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A Variety of Proofs of the Steiner-Lehmus Theorem

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A Variety of Proofs of the Steiner-Lehmus Theorem

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by

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ABSTRACT

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The Steiner-Lehmus Theorem has garnered much attention since its conception in the 1840s. A variety of proofs resulting from the posing of the theorem are still appearing today, well over 100 years later. There are some amazing similarities among these proofs, as different as they seem to be. These characteristics allow for some interesting groupings and observations.

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1 THE BEGINNINGS OF THE STEINER-LEHMUS THEOREM

Since approximately 300 B.C.E. Euclid struggled to collect and organize an axiomatic system for geometry, a field that had been developing empirically all over the learned world for some 280 years. He finally released his textbook, The Elements. As his creation spread, Euclid has been bombarded with criticism, suggestions, and new or newly stated theorems to add to the original publication. Over the following years, theorems upon theorems have been stated, proved and accepted and The Elements has grown to be the definitive basis of what we call Euclidean Geometry[5].

Hidden among Euclid's axioms and theorems are the descriptions of triangles, quadrilaterals, circles, polygons, and many of the relationships between these figures. These similarities and properties establish many specific types of triangles: scalene, equilateral, and of special interest to this work, isosceles (see Figure 1.)

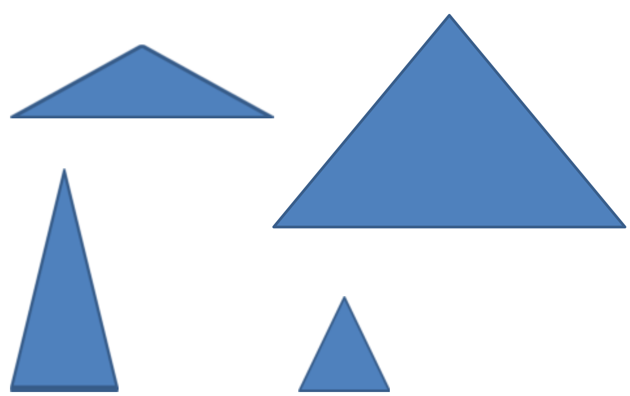

Figure 1: Various Isosceles Triangles

An isosceles triangle has two equal sides that oppose two angles of equal measure. The characteristics of its medians, cevians, internal bisectors, sines, cosines, etc. have been dissected and detailed within Euclid's Elements[5] and the work that followed. So it seems almost inconceivable that with all the attention, there is a property of isosceles triangles that is still being proven nearly 1500 years later.

The question of whether or not the angle bisectors of an isosceles triangle are congruent was itself a direct result of ASA (Angle-Side-Angle) congruency, which was shown in the first volume of The Elements[5]. But the converse seemed to have been forgotten or deliberately ignored for many years, and new proofs of this converse are still being published even today.

The first concrete evidence of an actual posing of the question is found in a letter from C.H. Lehmus penned to C. Sturm in 1840 looking for a geometric proof[1,14]. Sturm did not put forth a proof but he did pass the problem onto others. One of the first responses was from Jacob Steiner[14]. Thus the birth of what has come be to known as the Steiner-Lehmus Theorem: Any triangle with two angle bisectors of equal lengths is isosceles.

The Steiner-Lehmus Theorem has garnered attention since its conception and proofs have been put forth for over one hundred years, resulting in more than 80 accepted proofs[12]. We are going to look at a varied sample of these and discuss how they are constructed and the ways in which they both resemble and differ from each other. Then, at the end, we will expound on some of the more subtle characteristics of this particular theorem.

2 A CLOSE LOOK AT EACH PROOF INDIVIDUALLY

2.1 Seydel/Newman

This proof is a collaboration between an educator, Ken Seydel and his student, Carl Newman Jr.[13], and is itself the product of one of the many traits of the Steiner-Lehmus Theorem. The Steiner-Lehmus Theorem has long drawn the interest of educators because of the seemingly endless ways to prove the theorem (80 plus accepted different proofs.) This has made the it a popular challenge problem. This characteristic of the theorem has also drawn the attention of many mathematicians who are fascinated by puzzles, such as Martin Gardner[7].

This particular proof uses the following facts:

1.) Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (see Figure 2.)

3.) Double Angle Identity: sin 2A = 2sin Acos A

4.) Trigonometric Formula for Area of a Triangle: $\frac{1}{2}$(AB)(AC)sin∠A (see Figure 2.)

5.) Properties of Sine: specifically, increasing on the interval [0°,90°]

6.) Properties of Cosine: specifically, decreasing on the interval [0°,90°]

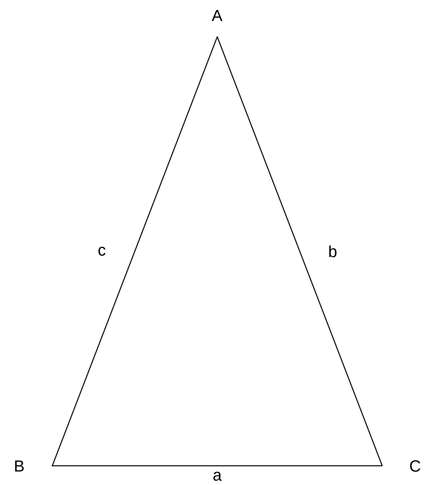

Figure 2: Typical labeling of a triangle

Let us start by stating the hypothesis: $\overline{AE}$, $\overline{CD}$ are angle bisectors of ∠A, ∠C respectively; AE = CD; $0° < \theta$, $\alpha < 90°$ (Figure 3.)

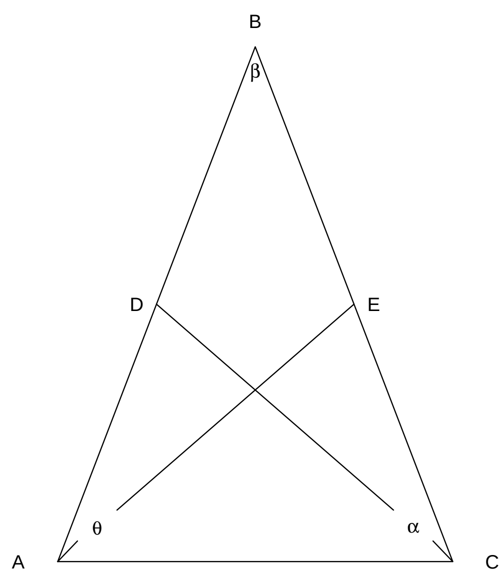

Figure 3: Figure for Seydel/Newman

It should be noted that this is the only proof discussed that limits the base angles of the triangle in question to less 90°. It does not change the outcome but it certainly simplifies some of the details by reducing the number of arguments needed.

Decompose △ABC into △ABE and △AEC (Figure 4) and then again into △CBD and △CDA. (Figure 5)

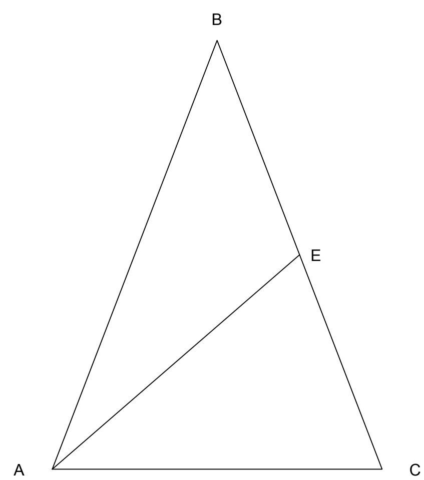

Figure 4: Decomposition into ABE and AEC

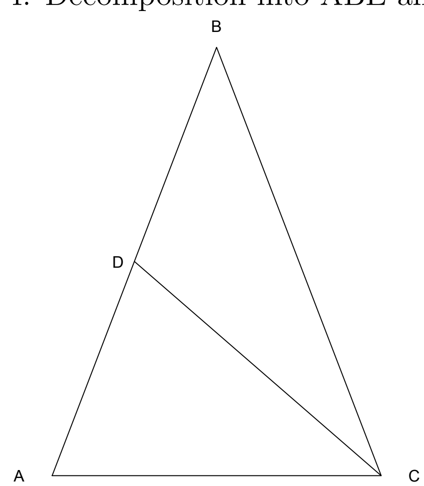

Figure 5: Decomposition into CBD and CDA

Apply the trigonometric formula for area of a triangle to each separate decomposition to achieve the following:

$$\begin{aligned} Area\triangle ABC &= Area\triangle ABE + Area\triangle AEC \\ &= \frac{1}{2}(AB)(AE)\sin\frac{\theta}{2} + \frac{1}{2}(AE)(AC)\sin\frac{\theta}{2}. \end{aligned}$$

And again,

$$\begin{aligned} Area\triangle ABC &= Area\triangle CBD + Area\triangle CDA \\ &= \frac{1}{2}(CB)(CD)\sin\frac{\alpha}{2} + \frac{1}{2}(CD)(AC)\sin\frac{\alpha}{2}. \end{aligned}$$

So, obviously,

$$\frac{1}{2}(CB)(CD)\sin\frac{\alpha}{2} + \frac{1}{2}(CD)(AC)\sin\frac{\alpha}{2} = \frac{1}{2}(AB)(AE)\sin\frac{\theta}{2} + \frac{1}{2}(AE)(AC)\sin\frac{\theta}{2}.$$

Now we establish the assumption, if $\theta = \alpha$, then △ABC is isosceles and the point of the proof is moot, so let us assume $\theta \neq \alpha$ is true. For example, let $\theta > \alpha$ be the assumption. Sine is an increasing function in this instance and $\overline{AE} \cong \overline{CD}$ by hypothesis; so it is accurate to say

$$\frac{1}{2}(AE)(AC)\sin\frac{\theta}{2} > \frac{1}{2}(CD)(AC)\sin\frac{\alpha}{2},$$

Then we can take the previous equation, namely,

$$\frac{1}{2}(CB)(CD)\sin\frac{\alpha}{2} + \frac{1}{2}(CD)(AC)\sin\frac{\alpha}{2} = \frac{1}{2}(AB)(AE)\sin\frac{\theta}{2} + \frac{1}{2}(AE)(AC)\sin\frac{\theta}{2},$$

and deduce the inequalities

$$\begin{aligned} \frac{1}{2}(CD)(AC)\sin\frac{\alpha}{2} &< \frac{1}{2}(AC)(AE)\sin\frac{\theta}{2}, \\ \frac{1}{2}(CD)(CB)\sin\frac{\alpha}{2} &< \frac{1}{2}(AB)(AE)\sin\frac{\theta}{2}. \end{aligned}$$

Apply the Law of Sines (Figure 3) to get

$$\frac{AB}{\sin\alpha} = \frac{AC}{\sin\beta} = \frac{CB}{\sin\theta},$$

and

$$AB = \frac{(AC)\sin\alpha}{\sin\beta}; \qquad CB = \frac{(AC)\sin\theta}{\sin\beta}.$$

So,

$$\frac{1}{2}(AC)\sin\theta\frac{1}{\sin\beta}\sin\frac{\alpha}{2} > \frac{1}{2}(AC)\sin\alpha\frac{1}{\sin\beta}\sin\frac{\theta}{2};$$

which reduces to

$$(\sin\theta)\left(\sin\frac{\alpha}{2}\right) > (\sin\alpha)\left(\sin\frac{\theta}{2}\right).$$

Using the double angle formulas to achieve some commonalty leads to,

$$(2)\left(\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}\right)\left(\sin\frac{\alpha}{2}\right) > (2)\left(\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}\right)\left(\sin\frac{\theta}{2}\right),$$

$$\cos\frac{\theta}{2} > \cos\frac{\alpha}{2}.$$

Cosine is a decreasing function in the setting of this particular proof implying that $\theta < \alpha$, which is a contradiction to the original assumption of $\alpha < \theta$. Using the assumption $\theta < \alpha$ will lead to a similar outcome, thus proving the only option left is that $\theta = \alpha$. That fact leads directly to the angle bisectors being congruent as well; therefore, the triangle is isosceles. $\square$

This proof is very similar to the proof attributed to Plachkey in a later paper[11]. In body and argument the papers are near identical but this proof is by contradiction and the Plachkey proof is considered direct. This is very nice proof using a construction and contradiction.

2.2 David Beran

This proof can be found in the article,“SSA and the Steiner-Lehmus Theorem” by David Beran[1]. In the article, Beran attributes this particular proof of the Steiner-Lehmus Theorem to F.G. Hesse and dates its publication to 1874, with a conception date of 1840[1]. That would make this one of the first proofs to appear in response to Sturm’s query for a proof. Before we get into the proof of the Steiner-Lehmus Theorem, a tool that Beran calls “side-side obtuse congruency” needs to be clarified.

Students are warned that using SSA (Side-Side-Angle) to show congruency is unwise in the first year of geometry. There are two distinct scenarios in which students are compelled to use SSA and then there are three distinct possibilities to the second scenario. In one of these possibilities it will become apparent the problem with SSA. The setting is that it is known that the two triangles in question have two congruent sides. The first scenario is that it is possible to show that the third sides are also congruent and SSS (Side-Side-Side) can be applied. The second scenario is that it is not possible to show any relationship between the third sides but there does exist enough information to show a relationship between a pair of angles.

One possibility to the second scenario is that this relationship is one of non-congruency and the triangles are in fact not congruent regardless of the location of the angles. The second possibility is that the relationship between the two angles is one of congruency and the angle is included (that is, it is located at the intersection of the two congruent sides) then SAS is possible and the triangles again are congruent. The third possibility is that the congruent angles are not included between the two congruent sides then the problem with SSA becomes apparent.

If the angle is an acute angle (less than 90 degrees) there is not enough information to assert that the triangles are congruent because a disturbing possibility appears. The two bottom triangles under consideration in Figure 6 are not congruent even though $\overline{BA} \cong \overline{BD}$, they share a common side $\overline{BC}$, and a common (acute) angle, namely ∠C. If the congruent angles are obtuse (greater than 90 degrees), then the triangles are congruent because no triangle can contain more than one obtuse angle. Beran refers to this as "side-side-obtuse" congruency[1].

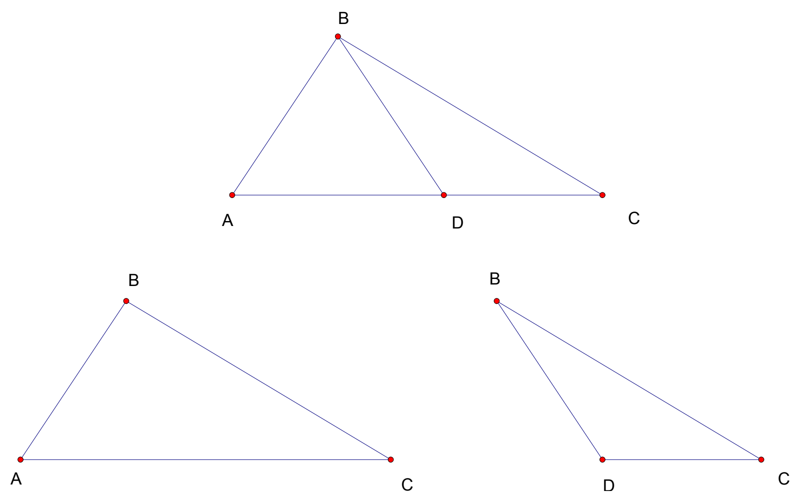

Figure 6: An example of the problem with SSA congruency

Beran presents the Steiner-Lehmus Theorem as the third example in his paper[1]. He uses Hesse's proof as a basis but adds his side-side-obtuse approach as part of the justification. Let us begin this version by assuming the following: $\overline{BD}$ bisects $\angle ABC$; $\overline{CE}$ bisects $\angle ACB$; and $\overline{BD} \cong \overline{CE}$.(Figure 7)

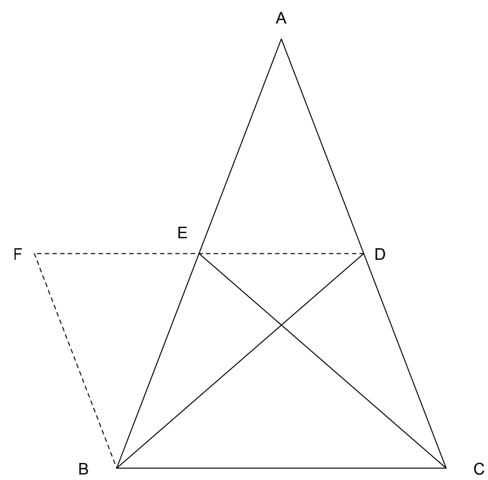

Figure 7: Initial set up for Beran proof

The goal of Beran is to show that $\overline{AB} \cong \overline{AC}$ or that the triangle is isosceles by showing that $\triangle BEC \cong \triangle CBD$ by SSS (Side-Side-Side Congruency.) To do that it is necessary to show that $\overline{EB} \cong \overline{DC}$ is true.

Let F be a point to the side of $\overline{BD}$ and opposite C so that $\overline{DF}$ can be constructed such that m∠BDF = $\frac{1}{2}$m∠C and $\overline{DF} \cong \overline{BC}$. Then △BDF ≅ △ECB by SAS, and ∠FBD ≅ ∠BEC, $\bar{BF} \cong \bar{EB}$ by CPCTC (Congruent Parts of Congruent Triangles are Congruent). Next construct $\overline{FC}$. (Figure 8)

Using this construction △BFC ≅ △DCF. Indeed,

m∠FBC = m∠FBD + m∠DBC, (by construction)

= m∠BEC + m$\frac{\angle B}{2}$, (by construction, original hypothesis)

= (m∠A + m$\frac{\angle C}{2}$) + m$\frac{\angle B}{2}$)(which is possible by accepting that the sum of the angles of a triangle is 180 degrees).

Hence,

m∠FDC = m∠FDB + m∠BDC, (by construction)

$= \frac{m\angle C}{2} + (\text{m}\angle\text{A} + \frac{m\angle B}{2})$,

m$\angle$FDC = m$\angle A + \frac{m\angle C}{2} + \frac{m\angle B}{2}$ = m$\angle$FBC. (by transitivity)

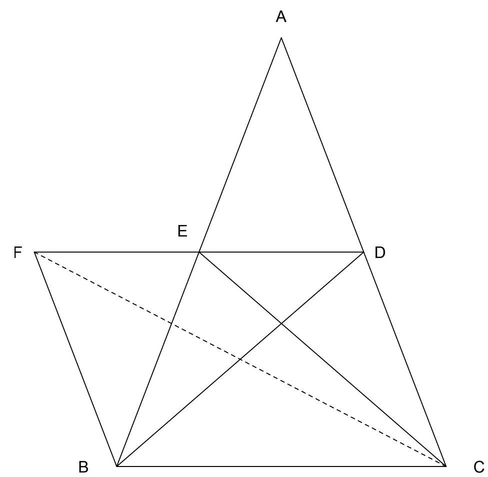

Figure 8: Construction of diagonal FC in the parallelogram

So, both are equivalent to each other and the following,

$= \text{m}\angle\text{A} + \frac{m\angle B}{2} + \frac{m\angle C}{2}$,

$= \frac{m\angle A + m\angle B + m\angle C}{2} + \frac{m\angle A}{2}$,

$= 90° + \frac{m\angle A}{2}$ is true.

So, m$\angle$FDC $\cong$ m$\angle$FBC $> 90°$; thus, both angles are obtuse.

Hence, $\overline{BF} \cong \overline{DC}$ (by CPCTC),

and $\overline{EB} \cong \overline{BF}$ (by construction);

therefore, $\overline{EB} \cong \overline{DC}$.

The result $\overline{EB} \cong \overline{DC}$ will allow us to show that $\triangle BEC \cong \triangle$DCF. It can be used to show that $\triangle$AED is isosceles, which can be further used to show that $\overline{AB} \cong \overline{AC}$

and finally that $\triangle$ABC is also isosceles. $\square$

Thus, this presentation of Hesse's proof is direct in that is does not use the proof style of contradiction, which most proofs of the Steiner-Lehmus Theorem utilize. It does; however, draw heavily on various constructions and many parts of the proof rely on the fact that the angles of any triangle sum to $180°$. Many of those constructions and the justifications of the summation of a triangle's angles are dependent on several theorems that are themselves limited to proofs by contradiction. To follow the most pure line of thought for direct proofs, a mathematician must avoid not only any indirect style of proof in his own proof but ensure that nothing used in his proof was proved using an indirect style either[4]. That line does not preclude the Steiner-Lehmus theorem with complete certainty but it certainly precludes this particular proof from being considered direct by a purist.

2.3 K.R.S. Sastry

The next 4 proofs are found in "A Gergonne Analogue of the Steiner-Lehmus Theorem" by K.R.S. Sastry[12]. This paper illustrates that the characteristics of angle bisectors used to show that a triangle is isosceles are also applicable to the Gergonne cevians of a triangle. A Gergonne cevian is a line segment from a vertex of a triangle to the point of contact/tangency of that triangle's incircle (on the side opposite the aforementioned vertex.)

The technique for this first proof is by contradiction. This is going to be one of the two main styles used for proofs of the Steiner-Lehmus Theorem. That is part of the unusual fascination associated with this theorem. Many mathematicians find the lack of directness in these proofs to be worthy of intense scrutiny. Why should the converse of such a straightforward theorem be itself lacking that same straight forwardness or directness? The converse of this theorem is, if a triangle is isosceles, then its (base) angle bisectors are congruent. This statement is very basic and direct, so much so that its proof is one of the first to show up following Euclid's initial axiomatic system of geometry.

Sastry begins by offering up a proof from 1880 by M. Descube[12]. He uses this proof as a lead to his own proof, which we will dissect momentarily. Descube's proof is a nice example of some of the existing, classic proofs. Let us begin by assuming $\overline{BE}$ and $\overline{CF}$ are angle bisectors of $\triangle$ABC; $\overline{BE} \cong \overline{CF}$. (Figure 9)

Proceeding by contradiction, assume that $\triangle$ABC is not isosceles even though its base angle bisectors are congruent. At the point of contradiction, the reader must concede that the original assumption of $\triangle$ABC not being isosceles must be incorrect, thus, leaving only the option that $\triangle$ABC is in fact isosceles.

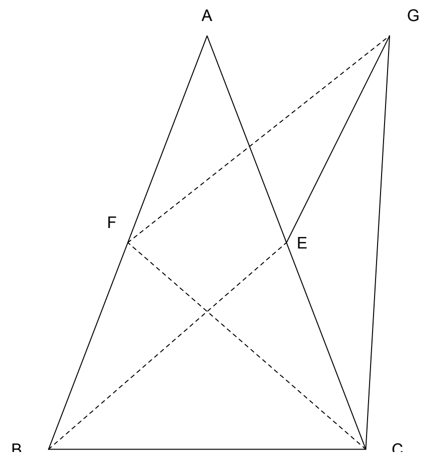

Figure 9: Initial figure for Sastry's presentation of Descube proof

Let AB $\neq$ AC be true. For example, let AB $<$ AC. The preceding inequality, AB $<$ AC leads to $\angle C < \angle$B, so $\frac{\angle C}{2} < \frac{\angle B}{2}$, ultimately implying that CE $>$ BF by the Hinge Theorem. This is illustrated by Figure 10.

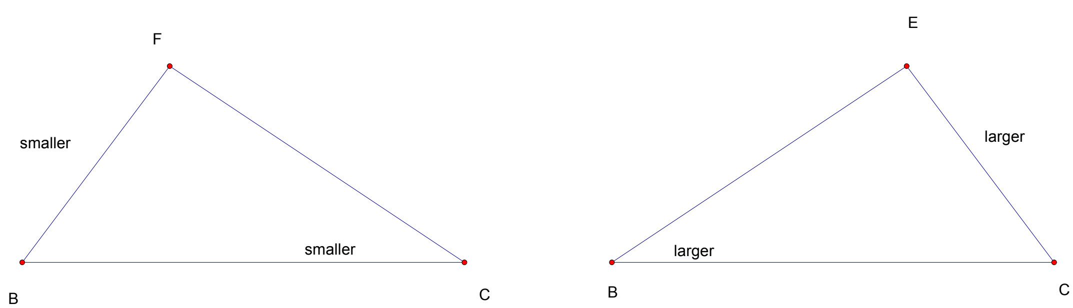

Figure 10: Decomposition for Descube's proof

$\angle FGE$ is constructed so that BFGE is a parallelogram. Drawing on the characteristics of parallelograms,

$\overline{EG} \cong \overline{BF}$, $\overline{FG} \cong \overline{BE}$,

$\angle$FGE = $\angle$FBE.

So, $\angle$FGE = $\frac{\angle B}{2}$, and FG = BE = CF.

But that implies that $\angle$FGC = $\angle$FCG.

Recall that part of the original assumption is that $\angle$FGE = $\frac{\angle B}{2} > \frac{\angle C}{2}$ = $\angle$FCE, which implied $\angle EGC <$ $\angle$ECG, and CE < GE.

But $\overline{GE} \cong \overline{BF}$ is true by construction and characteristics of parallelograms, which implies that CE < BF, contradicting a result of the initial assumption CE > BF.

So AB > AC is not true. A similar argument can be stated for the assumption AB > AC. Therefore, if neither of those statements is correct, the only possibility left is that $\overline{AB} \cong \overline{AC}$ has to be the case and the triangle is isosceles. □

The proof attributed to Descube[12] is nearly identical to the proof from Fetisov's[6] book that is profiled later. There is only the slightest of differences in the final arguments, but retaining both of these proofs is a good example of how similar some of the accepted proofs can be while still being considered original.

In Sastry's own proof he does not appeal to the results of the Steiner-Lehmus Theorem but instead focuses on using distance and area to prove the following: "If two Gergonne cevians of a triangle are equal, then the triangle is isosceles[12]."

Notice the labeling of Figure 2 and 11. Figure 2 is the standard labeling of sides and vertices unless otherwise noted. The additions to the sides of the triangle in Figure 11 are common in most proofs/discussions involving an inscribed circle. The 's' in Figure 11 is a standard variable used to represent the semi-perimeter ($\frac{a+b+c}{2}$.)

It is worthwhile to note that

$$BC = a = (s-b)+(s-c) = \left(\frac{a+b+c}{2}-b\right)+\left(\frac{a+b+c}{2}-c\right) = (a+b+c)-b-c = a.$$

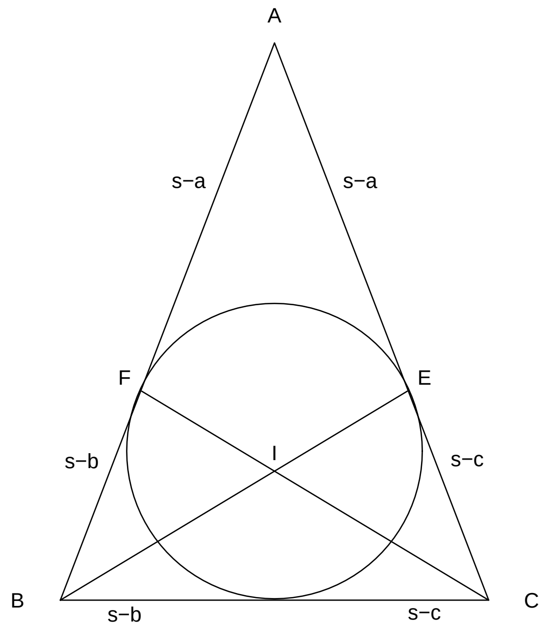

Figure 11: Sastry figure for cevian proof

Let us assume that $\overline{BE} \cong \overline{CF}$, where $\overline{BE}$, $\overline{CF}$ are the Gergonne cevians of $\triangle$ABC, as the hypothesis. Look at these two particular triangles in Figure 12.

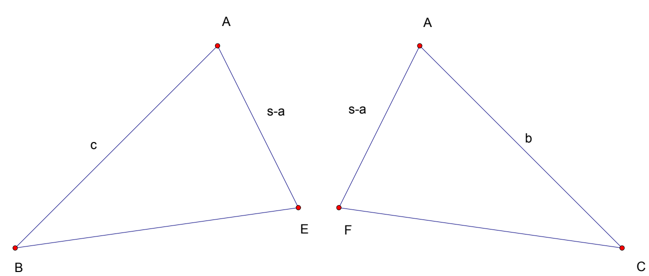

Figure 12: Breakdown of internal triangles for Sastry proof

Apply the law of cosines to the angle common to both triangles, ∠A. $(BE)^2 = c^2 + (s-a)^2 - 2c(s-a)\cos A$ and $(CF)^2 = b^2 + (s-a)^2 - 2b(s-a)\cos A$.

Since BE = CF, $c^2 + (s-a)^2 - 2c(s-a)\cos A = b^2 + (s-a)^2 - 2b(s-a)\cos A$ Some shifting and algebra leads to:

$$c^2 + (s-a)^2 - 2c(s-a)\cos A = b^2 + (s-a)^2 - 2b(s-a)\cos A,$$

$$c^2 - 2c(s-a)\cos A = b^2 - 2b(s-a)\cos A,$$

$$c^2 - b^2 - 2c(s-a)\cos A = -2b(s-a)\cos A,$$

$$c^2 - b^2 - 2c(s-a)\cos A + 2b(s-a)\cos A = 0,$$

$$c^2 - b^2 - 2(s-a)\cos A(c-b) = 0,$$

$$(c-b)(c+b) - 2(s-a)\cos A(c-b) = 0,$$

$$(c-b)((c+b) - 2(s-a)\cos A) = 0.$$

Then either Case 1:(c-b) = 0 or Case 2:((c+b) - 2(s-a)cos A) = 0.

Case 1: c-b = 0 or c=b, which means the triangle would be isosceles and the proof has come to an end.

We must look at Case 2 to either eliminate or accept it as a possibility. The following chain of equalities are equivalent:

$$(c+b) - 2(s-a)\cos A = 0,$$

$$(c+b) - 2(s-a)\Big(\frac{-a^2 + b^2 + c^2}{2cb}\Big) = 0,$$

(use law of cosines on △ ABC, and solve for cos A)

$$c + b - (s-a)(-a^2 + b^2 + c^2)\Big(\frac{1}{bc}\Big) = 0,$$

$$c+b-\left(\frac{a}{2}+\frac{b}{2}+\frac{c}{2}-a)(-a^2+b^2+c^2)\left(\frac{1}{bc}\right)=0,$$

$$c+b-\frac{1}{2bc}(-a+b+c)(-a^2+b^2+c^2)=0,$$

$$\frac{2bc^2}{2bc}+\frac{2b^2c}{2bc}-\frac{(-a+b+c)(-a^2+b^2+c^2)}{2bc}=0,$$

$$2bc^2+2b^2c-(-a+b+c)(-a^2+b^2+c^2)=0,$$

$$2bc^2+2b^2c-(a^3-ab^2-ac^2-a^2b+b^3+bc^2-a^c+b^2c+c^3)=0,$$

$$2bc^2+2b^2c-a^3+ab^2+ac^2+a^2b-b^3-bc^2+a^c-b^2c-c^3=0,$$

$$-a^3+ab^2+ac^2+a^2b-b^3+bc^2+a^2c+b^2c-c^3=0,$$

$$-a^3+a^2b+a^2c+ab^2-b^3+bc^2+a^2c+b^2c-c^3=0,$$

$$a^2(b+c-a)+b^2(a-b+c)+c^2(a+b-c)=0.$$

And by the triangle inequality and general rules of arithmetic all parts of the left side are positive and non-zero. This is an impossibility and Case 1 must be true. Thus △ABC is isosceles. □ This is the second direct proof thus far.

Sastry's third proof is a combination of pieces of the two previous proofs[12]. In fact, Figure 13 is a combination of Figures 9 and 11.

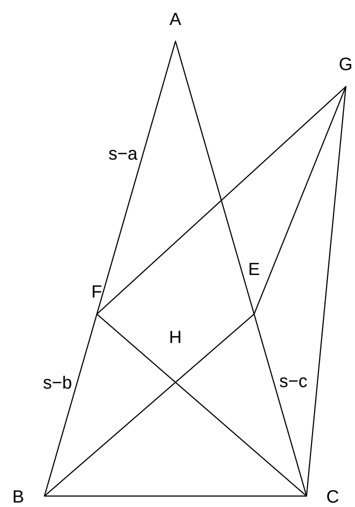

Figure 13: Sastry extension, figure for proof 3

This proof will take on the style of many Steiner-Lehmus proofs, a proof by contradiction. From the information in the paper, this is supposed to be a second proof of the statement, "If two Gergonne cevians of a triangle are equal, then the triangle is isosceles[12]." However, the cevians are not noted in the figure. Figure 14 is more reprensentative of the proof. This figure has only the addition of the inscribed circle and adds very little in the way of useful information. However, it does make clear that $\overline{BE}$ and $\overline{CF}$ are Gergonne cevians.

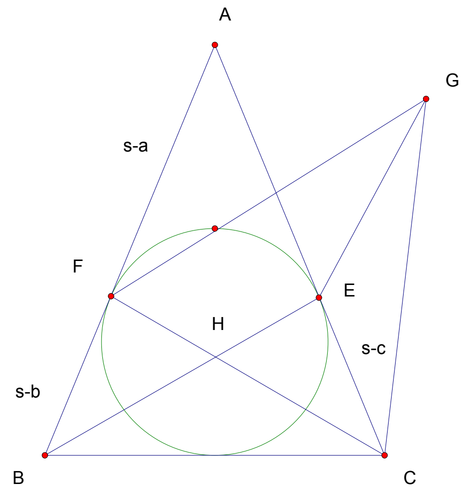

Figure 14: Addition of circle to denote cevians for Sastry extension proof 3

Let $\overline{BE}$ and $\overline{CF}$ be the Gergonne cevians associated with $\triangle$ABC and circle H; also $\overline{BE} \cong \overline{CF}$. Proceeding by contradiction assume that $\overline{AB} \neq \overline{AC}$, say $\overline{AB} < \overline{AC}$ or c < b. Recall, from the first proof, that if $\angle EBC > \angle$FCB is true then $\overline{CH} > \overline{BH}$ will also be true.

Since $\overline{BE} \cong \overline{CF}$ is true by hypothesis, then $\overline{FH} < \overline{EH}$ will also be true. Scrutinize $\triangle$s ABE and AFC (Figure 15) and notice that $\overline{AE} \cong \overline{AF}$ = (s-a), $\overline{BE} \cong \overline{CF}$, and $\overline{AB} < \overline{AC}$.

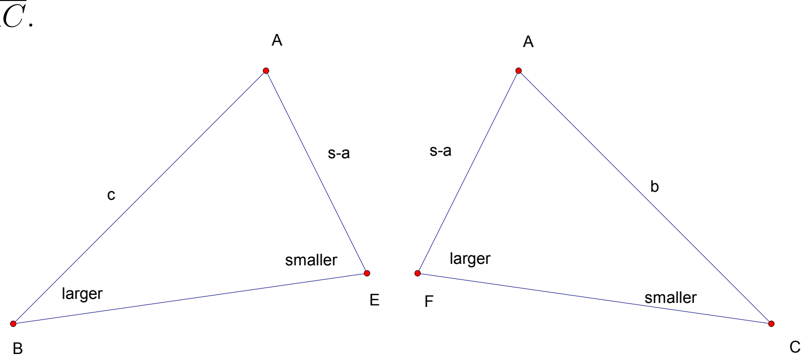

Figure 15: Decomposition to ABE and ACF

Scrutiny should lead to the conclusion that $\angle AEB < \angle$ACF. From that, the conclusion $\angle BEC > \angle$BFC or equivalently $\angle HEC > \angle$HFB should be clear. Now shift attention to $\triangle$BFH and $\triangle$EHC (Figure 16). Using the congruency of alternate angles, $\angle BHF \cong \angle$EHC and notice that $\angle FBH > \angle$ECH.

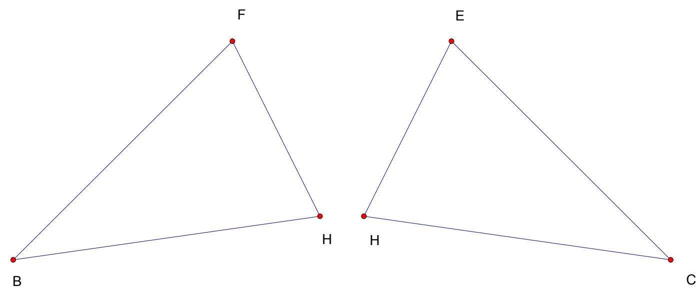

Figure 16: Decomposition into FBH and EHC

Sastry specifically states that we are using the same construction now as as the one used in the first proof, but at this point he calls our attention to the fact that $\triangle$FGC is isosceles by Descube's construction[12]. But, in fact, point G was established so that BFGE would be a parallelogram. So while it is true that $\triangle$FGC is isosceles, that was not the original purpose of the construction.

$\triangle$FBC is isosceles because $\overline{FG}$ was constructed as congruent to $\overline{BE}$ which is congruent to $\overline{CF}$ by hypothesis; therefore, $\angle$FGC = $\angle$FCG or equivalently, $\angle$FGE + $\angle$EGC = $\angle$HCE + $\angle$ECG. Since $\angle$FBH = $\angle FGE > \angle$HCE by both construction and previous statements, it stands to reason that $\angle EGC < \angle ECG$ must be true. Also, $\overline{EC} < \overline{EG}$ would be true by their relationship to their opposite angles. Which leads directly to

$$(s - c) < (s - b)$$

$$-c < -b$$

$$c > b.$$

Thus, contradicting our initial assumption that c < b. If the process is repeated assuming AB > AC or c > b, the result will again be similar, a contradiction. Therefore, if b < c is not true and c < b is not true, it must be that b = c and △ABC is isosceles. So, this is yet another proof involving contradiction and that will turn out to be the most prevalent style. □

Sastry wraps up his paper with an extension to his last proof noted in the following statement: The internal angle bisectors of the ∠ABC and ∠ACB of △ABC meet the Gergonne cevian $\overline{AD}$ at E and F respectively[12]. If $\overline{BE} \cong \overline{CF}$ then △ABC is isosceles. (Figure 17.)

So, let us assume that $\overline{BE} \cong \overline{CF}$ where $\overline{BE}, \overline{CF}$ are the angle bisectors of ∠B and ∠C, respectively. The gist of the argument is going to be that E and F are in fact the same point since Gergonne cevians and angle bisectors are concurrent. In an isosceles triangle not only are the angle bisectors concurrent, but they are also concurrent about the same point as the Gergonne cevians. Just to be clear, that means that the Gergonne cevians and angle bisectors are the same line in an isosceles triangle. The proof will follow the same path as most of Steiner-Lehmus proofs, namely a proof by contradiction.

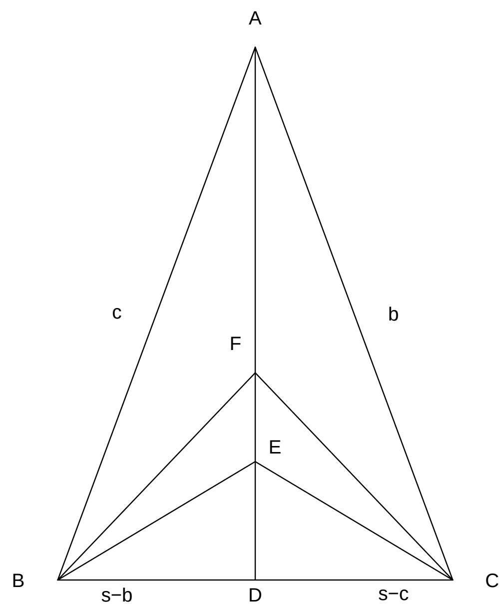

Figure 17: Sastry proof 4 extension figure

If AB $\neq$ AC then let AB $<$ AC which is equivalent to b $>$ c or (s-b) $<$ (s-c). Therefore, E lies below F on $\overline{AD}$. The assumption of AB $<$ AC implies $\angle ABC >$ $\angle$ACB. $\angle ECB > \angle FCD >$ $\angle$ECB is true by characteristics of angle bisectors and the parts are less than the whole. That implies that either CE $>$ BE or CE $>$ CF since CF $=$ BE.

However, $\angle$ADC $=$ $\angle EDC > \frac{\pi}{2}$ as noted in earlier proofs. And $\angle$FEC $=$ $\angle$EDC + $\angle$ECD, so $\angle FEC > \frac{\pi}{2}$. And $\angle EFC < \frac{\pi}{2}$ since the sum of the angles of any triangle are 180 degrees. Thus, CE $<$ CF which is a contradiction of our earlier result and furthermore implies that the original assumption of AB $<$ AC can not be true. As before, it is necessary to note that assuming AB $>$ AC will lead to a similar, albeit reversed situation. Therefore, our only choice is that AB $=$ AC or $\triangle$ABC is isosceles.

$\square$

Sastry has provided a total of four proofs of the Steiner-Lehmus Theorem. One is a direct proof and the other three are proofs by contradiction. This foreshadows that contradiction will ultimately be the most popular method for this theorem. When Sastry is introducing the paper the we have just discussed in this thesis, he brings up one of the more interesting characteristics of the Steiner-Lehmus Theorem. Sastry refers the Steiner-Lehmus Theorem as notorious because the record number of not only accepted proofs, but also due to the large number of incorrect proofs published and withdrawn.

2.4 Mowaffaq Hajja (I)

The style of this next proof will be very similar to the earlier ones in that it will use contradiction, but it will differ by depending heavily on trigonometric functions and identities. In fact, the title of the paper is "A Short Trigonometric Proof of the Steiner-Lehmus Theorem" authored by Mowaffaq Hajja[9].

Hajja has added more details to his figures to accommodate the many trigonometric formulas he uses. This is apparent by Figure 18. Note that: u = AB', U = B'C, v = AC', and V = BC'.

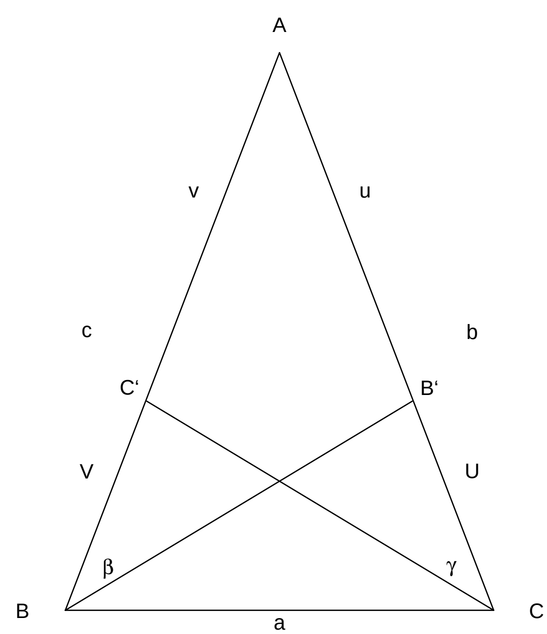

Figure 18: Figure for Hajja's first paper, initial

This first proof uses the contrapositive instead of the actual theorem as stated.

Hajja specifically notes that this proof is short, even titling the paper as such[9]. Maybe the only reason it is short is because Hajja has edited most of the algebra, and his reasoning and thought process, from the proof. His proof depends heavily on the Angle Bisector Theorem and the Law of Sines. Using Figure 18 for notation let us recall both results.

Angle Bisector Theorem: Relating the angle bisector $\overline{BB'}$, $\frac{u}{U} = \frac{c}{a}$ and relating the angle bisector $\overline{CC'}$, $\frac{v}{V} = \frac{b}{a}$.

The Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (see Figure 2 for notation.)

Let us begin by stating that $\overline{BB'}, \overline{CC'}$ are the angle bisectors of ∠B and ∠C respectively; m∠B = 2β; m∠C = 2γ. Let us assume that $\overline{BB'} = \overline{CC'}$ as the hypothesis of Steiner-Lehmus Theorem.

If △ABC is not isosceles then ∠C ≠ ∠B so let $\angle C >$ ∠B which implies to c > b.

So if c > b, then b < c obviously and $\frac{b}{u} < \frac{c}{v}$.

Since u+U = b and v+V = c, $\frac{u+U}{u} < \frac{v+V}{v}$.

$\frac{u}{u} + \frac{U}{u} < \frac{v}{v} + \frac{V}{v}$.

$1 + \frac{U}{u} < 1 + \frac{V}{v}$.

$\frac{U}{u} < \frac{V}{v}$.

Apply the angle bisector theorem and conclude that $\frac{a}{c} < \frac{a}{b}$.

So, we have taken the assumption c ≠ b or more specifically c > b and some clever substitution and the application of the angle bisector theorem to show that if c > b is true then $\frac{a}{c} - \frac{a}{b} < 0$ is also true.

The technique of establishing an assumption and building each step to reach a contradiction is more accepted. To present the proof with maximum clarity let us restate this previous section as such. Say c > b, then ...

$$
\begin{aligned}
\frac{1}{c} &< \frac{1}{b}, \\
\frac{a}{c} &< \frac{a}{b}, \\
\frac{c}{a} &> \frac{b}{a}, \\
\frac{u}{U} &> \frac{v}{V}, \text{(by applying the angle bisector theorem)} \\
\frac{U}{u} &< \frac{V}{v}, \\
1 + \frac{U}{u} &< 1 + \frac{V}{v}, \\
\frac{u}{u} + \frac{U}{u} &< \frac{v}{v} + \frac{V}{v}, \\
\frac{u+U}{u} &< \frac{v+V}{v}, \\
\frac{b}{u} &< \frac{c}{v}, \\
\frac{b}{u} - \frac{c}{v} &< 0.
\end{aligned}
$$

Continuing on with the initial assumption of c > b, the implication is

$$
\begin{aligned}
\gamma &< \beta, \\
\cos\gamma &< \cos\beta, \text{(because cosine is a decreasing function)} \\
\cos\beta &> \cos\gamma, \\
\cos\beta(\sin A) &> \cos\gamma(\sin A), \\
\frac{\cos\beta(\sin A)}{BB'} &> \frac{\cos\gamma(\sin A)}{CC'},
\end{aligned}
$$

(note CC' = BB' ; Law of Sines: $\frac{\sin A}{BB'} = \frac{\sin\beta}{u} = \frac{\sin\gamma}{v}$)

$$\frac{\cos\beta(\sin\beta)}{u} > \frac{\cos\gamma(\sin\gamma)}{v},$$
$$2\frac{\cos\beta(\sin\beta)}{u} > 2\frac{\cos\gamma(\sin\gamma)}{v},$$
$$\frac{\sin 2\beta}{u} > \frac{\sin 2\gamma}{v},$$
$$\frac{\sin B}{u} > \frac{\sin C}{v}, \text{(Law of Sines: sin B/C} = \frac{b/c\sin A}{a}\Big)$$
$$\frac{\frac{b\sin A}{a}}{u} > \frac{\frac{c\sin A}{a}}{v},$$
$$\frac{b}{u} > \frac{c}{v},$$
$$\frac{b}{u} - \frac{c}{v} > 0.$$

We have a contradiction. The same assumption has produced two possibilities that are incongruous, thus the assumption is considered false. Modifying the assumption to say c < b will result in a similar set of results, so the only option left is that c must be equal to b and △ABC must be isosceles. □

The remainder of the paper is spent on a discussion of the evolution of the Steiner-Lehmus Theorem[9]. Specifically, how the proofs presented since its conception seem to have shifted from the search for a direct geometric proof to the expansion of various other parts of the triangle; such as, cevians, in-circles, etc. Hajja also spends some time expanding his own opinions on applying Steiner-Lehmus Theorem to higher dimensions. His final thoughts are a digression into the possible existence of a direct proof of three equal angle bisectors implying an equilateral triangle.

2.5 Mowaffaq Hajja (II)

This paper is also by Mowaffaq Hajja. It is titled "Stonger Forms of the Steiner-Lehmus Theorem[10]". He has modified his labeling so we will have to forgo previous Figure for a new one, Figure 19.

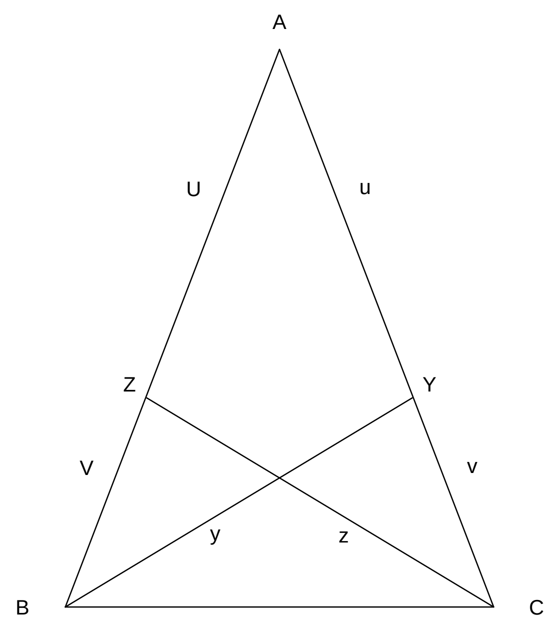

Figure 19: Hajja, 2nd paper, initial figure

This proof makes use of the following result (Breusch's Lemma)[3]:

$$\frac{p(ABC)}{a} = \frac{2}{1 - \tan\frac{B}{2}\tan\frac{C}{2}},$$

where P(ABC) is the perimeter (a+b+c) based on the standard labeling of a triangle (see Figure 2).

But more specifically for our purposes we will set the result up in the following format:

$$p(ABC) = \frac{2a}{1 - \tan\frac{B}{2}\tan\frac{C}{2}},$$

$$p(ABC)(1 - \tan\frac{B}{2}\tan\frac{C}{2}) = 2a,$$

$$\frac{p(ABC)(1 - tan\frac{B}{2}tan\frac{C}{2})}{2} = a.$$

Note that 'a' is the side included by the two 'base' angles used in the denominator.

And we begin as usual - y, z are angle bisectors of △ABC and are congruent to each other. Decompose △ABC into the two triangles in Figure 20.

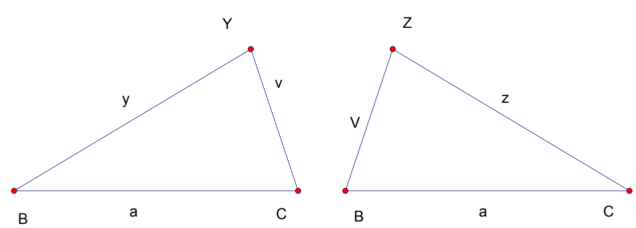

Figure 20: Decomposition into YBC and ZBC for application of Breusch's Lemma

So, $\frac{p(YBC)(1-\tan\frac{B}{4}\tan\frac{C}{2})}{2}$ = a = $\frac{p(ZBC)(1-\tan\frac{B}{2}\tan\frac{C}{4}}{2}$; therefore,

$$\frac{p(YBC)}{p(ZBC)} = \frac{(1 - \tan\frac{B}{2}\tan\frac{C}{4})}{(1 - \tan\frac{B}{4}\tan\frac{C}{2})}.$$

So, without loss of generality (WLOG), c > b will be the initial assumption and remember if this is true then the ratio of $\frac{p(YBC)}{p(ZBC)} > 1$.

$$c > b,$$

$$\angle C > \angle B,$$

$$\tan C > \tan B, \text{ (because tangent is an increasing function)}$$

$$\tan\frac{C}{2} > \tan\frac{B}{2}, \text{ (since c > b)},$$

$$\tan\frac{C}{2}\tan\frac{B}{4} > \tan\frac{B}{2}\tan\frac{C}{4}, \text{ (since } \frac{B}{4} < \frac{C}{4}\text{)}$$

$$\tan\frac{C}{4}\tan\frac{B}{4} > \tan\frac{B}{4}\tan\frac{C}{4}, \text{ (since } \frac{B}{4} > \frac{B}{4}\text{)}$$

$$2\tan\frac{C}{4}\tan\frac{B}{4} > 2\tan\frac{B}{4}\tan\frac{C}{4}$$

$$\tan\frac{C}{4} > \tan\frac{B}{4}, \text{ (because sin } \frac{C}{4} > \frac{B}{4}\text{)}$$

$$\tan^2\frac{C}{4} > \tan^2\frac{B}{4},$$

$$1 - \tan^2\frac{C}{4} < 1 - \tan^2\frac{B}{4}, \text{ (because } \tan^2\frac{C}{4}, \frac{B}{4} < 1\text{)}.$$

Hence, p(YBC) > p(ZBC).

Recall that

$$|BY| = y, |CZ| = z, |AZ| = U, |ZB| = V, |AY| = u, |YC| = v.$$

Simple substitution allows for p(YBC) = (YB + BC + CY) = (y + a + v)

and p(ZBC) = (ZB + BC + CZ) = (V + a + z).

We can now say that

$$y + a + v > V + a + z$$

$$y + v > V + z,$$

which is stronger. It is stronger because of a few facts. First, $y > v$, which is the same result one obtains with the Steiner-Lehmus Theorm. Second, we have to recall that $v < V$ from the initial setup of the proof. Third and lastly, we need to notice that those two inequalities are reversed. A small amount(v) was added to the larger side(y), and a large amount(V) was added to the smaller side(z). That made our inequality stronger because it did not change the original result.

In the previous statement, it is mentioned that V > v, which is easily proven by the following argument. By applying the angle bisector theorem to ∠C in the Figure 19, it is easy to establish that $\frac{V}{U} = \frac{a}{b}$ Then,

$$\begin{aligned}
\frac{V}{U} &= \frac{a}{b}, \\
\frac{U}{V} &= \frac{b}{a}, \\
1 + \frac{U}{V} &= \frac{b}{a} + 1, \\
\frac{V}{V} + \frac{U}{V} &= \frac{b}{a} + \frac{a}{a}, \\
\frac{V+U}{V} &= \frac{a+b}{a}, \\
\frac{V}{V+U} &= \frac{a}{a+b}, \\
\frac{V}{c} &= \frac{a}{a+b}, \text{(since U+V = c)} \\
V &= \frac{ac}{a+b}.
\end{aligned}$$

And for v, we can apply the angle bisector theorem to ∠B to get $\frac{v}{u} = \frac{a}{c}$, which is equivalent to $\frac{v}{v+u} = \frac{a}{a+c}$ and onto v = $\frac{ab}{a+c}$. Combining these,

$$
\begin{aligned}
V - v &= \frac{ac}{a+b} - \frac{ab}{a+c}, \\
&= \frac{ac(a+c)}{(a+b)(a+c)} - \frac{ab(a+b)}{(a+b)(a+c)}, \\
&= \frac{1}{(a+b)(a+c)}[a^2c + ac^2 - a^2b - ab^2], \\
&= \frac{1}{(a+b)(a+c)}[a(ac + c^2 - ab - b^2)], \\
&= \frac{1}{(a+b)(a+c)}[a(ac - ab + c^2 - b^2)], \\
&= \frac{1}{(a+b)(a+c)}[a(a(c-b) + (c-b)(c+b)], \\
&= \frac{1}{(a+b)(a+c)}a(c-b)(a+b+c), \\
&= \frac{a(c-b)(a+b+c)}{(a+b)(a+c)}.
\end{aligned}
$$

So, a>0; (a+b+c)>0; (a+b)>0; (a+c)>0 and if c>b (remember this is the original assumption upon which this whole argument is founded) then (c-b)>0 is also true and so V>v definitively.

If that is true, why can't we apply the angle bisector theorem and similar algebra to establish that U $= \frac{bc}{a+b}$ and u $= \frac{bc}{a+c}$? And carry that onto if U>u and c>b then a similar result of $y + u > z + U$. Combining these two could result in the following.

$$y + v > z + V \text{combined logically with} y + u \quad > z + U,$$

$$\text{to result in} y + (u + v) > z + (V + U),$$

$$y + b > z + c.$$

But that has been proved false by M. Tetiva in 2008[10], so there must be more to the Steiner-Lehmus Theorem than we are currently understanding.

The remainder of this paper is spent building algebraically on the result Hajja has achieved. For example, Hajja questions if y + b > z +c isn't true, does that mean that y + u > z +U is also false? Then he proceeds to build as such, noting that a proof of any of the 3 options listed would verify his conjecture:

$$y + u > z + U,$$

$$y - z > U - u,$$

$$\frac{y-z}{U-u} > 1,$$

$$\frac{(y-z)}{U-u}(y+z) > y + z,$$

$$\text{Option 1: } \frac{y^2-z^2}{U-u} > y + z,$$

$$\text{Option 2: } (\frac{y^2-z^2}{U-u})^2 > (y+z)^2,$$

$$(\frac{y^2-z^2}{U-u})^2 > y^2 + 2yz + z^2,$$

$$(\frac{y^2-z^2}{U-u})^2 > y^2 + z^2 + 2yz,$$

$$(\frac{y^2-z^2}{U-u})^2 > y^2 + z^2 + y^2 + z^2 > y^2 + z^2 + 2yz,$$

$$\text{Options 3: } (\frac{y^2-z^2}{U-u})^2 > (y^2+z^2)^2.$$

And combining these two results leads to:

$$c > b \Rightarrow y + u/v > z + U/V,$$

$$c > b \Rightarrow y + \frac{b}{2} > z + \frac{c}{2},$$

$$c > b \Rightarrow y - z > \frac{c}{2} - \frac{b}{2},$$

$$c > b \Rightarrow y - z > \frac{1}{2}(c - b).$$

Hajja also expands into multiplicative forms such as

$$c > b \Rightarrow y^2 b > z^2 c.$$

All of these forms are built algebraically from Hajja's original result. These proofs would be so similar to the one we have already scrutinized that it would not be productive to repeat them.

Hajja also includes an expansion on the implication $c > b \Rightarrow y - z > \frac{1}{2}(c - b)$. He uses an iterative computing program to attempt to improve the factor $\frac{1}{2}$ or 0.5. He has established a value of 0.8568 and indicates that as the coefficient 0.5 is increased, specifically as it is increased to 0.8568, the two sides of the inequality approach equality (the goal of the Steiner-Lehmus Theorem) and further work has resulted in a refinement of 0.856762. All of these were based on

$$lim_{(c-b)} \frac{y-z}{c-b} = \frac{(a^{+}ab + 2b^2)}{2b(a+b)^2}.$$

But, other than a passing mention, this is beyond the scope of this thesis as it is not a proof, nor is it portrayed as one. It is, in fact, a conjecture based on the exploration of the limitations of the theorem in certain settings. It is also an example of the widely varied work being done on the Steiner-Lehmus Theorem.

2.6 Oláh-Gál/Sándor

Leaving Hajja and his numerous proofs[9,10] let us discuss another paper from *Forum Geometricorum.* This paper is a collaboration by Róbert Oláh-Gál and József Sándor. Their paper is titled "On Trigonometric Proofs of the Steiner-Lehmus Theorem[11]". They introduce their paper with a very nice summary of the history of the Steiner-Lehmus Theorem.

The first proof offered in this paper is from a Romanian paper and attributed to V. Cristescu circa 1916[11]. Referring to Figure 21, BB', CC' are the angle bisectors of ∠B and ∠C, respectively, and we assume that $\overline{BB'} \cong \overline{CC'}$. By applying the Law of Sines to △BB'C, we reach the following: $\frac{BB'}{\sin C} = \frac{CC'}{\sin(C+\frac{B}{2})}$.

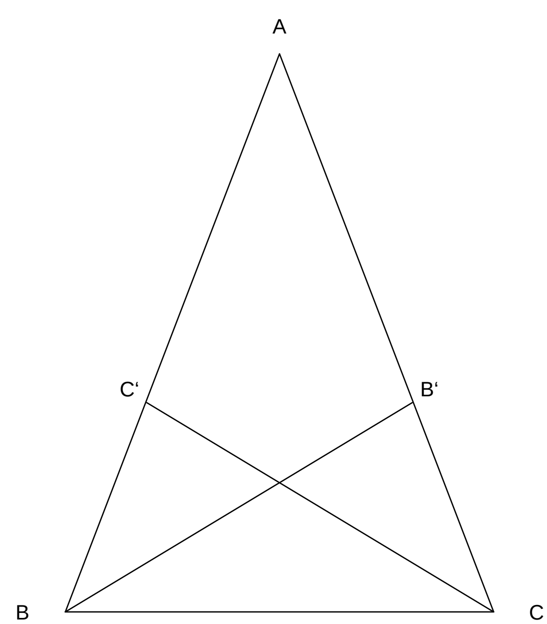

Figure 21: Figure for Cristescu proof

Note:

$$\angle C + \frac{\angle B}{2} = \angle C + \frac{180° - \angle C - \angle A}{2},$$
$$= \angle C + 90° \frac{\angle C}{2} - \frac{\angle A}{2},$$
$$= 90° - \frac{\angle A}{2} - \frac{\angle C}{2},$$
$$= 90° - \frac{\angle A - \angle C}{2}.$$

Then,

$$\frac{BB'}{\sin C} = \frac{a}{\sin(90° - \frac{\angle A - \angle C}{2})},$$
$$\frac{BB'}{\sin C} = \frac{a}{\cos(\frac{\angle A - \angle C}{2})},$$
$$BB' = a\frac{\sin C}{\cos(\frac{\angle A - \angle C}{2})}.$$

By a similar argument,

$$CC' = a\frac{\sin B}{\cos(\frac{\angle A - \angle B}{2})}.$$

At this point the proof makes use of the double and half angle formulas, so both will be stated as needed. (Use Figure 21 for notation.)

Double Angle Formula: sin C = 2 $\sin(\frac{C}{2})\cos(\frac{C}{2})$,

Half Angle Formulas: sin $\frac{C}{2} = \cos(\frac{A+B}{2})$; sin $\frac{B}{2} = \cos(\frac{A+C}{2})$,

Then,

$$a\frac{\sin C}{\cos(\frac{A-C}{2})} = a\frac{\sin B}{\cos(\frac{A-B}{2})},$$
$$\sin C\Big(\frac{1}{\cos(\frac{A-C}{2})}\Big) = \sin B\Big(\frac{1}{\cos(\frac{A-B}{2})}\Big),$$
$$2\sin\frac{C}{2}\cos\frac{C}{2}\frac{1}{\cos(\frac{A-C}{2})} = 2\sin\frac{B}{2}\cos\frac{B}{2}\frac{1}{\cos(\frac{A-B}{2})},$$
$$\cos\frac{C}{2}\cos\frac{A+B}{2}\frac{1}{\cos(\frac{A-C}{2})} = \cos\frac{B}{2}\cos\frac{A+C}{2}\frac{1}{\cos(\frac{A-B}{2})},$$

$$\cos\frac{C}{2}\cos\frac{A+B}{2}\cos\frac{A-C}{2} = \cos\frac{B}{2}\cos\frac{A+C}{2}\cos\frac{A-C}{2},$$

By applying the identity (cos (x+y))(cos (x -y)) = $\cos^2$ x + $\cos^2$ y - 1;

$$\cos\frac{C}{2}\Big(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-1\Big) = \cos\frac{B}{2}\Big(\cos^2\frac{A}{2}+\cos^2\frac{C}{2}-1\Big),$$

$$\cos\frac{C}{2}\cos^2\frac{A}{2}+\cos\frac{C}{2}\cos^2\frac{B}{2}-c\frac{C}{2}-\cos\frac{B}{2}\cos^2\frac{A}{2}-\cos\frac{B}{2}\cos^2\frac{C}{2}+\cos\frac{B}{2}=0,$$

$$\cos\frac{C}{2}\cos^2\frac{A}{2}+\cos\frac{B}{2}\cos^2\frac{A}{2}+\cos\frac{C}{2}\cos^2\frac{B}{2}-\cos\frac{B}{2}\cos^2\frac{C}{2}-\cos\frac{C}{2}+\cos\frac{B}{2}=0,$$

$$\cos^2\frac{A}{2}(\cos\frac{C}{2}-\cos\frac{B}{2})-\cos\frac{C}{2}\cos\frac{B}{2}(\cos\frac{C}{2}-\cos\frac{B}{2})-(\cos\frac{C}{2}-\cos\frac{B}{2})=0,$$

$$\Big(\cos\frac{C}{2}-\cos\frac{B}{2}\Big)\Big(\cos^2\frac{A}{2}-\cos\frac{C}{2}\cos\frac{B}{2}-1\Big)=0,$$

$$\Big(\cos\frac{C}{2}-\cos\frac{B}{2}\Big)\Big(\cos^2\frac{A}{2}-1-\cos\frac{C}{2}\cos\frac{B}{2}\Big)=0,$$

$$\Big(\cos\frac{B}{2}-\cos\frac{C}{2}\Big)\Big(1-\cos^2\frac{A}{2}+\cos\frac{C}{2}\cos\frac{B}{2}\Big)=0,$$

$$\Big(\cos\frac{C}{2}-\cos\frac{B}{2}\Big)\Big(\sin^2\frac{A}{2}-\cos\frac{C}{2}\cos\frac{B}{2}\Big)=0.$$

At this stage, either (cos $\frac{C}{2}$ - cos$\frac{B}{2}$) = 0 or ($\sin^2\frac{A}{2}$ - cos$\frac{C}{2}$cos $\frac{B}{2}$) = 0.

If $\sin^2\frac{A}{2}$ - cos$\frac{C}{2}$cos $\frac{B}{2}$ = 0 is true that would mean that cos $\frac{B}{2}$cos $\frac{C}{2}$ would have to be negative, which means that either C or B must have a measure greater than 180°. This is an impossibility for any triangle, isosceles or not. Therefore, cos $\frac{C}{2}$ - cos$\frac{B}{2}$ = 0 must be true, meaning that cos $\frac{C}{2}$ = cos$\frac{B}{2}$ is true. Which can only happen if ∠B = ∠C or if the triangle is isosceles. □

The second proof presented by Oláh-Gál/Sándor is attributed to Plachky in 2000[11], but it can be found in a similar form earlier in a 1983 publication as a collaboration of Ken Seydel and his student D. Carl Newman Jr[13]. However, that was a proof by contradiction and this proof is direct, so it will be profiled with equal importance.

Let us assume that $\overline{AA'}$, $\overline{BB'}$ are the angle bisectors of ∠s A and B respectively, and that $\omega_a = \omega_b$ where $\omega_a = \overline{AA'}$ and $\omega_b = \overline{BB'}$. Using the trigonometric area formula ($\frac{1}{2}$ab)sin γ (Figure 22) and applying it to the decomposition of △BB'C and △BB'A in Figure 23, we get

$$Area\triangle ABC = \frac{1}{2}a\omega_b \sin\frac{\beta}{2} + \frac{1}{2}c\omega_b \sin\frac{\beta}{2}.$$

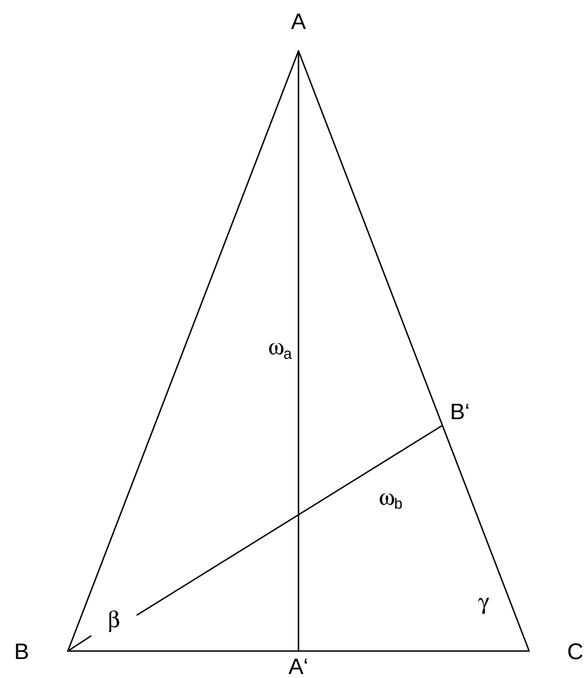

Figure 22: Figure for Plachky proof

In a similar fashion, using Figure 24 to apply the area formula to △AA'C and △AA'B

$$Area\triangle ABC = \frac{1}{2}b\omega_a \sin\frac{\alpha}{2} + \frac{1}{2}c\omega_a \sin\frac{\alpha}{2}.$$

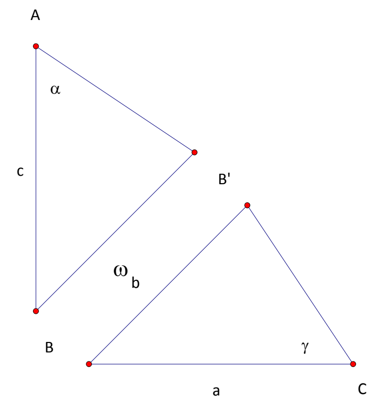

Figure 23: Decomposition based on ω_b

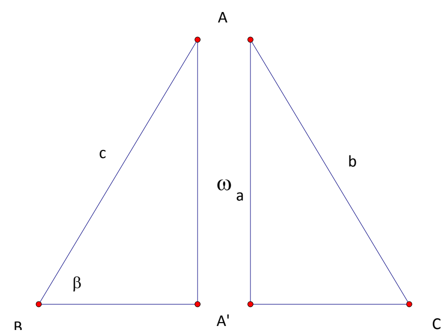

Figure 24: Decomposition based on ω_a

Using the Law of Sines and the identity sin (A-B) = sinA cosB - cosA sinB,

$$\frac{\sin\alpha}{a} = \frac{\sin(\pi) - (\alpha+\beta)}{c} = \frac{\sin\beta}{b},$$

$$a = \frac{c\sin\alpha}{\sin(\alpha+\beta)}.$$

Similarly it can be shown that

$$b = \frac{c\sin\beta}{\sin(\alpha+\beta)}.$$

Assume $\omega_a = \omega_b$, in other words assume the angle bisectors are congruent and combining the area formulations will result in the following:

$$\frac{1}{2}a\omega_b \sin\frac{\beta}{2} + \frac{1}{2}c\omega_b \sin\frac{\beta}{2} = \frac{1}{2}a\omega_a \sin\frac{\alpha}{2} + \frac{1}{2}c\omega_a \sin\frac{\alpha}{2},$$

$$\sin\frac{\beta}{2}(a+c) = \sin\frac{\alpha}{2}(b+c),$$

$$\sin\frac{\beta}{2}\Big(\frac{c\sin\alpha}{\sin(\alpha+\beta)} + c) = \sin\frac{\alpha}{2}\Big(\frac{c\sin\beta}{\sin(\alpha+\beta)} + c),$$

$$\sin\frac{\beta}{2}\frac{c\sin\alpha}{\sin(\alpha+\beta)} + \sin\frac{\beta}{2}c - \sin\frac{\alpha}{2}\frac{c\sin\beta}{\sin(\alpha+\beta)} - c\sin\frac{\alpha}{2} = 0,$$

$$c(\sin\frac{\beta}{2}\Big)\Big(\frac{\sin\alpha}{\sin(\alpha+\beta)}\Big) - c(\sin\frac{\alpha}{2}\Big)\Big(\frac{\sin\beta}{\sin(\alpha+\beta)}\Big) + c(\sin\frac{\beta}{2}\Big) - c(\sin\frac{\alpha}{2}\Big) = 0,$$

$$\sin\frac{\beta}{2}\Big(\frac{\sin\alpha}{\sin(\alpha+\beta)}\Big) - \sin\frac{\alpha}{2}\Big(\frac{\sin\beta}{\sin(\alpha+\beta)}\Big) + \sin\frac{\beta}{2} - \sin\frac{\alpha}{2} = 0,$$

$$\sin\frac{\beta}{2}\sin\alpha - \sin\frac{\alpha}{2}\sin\beta + \sin(\alpha+\beta)\sin\frac{\beta}{2} - \sin(\alpha+\beta)\sin\frac{\alpha}{2} = 0,$$

$$\sin\frac{\beta}{2}\sin\alpha - \sin\frac{\alpha}{2}\sin\beta + \sin(\alpha+\beta)\Big(\sin\frac{\beta}{2} - \sin\frac{\alpha}{2}\Big) = 0,$$

$$(\text{use sin u} = 2\text{sin}\frac{u}{2}\text{cos}\frac{u}{2})$$

$$2\sin\frac{\beta}{2}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2} + \sin(\alpha+\beta)\sin\frac{\beta}{2} - \sin(\alpha+\beta)\sin\frac{\alpha}{2} = 0,$$

$$2\sin\frac{\beta}{2}\sin\frac{\alpha}{2}(\cos\frac{\alpha}{2} - \cos\frac{\beta}{2}) + \sin(\alpha+\beta)(\sin\frac{\beta}{2} - \sin\frac{\alpha}{2}) = 0,$$

$$(\text{use sin u - sin v} = 2\ \text{sin}\frac{u-v}{2}\text{cos}\frac{u+v}{2})$$

$$(\text{and cos u - cos v} = \text{-2 sin}\frac{u-v}{2}\text{sin}\frac{u+v}{2})$$

$$2\sin\frac{\beta}{2}\sin\frac{\alpha}{2}(-2\sin\frac{\frac{\alpha}{2}-\frac{\beta}{2}}{2}\sin\frac{\frac{\alpha}{2}+\frac{\beta}{2}}{2}) + \sin(\alpha+\beta)(2\sin\frac{\frac{\beta}{2}-\frac{\alpha}{2}}{2}\cos\frac{\frac{\beta}{2}+\frac{\alpha}{2}}{2}) = 0,$$

$$-4\sin\frac{\beta}{2}\sin\frac{\alpha}{2}(\sin\frac{\alpha-\beta}{4}\sin\frac{\alpha+\beta}{4}) + 2\sin(\alpha+\beta)(\sin\frac{\beta-\alpha}{4}\cos\frac{\beta+\alpha}{4}) = 0,$$

$$4\sin\frac{\beta}{2}\sin\frac{\alpha}{2}\sin\frac{\alpha-\beta}{4}\sin\frac{\alpha+\beta}{4}) - 2\sin(\alpha+\beta)\sin\frac{\beta-\alpha}{4}\cos\frac{\beta+\alpha}{4} = 0,$$

$$2\sin\frac{\alpha-\beta}{4}(2\sin\frac{\beta}{2}\sin\frac{\alpha}{2}\sin\frac{\alpha+\beta}{4} + \sin(\alpha+\beta)\cos\frac{\alpha+\beta}{4}) = 0.$$

And, as before in the other direct proofs, either 2sin$\frac{\alpha-\beta}{4}$ = 0, which happens only when $\alpha = \beta$ (△ABC is isosceles), or 2sin$\frac{\beta}{2}$sin$\frac{\alpha}{2}$sin$\frac{\alpha+\beta}{4}$ + sin(α+β) cos $\frac{\alpha+\beta}{4}$ =0, which can never happen since (α+β) < π in this setting. This proof is complete. □

The third proof presented in the Oláh-Gál/Sándor paper[11] is an obscure proof from a Russian text with no specific author credited. It is another trigonometric proof that follows the style of contradiction. Recall that the area of a triangle is $\frac{1}{2}$bc sin A, which is one half the product of two sides and the sine of the angle formed by those two sides. So, to use ∠B as the angle, sides a and c must be used. Leading to

$$Area\triangle ABC = \frac{1}{2}(a)(c)(\sin B) = \frac{1}{2}(a)(c)(\sin\beta) = \frac{1}{2}(a)(c)(\sin 2 * \frac{\beta}{2}).$$

Decompose △ABC utilizing ω_b from Figure 25 into the following two triangles, △ABB' and △CBB'. Area = $\frac{1}{2}\omega_b$c(sin$\frac{\beta}{2}$) + $\frac{1}{2}\omega_b$a(sin$\frac{\beta}{2}$).

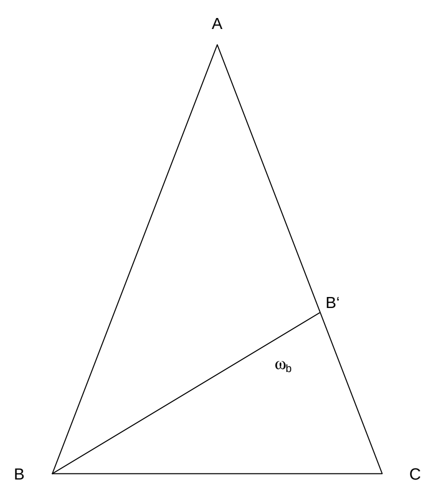

Figure 25: Figure for Russian proof offered in Oláh-Gál/Sándor paper

Therefore,

$$\frac{1}{2}\omega_b * c * \sin\frac{\beta}{2} + \frac{1}{2}\omega_b * a * \sin\frac{\beta}{2} = \frac{1}{2}a * c * \sin(2 * \frac{\beta}{2})$$

(use sin 2A = 2sinAcosA),

$$\frac{1}{2}\omega_b * c * \sin\frac{\beta}{2} + \frac{1}{2}\omega_b * a * \sin\frac{\beta}{2} = \frac{1}{2}a * c * 2\sin(\frac{\beta}{2})\cos(\frac{\beta}{2}),$$

$$\frac{1}{2}\omega_b * c + \frac{1}{2}\omega_b * a = a * c * \cos\frac{\beta}{2},$$

$$\omega_b * c + \omega_a = 2a * c * \cos\frac{\beta}{2},$$

$$\omega_b(a + c) = 2a * c * \cos\frac{\beta}{2},$$

$$\omega_b = \frac{2a * c}{a + c}\cos\frac{\beta}{2}.$$

Similarly, it can be established that $\omega_a = \frac{2b*c}{b+c}\cos\frac{\alpha}{2}$.

Now, let a > b, then $\alpha > \beta$ must be true in addition to $\frac{\alpha}{2} > \frac{\beta}{2}$. This a triangle, thus $\alpha, \beta < 180°$ is true as is $\frac{\alpha}{2}, \frac{\beta}{2} \in (0,\ \frac{\pi}{2})$.

So, if $\frac{\alpha}{2} > \frac{\beta}{2}$ then cos $\frac{\alpha}{2} < cos\frac{\beta}{2}$ because cosine is a positive decreasing function on $(0,\ \frac{\pi}{2})$.

Also, since a > b, $\frac{ac}{a+c} > \frac{bc}{b+c}$ would be accurate. Then the combination $\frac{2ac}{a+c}\cos\frac{\beta}{2} > \frac{2bc}{b+c}\cos\frac{\alpha}{2}$ is equivalent to stating $\omega_a > \omega_b$.

We have our contradiction to the hypothesis of the Steiner-Lehmus theorem that the angle bisectors are congruent. It is a similar argument to assume that a < b and reach the opposite contradiction. Thus ends the proof by reaching the conclusion that a = b must be true. □

This fourth proof presented by Oláh-Gál/Sándor[11] is inspired in part by the earlier proof presented by Hajja[10]. However, Oláh-Gál/Sándor have offered a proof using the Law of Sines and more elementary trigonometric facts. To follow the proof we will need to the notation in Figure 26.

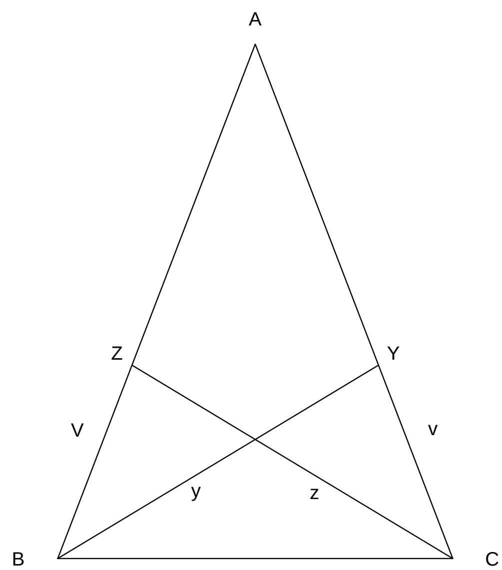

Figure 26: Figure for Oláh-Gál/Sándor proof

Using Figure 26 as the basis, extract Figures 27 and 28.

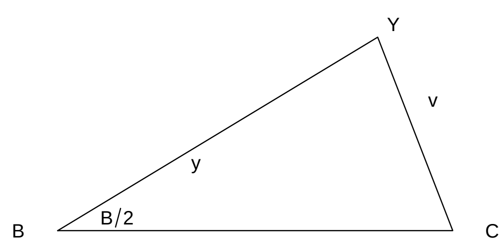

Figure 27: Decomposition to BYC for Oláh-Gál/Sándor proof

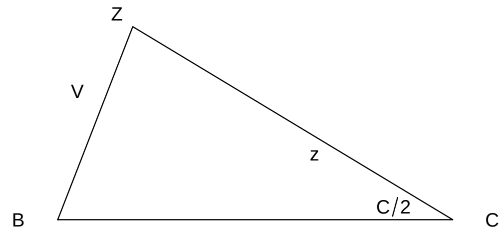

Figure 28: Decomposition to BZC for the Oláh-Gál/Sándor proof

Applying the law of sines to figure 27 results in

$$\frac{BC}{\sin(C+\frac{B}{2})} = \frac{CY}{\sin\frac{B}{2}} = \frac{BY}{\sin C}$$

or

$$\frac{a}{\sin(C+\frac{B}{2})} = \frac{v}{\sin\frac{B}{2}} = \frac{y}{\sin C}$$

which leads to

$$\frac{a}{\sin(C+\frac{B}{2})} = \frac{y+v}{\sin C+\sin\frac{B}{2}},$$

so y+v = $\frac{a(\sin C+\sin\frac{B}{2})}{\sin(C+\frac{B}{2})}$. Using Figure 28 will similarly result that z+V = $\frac{a(\sin B+\sin\frac{C}{2})}{\sin(B+\frac{C}{2})}$.

At this point results, from Hajja's[10] are used, instead of using the fact that if △ABC is isosceles then y+v = a+V would be true. Oláh-Gál/Sándor assume that y+v > z+V and start yet another proof by contradiction.

If y+v > z+V, then

$$\frac{a(\sin C+\sin\frac{B}{2})}{\sin(C+\frac{B}{2})} > \frac{a(\sin B+\sin\frac{C}{2})}{\sin(B+\frac{C}{2})},$$

$$\frac{\sin C+\sin\frac{B}{2}}{\sin(C+\frac{B}{2})} > \frac{\sin B+\sin\frac{C}{2}}{\sin(B+\frac{C}{2})},$$

(because $\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$)

$$\frac{1}{\sin(C+\frac{B}{2})}(2\sin\Big(\frac{C+\frac{B}{2}}{2}\Big)\cos\Big(\frac{C-\frac{B}{2}}{2}\Big)\Big) > \frac{1}{\sin(B+\frac{C}{2})}(2\sin\Big(\frac{B+\frac{C}{2}}{2}\Big)\cos\Big(\frac{B-\frac{C}{2}}{2}\Big)\Big),$$

$$\frac{2\sin(\frac{C}{2}+\frac{B}{4})\cos(\frac{C}{2}-\frac{B}{4})}{\sin(C+\frac{B}{2})} > \frac{2\sin(\frac{B}{2}+\frac{C}{4})\cos(\frac{B}{2}-\frac{C}{4})}{\sin(B+\frac{C}{2})},$$

(using sin 2A = 2 sinA cos A)

$$\frac{2\sin(\frac{C}{2}+\frac{B}{4})\cos(\frac{C}{2}-\frac{B}{4})}{\sin 2(\frac{C}{2}+\frac{B}{4})} > \frac{2\sin(\frac{C}{2}+\frac{C}{4})\cos(\frac{B}{2}-\frac{C}{4})}{\sin 2(\frac{B}{2}+\frac{C}{4})},$$

$$\frac{2\sin(\frac{C}{2}+\frac{B}{4})\cos(\frac{C}{2}-\frac{B}{4})}{2\sin(\frac{C}{2}+\frac{B}{4}\cos(\frac{C}{2}+\frac{B}{4})} > \frac{2\sin(\frac{C}{2}+\frac{C}{4})\cos(\frac{B}{2}-\frac{C}{4})}{2\sin(\frac{B}{2}+\frac{C}{4}\cos(\frac{B}{2}+\frac{C}{4})},$$

$$\frac{\cos(\frac{C}{2}-\frac{B}{4})}{\cos(\frac{C}{2}+\frac{B}{4})} > \frac{\cos(\frac{B}{2}-\frac{C}{4})}{\cos(\frac{B}{2}+\frac{C}{4})},$$

(note that $\cos\Big(\frac{C}{2}+\frac{B}{4}\Big)$, $\cos\Big(\frac{B}{2}+\frac{C}{4}\Big) > 0$),

$$\cos\Big(\frac{B}{2}+\frac{C}{4}\Big)\cos\Big(\frac{C}{2}-\frac{B}{4}\Big) > \cos\Big(\frac{C}{2}+\frac{B}{4}\Big)\cos\Big(\frac{B}{2}-\frac{C}{4}\Big),$$

(use cos u cos v = $\frac{\cos(u+v)}{2} + \frac{\cos(u-v)}{2}$)

$$\cos\Big(\frac{C}{2}-\frac{B}{4}\Big)\cos\Big(\frac{B}{2}+\frac{C}{4}\Big) > \cos\Big(\frac{B}{2}-\frac{C}{4}\Big)\cos\Big(\frac{C}{2}+\frac{B}{4}\Big)$$

(use cos u cos v = $\frac{1}{2}$cos(u-v)+$\frac{1}{2}$cos(u+v))

$$\frac{1}{2}\cos\Big(\frac{C}{2}-\frac{B}{4}-\frac{B}{2}-\frac{C}{4}\Big)+\frac{1}{2}\cos\Big(\frac{C}{2}-\frac{B}{4}+\frac{B}{2}+\frac{C}{4}\Big)$$

$$> \frac{1}{2}\cos\Big(\frac{B}{2}-\frac{C}{4}-\frac{C}{2}-\frac{B}{4}\Big)+\frac{1}{2}\cos\Big(\frac{B}{2}-\frac{C}{4}+\frac{C}{2}+\frac{B}{4}\Big),$$

$$\cos\Big(\frac{C}{4}-\frac{3B}{4}\Big)+\cos\Big(\frac{3C}{4}+\frac{B}{4}\Big) > \cos\Big(\frac{B}{4}-\frac{3C}{4}\Big)+\cos\Big(\frac{3B}{4}+\frac{C}{4}\Big),$$

$$\cos\Big(\frac{3C}{4}+\frac{B}{4}\Big)-\cos\Big(\frac{3B}{4}+\frac{C}{4}\Big) > \cos\Big(\frac{B}{4}-\frac{3C}{4}\Big)-\cos\Big(\frac{C}{4}-\frac{3B}{4}\Big),$$

(use cos u - cos v = -2sin$\frac{u-v}{2}$sin$\frac{u+v}{2}$)

$$-2\sin\frac{(\frac{3C}{4}+\frac{B}{4})-(\frac{3B}{4}+\frac{C}{4})}{2}\sin\frac{(\frac{3C}{4}+\frac{B}{4})+(\frac{3B}{4}+\frac{C}{4})}{2}$$

$$> -2\sin\frac{(\frac{B}{4}-\frac{3C}{4})-(\frac{C}{4}-\frac{3B}{4})}{2}\sin\frac{(\frac{B}{4}-\frac{3C}{4})+(\frac{C}{4}-\frac{3B}{4})}{2},$$

$$-2\sin\frac{(\frac{C}{2}-\frac{B}{2})}{2}\sin\frac{C+B}{2} > -2\sin\frac{(B-C)}{2}\sin\frac{(-\frac{B}{2}-\frac{C}{2})}{2},$$

$$-2\sin\Big(\frac{C}{4}-\frac{B}{4}\Big)\sin\Big(\frac{C}{2}+\frac{B}{2}\Big) > -2\sin\Big(\frac{B}{2}-\frac{C}{2}\Big)\sin\Big(-\frac{B}{4}-\frac{C}{4}\Big),$$

apply cos u - cos v = -2 $\sin\frac{u-v}{2}\sin\frac{u+v}{2}$

$$-\sin\frac{B}{2}\sin\frac{3C}{2} > -\sin\frac{C}{2}\sin\frac{3B}{2},$$

(use sin 3u = 3sin u - 4sin^{3}u)

$$-\sin\frac{B}{2}(3\sin\frac{C}{2}-4\sin^3\frac{C}{2}\Big) > -\sin\frac{C}{2}(3\sin\frac{B}{2}-4\sin^3\frac{B}{2}\Big),$$

$$-\frac{(3\sin\frac{C}{2}-4\sin^3\frac{C}{2})}{\sin\frac{C}{2}} > -\frac{(3\sin\frac{B}{2}-4\sin^3\frac{B}{2})}{\sin\frac{B}{2}},$$

$$-3+4\sin^{\frac{C}{2}} > -3+4\sin^{\frac{B}{2}},$$

$$4\sin^2\frac{C}{2} > 4\sin^2\frac{B}{2},$$

$$\sin^2\frac{C}{2} > \sin^2\frac{B}{2},$$

(Since $\frac{C}{2},\frac{B}{2}\in(0,\frac{\pi}{2})$ ensures that $\sin\frac{C}{2},\frac{B}{2} > 0$)

$$\sin\frac{C}{2} > \sin\frac{B}{2},$$

$$\frac{C}{2} > \frac{B}{2},$$

(because sine is an increasing function on $\Big(0,\frac{\pi}{2}\Big)\Big)$

$$\angle C > \angle B.$$

Thus, y+v > z+V $\Rightarrow c > b$ proving the contrapositive of Hajja's result[10] so that now it can be safely stated that y+v > z+V $\Leftrightarrow c > b$. □

2.7 A. I. Fetisov

The previous proofs have relied on trigonometric identities and functions, but this proof uses the method of construction. This proof appeared in Fetisov's own book[6]. As noted before, this proof is nearly identical to the proof attributed to Descube[12]. This proof will also follow the trend of utilizing contradiction. We want to get the result of AM = CN implies AB = BC after we assume that AB *neq* BC.

We need the application of the Hinge Theorem for this proof, so a short summary of that theorem will be offered. The Hinge Theorem states that if two triangles, for example $\triangle$s AMC and CNA from Figure 29, have two congruent sides (AM = AN; $\overline{AC}$ is common). And a relationship between the included angle is established, in our example $\beta > \alpha$. Then the respective third sides have the same relationship as the angles, namely AN > CM.

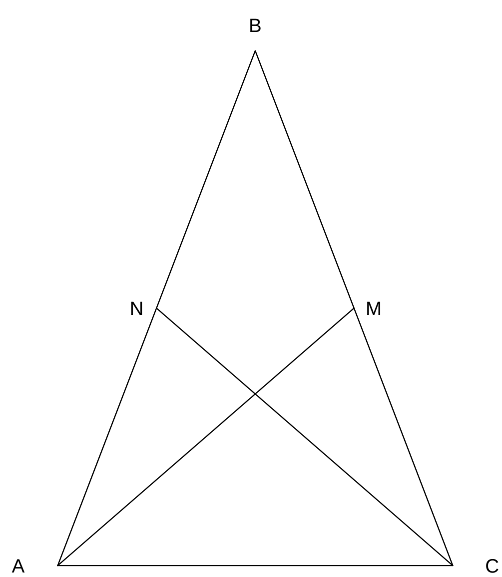

Figure 29: Initial figure for Fetisov proof

Let us say $\overline{AM}$ and $\overline{CN}$ are the angle bisectors of ∠A and ∠C. Proceeding by contradiction assume that AB > BC so $\angle C > \angle A$, using Figure 30 which is just Figure 29 with additional notation.

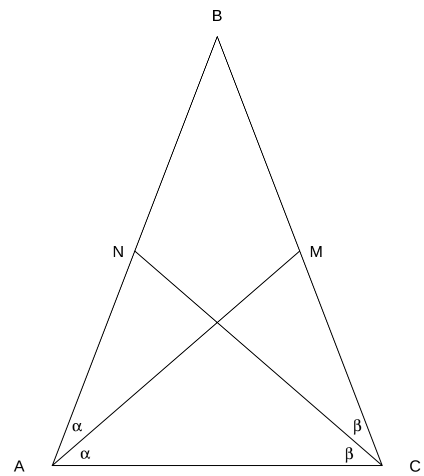

Figure 30: Addition of angle labels

Then, $2\beta > 2\alpha$ is equivalent to $\beta > \alpha$. Look at these triangles from Figure 31.

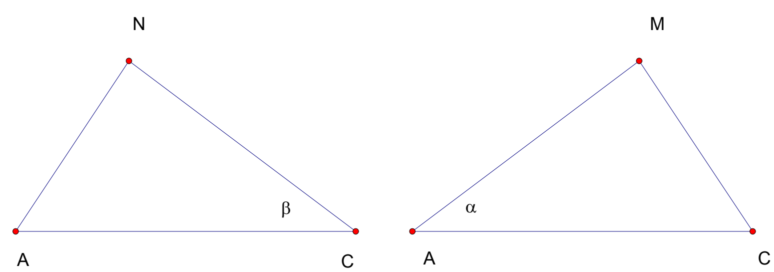

Figure 31: Decomposition for Fetisov proof

Notice that the two triangles have two sets of congruent sides. One is $\overline{AC} \cong \overline{AC}$ because it is the same side, and $\overline{AM} \cong \overline{CN}$ by hypothesis but since $\beta > \alpha$ the "Hinge

Theorem" allows us to assert that AN > CM. Now that we have established that AN > CM, we can construct and develop our contradiction.

Construct $\overline{ND}$ with the following characteristics: $\overline{ND}||\overline{AM}$ and $\overline{ND} \cong \overline{AM}$ (see Figure 32.)

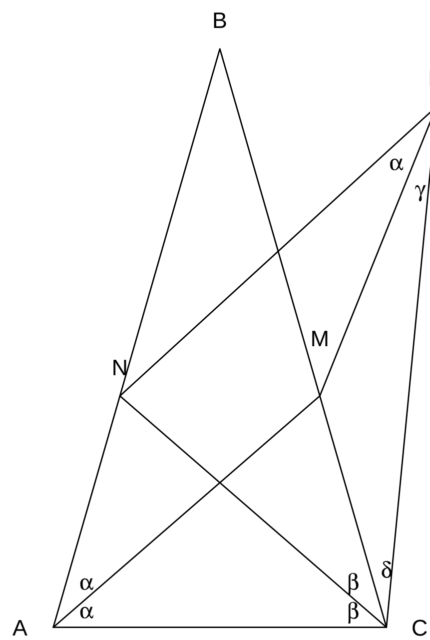

Figure 32: Complete figure for Fetisov proof

Now there are a pair of congruent and parallel sides, so ANDM is a parallelogram. Since ANDM is a parallellogram, MD = AN and $\angle\alpha \cong \angle$NDM. So, if $\overline{MD} \cong \overline{AN}$ and AN > CM then MD > CM must also be true.

Then look at Figure 33 to see that if MD > CM, then $\delta > \gamma$ is also true.

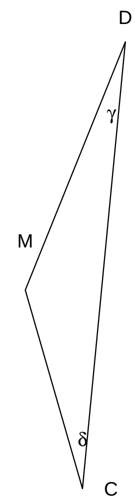

Figure 33: Triangle CMD

Then, let us look at $\triangle$CND in Figure 34.

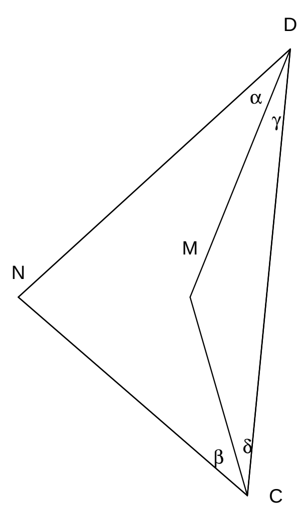

Figure 34: Triangles CND and CMD

Note that $\overline{AM} \cong \overline{CN}$ by hypothesis and $\overline{CN} \cong \overline{ND}$ by construction, so △CND is isosceles. So, $\angle NCD \cong$ ∠NDC or $\beta + \delta = \alpha + \gamma$. The result is a contradiction because $\beta > \alpha$ by assumption and $\delta > \gamma$. Therefore, $\beta + \delta > \alpha + \gamma$ should also have been true but it is not. Thus, another proof using contradiction is completed. □

2.8 Gilbert/MacDonnell

The next proof is attributed to G. Gilbert and D. MacDonnell[8]. This proof uses a property of concyclic points that we need to present initially for use later.

The specific property that we will need comes from the following proposition: If four points A, B, C, and D, are the vertices of a quadrilateral and angles α and β are congruent, then A, B, C, and D are concyclic. (Figure 35)

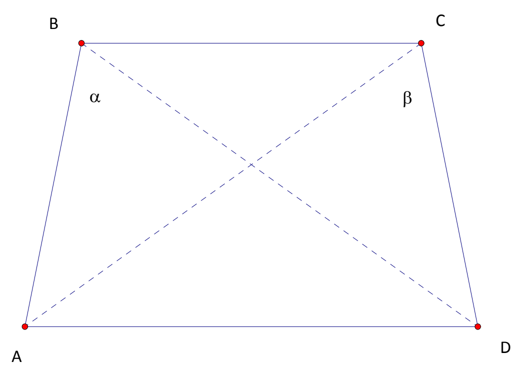

Figure 35: Concyclic points initial figure

Let A, B, C, and D be the vertices of a quadrilateral and assume that $\alpha \cong \beta$. Proceed by contradiction. Suppose that point C does not lie on the same circle as points A, B, and D. That would mean there are two distinct possibilities, either C is located in the interior of the circle (Figure 36) or C is located in the exterior of the circle (Figure 37).

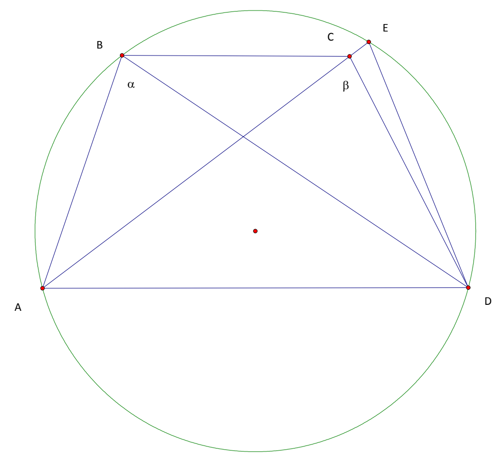

Figure 36: Concyclic points possibility 1

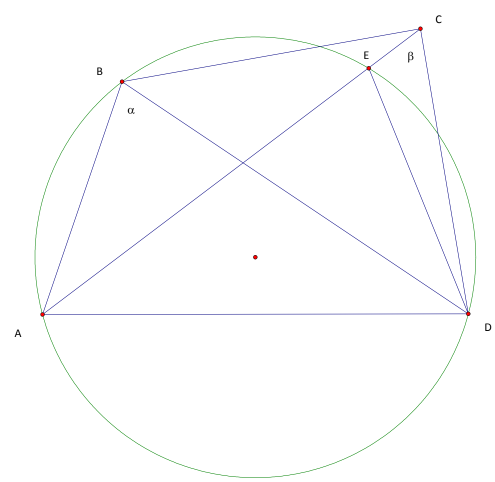

Figure 37: Concyclic points possibility 2

Let us look at the possibility that C is located in the interior of the circle first. Extend $\overline{AC}$ so that it intersects the circle and let that intersection be point E. Then construct $\overline{ED}$ to create ∠AED. But since ∠AED subtends the same arc as ∠ABD and both are inscribed (interior to the circle) then $\angle AED \cong \angle ABD$. However, by the exterior angle theorem (referring to △ECD) $\angle AED > \angle ACD$ must be true, which contradicts the original assumption. Thus point C can not be located on the interior of the circle.

Now let us look to the second possibility.

Say point C is located on the exterior of the circle. Then $\overline{AC}$ will intersect the circle at some point, say E. Construct $\overline{ED}$ to form ∠AED. As before ∠AED and ∠ABD are both inscribed and subtend the same arc; therefore, are congruent. As before we can use the Exterior Angle Theorem, but this time we apply it to △CED to illustrate that $\angle AED < \angle ACD$. Again, that contradicts the assumption that $\angle ABD = \angle ACD$. Hence, point C must lie on the circle and points A, B, C, and D are concyclic. □

Now, let us develop the Gilbert/MacDonnell proof of the Steiner-Lehmus Theorem. Assuming that $\overline{AE}, \overline{CD}$ are the angle bisectors and $\overline{AE} \cong \overline{CD}$, we wish to show that △ABC is isosceles. (Figure 38) Proceeding by contradiction, suppose that △ABC is not isosceles, say BC < AB. Then, $\angle BAC <$ ∠BCA, which in turn implies that $\alpha < \beta$.

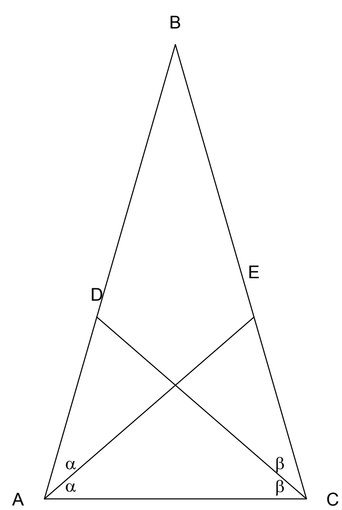

Figure 38: Initial figure for Gilbert and MacDonnell proof

Construct $\overline{CF}$ such that $\angle FCD \cong \alpha$ and then connect $\overline{DF}$ to create quadrilateral DFCA, illustrated by Figure 39.

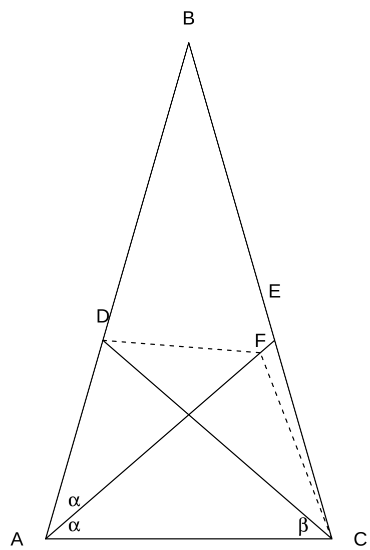

Figure 39: Construction for Gilbert and MacDonnell proof

Then AF < AE. Construct the circle that contains points D, F, C and A, illustrated by Figure 40.

Since $\alpha < \beta$, it follows that $\alpha + \alpha < \alpha + \beta$ is also true. So, DC < AF and since AF < AE we get DC < AF < AE or $\overline{DC} < \overline{AE}$, but $\overline{DC} \cong \overline{AE}$ was a central part of the hypothesis, thus a contradiction has been achieved. $\square$

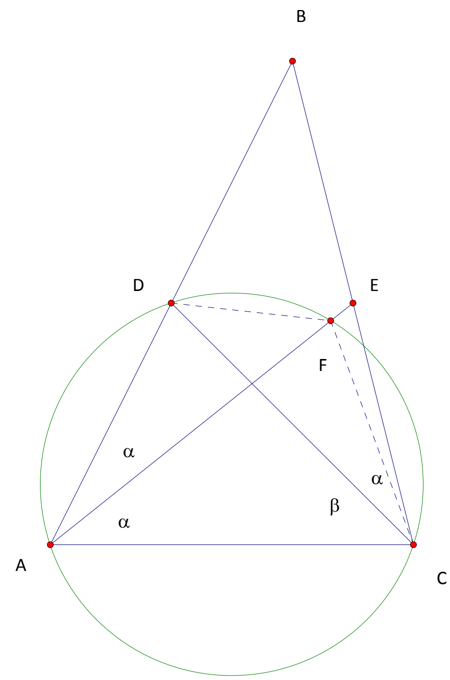

Figure 40: Quadrilateral DFCA with circle

Interestingly, this proof was also published by Martin Gardner in *Scientific American*[7]. As indicated earlier, the Steiner-Lehmus proof has long drawn the attention of many mathematicians and, in 1963, Martin Gardner included the theorem in his regular column. Hundreds of proofs were sent in response and this was the proof that Gardner selected as his favorite[7]. Even more interesting, this proof is very near to the proof that Steiner himself published in the 1840s[7].

2.9 Berele/Goldman

This proof is offered in *Geometry: Theorems and Constructions* by Allan Berele and Jerry Goldman[2]. It is also a proof by contradiction but, since it is not part of a published article, it has very little in the way of an introduction. It is under the heading, "An Old Chestnut (The Steiner-Lehmus Theorem)" with a short note mentioning the popularity this theorem enjoys in mathematical puzzles.

Let us begin by saying that $\overline{EC}$ and $\overline{BD}$ are angle bisectors of ∠C, ∠B respectively; $\overline{EC} \cong \overline{BD}$ in figure 41. This proof is also by contradiction, so let us say that AB $\neq$ AC. Then $\angle B \neq \angle C$, and without loss of generality, assume that $\angle B >$ ∠C.

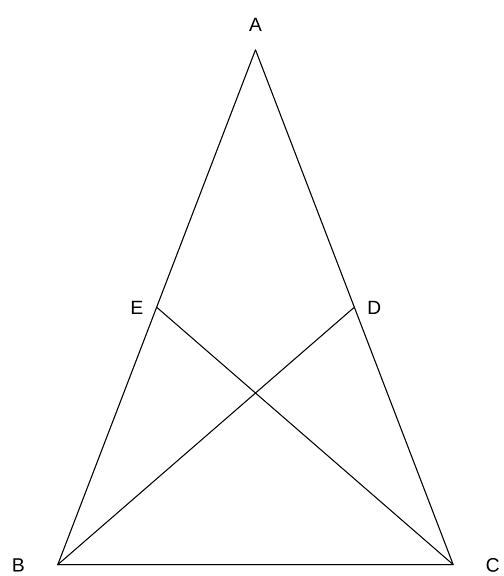

Figure 41: Berele/Goldman, initial figure

Since $\angle B >$ ∠C, then ∠ABD $= \frac{1}{2}\angle B > \frac{1}{2}$∠C is true and we can establish that a point F on $\overline{AC}$ exists so that $\overline{BF}$ can be constructed in such a way so that ∠FBD = $\frac{1}{2}$∠C. (See Figure 42 for illustration.)

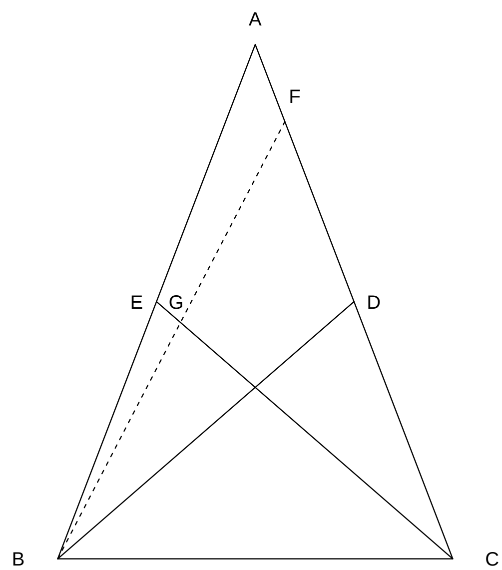

Figure 42: Berele/Goldman construction figure

Let the intersection of $\overline{BF}$ and $\overline{EC}$ be point G and notice that

a.) ∠GFC ≅ ∠BFD because they are the same angle with different notations.

b.) ∠FBD ≅ ∠FCG by construction. Ultimately, the conclusion is that △BFD ∼ △CFG by AA similarity.

Employing the characteristics of similarity, the ratio of the sides of those two triangles would be

$$\frac{CG}{BD} = \frac{CF}{BF}.$$

Since $\overline{CE} \cong \overline{BD}$ and CG + GE = CE; $\frac{CG}{BD} = \frac{CG}{CE} = \frac{CG}{CG+GE} < 1$ is true.

And if $\frac{CG}{BD} < 1$, then $\frac{CF}{BF} < 1$ which implies CF $<$ BF, leading to $\angle C > \angle$B, a contradiction to the assumption $\angle C < \angle$B.

The entire process can be repeated assuming $\angle C > \angle$B and reaching a similar contradiction of $\angle C < \angle$B. We complete our short study with a final example of the most prevalent style, contradiction. $\square$

2.10 The Converse

After all of these proofs of the Steiner-Lehmus Theorem, some of which are tedious and nearly all are lengthy contradictions, it seems beneficial to provide a proof of the converse of the theorem, namely, "If a triangle is isosceles, its angle bisectors are congruent."

Back to our standard figure, using Figure 43.

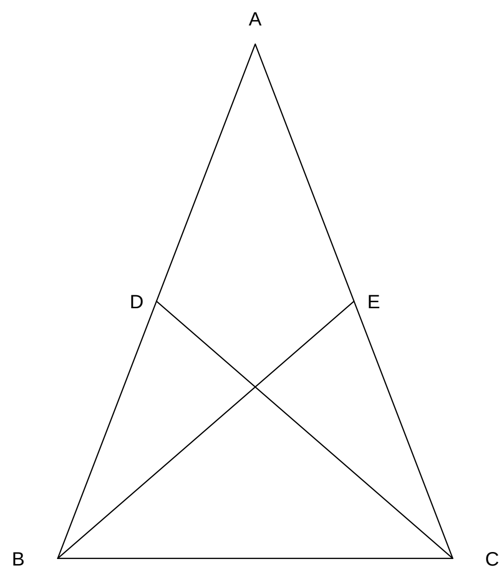

Figure 43: Typical labeling for Isosceles Triangle

Now the hypothesis is $\overline{AB} \cong \overline{AC}$ and $\overline{BE}, \overline{CD}$ are the angle bisectors of ∠B, ∠C respectively. The goal is to show that $\overline{BE} \cong \overline{CD}$.

Look at the following two triangles in Figure 44.

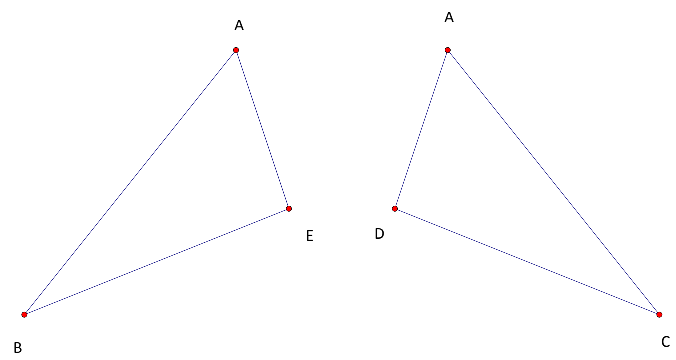

Figure 44: Decomposition into ABE and ACD

It is clear that $\angle$A is present in both triangles and $\overline{AB} \cong \overline{AC}$ by hypothesis. Since $\triangle$ABC is isosceles, $\angle B \cong \angle C$ which means that $\frac{\angle B}{2} \cong \frac{\angle C}{2}$ or $\angle$ABF $\cong$ $\angle$ACD. Thus, $\triangle$ABE $\cong$ $\triangle$ACD by ASA (Angle-Side-Angle) and $\overline{BE} \cong \overline{CD}$ by CPCTC.

It is hard to relate this proof of the converse with all the other proofs presented. Specifically, a relationship that is as near as theorem and converse seems almost inconceivable. The converse can be proven with approximately three lines, a direct argument and a single figure breakdown, whereas the Steiner-Lehmus Theorem arguably has no direct line of reasoning and generally requires some construction or trigonometric argument and certainly more that three lines.

3 SIMILARITIES, DIFFERENCES, AND GROUPINGS

In the previous pages there are twenty-one proofs are mentioned. Some are mentioned in more detail than others simply because they require a more thorough study. These are an estimated 25% of the proofs in existence for the Steiner-Lehmus Theorem. It is unusual for a single theorem to garner such attention. One aspect that has drawn such attention has been hinted at throughout the paper, namely the direct versus indirect argument. This seemingly innocuous theorem has no uncontested direct proof.

The direct versus indirect argument is long and well-documented. In his column in *Scientific American*, Martin Gardner[7] was one of several scholars who expressed a fascination with this theorem. Martin Gardner was not alone in this respect. The top most link in an internet search is a Wikipedia page where the third paragraph links the impossibility of a direct proof to an article by John Conway[14]. Within the next few links you will find several academic pages discussing the particulars of direct/indirect proofs of the Steiner-Lehmus Theorem. The discussion spirals out from there to include even more mathematicians, prominent websites, and publications. This direct/indirect discussion can be continued much longer but it is a digression from the purpose of this thesis. Which is to categorize and relate several proofs of the Steiner-Lehmus Theorem and their styles to one another. In that spirit, when the proofs presented here are considered direct it will not be in the strictest sense. We will simply accept a direct style as just that, direct.

There are several ways to classify these proofs. It has already been mentioned more than a few times, let us group the proofs into direct and indirect first. There are four direct proofs: the Beran proof[1], Sastry's proof using cevians[12], and two of the proofs presented in the Oláh-Gál/Sándor paper[11], specifically the proofs attributed to Plachkey and Cristescu. Of the fifteen proofs detailed in this paper, the remaining eleven are indirect. Two of those proofs are essentially the same, the proofs attributed to Descube[12] and Fetisov[6]. Even if we remove one, we are still left with ten proofs by contradiction. That is easily a majority of even our small sample.

It is interesting to note that there are two more proofs that are very similar. The proofs by Plachkey and by Seydel/Newman are nearly identical in argument but one is direct (Plachkey[10]) and the other is by contradiction (Seydel/Newman[13]). Because they differ in that distinct and important way, it is not really possible to remove either one from the work or to even let one overshadow the other.

There are a few more proofs mentioned in passing but it is difficult to place them in either the direct or indirect category. Hajja expanded his second proof to include six additional statements with only minimal support[10]. One of these is the conjecture based on an iterative computer program and does not fit either the direct mode or the indirect mode as it is not a proof anyway. As noted earlier, the remaining five of these results take Hajja's proof and build on that conclusion algebraically. At points where Hajja has a definitive and strictly positive formula on the right, he will use that as a benchmark for an additional form. At heart each of these is an indirect proof because the original proof was by contradiction, but this thesis has not included much more than just topical information on these proofs because they are all "children" of a single proof that was detailed.

The style of contradiction seems to be the most prevalent way to prove the Steiner-

Lehmus Theorem, which makes sense in that almost everyone with some experience in proving uses contradiction because it is one of the easiest methods. This is even more so in our case since, upon hearing this question, most mathematicians would respond with a 'how could it be otherwise' argument.

When assigning these proofs into body styles, all the proofs fit into two groups. The group with construction bodies contained eight of the fifteen proofs. That is slightly more than half. Leaving seven proofs with trigonometric body styles to create the second group.

When the proof were separated based on construction versus trigonometric body styyle, it became apparent that once an author completed a proof with either construction or trigonometry, he tended to stick with the same technique. Out of the authors who have papers with multiple proofs presented, all of Sastry's[12] proofs have a construction in them and all of Hajja's[9,10] and Oláh-Gál/Sándor's[11] proofs are trigonometry based. Even though all of Sastry's[12] proofs have a construction in them, he has three that are direct and one proof by contradiction. Hajja's[9,10] proofs all have trigonometric bodies but he has two proofs by contradiction and two direct. Oláh-Gál/Sándor[11] also have trigonometry proofs with two proofs by contradiction, one direct proof and one direct proof using the contrapositive. The obvious conclusion is that an author tends to adopt with a certain body style (construction or trigonometry) but will still use a variety of techniques (contradiction, direct, etc.)

Part of the original plan of analysis was to summarize the trigonometric identities used, but the list was quite long and not easily narrowed to a few key identities. Also planned was to try and limit the scope of the construction proofs to a few basic structures or theorems, but that was also too broad to be effective or meaningful. The reason for mentioning this is because both of these problems are a large part of the reason the Steiner-Lehmus Theorem has been and still it so popular. This theorem has proofs with breadth and variety. It is not the simple problem it first appears to be. There multiple approaches possible to reach the proof of the theorem, lending the theorem to different thinking and learning styles.

The Steiner-Lehmus Theorem is not a problem that will intimidate students or challenge seekers. It is simple in appearance. It is elegant and streamlined. Its converse is elementary, so it is easy for students to believe that the theorem is true and therefore possible. Overall, this is a very interesting theorem with proofs that range from the simple to the complex. In other words, it is an argument that can be enjoyed by all levels of mathematical talent. Any theorem/proof combination that is as inclusive as the Steiner-Lehmus Theorem will enjoy a long and productive life in the spotlight.

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