
\end{aligned}
$$

Let $Z_j :=$ number of occurrences of the jth sculpture. It follows that

$$
\begin{aligned}
P_j &= P(Z_j = 0), \text{ where} \\
E(Z_j) &= L \times \frac{1}{a^{k^3}} = \frac{\binom{n}{k}^3}{a^{k^3}}. \qquad (2)
\end{aligned}
$$

Applying Suen's Inequality

Suen's inequality (Theorem 2 of [1]) asserts that $P_j \leq e^{-\lambda_j + \Delta_j e^{2\delta_j}}$, where $\lambda_j = E(Z_j) = \frac{\binom{n}{k}^3}{a^{k^3}}$, $\delta_j = \max_{\ell} \sum_{m \sim \ell} P(I_m = 1)$, and $\ell \sim m$ if the ℓth and mth locations share at least one position (at least one row, one column and one face). Thus,

$$\begin{aligned}
\delta_j &= \max_{\ell} \begin{pmatrix} \text{number of} \\ \text{intersecting locations} \end{pmatrix} \frac{1}{a^{k^3}} \\
&= \sum_{r=1}^{k} \sum_{c=1}^{k} \sum_{d=1}^{k} \binom{k}{r} \binom{n-k}{k-r} \binom{k}{c} \binom{n-k}{k-c} \binom{k}{d} \binom{n-k}{k-d} \frac{1}{a^{k^3}} \\
&\text{(where } r + c + d < 3k\text{)} \\
&\leq \binom{k}{1}^3 \binom{n-1}{k-1}^3 \frac{1}{a^{k^3}}
\end{aligned}$$

so

$$\delta_j \leq k^3 \binom{n-1}{k-1}^3 \frac{1}{a^{k^3}} \tag{3}$$

As we notice, there is a similar pattern as with omnimosaics, and the difference is going to stand out while computing Δ_j. This part is going to be our main component. Unlike omnimosaics, we will consider a three-dimensional interaction when computing Δ_j.

We have

$$\Delta_j = \sum_{\ell=1}^{\binom{n}{k}^3} \sum_{\ell \sim m} P(I_m = 1, I_\ell = 1),$$

where $j \sim \ell$ if the jth and ℓth locations share at least one position.

$$\begin{aligned}
&\leq \frac{\binom{n}{k}^3}{a^{2k^3}} \sum_{\substack{r,c,d=1 \\ r+c+d<3k}}^{k} \binom{k}{r}\binom{k}{c}\binom{k}{d}\binom{n}{k-r}\binom{n}{k-c}\binom{n}{k-d} a^{rcd} \\
&= \frac{\binom{n}{k}^3}{a^{2k^3}} \sum_{\substack{r,c,d=1 \\ r+c+d<3k}}^{k} \Phi(r,c,d), \\
&\leq \frac{\binom{n}{k}^3}{a^{2k^3}} k^3 \max\{\Phi(r,c,d) : 1 \leq r,c,d \leq k; r+c+d \leq 3k\}, \qquad (4)
\end{aligned}$$

where $\Phi(r,c,d) = \binom{k}{r}\binom{k}{c}\binom{k}{d}\binom{n}{k-r}\binom{n}{k-c}\binom{n}{k-d} a^{rcd}$.

We have done some further analysis of $\Phi(r,c,d)$ and we discovered the following lemmas.

Lemma 1.1 $\Phi(1,1,1) \geq \Phi(2,1,1)$, *if* $n \geq \frac{(k-1)^2}{2} \times a + k - 2$.

Proof. This can be shown easily by plugging the given values into the Φ function.

$$\Phi(1,1,1) \geq \Phi(2,1,1) \text{ iff}$$

$$\binom{k}{1}\binom{k}{1}\binom{k}{1}\binom{n}{k-1}\binom{n}{k-1}\binom{n}{k-1} a \geq \binom{k}{2}\binom{k}{1}\binom{k}{1}\binom{n}{k-2}\binom{n}{k-1}\binom{n}{k-1} a^2.$$

Cancelling same terms, we see that we need to prove that

$$\binom{k}{1}\binom{n}{k-1} \geq \binom{k}{2}\binom{n}{k-2}a, \text{ or}$$

$$k \times \frac{n!}{(n-k+1)!(k-1)!} \geq \frac{k \times (k-1)}{2} \times \frac{n!}{(n-k+2)!(k-2)!} \times a, \text{ or}$$

$$1 \geq \frac{(k-1)^2}{2(n-k+2)} \times a.$$

This completes the proof. □

Lemma 1.2 $\Phi(k,k,k) \geq \Phi(k-1,k,k)$, if $a^{k^2} \geq k \times n$.

Proof. This can also be shown with a similar method as that used in the previous lemma. That is

$$\Phi(k,k,k) \geq \Phi(k-1,k,k) \text{ iff}$$

$$\binom{k}{k}\binom{k}{k}\binom{k}{k}\binom{n}{0}\binom{n}{0}\binom{n}{0}a^{k^3} \geq \binom{k}{k-1}\binom{k}{k}\binom{k}{k}\binom{n}{1}\binom{n}{0}\binom{n}{0}a^{k^3-k^2} \text{ iff}$$

$$\binom{k}{k}\binom{k}{k}\binom{k}{k}\binom{n}{0}\binom{n}{0}\binom{n}{0}a^{k^3} \geq \binom{k}{k-1}\binom{k}{k}\binom{k}{k}\binom{n}{1}\binom{n}{0}\binom{n}{0}a^{k^3-k^2}$$

Doing the necessary canceling, we see that we must have

$$a^{k^3} \geq \binom{k}{k-1}\binom{n}{1}a^{k^3-k^2}, \text{ or}$$

$$a^{k^2} \geq k \times n.$$

This completes the proof. □

Lemma 1.3 $\Phi(k-1,k,k) \geq \Phi(1,1,1)$ provided that k is large enough.

Proof.

$$\Phi(k-1,k,k) \geq \Phi(1,1,1) \quad \text{iff}$$

$$\binom{k}{k-1}\binom{k}{k}\binom{k}{k}\binom{n}{1}\binom{n}{0}\binom{n}{0}a^{k^3-k^2} \geq \binom{k}{1}\binom{k}{1}\binom{k}{1}\binom{n}{k-1}\binom{n}{k-1}\binom{n}{k-1}a \;\text{ iff}$$

$$kna^{k^3-k^2} \geq k^3\binom{n}{k-1}^3 a \;\text{ only if}$$

$$na^{k^3-k^2-1} \geq k^2\left(\frac{ne}{k-1}\right)^{3k-3}, \;\text{ by Stirling's approximation.}$$

This holds iff

$$\frac{a^{k^3-k^2-1}}{k^2} \times \frac{(k-1)^{3k-3}}{e^{3k-3}} \geq n^{3k-4}, \text{and plugging in } n = \frac{ka^{\frac{k^2}{3}}}{e}(1+\epsilon),$$

we need to show

$$\frac{a^{k^3-k^2-1}}{k^2} \times \frac{(k-1)^{3k-3}}{e^{3k-3}} \geq \frac{k^{3k-4}}{e^{3k-4}}a^{\frac{k^2}{3}(3k-4)}(1+\epsilon)^{3k-4}, \;\text{ or}$$

$$a^{k^3-k^2-1}\left(\frac{k-1}{k}\right)^{3k-3} \geq kea^{(k^3-\frac{4}{3}k^2)}(1+\epsilon)^{3k-4}, \;\text{ or}$$

$$a^{\frac{k^2}{3}} \geq ke(1+\epsilon)^{3k-4}.$$

This completes the proof. $\square$

Lemma 1.4 A critical point of $\Phi = \Phi(r,c,d)$ is when $r = c = d$.

Proof. Since $\Phi(r,c,d)$ is discrete we must check when $\Phi(r+1,c,d) - \Phi(r,c,d) = 0$.

$$\Phi(r+1,c,d) - \Phi(r,c,d) = 0 \;\text{ iff}$$

$$\Phi(r+1,c,d) \approx \Phi(r,c,d)$$

Similarly the following must also hold:

$$\Phi(r, c+1, d) \approx \Phi(r, c, d),$$

and

$$\Phi(r, c, d+1) \approx \Phi(r, c, d).$$

Consider the equation $\Phi(r+1, c, d) \approx \Phi(r, c, d)$, which is satisfied when $\dfrac{\Phi(r+1, c, d)}{\Phi(r, c, d)} \approx 1$.

$$\text{Since} \frac{\Phi(r+1, c, d)}{\Phi(r, c, d)} = \frac{(k-r)^2 a^{cd}}{(r+1)(n-k+r+1)}, \tag{5}$$

we need to check when

$$\frac{(k-r)^2 a^{cd}}{(r+1)(n-k+r+1)} = 1,$$

$$\frac{(k-c)^2 a^{rd}}{(c+1)(n-k+c+1)} = 1,$$

and

$$\frac{(k-d)^2 a^{rc}}{(d+1)(n-k+d+1)} = 1.$$

In other words, we need to solve

$$\frac{(k-r)^2 a^{cd}}{(r+1)(n-k+r+1)} = \frac{(k-c)^2 a^{rd}}{(c+1)(n-k+c+1)} = \frac{(k-d)^2 a^{rc}}{(d+1)(n-k+d+1)} = 1$$

We can observe that the above mentioned equality is true only when $r = c = d$. This is the end of this proof. $\square$

Lemma 1.5 $\Phi(r, r, r)$ is first decreasing then increasing.

Proof. Let $\sqrt[3]{\Phi(r, r, r)} = \alpha(r) = \binom{k}{r}\binom{n}{k-r} a^{\frac{r^3}{3}}$. We know $\alpha(r)$ is increasing when $\dfrac{\alpha(r+1)}{\alpha(r)} = \dfrac{(k-r)^2 a^{(\frac{r+1^3}{3} - \frac{r^3}{3})}}{(r+1)(n-k+r+1)} \geq 1$. Taking logarithms to both sides, we see that we must have

$$2\log(k-r) + (\frac{(r+1)^3}{3} - \frac{r^3}{3})\log(a) - \log(r+1) - \log(n-k+r+1) \geq 0, \quad \text{or}$$

$$\begin{aligned} 2\log(k-r) - \log(r+1) - \log(n-k+r+1) &\geq -(\frac{(r+1)^3}{3} - \frac{r^3}{3})\log(a) \\ &= \frac{-r^3 - 3r^2 - 3r - 1 + r^3}{3}\log(a) \\ &= -(r^2 + r + \frac{1}{3})\log(a) \end{aligned}$$

Note that the right hand side of the inequality above is less than that in [1], Lemma 3.6. Therefore we claim the same argument. This finishes the proof. □

Unlike omnimosaics we are dealing with a three dimensional data set. Notice also that so far we have only covered the behavior of $\Phi(r, c, d)$ along the diagonal of the $n \times n \times n$ sculpture. We will also need to investigate the trend along its surfaces. For instance, what happens to $\Phi(r, c, d)$ on the front face of the $n \times n \times n$ cube when the rows increase? The following lemmas will cover all the surface analysis of $\Phi(r, c, d)$ needed to compare the critical points with those on the boundary so as to identify a maximum.

Lemma 1.6 On the front facing surface of the $n \times n \times n$ sculpture $\Phi(r+1, c_0, 1) \leq \Phi(r, c_0, 1)$ for each fixed column c_0.

Proof.

$$\Phi(r+1, c_0, 1) \leq \Phi(r, c_0, 1) \text{ iff}$$
$$\frac{\Phi(r+1, c_0, 1)}{\phi(r, c_0, 1)} = \frac{(k-r)^2 a^{c_0}}{(r+1)(n-k+r+1)} \leq 1.$$

Note $c_0 \leq k$. Also note that to maximize the above inequality c needs to be maximum and $(k-r)^2$ needs to be maximum by setting r to its minimum value. Also, we will set $r = 1$ in the denominator. Our conclusion will thus be valid if

$$\frac{(k-1)^2 a^k}{(1+1)(n-k+1+1)} \approx \frac{k^2 a^k}{2n} \leq 1$$

However, we may assume without loss of generality that $n \geq \frac{ka^{\frac{k^2}{3}}}{e} \geq \frac{k^2 a^k}{2}$. This is the end of the proof. □

Lemma 1.7 On the top surface of the $n \times n \times n$ sculpture, $\Phi(1, c_0, d+1) \leq \Phi(1, c_0, d)$.

Proof. Similar to Lemma 1.6

$$\Phi(1, c_0, d+1) \leq \Phi(1, c_0, d) \text{ iff}$$
$$\frac{\Phi(1, c_0, d+1)}{\phi(1, c_0, d)} = \frac{(k-d)^2 a^{c_0}}{(d+1)(n-k+d+1)} \leq 1$$

Note $c_0 \leq k$. Also note that, to maximize the above mentioned inequality c needs to be maximum and $(k-d)^2$ needs to be maximum by setting d to its minimum value.

So, we check when

$$\frac{(k-1)^2 a^k}{(1+1)(n-k+1+1)} \approx \frac{k^2 a^k}{2n} \leq 1.$$

We know that $n \geq \dfrac{ka^{\frac{k^2}{3}}}{e} \geq k^2 a^k/2$.

This is the end of this proof. □

Lemma 1.8 On the left facing surface of the $n \times n \times n$ sculpture, $\Phi(r+1, 1, d_0) \leq \Phi(r, 1, d_0)$

Proof. Similarly to the previous two lemmas. The following lemmas will analyze the remaining 3 surfaces.

Lemma 1.9 On the right facing surface of $n \times n \times n$ sculpture, $\Phi(r_0, k, d+1) \leq \Phi(r_0, k, d)$ when $r_0 \leq \frac{k}{3}$, and furthermore $\Phi(r_0, k, d+1) \geq \Phi(r_0, k, d)$ when $r_0 > \frac{k}{3}$.

Proof.

$$\Phi(r_0, k, d+1) \leq \Phi(r_0, k, d) \text{ iff}$$

$$\frac{\Phi(r_0, k, d+1)}{\Phi(r_0, k, d)} = \frac{(k-d)^2 a^{rk}}{(d+1)(n-k+d+1)} \leq 1.$$

Now as before

$$\frac{(k-d)^2 a^{rk}}{(d+1)(n-k+d+1)} \approx \frac{(k-d)^2 a^{rk}}{(d+1)n} \leq 1$$

if

$$\frac{(k-d)^2 a^{rk}}{(d+1)n} \leq \frac{(k-1)^2 a^{rk}}{(1+1)n} \leq 1$$

or when

$$\frac{(k-1)^2 a^{rk}}{(1+1)n} \leq \frac{k^2 a^{rk}}{n} \leq 1,$$

i.e., when

$$k^2 a^{rk} \leq n.$$

Since

$$\frac{ka^{\frac{k^2}{3}}}{e} \leq n,$$

we see that the above will hold if $r \leq \frac{k}{3}$; when $r > \frac{k}{3}$ the inequality will reverse and thus $\Phi(r_0, k, d+1) \geq \Phi(r_0, k, d)$. This completes the proof. $\Box$

Lemma 1.10 On the bottom surface of the $n \times n \times n$ sculpture, $\Phi(k, c_0, d+1) \leq \Phi(k, c_0, d)$ when $c_0 \leq \frac{k}{3}$ and furthermore $\Phi(k, c_0, d+1) \geq \Phi(k, c_0, d)$ when $c_0 > \frac{k}{3}$.

Proof. Similar to that of Lemma 1.9. $\Box$

Lemma 1.11 On the bottom surface of the $n \times n \times n$ sculpture, $\Phi(r+1, c_0, k) \leq \Phi(r, c_0, k)$ when $c_0 \leq \frac{k}{3}$ and furthermore $\Phi(r+1, c_0, k) \geq \Phi(r, c_0, k)$ when $c_0 > \frac{k}{3}$.

Proof. Similiar to that of Lemmas 1.9 & 1.10. Omitted. $\Box$

As a result of these lemmas, we conclude that the maximum of $\Phi(r,c,d)$ is $\Phi(k,k,k-1)$. Recall (4), that is

$$\Delta_j \le \frac{\binom{n}{k}^3}{a^{2k^3}} k^3 \max\{\Phi(r,c,d)\}. \tag{6}$$

It follows that

$$\begin{aligned}\Delta_j &\le \frac{\binom{n}{k}^3}{a^{2k^3}} k^3 \Phi(k,k,k-1)\\ &= \frac{\binom{n}{k}^3}{a^{2k^3}} nk^4 a^{k^2(k-1)}.\end{aligned} \tag{7}$$

Since $\lambda = \dfrac{\binom{n}{k}^3}{a^{k^3}}$, (7) yields

$$\Delta_j \le \frac{nk^4\lambda}{a^{k^2}}. \tag{8}$$

We see from (3) that

$$\delta_j \le k^3 \binom{n-1}{k-1}^3 \frac{1}{a^{k^3}} \le k^3 \binom{n}{k-1}^3 \frac{1}{a^{k^3}}.$$

Multiplying top and bottom by $\dbinom{n}{k}^3$,

we see that

$$\begin{aligned}\delta_j &\le k^3 \binom{n}{k-1}^3 \frac{1}{a^{k^3}} \frac{\binom{n}{k}^3}{\binom{n}{k}^3}\\ &= k^3 \frac{\binom{n}{k-1}^3}{\binom{n}{k}^3} \lambda\\ &\le \frac{2k^6}{n^3}\lambda.\end{aligned} \tag{9}$$

Now we can plug (8) and (9) into Suen's inequality to get

$$P_j \le e^{\left(-\lambda + \lambda \frac{nk^4}{a^{k^2}} e^{\frac{4k^6\lambda}{n^3}}\right)}.$$

Noting that $e^{\dfrac{4k^6\lambda}{n^3}} \to 1$ since $\dfrac{4k^6\lambda}{n^3} \to 0$, we conclude that

$$\begin{aligned} P(\text{Structure is not an Omnisculpture}) &\leq \sum_j P_j \\ &\leq a^{k^3} \times e^{-\lambda+\lambda\dfrac{2nk^4}{a^{k^2}}} \end{aligned} \tag{10}$$

Since we can assume that $n \leq ka^{k^2/3}$,

$$\begin{aligned} \text{P(not omni) } &\leq a^{k^3} \times e^{(-\lambda+2\lambda(\dfrac{ka^{k^2/3}k^4}{a^{k^2}}))} \\ &= a^{k^3} \times e^{(-\lambda+2\lambda\dfrac{k^5}{a^{2k^2/3}})}. \end{aligned}$$

Assuming

$$\frac{ka^{k^2/3}}{e} \leq n \leq \frac{k \times a^{k^2/3}}{e}(1+\epsilon),$$

we have

$$\lambda \leq (\frac{ne}{k})^{3k}\frac{1}{a^{k^3}} = (1+\epsilon)^{3k} \leq k^5,$$

for suitably chosen ϵ, so that

$$\text{P(not omni) } \leq a^{k^3}e^{(-\lambda+2\dfrac{k^{10}}{a^{2k^2/3}})} \to 0. \tag{10}$$

This completes the proof of the main result. $\square$

5 CONCLUSION

With this thesis, we carried the study of omnipatterns to a higher dimension. First, we reviewed previous results on the omnipatterns in one and two dimensions. Then, we investigated the three dimensional case and the probability of omnipatterns in the three dimensional space. We proved that the probability of the existence of a minimal omnisculpture is guaranteed when

$$n \geq \frac{ka^{\frac{k^2}{3}}}{e}(1+\epsilon)$$

and $\epsilon = \epsilon_k \to 0$ is a sufficiently small function of k.

We believe this study can be extended to more dimensions. For example, four, five even d-dimensional for $d \geq 6$ omni patterns may exist. We wish to continue our work on this topic and extend our study to $d-$dimensional omni patterns.

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VITA

CIHAN EROGLU

Education:	B.S. Mathematics, East Tennessee State University, Johnson City, Tennessee, 2008
	M.S. Mathematical Sciences, East Tennessee State University, Johnson City, Tennessee, 2011
Professional Experience:	Graduate Assistant, East Tennessee State University, Johnson City, Tennessee, 2009–2011