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Solving the Differential Equation for the Probit Function Using a Variant of the Carleman Embedding Technique.

Kelechukwu Iroajanma Alu
East Tennessee State University

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by

Kelechukwu Iroajanma Alu

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Jeff R. Knisley, Ph.D., Chair

Robert A. Beeler, Ph.D.

Ariel Cintrón-Arias, Ph.D.

Robert B. Gardner, Ph.D.

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ABSTRACT

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The probit function is the inverse of the cumulative distribution function associated with the standard normal distribution. It is of great utility in statistical modelling. The Carleman embedding technique has been shown to be effective in solving first order and, less efficiently, second order nonlinear differential equations. In this thesis, we show that solutions to the second order nonlinear differential equation for the probit function can be approximated efficiently using a variant of the Carleman embedding technique.

DEDICATION

This work is dedicated to my beloved sons, Kelechukwu Okpani Alu and Prince Onyinyechukwu Alu. Dare to achieve all that you want to achieve, my sons. There is no limit to what you can do and be. I know that you are both destined for greatness!

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1 INTRODUCTION AND BASIC DEFINITIONS

1.1 Introduction

Many important problems in engineering, the physical sciences, and the social sciences, when formulated in mathematical terms, require the study of a function satisfying a differential equation [9]. The theory of differential equations is therefore an indispensable tool in mathematics and other mathematically dependent sciences. Most physical systems are inherently nonlinear in nature, and many natural laws and models of natural phenomena are described using nonlinear finite autonomous systems of differential equations [34, 17]. These may be first order, second order, or higher order differential equations. Examples of nonlinear differential equations are the Navier-Stokes equations [14] in fluid dynamics, the Lotka-Volterra equations [14] in biology, the Black-Scholes equation [16] in finance, the Van der Pol equation [19] in physics, and the Duffing equation [19] in physics.

Second and higher order ordinary differential equations (generally, systems of nonlinear equations) seldom yield closed form solutions, although implicit solutions and solutions involving nonelementary integrals are often obtained. In fact, finding the closed form of the solutions of finite autonomous systems of differential equations is nearly impossible [22]. There are a great variety of problems involving nonlinear differential equations, and methods of solution or analysis are problem-dependent. Some common methods for the qualitative analysis of nonlinear ordinary differential equations (ODE's) are linearization [34, 55], bifurcation theory [31, 55], and perturbation methods [30, 31, 55].

One approach to working with second order differential equations involves rephrasing the problem in terms of a system of first order equations. When second and higher order differential equations are transformed into a system of first order differential equations, there is no loss of information or generality [12, 7]. This fact validates the Carleman embedding technique and justifies the use of its modified version in this research.

When a differential equation has a polynomial nonlinearity, the technique of linear embedding is sometimes preferred to perturbation and other techniques [25]. The idea of applying the theory of linear integral equations in the study of nonlinear ordinary differential equations was proposed by Henri Poincaré in 1908 [33, 17]. The first attempt in this direction was made by Ivar Fredholm in 1920 [33]. The algorithm of embedding of finite nonlinear dynamical systems, $x^{'} = P(x)$, where $x \in \mathbb{R}^k$ and the $P(x)$ is a k-tuple of polynomials in x, into an infinite system of linear differential equations, was introduced by Torsten Carleman in 1932 [13]. The Carleman approach, which is known today as Carleman linearization or Carleman embedding, has been successfully used to solve numerous nonlinear problems [33, 22]. The original Carleman approach dealt with autonomous polynomial systems [34].

In 1982, Wong [52] showed that Carleman embedding can be viewed as a reduction of nonlinear dynamical systems, $x^{'} = F(x, t)$, where F is analytic in x, to a linear evolution equation in Banach space. The Wong approach for the mathematical foundations of Carleman embedding is of great significance. However, its formalism is complicated and its practical application in the study of concrete nonlinear equations is quite limited [33].

Carleman embedding is a procedure that allows us to embed a finite dimensional system of nonlinear differential equations, with analytic or polynomial data, into a system of infinite dimensional linear differential equations [39, 17]. Thus, we trade polynomials (or analytic functions) that describe the system for the infinite matrices of the Carleman Linearization [39]. This technique works well when dealing with first order nonlinear differential equations. However, for higher order nonlinear ordinary differential equations, it is difficult to use the Carleman embedding method [17].

Azamed Gazaghane, in his master's thesis [22], applied Carleman embedding on the Van der Pol equation and the Fitzhugh Nagumo model. Gazaghane's work exposed the difficulty in using Carleman linearization on second order nonlinear differential equations. The Carleman embedding technique, when used on second order nonlinear differential equations, has the following shortcomings: (i) the matrix of the linear system is unbounded, thereby making truncation to a finite system nearly impossible; (ii) it is difficult to extend this technique to higher order differential equations [17].

A variant of the Carleman embedding technique, developed by Dr. Jeff Knisley [17, 25], was successfully used by Charles Dzacka [17] to solve Duffing's equation, $x'' + x = 2\epsilon x^3$. This variant of the Carleman embedding technique addressed some of the deficiencies of the original Carleman embedding technique. It proved to be easier and more efficient because the solution to the Duffing equation obtained using the new method was bounded [17].

In this thesis, we apply a variant of the Carleman technique to a quantile function. The quantile function of a probability distribution is the inverse F^{-1} of F, which is its cumulative distribution function (CDF) [24]. The first paper to systematically develop quantile functions was by Parzen in 1979 [41]. Gilchrist systematically examined many issues associated with the steps of the statistical modelling process, using an approach based on what he termed *quantile methods* [24].

Quantile functions are used extensively in statistical modelling [51]. In stochastic analysis as well as in traditional probability and statistics, the quantile function provides a helpful way of characterizing a static or dynamic distribution [49]. The quantile function provides benefits that are not available from the density or distribution function. For instance, the simplest way of simulating any non-uniform random variable is applying its quantile function to uniform deviates [49]. To an increasing degree, quantile functions are being used in Monte Carlo simulation. Quantile functions also work well with copula methods and low-discrepancy sequences, for instance, for sampling of the normal [49].

The quantile function or inverse cumulative distribution function associated with the standard normal distribution is called the probit function. The probit function is a nonlinear function for which no closed form solution exists. The function is continuous, monotonically increasing, infinitely differentiable, and maps the open interval (0,1) to the whole real line [2]. The probit function has several applications. It provides benefits not available from the normal probability density function or normal distribution [49]. The probit function is increasingly being used in Monte Carlo simulation [49]. In addition, the probit function works very well with copula methods and low-discrepancy sequences, for example, for sampling of the normal [49].

The use of differential equations and series methods for the analysis of quantile functions can be traced to the earlier work of Hill and Davis [28], as well as Abernathy and Smith [1]. This earlier work developed differential recursions with emphasis on Cornish-Fisher expansions [49]. Steinbrecher and Shaw [49] derived a nonlinear differential equation for the probit function, the quantile function associated with the normal distribution, and solved it using power series.

The aim of this research is to show that a variant of the Carleman embedding technique can be used to solve the nonlinear differential equation for the probit function,

$$\frac{d^2w}{dp^2} = w\left(\frac{dw}{dp}\right)^2,$$

subject to the conditions

$$w(0.5) = 0, w'(0.5) = \sqrt{2\pi},$$

where $w = w(p)$ is the probit function, and $0 \leq p \leq 1$.

This thesis is divided into five chapters. In this first introductory chapter, we provide a preamble. We also give basic definitions and explanations of some terms necessary for a proper understanding of this thesis. Following this introductory chapter is Chapter 2 in which we discuss quantile functions, the probit function, and the ordinary differential equation for the probit function. Also featured in this chapter is the already published general power series solution of the ordinary differential equation for the probit function. In Chapter 3, we give a brief history of the Carleman embedding technique and present an explanation of Carleman linearization.

In Chapter 4, we present a variant of the Carleman embedding technique. We review earlier work on a variant of the Carleman embedding technique for second order systems. We then find approximate solutions to the second order nonlinear ordinary differential equation for the probit function, using a variant of the Carleman embedding technique. Afterwards, we compare our solutions with already published power series solutions. We make significant use of the software *Maple* in this chapter. In our last chapter, Chapter 5, we present our conclusion.

1.2 Basic Definitions

In this section, we define and explain some important terms, mostly from probability and statistics. A knowledge of these terms is essential for a proper understanding of this thesis. Most of the following definitions are, sometimes with minor changes, from the STEPS Statistics Glossary v1.1 by Valerie J. Easton and John H. McColl [18].

Probability: The probability p of an event E is a quantitative measure of the likelihood of occurrence of that event. The probability of an event occurring must have a value between 0 and 1.

Random Variable: A random variable X is a function that associates a unique numerical value with every outcome of an experiment. There are two types of random variables, discrete and continuous. A discrete random variable has an associated probability distribution, while a continuous random variable has an associated probability density function.

Discrete Random Variable: A discrete random variable is one which may take on only a countable number of distinct values such as $0, 1, 2, 3, ...$ Examples of discrete random variables are the number of students in a class, the number of customers in a grocery store, and the number of defective items in a box.

Continuous Random Variables: A continuous random variable is one which takes on an infinite number of possible values. Continuous random variables are usually measurements. An example of a continuous random variable is the time taken to drive from one town to the other.

Expected value: The expected value (or population mean), $E(X)$ or μ, of a random variable X is a value (a number) that indicates the average or central value of that random variable. It is a useful summary value of the variable's distribution.
If X is a discrete random variable with possible values $x_1, x_2, x_3, ..., x_n$ and $P(x_i)$ denotes $P(X = x_i)$, then the expected value of X is defined by:

$$\mu = E(X) = \sum_{i=1}^{n} x_i p(x_i).$$

If X is a continuous random variable with probability density function $f(x)$, then the expected value of X is defined by:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

Variance: The (population) variance σ^2 of a random variable X is a non-negative number which gives us an idea of how widely spread the values of the random variable are likely to be. The more the variance, the more scattered the observations are on average. The variance gives us an idea of how closely concentrated around the expected value (or mean) the distribution is. It is a measure of the "spread" of a distribution about its average value.
The variance of a random variable X is given by:

$$\sigma^2 = Var(X) = E[(X - E(X)]^2 = E(X^2) - E(X)^2 = E(X^2) - \mu^2,$$

where $\mu = E(X)$ is the mean or expected value of the random variable X.

Standard Deviation: Standard deviation is a measure of the spread or dispersion of a set of data. It is calculated by taking the square root of the variance.
$Standard\ deviation = \sqrt{Var(X)} = \sqrt{\sigma^2} = \sigma.$

Probability Distribution: The probability distribution of a discrete random variable X is a function which gives the probability $P(x_i)$ that the random variable equals x_i, for each value x_i. That is, $P(x_i) = P(X = x_i)$. The probability distribution of a discrete random variable can also be defined as a list of probabilities associated with each of the possible values of the random variable. The probability distribution of a discrete random variable satisfies the following conditions:

$$(a) 0 \leq p(x_i) \leq 1;$$
$$(b) \sum_{i=1}^{n} p(x_i) = 1.$$

A probability distribution is sometimes called a probability function or probability mass function.

Probability Density Function: The probability density function, *pdf*, of a continuous random variable is a function which can be integrated to obtain the probability that the random variable takes a value in a given interval.
The *pdf*, $f(x)$, of a continuous random variable X is essentially the derivative of the cumulative distribution function, $F(x)$. That is,

$$f(x) = \frac{d}{dx}F(x).$$

Since $F(x) = p(X \leq x)$, it follows that:

$$\int_a^b f(x)dx = F(b) - F(a) = p(a < X < b).$$

A probability density function $f(x)$ must satisfy two conditions:

$$\begin{aligned} (a) \int f(x)dx &= 1; \\ (b) f(x) &> 0 \forall x. \end{aligned}$$

Cumulative Distribution Function: The cumulative distribution function (CDF) of a random variable X is a function giving the probability that the random variable is less than or equal to x, for every value x. The cumulative distribution function $F(x)$ of a random variable X is given by:

$$F(x) = P(X \leq x), (-\infty < x < \infty).$$

In terms of the probability density function $f(x)$, the CDF of X is given by:

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Normal Distribution: A continuous random variable X, taking all real values in the range $(-\infty, \infty)$, is said to follow a normal distribution with parameters μ and σ if it has a probability density function

$$\begin{aligned} f(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \\ &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right]. \end{aligned}$$

This probability density function is a symmetrical, bell-shaped curve that is centered at its expected value (or mean), μ.

Standard Normal Distribution: This is the simplest case of the normal distribution, with expected value (or mean) 0 and variance 1. This is written as $N(0,1)$.

Hence, the standard normal distribution is given by:

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), or \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Standard Normal Cumulative Distribution Function: This is the cumulative distribution associated with the standard normal distribution function. It is given by:

$$\Phi(x) = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2} dt.$$

Error Function: The error function is a special function of sigmoid shape which occurs in probability, statistics, and partial differential equations. It is essentially identical to the standard normal cumulative distribution function, and differs from it only in translation and scaling. The error function is given by:

$$\text{erf}(x) = \frac{2}{\sqrt{2\pi}} \int_{0}^{x} e^{-t^2} dt.$$

The error function is an odd function. This means that:

$$\text{erf}(-x) = -\text{erf}(x).$$

When the results of a series of measurements are described by a normal distribution with standard deviation σ and expected value 0, then erf $\left(\frac{a}{\sigma\sqrt{2}}\right)$ is the probability that the error of a single measurement lies between $-a$ and a, for positive a. This is useful, for instance, in determining the bit error rate of a digital communication system. The error function also occurs in solutions of the heat equation when boundary conditions are given by the Heaviside step function.

The error function is related to the standard normal cumulative distribution function, Φ, by the following:

$$\Phi(x) = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2} dt = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$$

or,

$$\operatorname{erf}(x) = 2\Phi(x\sqrt{2}) - 1.$$

The standard normal CDF is used more often in probability and statistics, and the error function is used more often in other branches of mathematics.

Inverse Error Function: The inverse error function, erf^{-1}, has the series:

$$\operatorname{erf}^{-1}(x) = \sum_{k=0}^{\infty} \frac{c_k}{(2k+1)} \left(\frac{\sqrt{\pi}}{2} x\right)^{2k+1},$$

where $c_0 = 1$, and

$$c_k = \sum_{m=0}^{k-1} \frac{c_m c_{k-1-m}}{(m+1)(2m+1)}$$

[18, 46].

2 THE PROBIT FUNCTION AND ITS DIFFERENTIAL EQUATION

2.1 Quantile Functions

The quantile function of a probability distribution is the inverse F^{-1} of its cumulative distribution function (CDF), F. For a continuous and strictly monotonic distribution function, $F : \mathbb{R} \to (0, 1)$, the quantile function returns the value below which random draws from the given distribution would fall, $p \cdot 100$ percent of the time. That is, it returns the value of x such that

$$F(x) = P(X \leq x) = p. \tag{1}$$

For discrete as well as continuous distributions, the quantile function is generally given by

$$w(p) = F^{-1}(p) = inf\{x \in \mathbb{R} : p \leq F(x)\} \tag{2}$$

for a probability $0 < p < 1$, and the quantile function returns the minimum value of x for which the probability statement (1) holds [54, 23].

The first paper to systematically develop quantile functions was by Parzen in 1979 [41]. Gilchrist systematically examined many issues associated with the steps of the statistical modelling process, using an approach based on what he termed *quantile methods* [23].

Quantile functions are used extensively in statistical modelling [51]. In stochastic analysis as well as in traditional probability and statistics, the quantile function provides a helpful way of characterizing a static or dynamic distribution. The quantile function provides benefits that are not available from the density or distribution function. For example, the simplest way of stimulating any non-uniform random variable is applying its quantile function to uniform deviates [49]. To an increasing degree, quantile functions are being used in Monte Carlo simulation. Monte Carlo simulation methods are among the most powerful and widely applicable tools available for valuing derivatives and other financial securities [37].

Quantile functions also work well with copula methods [49]. The term *copula* is a Latin noun which means a *link, tie, bond*, referring to joining together [40, 35]. A copula is therefore a function that joins multivariate distribution functions to their one-dimensional marginal distribution functions. It is a multivariate distribution function defined on the unit $n - cube$ $[0, 1]^n$, with uniformly distributed marginals [35]. Quantile functions are also used for low-discrepancy sequences. Low-discrepancy sequences in turn are used in quasi-Monte Carlo methods for numerical integration, in simulation and optimization, and in related applications [10].

The quantile function or inverse cumulative distribution function associated with the standard normal distribution is called the probit function.

2.2 The Probit Function

The term *probit* is an abbreviation for *probability unit* [3]. In probability and statistics, the probit function is the inverse cumulative distribution function (CDF) or quantile function associated with the standard normal distribution, Φ [43]. The probit function is a nonlinear function for which no closed form solution exists. The function is continuous, monotonically increasing, infinitely differentiable, and maps the open interval (0,1) to the whole real line [2].

The probit function may be expressed in terms of the inverse error function as

$$Probit(p) = w(p) = \Phi^{-1}(p) = \sqrt{2}\mathrm{erf}^{-1}(2p-1),$$

for $0 \leq p \leq 1$ [42].

The probit concept was published in 1934 by Chester Bliss in an article in *Science* on how to treat data such as the percentage of a pest killed by a pesticide [8]. He proposed converting the percentage killed into a *probability unit* (or *probit*). Bliss included a table to help other researchers convert their kill percentages to his probit. They could then plot the probit values against the logarithm of the dose and, hopefully, get a more or less straight line. Such a so-called probit model is still of importance in toxicology and other fields. Bliss's method was continued in an important text on toxicological applications by D. J. Finney [20].

The cumulative distribution function (CDF) and the inverse cumulative distribution function (the probit function) associated with the standard normal distribution are not available in closed form [15, 45]. Their computations are normally carried out using numerical procedures. These functions can be found in statistics software, in probability modeling software, and in spreadsheets. For example, the probit function is available as $normsinv(p)$ in Microsoft Excel. In computing environments where numerical implementations of the inverse error function are available, the probit function may be obtained as $prob(p) = \sqrt{2}\mathrm{erf}^{-1}(2p-1)$, where p is a probability between 0 and 1 and erf^{-1} is the inverse error function. For example, MATLAB implements

$erfinv$, while the language Mathematica implements $InverseErf$. An alternative method of computation involves forming a nonlinear ordinary differential equation for the probit function Steinbrecher and Shaw [49] derived a nonlinear differential equation for the probit function and solved it using power series.

2.3 An Ordinary Differential Equation for the Probit Function

The use of differential equations and series methods for the analysis of quantile functions originated from the earlier work of Hill and Davis [28], and Abernathy and Smith [1]. Significant contributions have more recently been made by Steinbrecher and Shaw [49]. Steinbrecher and Shaw derived non-linear ordinary differential equations for the quantile functions of some key distributions (the student, beta, normal, and gamma distributions). They also gave power series solutions for these quantile ODE's.

Generally, we can express the derivative of a quantile function w as the reciprocal of the usual density function expressed in terms of w, then keep differentiating until we obtain a closed differential relation, which is generally nonlinear [49].

If $f(x)$ is the probability density function, the first order quantile ODE is

$$\frac{dw}{dp} = \frac{1}{f(w)},$$

where $w(p)$ is the quantile function considered as a function of p, with $0 \leq p \leq 1$. We then differentiate again to find a simple second order non-linear ODE [49].

Steinbrecher and Shaw [49] derived a nonlinear differential equation for the probit function using the above procedure, and solved it using power series. We now give that derivation in greater detail.

We know that the standard normal distribution is given by:

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

If $w(p)$ denotes the quantile function of p, then

$$f(w) \quad = \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}.$$

Hence,

$$\frac{dw}{dp} \quad = \quad \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}},$$

and the first order quantile ODE is given by,

$$\frac{dw}{dp} \quad = \quad \sqrt{2\pi} e^{\frac{w^2}{2}}.$$

Differentiating again,

$$\begin{aligned} \frac{d^2w}{dp^2} &= \sqrt{2\pi} e^{\frac{w^2}{2}} \cdot \frac{2w}{2} \frac{dw}{dp} \\ &= \sqrt{2\pi} e^{\frac{w^2}{2}} \cdot w \frac{dw}{dp} \\ &= \frac{dw}{dp} \cdot w \frac{dw}{dp} \\ &= w \left(\frac{dw}{dp} \right)^2. \end{aligned}$$

We thus have the second order nonlinear ODE for the probit function, together with its boundary conditions [49]:

$$\frac{d^2w}{dp^2} \quad = \quad w \left(\frac{dw}{dp} \right)^2, \qquad (3)$$

$$w(0.5) = 0, w'(0.5) = \sqrt{2\pi},$$

where $0 \leq p \leq 1$.

2.4 General Power Series Solution of the Second Order ODE for the Probit Function

The second order nonlinear ODE for the Probit Function can be solved using various methods, including the power series approach used by Steinbrecher and Shaw [49]. The general power series solution to this differential equation is:

$$w(p) = \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{d_k}{(2k+1)} (2p-1)^{2k+1}, \tag{4}$$

where the coefficients d_k satisfy the non-linear recurrence

$$d_{k+1} = \frac{\pi}{4} \sum_{j=0}^{k} \frac{d_j d_{k-j}}{(j+1)(2j+1)}, \tag{5}$$

with $d_0 = 1$ [49]. In this form the ratio $\frac{d_{k+1}}{d_k} \to 1$ as $k \to \infty$. This implies slow convergence.

We will compare this to solutions that we shall later obtain using a variant of the Carleman embedding technique.

3 THE CARLEMAN EMBEDDING TECHNIQUE

3.1 History of the Carleman Embedding Technique

The following history of the Carleman embedding technique was adapted, with minor modifications, from the work of Gaude [21], Gazaghane [22], and Dzacka [17].

The mathematician Torsten Carleman, in the 1930's, developed a theoretical technique to globally linearize systems of nonlinear differential equations. His 1932 article, entitled "Application of the Theory of Linear Integral Equations to Systems of Nonlinear Differential Equations", introduced the linearization method [13]. Carleman's ideas were motivated by remarks made by Henri Poincaré [21]. Poincaré remarked at a 1908 conference in Rome, that one should be able to apply the theory of linear integral equations to the study of ordinary nonlinear differential equations. Motivated by that remark, Carleman worked on an approach to embed a system of nonlinear differential equations into an infinite set of linear equations [22].

The Carleman technique basically remained unused for over thirty years before Bellman and Richardson applied the method to approximate solutions of a nonlinear ordinary differential equation [6]. Thirteen years later, Montroll and Hellman [38] studied the embedding technique in relation to small denominators and secular terms. In 1980, Steeb and Wilhelm [48] used Carleman embedding to approximate the solution of the Lotka-Volterra problem. The Lotka-Volterra model is represented by systems of nonlinear equations that have periodic solutions. The Carleman technique was successfully applied to solve the Lotka-Volterra problem [21].

In 1981, Kerner [32] studied the technique for embedding nonlinear systems into polynomial systems. Also in 1981, Andrade and Rauh [4], and Brenig and Fairen [11] studied the Lorenz model and power series expansions for nonlinear systems, respectively, using the Carleman Embedding technique. In 1982, Wong [52] showed that a linear operator acting on a Banach space could be related to analytical vector fields. This became known as the Carleman linearization or transformation of a vector field.

There were some other results with linearization. In 1987, Kowalski [34] related finite dimensional nonlinear systems to problems in a Hilbert space. Tsiligiannis and Lyberatos [50] studied steady state bifurcation and exact multiplicity conditions using the Carleman method. Finally, by 1989, Steeb showed that there is a one-to-one correspondence between solutions of the infinite linear system and solutions of the associated nonlinear finite system for the analytic solutions [21]. Kowalski and Steeb summarized a large portion of this work into one book, *A Note on Carleman Linearization* [47]. Most of the history of the Carleman method is outlined in that book.

3.2 Carleman Embedding

Carleman embedding or Carleman linearization is a procedure that allows us to embed a finite-dimensional system of nonlinear differential equations, with analytic or polynomial data, into a system of infinite-dimensional linear differential equations [39, 17]. We therefore trade polynomials (or analytic functions) that describe the system for the infinite matrices of the Carleman Linearization [39].

The following brief explanation of the Carleman linearization (or Carleman embedding) procedure is from a paper by Gralewicz and Kowalski [26].

Let us consider the system

$$\dot{x} = F(x), \tag{6}$$

where

$$F : \mathbb{R}^k \to \mathbb{R}^k$$

and F is analytic in x. Consider the case with $k = 1$, that is, the ordinary differential equation (6). If we let

$$x_j := x^j, j = 1, 2, ... \tag{7}$$

where x fulfills (6), then we arrive at the infinite linear system

$$\dot{x_j} = \sum_{k=0}^{\infty} L_{jk} x_k, \tag{8}$$

with the constant coefficient matrix L_{jk}. In view of (7), the finite system (6) is embedded into the infinite system (8). As shown by Steeb [47], Carleman embedding can easily be generalized to the case with nonlinear recurrences of the form

$$x_{n+1} = f(x_n), \tag{9}$$

where f is analytic in x_n. Similar to what we did in (7), we set

$$x_{jn} := x_n^j, \tag{10}$$

where x_n fulfills (9). This leads to the infinite-dimensional linear system of difference equations such that

$$x_{jn+1} = \sum_{k=0}^{\infty} M_{jk} x_{kn}. \tag{11}$$

Hence, the finite-dimensional recurrence (9) is embedded into the infinite linear system (11), just as in the case of ordinary differential equations [26].

4 A VARIANT OF THE CARLEMAN EMBEDDING TECHNIQUE FOR SECOND ORDER SYSTEMS

4.1 Introduction

As noted in Section 1.1, nonlinear systems of second order or higher are difficult to solve with the original Carleman embedding method [17]. In an attempt to overcome the earlier-mentioned problems associated with the original Carleman embedding, a variant of the Carleman embedding technique was developed by Dr. Jeff Knisley [17, 25] to solve second order nonlinear systems. This variant of Carleman embedding was implemented by Dzacka [17] in his master's research.

4.2 Review: Solution of the Duffing Equation Using a Variant of the Carleman Embedding Technique

In Dzacka [17], the Duffing equation,

$$\frac{d^2x}{dt^2} = -x + 2\epsilon x^3, \quad x(0) = 1, x^{'}(0) = 0, \tag{12}$$

was solved using a variant of the Carleman embedding technique. The transformation

$$u_n(t) = [x(t)]^n \tag{13}$$

was used. An infinite-dimensional system of equations was generated by substituting (13) and its derivatives in (12). Truncation was done at $n = 10$ and the resulting system of equations solved using matrix algebra. The result obtained using this variant of Carleman embedding was compared to results obtained using classical perturbation techniques as well as those obtained using perturbation combined with Carleman embedding. The comparisons showed that classical perturbation and perturbation combined with Carleman embedding produced similar approximations, which were unbounded. In contrast, the approximation obtained using a variant of Carleman embedding was bounded. This was an indication that the variant of Carleman embedding used in solving the Duffing equation was better than other methods [17].

4.3 Solving the Second Order ODE for the Probit Function using a Variant of the Carleman Embedding Technique

It might be difficult to solve the ODE for the probit function using the original Carleman embedding technique. This is because the coefficient matrix of the linear system generated using this technique may be too large and unbounded, thereby making truncation to a finite system nearly impossible [17]. We aim to obtain a better solution with the variant of the Carleman embedding technique.

Consider equation (3), the differential equation for the probit function:

$$w'' = w(w')^2, \quad w(0.5) = 0, \quad w'(0.5) = \sqrt{2\pi}.$$

For our transformation, let

$$u_{m,n} = w^m (w')^n. \tag{14}$$

We obtain the first derivative of $u_{m,n}$ as follows:

$$\begin{aligned} u'_{m,n} &= w^m \cdot n(w')^{n-1} \cdot w'' + (w')^n \cdot m w^{m-1} \cdot w' \\ &= n w^m (w')^{n-1} w'' + m w^{m-1} (w')^n \cdot w'. \end{aligned}$$

But $w'' = w(w')^2$. Hence,

$$u'_{m,n} = n w^m (w')^{n-1} \cdot w(w')^2 + m w^{m-1} (w')^{n+1}.$$

We therefore have:

$$u'_{m,n} = n w^{m+1} (w')^{n+1} + m w^{m-1} (w')^{n+1}. \tag{15}$$

Applying (14) on (15), the latter becomes:

$$u'_{m,n} = n u_{m+1,n+1} + m u_{m-1,n+1}. \tag{16}$$

We now derive several other important relations mainly from (14) and (16).

From (14), we have:

$$u_{m,0} = w^m (w')^0 = w^m.$$

In other words, we have the relation:

$$u_{m,0} = w^m. \tag{17}$$

Also from (14),

$$u_{0,n} = w^0 (w')^n = (w')^n.$$

Hence, we have the relation:

$$u_{0,n} = (w')^n. \tag{18}$$

From (17), we have:

$$u_{1,0} = w^1 = w.$$

In other words, we have the relation:

$$u_{1,0} = w. \tag{19}$$

From (18), we have:

$$u_{0,1} = (w')^1 = w'.$$

We therefore have the relation

$$u_{0,1} = w'. \tag{20}$$

From (16), we have:

$$\begin{aligned} u'_{m,0} &= 0 \cdot u_{m+1,0+1} + mu_{m-1,0+1} \\ &= 0 + mu_{m-1,1}. \end{aligned}$$

We therefore have the relation:

$$u'_{m,0} = mu_{m-1,1}. \tag{21}$$

Also from (16),

$$\begin{aligned} u'_{0,n} &= n \cdot u_{0+1,n+1} + 0 \cdot u_{0-1,n+1} \\ &= nu_{1,n+1} + 0. \end{aligned}$$

We therefore have:

$$u'_{0,n} = nu_{1,n+1}. \tag{22}$$

We can generate an infinite-dimensional system of equations from (16). For the purpose of this thesis, we work with the truncated system in which $0 \leq m \leq M$, $0 \leq n \leq N$. The order of truncation is MN. In our *Maple* code, *CutOff* is used for M, N. For convenience, let us first truncate at $m = 1$, $n = 1$, and set the RHS (right hand side) equal to zero whenever either $m+1$ or $n+1$ is greater than our cut-off value, 1. We then have the following system of equations:

$$\begin{aligned} u'_{0,0}(p) &= 0; \\ u'_{0,1}(p) &= 0; \\ u'_{1,0}(p) &= u_{0,1}(p); \\ u'_{1,1}(p) &= 0. \end{aligned}$$

Our original initial conditions were $w(0.5) = 0, \quad w'(0.5) = \sqrt{2\pi}$. In order to obtain the initial conditions for our new system of equations, recall equations (19) and (20), $u_{1,0} = w$, and $u_{0,1} = w'$, respectively. Our initial conditions therefore become $u_{1,0}(0.5) = 0$ and $u_{0,1}(0.5) = \sqrt{2\pi}$. Solving the above system with the initial conditions expressed as sequences, we have:

$$\begin{aligned} u_{0,0}(p) &= 1; \\ u_{0,1}(p) &= \sqrt{2\pi}; \\ u_{1,0}(p) &= \sqrt{2\pi}p - \frac{1}{2}\sqrt{2\pi}; \\ u_{1,1}(p) &= 0. \end{aligned}$$

Recall that $u_{1,0} = w$. The value of $u_{1,0}$ above therefore gives us a solution, which we shall call w_1:

$$w_1(p) = \sqrt{2\pi}p - \frac{1}{2}\sqrt{2\pi}.$$

When we evaluate w_1 at the points $p = 0$, $p = 0.5$, and $p = 1$, we have its values as $-\frac{1}{2}\sqrt{2\pi}$, 0, and $\frac{1}{2}\sqrt{2\pi}$ respectively. These approximate to -1.253314137, 0, and 1.253314137, respectively.

By changing the values of our cut-off (varying the values of m and n), we can obtain other possible solutions. Suppose we let $m = 4$, $n = 4$, and set the RHS equal to zero whenever either $m+1$ or $n+1$ is greater than the cut-off value, 4. We obtain the following system of equations:

$$u'_{0,0}(p) = 0;$$

$$u'_{0,1}(p) = u_{1,2}(p);$$

$$u'_{0,2}(p) = 2u_{1,3}(p);$$

$$u'_{0,3}(p) = 3u_{1,4}(p);$$

$$u'_{0,4}(p) = 0;$$

$$u'_{1,0}(p) = u_{0,1}(p);$$

$$u'_{1,1}(p) = u_{2,2}(p) + u_{0,2}(p);$$

$$u'_{1,2}(p) = 2u_{2,3}(p) + u_{0,3}(p);$$

$$u'_{1,3}(p) = 3u_{2,4}(p) + u_{0,4}(p);$$

$$u'_{1,4}(p) = 0;$$

$$u'_{2,0}(p) = 2u_{1,1}(p);$$

$$u'_{2,1}(p) = u_{3,2}(p) + 2u_{1,2}(p);$$

$$u'_{2,2}(p) = 2u_{3,3}(p) + 2u_{1,3}(p);$$

$$u'_{2,3}(p) = 3u_{3,4}(p) + 2u_{1,4}(p);$$

$$u'_{2,4}(p) = 0;$$

$$u'_{3,0}(p) = 3u_{2,1}(p);$$

$$u'_{3,1}(p) = u_{4,2}(p) + 3u_{2,2}(p);$$

$$u'_{3,2}(p) = 2u_{4,3}(p) + 3u_{2,3}(p);$$

$$u'_{3,3}(p) = 3u_{4,4}(p) + 3u_{2,4}(p);$$

$$u'_{3,4}(p) = 0;$$

$$u'_{4,0}(p) = 4u_{3,1}(p);$$

$$u'_{4,1}(p) = 4u_{3,2}(p);$$

$$u'_{4,2}(p) = 4u_{3,3}(p);$$

$$u'_{4,3}(p) = 4u_{3,4}(p);$$

$$u'_{4,4}(p) = 0.$$

Our original initial conditions were $w(0.5) = 0$, $w'(0.5) = \sqrt{2\pi}$. In order to obtain the initial conditions for our new system of equations, recall equations (19) and (20), $u_{1,0} = w$, and $u_{0,1} = w'$, respectively. Our initial conditions therefore become $u_{1,0}(0.5) = 0$, $u_{0,1}(0.5) = \sqrt{2\pi}$. Solving the above system with the initial conditions expressed as sequences, we have:

$$\begin{aligned}
u_{0,0}(p) &= 1; \\
u_{0,1}(p) &= p^2\sqrt{2}\pi^{3/2} - \sqrt{2}\pi^{3/2}p + \frac{1}{4}\sqrt{2}\pi^{3/2} + \sqrt{2\pi}; \\
u_{0,2}(p) &= 4p^2\pi^2 - 4\pi^2 p + \pi^2 + 2\pi; \\
u_{0,3}(p) &= 2\sqrt{2}\pi^{3/2}; \\
u_{0,4}(p) &= 4\pi^2; \\
u_{1,0}(p) &= \frac{1}{3}\sqrt{2}\pi^{3/2}p^3 - \frac{1}{2}p^2\sqrt{2}\pi^{3/2} + (\frac{1}{4}\sqrt{2}\pi^{3/2} + \sqrt{2\pi})p - \frac{1}{24}\sqrt{2}\pi^{3/2} - \frac{1}{2}\sqrt{2\pi}; \\
u_{1,1}(p) &= \frac{8}{3}p^3\pi^2 - 4p^2\pi^2 + \pi^2 p + (\pi^2 + 2\pi)p - \frac{1}{3}\pi^2 - \pi; \\
u_{1,2}(p) &= 2\sqrt{2}\pi^{3/2}p - \sqrt{2}\pi^{3/2}; \\
u_{1,3}(p) &= 4\pi^2 p - 2\pi^2; \\
u_{2,0}(p) &= \frac{4}{3}p^4\pi^2 - \frac{8}{3}p^3\pi^2 + p^2\pi^2 + p^2(\pi^2 + 2\pi) + 2(-\frac{1}{3}\pi^2 - \pi)p + \frac{1}{12}\pi^2 + \frac{1}{2}\pi; \\
u_{2,1}(p) &= 2p^2\sqrt{2}\pi^{3/2} - 2\sqrt{2}\pi^{3/2}p + \frac{1}{2}\sqrt{2}\pi^{3/2}; \\
u_{2,2}(p) &= 4p^2\pi^2 - 4\pi^2 p + \pi^2; \\
u_{3,0}(p) &= 2\sqrt{2}\pi^{3/2}p^3 - 3p^2\sqrt{2}\pi^{3/2} + \frac{3}{2}\sqrt{2}\pi^{3/2}p - \frac{1}{4}\sqrt{2}\pi^{3/2}; \\
u_{3,1}(p) &= 4p^3\pi^2 - 6p^2\pi^2 + 3\pi^2 p - \frac{1}{2}\pi^2; \\
u_{4,0}(p) &= 4p^4\pi^2 - 8p^3\pi^2 + 6p^2\pi^2 - 2\pi^2 p + \frac{1}{4}\pi^2.
\end{aligned}$$

The rest are zero.

Recall that $u_{1,0} = w$. The value of $u_{1,0}$ above therefore gives us a solution, which we shall call w_2:

$$w_2(p) = \frac{1}{3}\sqrt{2}\pi^{3/2}p^3 - \frac{1}{2}p^2\sqrt{2}\pi^{3/2} + \left(\frac{1}{4}\sqrt{2}\pi^{3/2} + \sqrt{2\pi}\right)p - \frac{1}{24}\sqrt{2}\pi^{3/2} - \frac{1}{2}\sqrt{2\pi}.$$

When we evaluate w_2 at the points $p = 0$, $p = 0.5$, and $p = 1$, we have its values as $-\frac{1}{24}\sqrt{2}\pi^{3/2} - \frac{1}{2}\sqrt{2\pi}$, 0, and $\frac{1}{24}\sqrt{2}\pi^{3/2} + \frac{1}{2}\sqrt{2\pi}$, respectively. These approximate to -1.581431011, 0, and 1.581431011, respectively.

Figure 1 is a plot of w_2, a solution obtained using the variant of the Carleman embedding technique, for $0 \leq p \leq 1$.

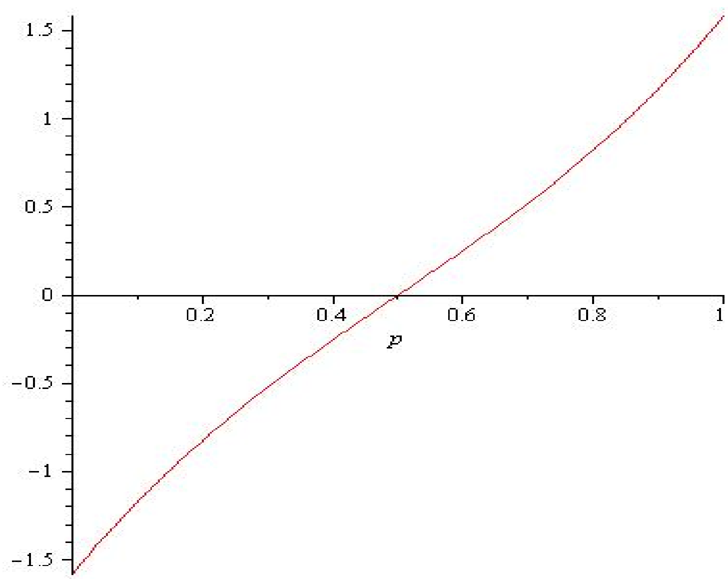

Figure 1: A second solution of the differential equation for the probit function using a variant of Carleman embedding.

Suppose we let $m = 8$, $n = 8$, and set the RHS equal to zero whenever either $m+1$ or $n+1$ is greater than the cut- off value, 8. By the same process outlined above, we obtain a system of equations and solve the system. As before, we are interested in the value of $u_{1,0}$. This time,

$$
\begin{aligned}
u_{1,0}(p) &= \frac{1}{3}p^3\left(\frac{11}{12}\sqrt{2}\pi^{\frac{7}{2}} + \sqrt{2}\pi^{\frac{5}{2}}\right) + \frac{1}{6}\left(\frac{13}{16}\sqrt{2}\pi^{\frac{7}{2}} + \frac{3}{2}\sqrt{2}\pi^{\frac{5}{2}} + 2\sqrt{2}\pi^{\frac{3}{2}}\right)p^3 \\
&+ \frac{127}{630}p^7\sqrt{2}\pi^{\frac{7}{2}} - \frac{127}{180}p^6\sqrt{2}\pi^{\frac{7}{2}} + \frac{23}{30}p^5\sqrt{2}\pi^{\frac{7}{2}} + \frac{7}{120}p^5\left(5\sqrt{2}\pi^{\frac{7}{2}} + 4\sqrt{2}\pi^{\frac{5}{2}}\right) \\
&+ \frac{7}{24}p^4\left(-\frac{13}{6}\sqrt{2}\pi^{\frac{7}{2}} - 2\sqrt{2}\pi^{\frac{5}{2}}\right) - \frac{1}{4}p^4\sqrt{2}\pi^{\frac{7}{2}} \\
&+ \frac{1}{2}p^2\left(-\frac{127}{480}\sqrt{2}\pi^{\frac{7}{2}} - \frac{7}{12}\sqrt{2}\pi^{\frac{5}{2}} - \sqrt{2}\pi^{\frac{3}{2}}\right) + \left(\frac{127}{5760}\sqrt{2}\pi^{\frac{7}{2}} + \frac{7}{96}\sqrt{2}\pi^{\frac{5}{2}}\right) \\
&+ \left(\frac{1}{4}\sqrt{2}\pi^{\frac{3}{2}} + \sqrt{2\pi}\right)p - \frac{127}{80640}\sqrt{2}\pi^{\frac{7}{2}} - \frac{7}{960}\sqrt{2}\pi^{\frac{5}{2}} - \frac{1}{24}\sqrt{2}\pi^{\frac{3}{2}} - \frac{1}{2}\sqrt{2\pi}.
\end{aligned}
$$

Hence, another possible solution, w_3, is given by:

$$
\begin{aligned}
w_3(p) &= \frac{1}{3}p^3\left(\frac{11}{12}\sqrt{2}\pi^{\frac{7}{2}} + \sqrt{2}\pi^{\frac{5}{2}}\right) + \frac{1}{6}\left(\frac{13}{16}\sqrt{2}\pi^{\frac{7}{2}} + \frac{3}{2}\sqrt{2}\pi^{\frac{5}{2}} + 2\sqrt{2}\pi^{\frac{3}{2}}\right)p^3 \\
&+ \frac{127}{630}p^7\sqrt{2}\pi^{\frac{7}{2}} - \frac{127}{180}p^6\sqrt{2}\pi^{\frac{7}{2}} + \frac{23}{30}p^5\sqrt{2}\pi^{\frac{7}{2}} + \frac{7}{120}p^5\left(5\sqrt{2}\pi^{\frac{7}{2}} + 4\sqrt{2}\pi^{\frac{5}{2}}\right) \\
&+ \frac{7}{24}p^4\left(-\frac{13}{6}\sqrt{2}\pi^{\frac{7}{2}} - 2\sqrt{2}\pi^{\frac{5}{2}}\right) - \frac{1}{4}p^4\sqrt{2}\pi^{\frac{7}{2}} \\
&+ \frac{1}{2}p^2\left(-\frac{127}{480}\sqrt{2}\pi^{\frac{7}{2}} - \frac{7}{12}\sqrt{2}\pi^{\frac{5}{2}} - \sqrt{2}\pi^{\frac{3}{2}}\right) + \left(\frac{127}{5760}\sqrt{2}\pi^{\frac{7}{2}} + \frac{7}{96}\sqrt{2}\pi^{\frac{5}{2}}\right) \\
&+ \left(\frac{1}{4}\sqrt{2}\pi^{\frac{3}{2}} + \sqrt{2\pi}\right)p - \frac{127}{80640}\sqrt{2}\pi^{\frac{7}{2}} - \frac{7}{960}\sqrt{2}\pi^{\frac{5}{2}} - \frac{1}{24}\sqrt{2}\pi^{\frac{3}{2}} - \frac{1}{2}\sqrt{2\pi}.
\end{aligned}
$$

When we evaluate w_3 at the points $p = 0$, $p = 0.5$, and $p = 1$, we have its values as $-\frac{127}{80640}\sqrt{2}\pi^{\frac{7}{2}} - \frac{7}{960}\sqrt{2}\pi^{\frac{5}{2}} - \frac{1}{24}\sqrt{2}\pi^{\frac{3}{2}} - \frac{1}{2}\sqrt{2\pi}$, $-2.6 \cdot 10^{-11}\sqrt{2}\pi^{\frac{7}{2}} + 1.7 \cdot 10^{-11}\sqrt{2}\pi^{\frac{5}{2}}$, and $\frac{127}{80640}\sqrt{2}\pi^{\frac{7}{2}} + \frac{7}{960}\sqrt{2}\pi^{\frac{5}{2}} + \frac{1}{24}\sqrt{2}\pi^{\frac{3}{2}} + \frac{1}{2}\sqrt{2\pi}$, respectively. These approximate to -1.884225879, $-1.600181154 \cdot 10^{-9}$, and 1.884225879, respectively.

Figure 2 is a plot of w_3, another solution obtained using the variant of the Carleman embedding technique, for $0 \leq p \leq 1$.

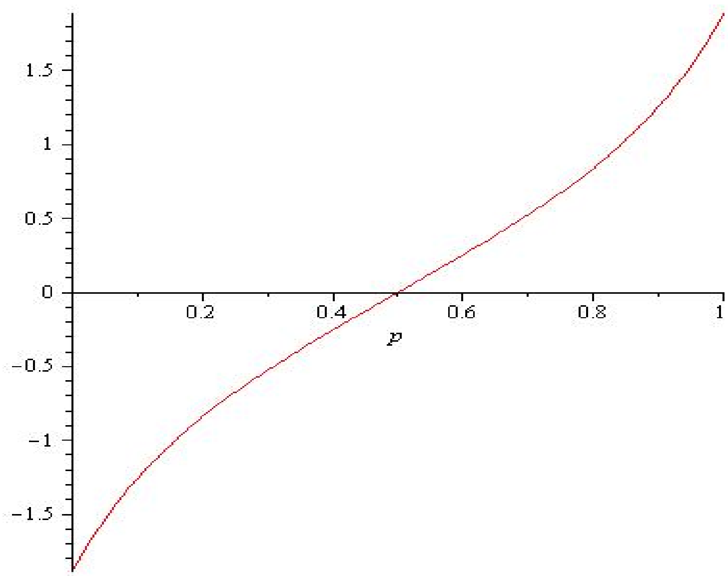

Figure 2: A third solution of the differential equation for the probit function using a variant of Carleman embedding.

Finally, let $m = 10$, $n = 10$. Following the earlier- described procedure, we obtain $u_{1,0}$ and, by implication, another possible solution, w_4. Here,

$$
\begin{aligned}
w_4(p) \;=\;& \frac{1}{5}\sqrt{2}\pi^{\frac{9}{2}}p^5 + \frac{29}{21}\sqrt{2}\pi^{\frac{9}{2}}p^7 - \frac{163}{180}\sqrt{2}\pi^{\frac{9}{2}}p^6 - \frac{4369}{5040}p^8\sqrt{2}\pi^{\frac{9}{2}} \\
+\;& \frac{4369}{22680}p^9\sqrt{2}\pi^{\frac{9}{2}} - \frac{1}{2}\sqrt{2\pi} - \frac{1}{24}\sqrt{2}\pi^{\frac{3}{2}} - \frac{4369}{11612160}\sqrt{2}\pi^{\frac{9}{2}} - \frac{7}{960}\sqrt{2}\pi^{\frac{5}{2}} \\
-\;& \frac{127}{80640}\sqrt{2}\pi^{\frac{7}{2}} + \frac{1}{6}p^3(\frac{367}{960}\sqrt{2}\pi^{\frac{9}{2}} + \frac{13}{16}\sqrt{2}\pi^{\frac{7}{2}} + \frac{3}{2}\sqrt{2}\pi^{\frac{5}{2}} + 2\sqrt{2}\pi^{\frac{3}{2}}) \\
+\;& \frac{1}{3}\left(\frac{817}{1440}\sqrt{2}\pi^{\frac{9}{2}} + \frac{11}{12}\sqrt{2}\pi^{\frac{7}{2}} + \sqrt{2}\pi^{\frac{5}{2}}\right)p^3 \\
+\;& \frac{1}{2}\left(-\frac{4369}{40320}\sqrt{2}\pi^{\frac{9}{2}} - \frac{127}{480}\sqrt{2}\pi^{\frac{7}{2}} - \frac{7}{12}\sqrt{2}\pi^{\frac{5}{2}} - \sqrt{2}\pi^{\frac{3}{2}}\right)p^2 \\
+\;& \frac{1}{4}\left(-\frac{5}{4}\sqrt{2}\pi^{\frac{9}{2}} - \sqrt{2}\pi^{\frac{7}{2}}\right)p^4 + \frac{7}{24}\left(-\frac{367}{240}\sqrt{2}\pi^{\frac{9}{2}} - \frac{13}{6}\sqrt{2}\pi^{\frac{7}{2}} - 2\sqrt{2}\pi^{\frac{5}{2}}\right)p^4 \\
+\;& \frac{127}{720}\left(-\frac{19}{3}\sqrt{2}\pi^{\frac{9}{2}} - 4\sqrt{2}\pi^{\frac{7}{2}}\right)p^6 + \frac{7}{120}\left(\frac{95}{24}\sqrt{2}\pi^{\frac{9}{2}} + 5\sqrt{2}\pi^{\frac{7}{2}} + 4\sqrt{2}\pi^{\frac{5}{2}}\right)p^5 \\
+\;& p(\frac{4369}{645120}\sqrt{2}\pi^{\frac{9}{2}} + \frac{127}{5760}\sqrt{2}\pi^{\frac{7}{2}} + \frac{7}{96}\sqrt{2}\pi^{\frac{5}{2}} + \frac{1}{4}\sqrt{2}\pi^{\frac{3}{2}} + \sqrt{2\pi}) \\
+\;& \frac{127}{5040}\left(14\sqrt{2}\pi^{\frac{9}{2}} + 8\sqrt{2}\pi^{\frac{7}{2}}\right)p^7 + \frac{23}{60}p^5\left(\frac{17}{6}\sqrt{2}\pi^{\frac{9}{6}} + 2\sqrt{2}\pi^{\frac{7}{2}}\right).
\end{aligned}
$$

w_4 evaluated at the points $p = 0$, $p = 0.5$ and $p = 1$, yields $-\frac{4369}{11612160}\sqrt{2}\pi^{\frac{9}{2}} - \frac{127}{80640}\sqrt{2}\pi^{\frac{7}{2}} - \frac{7}{960}\sqrt{2}\pi^{\frac{5}{2}} - \frac{1}{24}\sqrt{2}\pi^{\frac{3}{2}} - \frac{1}{2}\sqrt{2\pi}$, $1\cdot 10^{-11}\sqrt{2}\pi^{\frac{9}{2}} + 2\cdot 10^{-11}\sqrt{2}\pi^{\frac{5}{2}} - 3\cdot 10^{-11}\sqrt{2}\pi^{\frac{7}{2}}$, and $\frac{4369}{11612160}\sqrt{2}\pi^{\frac{9}{2}} + \frac{127}{80640}\sqrt{2}\pi^{\frac{7}{2}} + \frac{7}{960}\sqrt{2}\pi^{\frac{5}{2}} + \frac{1}{24}\sqrt{2}\pi^{\frac{3}{2}} + \frac{1}{2}\sqrt{2\pi}$, respectively. These approximate to -1.976092652, $6.04836112 \cdot 10^{-10}$, and 1.976092651, respectively.

Figure 3 is a plot of w_4, for $0 \le p \le 1$.

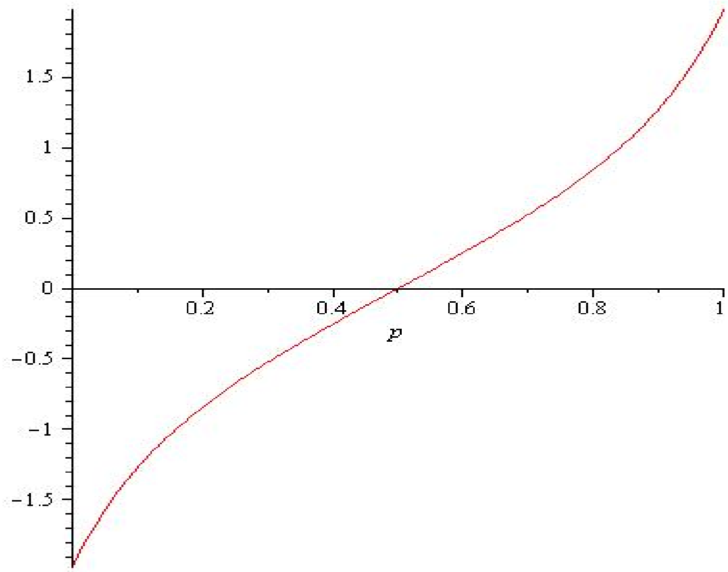

Figure 3: A fourth solution of the differential equation for the probit function using a variant of Carleman embedding.

The procedure discussed in this section not only gives us a value for w (our main aim), but also gives us values for different powers of w. Recall (17), $u_{m,0} = w^m$, or $w^m = u_{m,0}$, which helped us pick our desired solution from the solution set (gotten from solving the system of difference equations generated from (16)), since $w = u_{1,0}$. Likewise,

$$\begin{aligned} w^2 &= u_{2,0}; \\ w^3 &= u_{3,0}; \\ w^4 &= u_{4,0}... \end{aligned}$$

We can also immediately obtain values for w', $(w')^2$, $(w')^3$, $(w')^4$, ..., from our solution set. To do this, recall (18), $u_{0,n} = (w')^n$, or $(w')^n = u_{0,n}$. From this, we have:

$$\begin{aligned} w' &= u_{0,1}; \\ (w')^2 &= u_{0,2}; \\ (w')^3 &= u_{0,3;} \\ (w')^4 &= u_{0,4}... \end{aligned}$$

Furthermore, we can also determine the values of w'', w''', ..., at different points (for different values of p). Recall our differential equation for the probit function,

$$w''(p) = w(p)(w'(p))^2.$$

Let us find the value of $w''(p)$ at, say, $p = 0.5$. Now, $w(0.5) = 0$ and $w'(0.5) = \sqrt{2\pi}$ are already given as initial conditions. Hence,

$$\begin{aligned} w''(0.5) &= w(0.5)(w'(0.5))^2 \\ &= 0 \cdot (\sqrt{2\pi})^2 \\ &= 0. \end{aligned}$$

Similarly,

$$w''' = w \cdot 2w' \cdot (w'') + (w')^2 \cdot w' = 2ww'w'' + (w')^3.$$

Hence,

$$w'''(0.5) = 2w(0.5)w'(0.5)w''(0.5) + (w'(0.5))^3$$

$$= 2 \cdot 0 \cdot \sqrt{2\pi} \cdot 0 + (\sqrt{2\pi})^3$$

$$= 2\sqrt{2}\pi^{3/2}.$$

Finally, notice that we have some other interesting relationships from the preceding expressions. Notice, for instance, that

$$(w'(0.5))^3 = w'''(0.5).$$

Both equal $2\sqrt{2}\pi^{3/2}$.

4.4 Comparison of Solutions Obtained Using A Variant of the Carleman Embedding Technique with Power Series Solutions

In this section, we compare our solutions obtained using a variant of the Carleman embedding technique with the power series solution of Steinbrecher and Shaw [49]. For ease of comparison, let us denote the general power series solution by $w_{pss}(p)$, and that obtained using a variant of the Carleman embedding technique by $w_{vce}(p)$.

Let us first obtain some particular solutions from the general power series solution. Recall the general power series solution, given by equations (4) and (5). With our new notation, the general power series solution is given by:

$$w_{pss}(p) = \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{d_k}{(2k+1)} (2p-1)^{2k+1},$$

where the coefficients d_k satisfy the nonlinear recurrence:

$$d_{k+1} = \frac{\pi}{4} \sum_{j=0}^{k} \frac{d_j d_{k-j}}{(j+1)(2j+1)},$$

with $d_0 = 1$.

If we stop at $k = 0$ when we expand the power series, we obtain a first degree solution:

$$\begin{aligned} w_{pss}(p) &= -\frac{1}{2}\sqrt{2\pi}d_0 + \sqrt{2\pi}d_0 p \\ &= -\frac{1}{2}\sqrt{2\pi} + \sqrt{2\pi}p. \end{aligned}$$

We again expand the general power series solution, this time stopping at $k = 1$. This time, we will need d_1. Let us first find d_1 from equation (5),

$$d_{k+1} = \frac{\pi}{4}\sum_{j=0}^{k} \frac{d_j d_{k-j}}{(j+1)(2j+1)}.$$

$$\begin{aligned} d_1 &= d_{0+1} = \frac{\pi}{4}\sum_{j=0}^{0} \frac{d_j d_{0-j}}{(j+1)(2j+1)} \\ &= \frac{\pi}{4}\left[\frac{d_0 d_{0-0}}{(0+1)(2(0)+1)}\right] \\ &= \frac{\pi}{4}\left[\frac{d_0 d_0}{(1)(0+1)}\right] \\ &= \frac{\pi}{4}\left[\frac{(1)(1)}{(1)(1)}\right] \\ &= \frac{\pi}{4}. \end{aligned}$$

When we expand the power series, stopping at $k = 1$, we have:

$$\begin{aligned} w_{pss}(p) &= \frac{\sqrt{2\pi}}{2}\left(-d_0 - \frac{d_1}{3}\right) + \frac{\sqrt{2\pi}}{2}(2d_0 + 2d_1)p - 2\sqrt{2\pi}d_1p^2 + \frac{4}{3}\sqrt{2\pi}d_1p^3 \\ &= \frac{\sqrt{2\pi}}{2}\left[-1 - \frac{1}{3}\left(\frac{\pi}{4}\right)\right] + \frac{\sqrt{2\pi}}{2}\left[(2(1) + 2\left(\frac{\pi}{4}\right)\right]p - 2\sqrt{2\pi}\left(\frac{\pi}{4}\right)p^2 \\ &+ \frac{4}{3}\sqrt{2\pi}\left(\frac{\pi}{4}\right)p^3 \\ &= \frac{\sqrt{2\pi}}{2}\left(-1 - \frac{\pi}{12}\right) + \sqrt{2\pi}\left(1 + \frac{\pi}{4}\right)p - \pi\frac{\sqrt{2\pi}}{2}p^2 + \pi\frac{\sqrt{2\pi}}{3}p^3. \end{aligned}$$

Figure 4 is a plot of w_{pss2}, a third degree power series solution, for $0 \leq p \leq 1$.

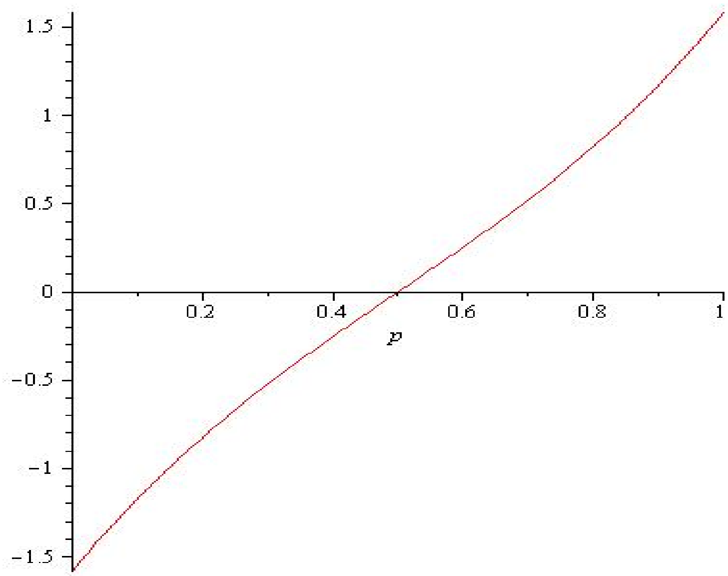

Figure 4: A Second Power Series Solution of the Differential Equation for the Probit Function.

Theorem 4.1: *The solution $u_{1,0}$ to the truncated system of order MN is the Taylor polynomial of the probit function.*

Proof: The truncated system of order MN can be written as:

$$\dot{\mathbf{v}} = A\mathbf{v},$$

where $\mathbf{v}$ is an MN x 1 vector of $u_{i,j}$ and A is an MN x MN matrix.

The matrix A is a banded matrix. This is because none of the equations in our truncated system involve more than two of the $u_{i,j}$ on the RHS. Also, $a_{i,i} = 0 \ \forall \ i$, since $u'_{i,j}$ is independent of $u_{i,j} \ \forall \ i, j$. Moreover, both terms on the RHS of the equation

$$u'_{m,n} = nu_{m+1,n+1} + mu_{m-1,n+1}$$

are dependent on $n+1$. We can therefore define the vector $\mathbf{v}$ in such a way that the resulting matrix A is an upper triangular matrix.

Since A is upper triangular with zeros on its diagonal, the characteristic polynomial of A is given by

$$X(A) = A^{MN}.$$

Our truncated system is therefore a degenerate system of ordinary differential equations.

By setting $X(A) = 0$, we see that there is a single eigenvalue 0 with multiplicity MN. Thus, the general solution is a polynomial of degree MN.

By applying the initial conditions for the vector $\mathbf{v}$, we obtain a unique solution. Furthermore, as $M, N \to \infty$, the solution becomes the power series solution to the probit ODE.

Now, Taylor polynomials are unique. Also, the solution to the ODE is unique. Therefore, the solution $u_{1,0}$ to the truncated system of order MN is the Taylor polynomial of the probit function. *QED*.

The above argument implies that $u_{n,o}$ converges to the n^{th} power of the probit, so the solution to the system of equations yields Taylor Polynomials of powers of the probit function.

We now compare our solutions obtained using a variant of the Carleman embedding technique with the power series solution of Steinbrecher and Shaw [49]. To do this, we shall compare solutions of the same degree (those in which the highest power of p is the same) from both methods. We shall compare constant terms and coefficients of p in those solutions.

The power series solution of first degree is given by:

$$w_{pss}(p) = -\frac{1}{2}\sqrt{2\pi} + \sqrt{2\pi}p.$$

From earlier results, the first degree solution obtained using the variant of Carleman embedding technique is given by:

$$\begin{aligned} w_{vce}(p) &= \sqrt{2\pi}p - \frac{1}{2}\sqrt{2\pi} \\ &= -\frac{1}{2}\sqrt{2\pi} + \sqrt{2\pi}p. \end{aligned}$$

We see immediately that both solutions are identical.

Let us consider third degree solutions from both methods. The third degree power series solution is given by:

$$w_{pss}(p) = \frac{\sqrt{2\pi}}{2}\left(-1 - \frac{\pi}{12}\right) + \sqrt{2\pi}\left(1 + \frac{\pi}{4}\right)p - \pi\frac{\sqrt{2\pi}}{2}p^2 + \pi\frac{\sqrt{2\pi}}{3}p^3.$$

The third degree solution using the variant of Carleman embedding is that obtained with a cut-off of 4 (see Section 4.3 above). It is given by:

$$w_{vce}(p) = \frac{1}{3}\sqrt{2}\pi^{3/2}p^3 - \frac{1}{2}p^2\sqrt{2}\pi^{3/2} + \left(\frac{1}{4}\sqrt{2}\pi^{3/2} + \sqrt{2\pi}\right)p - \frac{1}{24}\sqrt{2}\pi^{3/2} - \frac{1}{2}\sqrt{2\pi}.$$

Rearranging terms in increasing powers of p, we have:

$$w_{vce}(p) = -\frac{1}{24}\sqrt{2}\pi^{3/2} - \frac{1}{2}\sqrt{2\pi} + \left(\frac{1}{4}\sqrt{2}\pi^{3/2} + \sqrt{2\pi}\right)p - \frac{1}{2}\sqrt{2}\pi^{3/2}p^2 + \frac{1}{3}\sqrt{2}\pi^{3/2}p^3.$$

We now compare constant terms and coefficients of p, p^2, and p^3 in w_{pss} and w_{vce}. Let us first compare constant terms.

In w_{pss}, the constant term $=\frac{\sqrt{2\pi}}{2}\left(-1 - \frac{\pi}{12}\right) \approx -1.58143101$.

In w_{vce}, the constant term $=-\frac{1}{24}\sqrt{2}\pi^{3/2} - \frac{1}{2}\sqrt{2\pi} \approx -1.58143101$.

We see that the constant terms are equal in both solutions.

Next, we compare coefficients of p.

In w_{pss}, the coefficient of $p =\sqrt{2\pi}\left(1 + \frac{\pi}{4}\right) \approx 4.475329517$.

In w_{vce}, the coefficient of $p =\left(\frac{1}{4}\sqrt{2}\pi^{3/2} + \sqrt{2\pi}\right) \approx 4.475329517$.

Hence, the coefficients of p are the same in both solutions.

Next, we compare coefficients of p^2.

In w_{pss}, the coefficient of $p^2 = -\pi\frac{\sqrt{2\pi}}{2} \approx -3.937402486$

In w_{vce}, the coefficient of $p^2 = -\frac{1}{2}\sqrt{2}\pi^{3/2} \approx -3.937402486$

Hence, the coefficients of p^2 are the same in both solutions.

Finally, we compare coefficients of p^3.

In w_{pss}, the coefficient of $p^3 = \pi\frac{\sqrt{2\pi}}{3} \approx 2.624934990$

In w_{vce}, the coefficient of $p^3 = \frac{1}{3}\sqrt{2}\pi^{3/2} \approx 2.624934990$

We also see that the coefficients of p^3 are the same in both solutions.

Since the constant terms and coefficients of powers of p are equal in solutions obtained using either method, it follows that solutions obtained using either method are equal for all values of p. We expect this to hold for solutions of any degree, for both methods.

The third degree solutions from both methods are compared graphically in Figure 5:

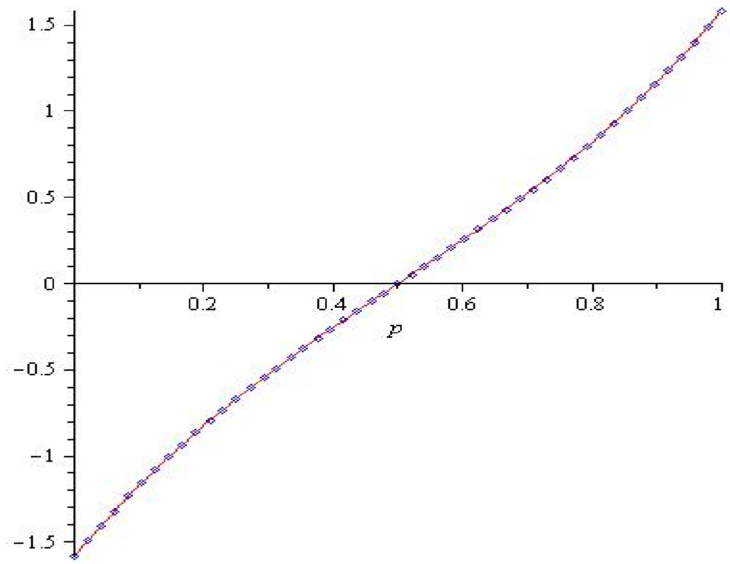

Figure 5: Comparison of third degree power series solution (red) with third degree variant of Carleman embedding solution (blue).

It is clear that both the power series method and the variant of Carleman embedding technique yield more or less the same results when used to solve the second order ordinary differential equation for the probit function. Clearly also, our solutions using the variant of the Carleman embedding technique are bounded and continuous. This is because the solutions are polynomial in p.

5 CONCLUSION

In this research, we used a variant of the Carleman embedding technique with an appropriate transformation to approximate solutions to the second order nonlinear ordinary differential equation for the probit function. We obtained polynomial solutions of different degrees by varying the cut-off values of the indices in our transformation. Our solutions were bounded and continuous. We later compared our solutions with the published power series solutions of Steinbrecher and Shaw [49].

Our comparisons showed that the variant of the Carleman embedding technique used in this research yielded results that are in very good agreement with those of the power series method of Steinbrecher and Shaw [49]. The variant of Carleman embedding technique is especially convenient because we can obtain solutions of different degrees by varying the values of our cut-off for the indices in our transformation. We can also find different powers of the probit function w by varying the values of the indices in our transformation.

Powers of the probit function, just like the probit itself, are important in applications. The method used in this research produces not only Taylor polynomials of the probit function, but also Taylor polynomials of powers of the probit function.

In conclusion, the variant of Carleman embedding technique used in this research is a convenient alternative to power series methods for approximating solutions to the second order nonlinear ordinary differential equation for the probit function. This technique merits further research, to determine its suitability in approximating solutions to other second order and higher order nonlinear differential equations.

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APPENDIX: Maple Code

In the following Maple code, the value of "Cutoff" determines the size of the truncated system of equations generated. Here, the Cutoff value, 10, is arbitrary.

```
U := proc(m,n,Cutoff)
   if( m > Cutoff or n > Cutoff) then
      return 0
   else
      return ( w(t) )^m * ( D(w)(t) ) ^ n
   end if:
end proc:
CutOff:=10:
NewEqs := {}:
for n from 0 to CutOff by 1 do
   for m from 0 to CutOff by 1 do
      if( m+1 > CutOff ) then
         Eq := Diff(u[m,n](p),p) =   m*u[m-1,n+1](p)
      end if:
      if( n+1 > CutOff ) then
         Eq := Diff(u[m,n](p),p) = 0
      end if:
      if( n+1 <= CutOff and m+1 <= CutOff ) then
         Eq := Diff(u[m,n](p),p) = n*u[m+1,n+1](p) + m*u[m-1,n+1](p)
```

```
      end if:
      NewEqs := NewEqs union {Eq}:
   end do
end do:
NewEqs ;
for i in NewEqs do
   print(i)
end do:

ANS := dsolve(NewEqs)
for i in ANS do
print(i)
end do:
IC1 := {seq( seq( u[m,n](0.5) = 0, m=1..10),n=0..10)};
IC2:={seq( seq( u[m,n](0.5) = (sqrt(2*Pi))^n, m=0..0),n=0..10)};
IC:=IC1 union IC2;

SOLN:=dsolve(NewEqs union IC);
for i in SOLN do
print(i)
end do:

## We pick the desired solution, u_{1,0}(p), from the solution set,
```

```
## and use it in the next statement:
## u_{1,0}=...

## For convenience, we assign the value of u_{1,0}(p) (the RHS)
## to w(p), then plot the latter:

w(p):=rhs(%);
plot(w(p), p = 0..1)
```

VITA

KELECHUKWU IROAJANMA ALU

Education:	M.S. Mathematical Sciences, East Tennessee State University, Johnson City, Tennessee, USA, 2011.
	B.Tech. Mathematics and Computer Science, Federal University of Technology Owerri, Imo State, Nigeria, 1998.
Professional Experience:	Graduate Teaching Associate, East Tennessee State University, Johnson City, Tennessee, USA, 2008-2009.
	Graduate Research Assistant, East Tennessee State University, Johnson City, Tennessee, USA, 2008.
	Graduate Teaching Assistant, East Tennessee State University, Johnson City, Tennessee, USA, 2007.
	Lecturer III, Federal Polytechnic Bauchi, Bauchi State, Nigeria, 2003-2005.
	Assistant Lecturer, Federal Polytechnic Bauchi, Bauchi State, Nigeria, 2000-2003.