5-2013

Very Cost Effective Partitions in Graphs

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Very Cost Effective Partitions in Graphs

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

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May 2013

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Keywords: very cost effective partition, Cartesian product, cactus graph

ABSTRACT

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For a graph $G = (V, E)$ and a set of vertices $S \subseteq V$, a vertex $v \in S$ is said to be very cost effective if it is adjacent to more vertices in $V \setminus S$ than in S. A bipartition $\pi = \{S, V \setminus S\}$ is called very cost effective if both S and $V \setminus S$ are very cost effective sets. Not all graphs have a very cost effective bipartition, for example, the complete graphs of odd order do not. We consider several families of graphs G, including Cartesian products and cacti graphs, to determine whether G has a very cost effective bipartition.

DEDICATION

I lovingly dedicate this thesis to my mother, Larisa Vasilieva, for her kindness and endless support throughout my whole life. I could never thank you enough for your love and selflessness. You have always been my inspiration and I am eternally grateful to have you as my parent.

ACKNOWLEDGMENTS

This thesis would not have been possible without the guidance and the help of several individuals who in one way or another contributed and extended their valuable assistance in the preparation and completion of this work.

First and foremost, I would like to express my sincere gratitude to my advisor, Dr. Teresa Haynes, for the continuous support of my study and research, for her motivation, and for her patience. I am thankful for her enthusiasm and dedication to graph theory. It is a great honor to work with Dr. Haynes and I will always remember this valuable experience. I would like to also thank Dr. Stephen Hedetniemi for his many ideas that made this thesis possible. In addition, a thank you to my committee members, Dr. Robert Beeler and Dr. Anant Godbole, for their patience and support and would like to laud them for their inspirational teaching and excellence in their respective fields.

It gives me great pleasure in acknowledging the support and help of professor from my previous university, Yuriy Zakhariychenko. His outstanding ways of teaching mathematics has inspired me to pursue a career in that field. I would also like to thank my family and friends for their encouragement and support during my life and, in particular, my graduate school career. A special thank you to my best friend Sophia. I cannot imagine having more kind, sincere, thoughtful and supportive friends than mine.

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1 INTRODUCTION

1.1 Basic Terminology of Graph Theory

As defined in [4], a graph $G = (V, E)$ is a nonempty, finite set of elements called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges* . The vertex set of G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. When there is no risk of ambiguity, these are denoted V and E, respectively. In Figure 1, we have an example of a graph.

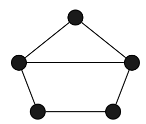

Figure 1: House graph

In this thesis, we will be studying *simple graphs*, that is, graphs for which there exists at most one edge between any two vertices and for which the endpoints of any edge are distinct. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent* vertices, while u and e are *incident*, as are v and e. Two adjacent vertices are called *neighbors* of each other. The *degree* of a vertex v in a graph G is the number of edges incident to v. A vertex of degree 0 in G is called an *isolated vertex*, while a vertex of degree of 1 is referred to as a *leaf* or *pendant*. A vertex v is said to be *even* or *odd*, according to whether its degree in G is even or odd.

Given any graph G, the *order* of G, denoted $n(G) = |V(G)|$, is the number of vertices in G. The *size* of G, denoted $m(G) = |E(G)|$, is the number of edges in G. If there is no risk of ambiguity, these are written as n and m, respectively. For example, for the graph G in Figure 1, the order $n(G) = 5$ and the size $m(G) = 6$. A graph of order 1 is called a *trivial graph*, and a graph of order at least 2 is called a *nontrivial graph*. A graph of size 0 is called an *empty graph*. A nonempty graph has one or more edges. A *u-v walk* W of G is a finite, alternating sequence $W : u = u_0, e_1, u_1, e_2, ..., u_{k-1}, e_k = v$ of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1}u_i$ for $i = 1, 2, ..., k$. The number k (the number of occurrences of edges) is called the *length* of W. A u-v walk is *closed* or *open* depending on whether $u = v$ or $u \neq v$. A *u-v trail* is a u-v walk in which no edge is repeated. A *u-v path* is a u-v walk in which no vertex is repeated. A nontrivial closed trail of a graph G is referred to as a *circuit* of G. A circuit $v_1, v_2, ..., v_n, v_1 (n \geq 3)$ whose n vertices v_i are distinct is called a *cycle*. Paths on n vertices are denoted P_n and cycles on n vertices are denoted C_n. A *wheel graph* W_n on n vertices is a graph consisting of a cycle C_n and a single vertex which is adjacent to all vertices in the cycle (see Figure 2).

A graph of n vertices is *complete* if every two of its vertices are adjacent. This is denoted K_n. A graph G is *k-partite*, $k \geq 1$, if it is possible to partition $V(G)$ into k subsets $V_1, V_2, ..., V_k$ (called partite sets) such that every element of $E(G)$ joins a vertex of V_i to a vertex of $V_j, i \neq j$. For $k = 2$, such graphs are called *bipartite*. A *complete k-partite graph* G is a k-partite graph with partite sets $V_1, V_2, ..., V_k$ having the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then $uv \in E(G)$. A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = r$ and $|V_2| = s$ is denoted by $K_{r,s}$. The graph $K_{1,s}$ is called a *star*.

A vertex v of a connected graph G is called a *cut vertex* of G if its removal produces a disconnected graph. A nontrivial connected graph with no cut vertices is called a *nonseparable graph.* A labeled graph H is a *subgraph* of a labeled graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *block* of a graph G is a maximal nonseparable subgraph of G. A *cactus graph* is a connected graph where each block is either an edge or a cycle. A cactus graph having one cycle is called a *unicyclic graph* and a connected cactus graph with no cycle is called a *tree.*

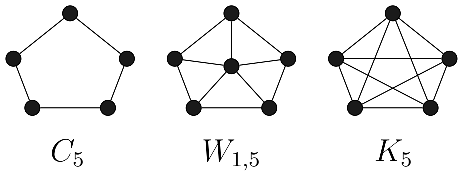

Figure 2: C_5, W_5, K_5

The *complement* of G, denoted $\overline{G}$, is a graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{ab \mid ab \notin E(G)\}$. In other words, the complement G of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in G if and only if these vertices are not adjacent in G. This means that both G and its complement G have the same vertices, but G has precisely the edges that G lacks.

For any vertex $v \in V(G)$, the *open neighborhood* of v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, its *open neighborhood* $N_G(S) = \cup_{v \in S} N_G(v)$, and its *closed neighborhood* $N_G[S] = N_G(S) \cup S$. These are sometimes denoted $N(v)$, $N[v]$, $N(S)$ or $N[S]$, respectively, if there is no risk of ambiguity.

A set $S \subseteq V(G)$ is a *dominating set* (abbreviated DS) if $N[S] = V(G)$. In other words, every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. Every graph has a DS since $V(G)$ is such a set. Among all DS of G, a DS with minimum cardinality is said to be a *$\gamma(G)$-set*. Its cardinality is known as the *domination number* of G and it is denoted by $\gamma(G)$. An *independent dominating set*, abbreviated IDS, of a graph G is a set $S \subseteq V(G)$ that is a DS of G and is an independent set. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an IDS of G. An IDS of G of cardinality $i(G)$ is called an *$i(G)$-set*.

1.2 Very Cost Effective Partitions in Graphs

Very cost effective partitions in graphs were first introduced in [11] and were motivated by the studies of unfriendly partitions (i.e. [1, 8, 19, 20]). According to [11], a vertex v in a set S is said to be *cost effective* if it is adjacent to at least as many vertices in $V \setminus S$ as in S. A vertex v is *very cost effective* if it is adjacent to more vertices in $V \setminus S$ than in S. A set S is a (very) cost effective set if every vertex $v \in S$ is (very) cost effective.

A bipartition $\pi = \{S, V \setminus S\}$ is called *cost effective* if (i) for every vertex $i \in S$, $|N(i) \cap (V \setminus S)| \geq |N(i) \cap S|$, and (ii) for every vertex $j \in V \setminus S$, $|N(j) \cap S| \geq |N(j) \cap (V \setminus S)|$. Thus, given a cost effective partition $\pi = \{S, V \setminus S\}$, every vertex in S is cost effective with respect to the set S, and every vertex in $V \setminus S$ is cost effective with respect to the set $V \setminus S$.

Theorem 1.1 [11] *Every connected graph G of order $n \geq 2$ has a cost effective partition.*

Proof. Let $\pi = \{S, V \setminus S\}$ be any bipartition of $V(G)$ having the property that the number of edges between S and $V \setminus S$ is a maximum. To show that π is a cost effective partition, we assume to the contrary, that it is not. Then, without loss of generality, we may assume S is not cost effective. Hence, there is a vertex, say $v \in S$, having more neighbors in S than in $V \setminus S$. In this case, moving v to $V \setminus S$ will increase the number of edges between S and $V \setminus S$, contradicting the assumption that π has a maximum number of edges between the two sets. □

A bipartition $\pi = \{S, V \setminus S\}$ of the vertices V of a graph $G = (V, E)$ is *very cost effective* if (i) for every vertex $u \in S$, $|N[u] \cap (V \setminus S)| \geq |N[u] \cap S|$, and (ii) for every vertex $v \in V \setminus S$, $|N[v] \cap S| \geq |N[v] \cap (V \setminus S)|$. Equivalently, every vertex in S is very cost effective with respect to S, and every vertex in $V \setminus S$ is very cost effective with respect to $V \setminus S$. A graph G is called *very cost effective* if it has a very cost effective bipartition.

Thus, we have seen that every nontrival graph G has a cost effective bipartition, but not every graph has a very cost effective bipartition. In this work, we study several families of graphs and determine whether they have a very cost effective bipartition.

1.3 Motivation and Applications

In terms of an application, we assume that maintaining edges in a network has an associated cost, and thus they should be used effectively. We assume that an edge between a vertex in a set S and a vertex in $V \setminus S$ is being used effectively, while an edge between two vertices in S is not necessarily being used cost effectively. Thus, a vertex is considered to be cost effective if at least as many edges incident to it are being used cost effectively as not.

Also, (very) cost effective partitions are motivated by business applications. For example, a company that offers service to both customers and employees would want to be certain to make more money than it is spending. Let the edges inside S represent services that employees are using (internal cost), and let edges between S and $V \setminus S$ represent income from customers paying the company for services. If the company allows employees to use the services it offers for free or at a discounted price, then the company needs to have more edges between S and $V \setminus S$ to be able to make a profit. Thus, for each vertex, $v \in S$ it would be necessary for v to have at least as many neighbors in $V \setminus S$ as in S in order for the company to make money.

2 LITERATURE SURVEY

2.1 Unfriendly Partitions

Very cost effective partitions are derived from the study of unfriendly partitions of graphs, as follows. Let C be a two-coloring of the vertices of a graph G, $C : V \to \{Red, Blue\}$. For every vertex $i \in V$, define $B(i) = \{j \in N(i), C(j) = Blue\}$ and $R(i) = \{j \in N(i), C(j) = Red\}$. Similarly, define $B(V) = \{j \in V, C(j) = Blue\}$ and $R(V) = \{j \in V, C(j) = Red\}$. A two-coloring produces a bipartition of V, $\pi = \{B(V), R(V)\}$. Given such a bipartition π, we say that an edge $uv \in E$ is *bicolored* if $C(u) \neq C(v)$. A bipartition π is called an *unfriendly partition* if every vertex $i \in B(V)$ has at least as many neighbors in $R(V)$ as it does in $B(V)$, and every vertex $j \in R(V)$ has at least as many neighbors in $B(V)$ as it does in $R(V)$. That is, if $C(i) = Blue$, then $|B(i)| \leq |R(i)|$, and if $C(i) = Red$, then $|R(i)| \leq |B(i)|$. These types of partitions were defined and studied by Borodin and Koshtochka [3], Aharoni, Milner, and Prikry [1], and Shelah and Milner [20], who called these *unfriendly partitions.* They observed the following, simple proof of which we provide here since it formed the basis for Theorem 1.1.

Theorem 2.1 [1] *Every finite connected graph G of order $n \geq 2$ has an unfriendly partition.*

Proof. Let $\pi = \{B(V), R(V)\}$ be any bipartition of $V(G)$ having the property that the number of bicolored edges is a maximum. It is easy to see that such a partition is unfriendly. If π is not an unfriendly partition then there must exist a vertex, say $v \in R(V)$, having more red neighbors than blue neighbors. In this case, moving v to

$B(V)$ will increases the number of bicolored edges, contrary to the assumption that π has a maximum number of bicolored edges. $\square$

Unfriendly partitions have shown up indirectly in several other lines of research. In [5, 6] the concept of α-domination in graphs is defined and studied. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called an *α-dominating set* if for every vertex $v \in V \setminus S$, $|N(v) \cap S|/|N[v]| \geq \alpha$, where $0 \leq \alpha < 1$. In the case where $\alpha \geq 1/2$ every vertex in $V \setminus S$ meets the *unfriendly condition* in that it has at least as many neighbors in S as it has in $V \setminus S$. However, no unfriendly condition is imposed on the vertices in S.

Similarly, in [7, 12, 13, 15, 17] global offensive alliances in graphs are defined and studied. A set $S \subseteq V$ of vertices is called a *global offensive alliance* if for every vertex $v \in V \setminus S$, $|N(v) \cap S| \geq |N[v] \cap (V \setminus S)|$. As with α-domination, if S is a global offensive alliance, then every vertex $v \in V \setminus S$ satisfies the unfriendly condition, in that it has at least as many neighbors in S as it has in $V \setminus S$, if you count the vertex v as one of its own neighbors. But no unfriendly condition is imposed on the vertices in S.

A partition that is in some sense dual to an unfriendly partition is a bipartition $\pi = \{B(V), R(V)\}$ called a *satisfactory partition* such that every vertex $i \in B(V)$ has at least as many neighbors in $B(V)$ as it does in $R(V)$, and every vertex $i \in R(V)$ has at least as many neighbors in $R(V)$ as it has in $B(V)$. That is, if $C(i) = Blue$, then $|B(i)| \geq |R(i)|$, and if $C(i) = Red$, then $|R(i)| \geq |B(i)|$. Satisfactory partitions have been studied in [8, 9, 10, 19]. However, unlike unfriendly partitions, not every graph has a satisfactory partition. In fact, it is an NP-complete problem to decide if an arbitrary graph has a satisfactory partition [2].

2.2 Cost Effective Domination

A related topic to very cost effective partitions is (very) cost effective domination. Cost effective domination was introduced in [11] and studied further in [14, 16].

Definition 2.2 *A set S is a cost effective dominating set if S is both a cost effective set and a dominating set.*

Definition 2.3 *The cost effective domination number $\gamma_{c\epsilon}(G)$ of a graph G equals the minimum cardinality of a cost effective dominating set in G. The upper cost effective domination number $\Gamma_{c\epsilon}(G)$ equals the maximum cardinality of a cost effective dominating set in G. A cost effective set of G with cardinality $\gamma_{c\epsilon}(G)$ is called a $\gamma_{c\epsilon}$-set of G. The very cost effective domination number and the upper very cost effective domination number are defined similarly.*

It should be pointed out that while the property of being a dominating set is *superhereditary*, that is, every superset of a dominating set is also a dominating set, the property of being a cost effective dominating set is not superhereditary. This explains why the definition of the upper cost effective domination number does not include the word "minimal" as it does in the definition of the upper domination number. Without the word *minimal* in the definition of $\Gamma(G)$, the value of $\Gamma(G)$ would equal $n = |V|$ for all graphs.

Proposition 2.4 [11] *Every independent dominating set S in a graph G is a cost effective dominating set.*

Proof. Let $S \subseteq V$ be an independent dominating set of G. Then for each $v \in S$, $|N(v) \cap S| = 0 \leq |N(v) \cap (V \setminus S)|$. Thus, every vertex $v \in S$ is cost effective. $\square$

Corollary 2.5 [11] *For any graph G, $\gamma(G) \leq \gamma_{ce}(G) \leq i(G)$.*

The *corona* of graphs G and H, denoted $G \circ H$, is the graph formed from one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex in $V(G)$ is adjacent to every vertex in the i^{th} copy of H. We note that all four combinations of the inequalities in Corollary 2.5 are possible (see Figure 3 where the darkened vertices represent a γ_{ce}-set). In particular, for $\gamma(G) < \gamma_{ce}(G) < i(G)$, let G be the corona $G = K_t \circ \overline{K}_{t-3}$, for $t \geq 5$. Then $\gamma(G) = t$, $\gamma_{ce}(G) = 2t - 4$, and $i(G) = t^2 - 4t + 4$. Figure 1(a) illustrates $K_5 \circ \overline{K}_2$. For graphs G having $\gamma(G) < \gamma_{ce}(G) = i(G)$, let G be the tree $K_{1,3} \circ \overline{K}_2$. That is, the tree formed by adding two leaf vertices adjacent to each vertex of the claw $K_{1,3}$. This graph has $\gamma(G) = 4 < \gamma_{ce}(G) = i(G) = 5$ (see Figure 3(b)). The cycles C_n have $\gamma(C_n) = \gamma_{ce}(C_n) = i(C_n) = \lceil n/3 \rceil$ (see Figure 3(c) for the example, C_{10}). The *doublestar* is the tree having exactly two non-leaf vertices. Finally, for a graph G with $\gamma(G) = \gamma_{ce}(G) < i(G)$, let G be the doublestar where each non-leaf vertex is adjacent to $k \geq 2$ leaves, that is, $G = K_2 \circ \overline{K}_k$. Here $\gamma(G) = \gamma_{ce}(G) = 2 < k + 1 = i(G)$ (see Figure 3(d)).

The trees in Figure 3(b) show that not all trees have cost effective γ-sets. Similarly, we can consider characterizing the graphs obtaining equalities in any of the bounds. Also, [14] provides us with upper bound on the cost effective domination number of trees and characterizes the trees obtaining this bound.

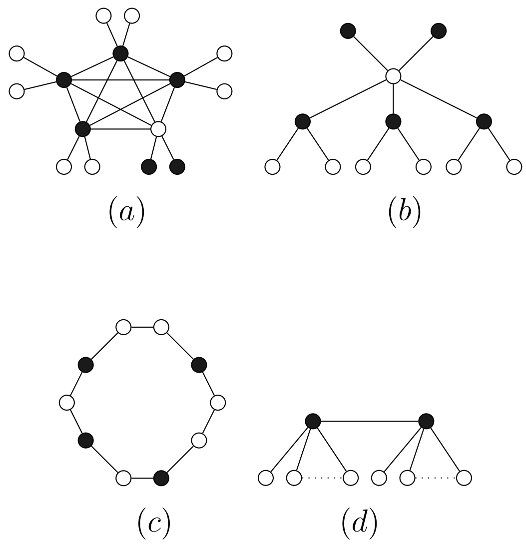

Figure 3: Inequalities of Corollary 2.5.

Theorem 2.6 [14] *If T is a tree with $\gamma(T) \geq 3$, then $\gamma(T) \leq \gamma_{ce}(T) \leq 2\gamma(T) - 3$, and these bounds are sharp.*

In order to characterize the trees obtaining the bound, in [14] the authors define the family F of trees T_t, which are obtained from the star $K_{1,t}$ with center x and leaves $x_1, x_2, ..., x_t$ as follows. Add exactly $t-1$ new vertices adjacent to x, and for $1 \leq i \leq t$, add at least $t-1$ new vertices adjacent to x_i.

In general, we use notation and terminology of [11]. We will employ the following terminology and results from [16].

Definition 2.7 [16] *The minimum (very) cost effectiveness of a graph G, denoted $(vce(G))$, $ce(G)$ equals the minimum order of a maximal (very) cost effective set in G, and the maximum (very) cost effectiveness of G, $(VCE(G))$, $CE(G)$ equals the maximum cardinality of a (very) cost effective set in G.*

Proposition 2.8 [16] *For any connected graph G of order $n > 1, 1 \leq vce(G) \leq VCE(G) \leq n-1$, and these bounds for $vce(G)$ and $VCE(G)$ are sharp.*

Proof. Trivially, $vce(K_1) = vce(K_3) = 1$. In order to show that the upper bound for $VCE(G)$ is sharp note that for the star $K_{1,n}$ of order $n+1, VCE(K_{1,n}) = n$. $\square$

It is natural to ask: for which classes of graphs is $vce(G) = 1$ and for which classes of graphs is $VCE(G) = n-1$?

Theorem 2.9 [16] *Let G be a connected graph of order $n > 1$. Then (i) $vce(G) = 1$ if and only if $G = K_1 + (rK_2 \cup sK_1)$, for $r + s \geq 1$, and (ii) $VCE(G) = n-1$ if and only if $G = K_{1,n_1}$.*

Proposition 2.10 [16] *Every independent set S in a graph G without isolated vertices is a very cost effective set.*

The proof is similar to a proof of Proposition 1.4.

Observation 2.11 *All bipartite graphs are very cost effective.*

Corollary 2.12 [16] *For any connected graph G of order $n > 1, VCE(G) \leq CE(G)$.*

An interesting relation between very cost effective sets and the dominating sets is established in the next proposition.

Proposition 2.13 [16] *Every maximal very cost effective set in a connected graph G is a dominating set.*

Proof. Let $S \subseteq V$ be a maximal very cost effective set, and assume that S is not a dominating set. Let $v \in V \setminus N[S]$ be a vertex that is not dominated by any vertex in S. Then it follows that $S \cup \{v\}$ is a very cost effective set since $|N(v) \cap S = 0 < |N(v) \cap (V \setminus S)|$, contradicting the assumption that S is a maximal very cost effective set. □

We have already discussed the properties of cost effective domination. At this point, we turn our attention to very cost effective domination since it is more closely related to the field of our research. Thus, [16] suggests the following definitions.

Definition 2.14 *A set S is a very cost effective dominating set if S is both a very cost effective set and a dominating set.*

Definition 2.15 *The very cost effective domination number of a graph G, denoted $\gamma_{vce}(G)$, equals the minimum cardinality of a very cost effective dominating set in G. The upper very cost effective domination number $\Gamma_{vce}(G)$ equals the maximum cardinality of a minimal dominating set in G that is very cost effective.*

It should be pointed out that while the property of being a dominating set is superhereditary, that is, every superset of a dominating set is also a dominating set, the property of being a very cost effective dominating set is not superhereditary. It is easy to see that every isolate-free graph has a very cost effective dominating set since every maximal independent set is both dominating and very cost effective.

Proposition 2.16 [16] *For any graph G without isolated vertices, $\gamma(G) \leq \gamma_{ce}(G) \leq \gamma_{vce}(G) \leq \{vce(G), i(G)\}$, and all of these inequalities can be strict.*

Also, in terms of Γ we have the following inequalities.

Corollary 2.17 [16] *For any connected graph G without isolated vertices,* $\Gamma_{vce}(G) \leq \Gamma_{ce}(G) \leq \Gamma(G)$*, and these bounds are sharp.*

2.3 Variations of the Aharoni, Milner, Prikry Theorem

In Section 2, we gave a simple proof of Theorem 2.1 that every graph of order $n \geq 2$ has an unfriendly partition. This theorem can be represented equivalently by the following algorithm, which was given in [1].

Algorithm Cost Effective 2-Coloring

Input: A graph G of order $n \geq 2$.

Output: A partition $V(G)$ into two cost effective sets.

1. Arbitrarily color the vertices red and blue.

2. **While** there exists a vertex $v \in V$ having more neighbors of its own color than the other color **do** change v's color to the other color **od**.

This algorithm must terminate, since every change of color increases the number of bi-colored edges. When this algorithm is finished, the set of vertices colored red and the set of vertices colored blue will both be cost effective sets. It is easy to see that one of these two sets must have cardinality at least $n/2$.

Corollary 2.18 [16] *For any graph of order* $n \geq 2, CE(G) \leq n/2$.

The following results hold for odd-regular graphs. They represent a family of very cost effective graphs.

Corollary 2.19 [16] *The vertices of every graph of order $n \geq 2$, in which every vertex has odd degree, can be partitioned into two maximal very cost effective sets.*

Corollary 2.20 [14] *If G has a $\gamma_{ce}(G)$-set that consists of only odd vertices, then $\gamma_{ce}(G) = \gamma_{vce}(G)$.*

Note that in particular, $\gamma_{ce}(G) = \gamma vce(G)$ for cubic graphs.

We have previously observed that every nontrivial graph can be partitioned into two cost effective sets, but not every graph can be partitioned into two very cost effective sets, for example the odd order complete graphs K_{2n+1}. However most graphs can be partitioned into three very cost effective sets, using the following simple algorithm given in [16].

Algorithm Very Cost Effective 3-Coloring (VCE3)

Input: A graph G of order $n \geq 3$ having no isolated vertices.

Output: A partition $V(G)$ into three non-empty, very cost effective sets.

1. Color one vertex red, one vertex white and one vertex blue. Then,
arbitrarily color each remaining vertex red, white or blue.

2. **While** there exists a vertex $v \in V$ that has more neighbors of its own color
than it has of one of the other two colors **do** change v's color to the color
appearing least frequently in $N(v)$. **od**

Algorithm VCE3 must terminate since every change of color increases the number of bicolored edges, or equivalently, decreases the number of mono-colored edges.

Corollary 2.21 [16] *The vertices of any graph having order $n \geq 3$ and no isolated vertices can be partitioned into three very cost effective sets.*

Corollary 2.22 [16] *For any graph having order* $n \geq 3$ *and no isolated vertices,* $VCE(G) \geq n/3$*, and this bound is sharp. Notice that for cycles of order* $n = 3k, VCE(G) = n/3 = k$*.*

3 PRELIMINARY RESULTS

The research for this thesis was conducted in two phases. Results from the first phase have already been published in [11]. In this section, we present results that serve as an introduction to the main results of this thesis found in Section 4.

As we have observed, all nontrivial, connected graphs are cost effective, in that they have a cost effective bipartition, but not all graphs are very cost effective. The class of graphs that are not very cost effective includes the complete graphs of odd order. We first establish four families of very cost effective graphs.

Proposition 3.1 *The following classes of graphs are very cost effective:*

(i) connected bipartite graphs $G = (X, Y, E)$ of order $n \geq 2$,

(ii) complete graphs K_{2k} of even order,

(iii) the corona $G \circ K_1$, where G is a nontrivial connected graph, and

(iv) the wheel $W_{1,k}$ for $k \neq 4$.

Proof.

(i) Since $G = (X, Y, E)$ has no isolated vertices, it is easy to see that the natural bipartition $\pi = \{X, Y\}$ into two disjoint independent sets X and Y is very cost effective.

(ii) Simply partition the vertices of K_{2k} into two sets X and Y of order k.

(iii) Since G is assumed to be connected and of order $n \geq 2$, by Theorem 1.1, G has a cost effective bipartition $\pi = \{R, B\}$. But this may not be a very cost effective bipartition. However, if each leaf is colored differently than the vertex to which it is adjacent, the resulting bipartition is very cost effective.

(iv) For the wheel, $W_{1,k}$, we consider two cases based on the parity of k.

(**Case 1**) Let $k = 2t+1$. Color the vertices of the cycle C_{2t+1}, $1, 2, 1, 2, \ldots, 1, 2, 3$, and color the central vertex x with color 4. Let R be the set of vertices colored 1 together with the single vertex, say y, colored 3, and let B be the set of vertices colored 2 together with the central vertex x colored 4. We show that the bipartition $\pi = \{R, B\}$ is very cost effective.

Notice that the set of vertices colored 1 is independent. Thus, if v is a vertex colored 1, then $1 \leq |N[v] \cap R| \leq 2$, while $2 \leq |N(v) \cap B| \leq 3$. Also for y, the only vertex colored 3, $2 = |N[y] \cap R| = |N(y) \cap B|$. Similarly, the set of vertices colored 2 is independent. Thus, if w is a vertex colored 2, then $|N[w] \cap B| = |N(w) \cap R| = 2$. Also for x, the only vertex colored 4, $2 \leq |N[x] \cap B| = |N(x) \cap R| = t + 1$. Thus, the bipartition $\pi = \{R, B\}$ is very cost effective.

(**Case 2**) Let $k \geq 6$ be even, and let the vertices of the cycle C_k be labeled in order $v_1, v_2, \ldots, v_k$. First color the center vertex x red. For the vertices v_i, for

$1 \le i \le 6$, color v_1 and v_4 red and v_2, v_3, v_5, v_6 blue. Then for v_i, $7 \le i \le k$, alternately color the vertices red and blue (beginning with red). This coloring forms a bipartition $\pi = \{R, B\}$, where R is the set of red vertices and B is the set of blue vertices. Consider the set R. For the vertex x, $|N[x] \cap R| = (k-6)/2 + 3 < |N(x) \cap B| = (k-6)/2 + 4$. For every other vertex, say $v \in R$, $|N[v] \cap R| = 2 = |N(v) \cap B|$. For $v \in B$, $1 \le |N[v] \cap B| \le 2$ and $2 \le |N(v) \cap R| \le 3$. Thus, the bipartition $\pi = \{R, B\}$ is a very cost effective partition. □

We note that the wheel $W_{1,4}$ is not very cost effective. We next consider other families of graphs that are not very cost effective.

Proposition 3.2 *No cycle C_{2k+1} of odd order is very cost effective.*

Proof. Any bipartition $\pi = \{R, B\}$ of the vertices of an odd cycle into two non-empty sets must place two adjacent vertices into either set R or set B. This follows since odd cycles do not have partitions into two independent sets. Suppose that two adjacent vertices, say u and v, are in the same set, say R. Let vertex w be the second vertex adjacent to v on the cycle. If $w \in R$, then vertex v does not meet the very cost effective condition, having no neighbors in B. While if $w \in B$, then $|N[v] \cap R| = 2 > |N(v) \cap B| = 1$. Thus, C_{2k+1} does not have a very cost effective bipartition. □

Proposition 3.3 *Let $H = C_5 - C_5$ be a graph obtained from two disjoint cycles C_5 by adding one edge between any vertex in one C_5 and any vertex in the other C_5. Then H is not very cost effective.*

Proof. Let the vertices of the first five-cycle of H be labelled in order u_1, u_2, u_3, u_4, u_5, and those of the second five cycle of H be labelled v_1, v_2, v_3, v_4, v_5. Assume that vertices u_1 and v_1 are joined by an edge. We must show that this graph does not have a very cost effective bipartition.

Assume that $\pi = \{R, B\}$ is a very cost effective bipartition of the graph H. Assume without loss of generality that $u_1 \in R$. Vertex u_1 is adjacent to vertices u_2 and u_5. If $u_2 \in R$, then it cannot be very cost effective, having two vertices in its closed neighborhood colored red, but at most one vertex in its neighborhood colored blue. Thus, vertex u_2 must be colored blue. The same argument applies to vertex u_5, and thus vertex u_5 must be colored blue. Now by the same reasoning, vertex u_3 cannot be colored blue, else vertex u_2 is not very cost effective, having two vertices in its closed neighborhood colored blue, and at most one colored red. Therefore, vertex u_3 must be colored red. But the same argument can be used to show that vertex u_4 must also be colored red. Thus, neither u_3 nor u_4 is very cost effective. □

Corollary 3.4 *No graph G containing a C_5 attached to the rest of the graph by a connecting edge is very cost effective.*

By much the same reasoning, it can be shown that no graph containing an attached odd cycle has a very cost effective bipartition. This result is generalized even further in Section 4 (see Theorem 4.5). A *cactus* is any connected graph having the property that no edge is contained in two or more cycles. Thus, all trees are cacti, as are all cycles. The class of cacti provide more examples of graphs that are not very cost effective. Since all bipartite cacti are very cost effective, a cactus that is not very cost effective must contain an odd cycle. First note that a *unicyclic graph* is any connected graph G of order n and size n. Equivalently, it is a graph obtained from a tree by adding an edge between any two non-adjacent vertices of T. Of course, every unicyclic graph is a cactus. It follows that if a unicyclic graph is not very cost effective, then its only cycle has odd order. However, as we shall see in Section 4, a unicyclic graph with an odd cycle can be very cost effective.

Recall that the *Cartesian product* of two graphs, G and H, is the graph denoted $G\Box H = (V(G) \times V(H),\ E(G\Box H))$, where two vertices (u, v) and (w, x) are adjacent in $G\Box H$ if and only if either $u = w$ and v is adjacent to x in H, or u is adjacent to w in G and $v = x$. In $G\Box H$, there exists a copy of H for each vertex in $V(G)$ and a copy of G for each vertex in H. To aid in our discussion, we let H_v represent the copy of H in $G\Box H$ corresponding to the vertex v in G. In other words, H_v is the subgraph induced by the vertices of $V(G\Box H)$ whose first coordinate is v.

Theorem 3.5 *Let $G = (X, Y, E)$ be a bipartite graph with no isolated vertices and H be a nontrivial, connected graph. Then the Cartesian product $G\Box H$ is very cost effective.*

Proof. Let G be an isolate-free, bipartite graph with partite sets X and Y, and let H be a nontrivial, connected graph. Since H is connected, Theorem 1.1 implies that H has a cost effective bipartition $\pi = (V_1, V_2)$. We build a very cost effective partition of $G\Box H$ as follows. Let $(u, v) \in V(G\Box H)$. First, assume that u is the partite set X in G. If $v \in V_1(H)$, then color (u, v) red; and if $v \in V_2(H)$, then color (u, v) blue. Next, let $u \in Y(G)$. If $v \in V_1(H)$, then color (u, v) blue; and if $v \in V_2(H)$, then color (u, v) red. In other words, for a copy of H corresponding to a vertex in the partite set X of G, we use the cost effective partition of H, $\pi = (V_1, V_2)$; and for a copy of H corresponding to a vertex in the other partite set Y, we reverse the colors of those in π. We note that since $\pi = (V_1, V_2)$ is a cost effective partition for H, each vertex (x, y) is cost effective in H_x. Also, every neighbor of (x, y) that is not in H_x is colored a different color than (x, y). Since G is isolate-free, each vertex (x, y) has at least one neighbor outside of H_x. Hence, each vertex (x, y) is very cost effective with respect to its color class in $G \square H$. It follows that the bipartition (R, B), where R is the set of red vertices and B is the set of blue vertices, is a very cost effective bipartition of $G \square H$. $\square$

This result from our preliminary work raised several questions that we were able to answer in the second phase of our research. Answers to the following questions can be found in Section 4.

Question 1 *If G is very cost effective, is $G \square H$ very cost effective for all graphs H?*

Question 2 *If G and H are both very cost effective, is $G \square H$ very cost effective?*

Question 3 *Is every Cartesian product $G \square H$ very cost effective?*

4 MAIN RESULTS

In this section we present the results from phase 2 of our research. In particular, we present results on very cost effective partitions and explore families of graphs, such as Cartesian products and cacti graphs. To aid in our discussion, given a partition $\pi = \{R, B\}$ of a graph G, we say the vertices of R are colored red and the vertices of B are colored blue under π. If π is a very cost effective partition of G, then we say that G is very cost effective under π.

4.1 General Results

The following result is shown in [16].

Theorem 4.1 [16] *Every connected graph G of order $n \geq 3$ can be partitioned into three very cost effective sets.*

Theorem 4.2 *No very cost effective partition of G is a very cost effective partition of its complement $\overline{G}$.*

Proof. Let $\pi = \{V_1, V_2\}$ be a very cost effective partition of G. Assume to the contrary that π is a very cost effective partition of $\overline{G}$. Without loss of generality, let u be a vertex in V_1. Let $|N_G(u) \cap V_1| = k_1$ and $|N_G(u) \cap V_2| = k_2$. Then, $k_1 < k_2$ because u is very cost effective in G. Also, $|N_{\overline{G}}(v) \cap V_1| = |V_1| - k_1 - 1 < |V_2| - k_2 = |N_{\overline{G}}(v) \cap V_2|$, because according to our assumption, u is very cost effective in $\overline{G}$ under π. Since $k_1 < k_2$, the last inequality can be written as $|V_1| - k_2 < |V_2| - k_2$. Hence, $|V_1| < |V_2|$. On the other hand, an analogous argument for an arbitrary vertex in V_2 shows that $|V_2| < |V_1|$, a contradiction. □

As promised in Section 3, our next result generalizes Corollary 3.4.

Lemma 4.3 *If G is a graph that has an odd cycle as an endblock, then G is not very cost effective.*

Proof. Let G contain an odd cycle endblock C, and label the vertices on C, $u_1, u_2, ..., u_{2k+1}$ such that u_1 is the cut vertex on C. Suppose to the contrary that $\pi = \{R, B\}$ is a very cost effective bipartition of the graph G. Assume, without loss of generality, that $u_1 \in R$. Since C is an endblock and u_1 is the cut vertex on C, $deg(u_j) = 2$ for $2 \leq j \leq 2_{k+1}$. Since π is a very cost effective partition, it follows that $u_2 \notin R$, that is, $u_2 \in B$. Further, $u_{2i+1} \in R$ and $u_{2i} \in B$ for $1 \leq i \leq k$. But then $u_{2k+1} \in R$ and $|N(u_{2k+1}) \cap R| = |N(u_{2k+1}) \cap B| = 1$, so u_{2k+1} is not very cost effective under π, a contradiction. □

Figure 4 is an example of a graph G that has an odd cycle endblock.

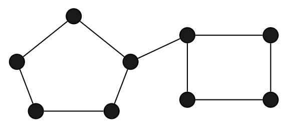

Figure 4: Graph G has an odd cycle endblock

4.2 Cartesian Product and Cacti Graphs

Here we answer the questions raised in Section 3 concerning Cartesian products. Not every Cartesian product is very cost effective. For example, the $C_3 \Box C_3$ shown in Figure 5 is not very cost effective.

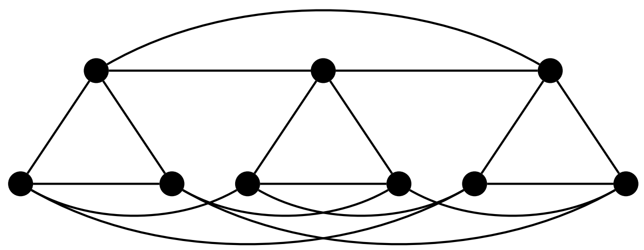

Figure 5: $C_3 \square C_3$

On the other hand, in our next result, we prove that the Cartesian product of a very cost effective graph and an arbitrary connected graph is very cost effective.

Theorem 4.4 *Let G be a very cost effective graph. Then $G \square H$ is a very cost effective graph for any connected graph H.*

Proof. Let G be a graph with a very cost effective partition $\pi_1 = (V_1(G), V_2(G))$. If H is the trivial graph, then $G \square H = G$, and, clearly, the result holds. Let H be a non-trivial, connected graph. By Theorem 1.1, H has a cost effective partition $\pi_2 = (V_1(H), V_2(H))$. We define two colorings, C_1 and C_2, of H. Let C_1 color the vertices of $V_1(H)$ red and the vertices of $V_2(H)$ blue; and let the coloring C_2 swap the colors of C_1. We consider $u \in V(G)$ to determine the coloring of H_u in $G \square H$. If $u \in V_1(G)$, then use the coloring C_1 for H_u; while if $u \in V_2(G)$, then use the coloring C_2 for H_u. Let R be the set of red vertices in $G \square H$, and let B be the set of blue ones. To see that $\pi = (R, B)$ is a very cost effective partition of $G \square H$, consider an arbitrary vertex $(u, v) \in V(G \square H)$. Without loss of generality, assume that (u, v) is colored red and H_u is colored by C_1. Since C_1 is a red-blue cost effective partition of H_u, (u, v) has at least as many blue neighbors in H_u as red ones. It suffices to show that (u, v) has more blue neighbors than red ones in G_v. Since (u, v) is red under C_1 of H_u, it follows that (x, v) is red for all $x \in V_1(G_v)$. Moreover, (y, v) is blue for all

$y \in V_2(G_v)$. Since $\pi_2 = (V_1(G_v), V_2(G_v))$ is a very cost effective partition of G_v, we have that (u, v) has more blue neighbors than red ones in G_v. Hence, $\pi = (R, B)$ is a very cost effective partition of $G \Box H$. $\Box$

Figure 6 illustrates the Cartesian product $K_2 \Box C_3$, in which K_2 is a very cost effective graph and C_3 is not.

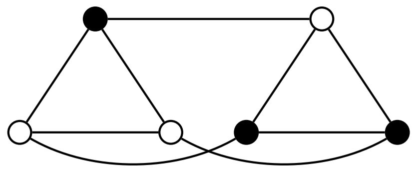

Figure 6: $K_2 \square C_3$

Recall that a cactus graph is a connected graph where each block is either an edge or a cycle. The cacti is a family of graphs that has many applications. According to [18], cacti graphs are used as a data structure for comparing sets of related genomes. Cacti graphs can represent duplications and general genomic rearrangements. Additionally, they naturally decompose the common substructures in a set of related genomes into a hierarchy of chains that can be visualized as two-dimensional multiple alignments and nets that can be visualized in circular genome plots.

Given a block B and a vertex v in B of a cactus graph G, we define a *v-branch* B_v to be a connected subgraph such that $V(B) \cap V(B_v) = \{v\}$, and B_v is maximal with this property. We say that v is the root of the v-branch B_v of B, and that B *supports* B_v. To aid in our discussion, we say that a branch is *bad* if it is not very cost effective. In Figure 7, we give an example of a v-branch. Let G be the graph in Figure 7 and B be a block of G induced by the set $\{v_6, v_7, v_8\}$. The v_6-branch of B is the subgraph induced by the set $\{v_6, v_9, v_{10}, ..., v_{20}\}$, and the v_7-branch of B is the subgraph induced by the set $\{v_7, v_1, v_2, ...v_5\}$. Also note that if B' is the block induced by the set $\{v_6, v_9, v_{10}, v_{11}, v_{12}\}$. Then the v_6-branch of B' is the graph $G \setminus \{v_9, v_{10}, v_{11}, v_{12}\}$.

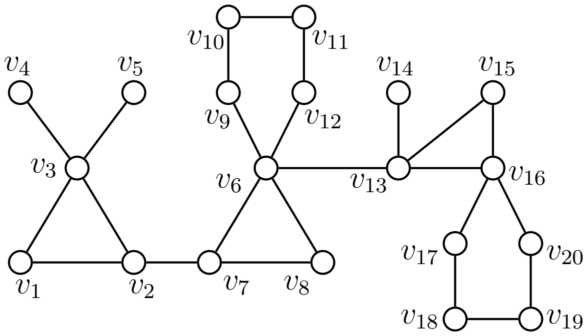

Figure 7: Example of a branch

Theorem 4.5 *A cactus graph G is very cost effective if and only if every odd cycle block C of G supports two branches C_u and C_v, where u and v are adjacent vertices on C and each of C_u and C_v is a very cost effective cactus.*

Proof. We say that a cactus G has Property P if every odd cycle block C of G supports two branches C_u and C_v, where u and v are adjacent vertices on C and each of C_v and C_u is a very cost effective cactus.

We show that if G is a very cost effective cactus graph, then G has Property P by proving the contrapositive. Assume that G is a cactus graph that does not have Property P. Since Property P vacuously holds for graphs with no odd cycles, G has an odd cycle, say C, that does not comply with Property P.

Suppose, for the purpose of a contradiction, that G is very cost effective, and let $\pi = \{R, B\}$ be a very cost effective partition of G. Since C is an odd cycle, under π at least two adjacent vertices, say u and v, of C are colored the same color, say red. Assume first that adjacent vertices u and v are roots of branches C_u and C_v, respectively. Since G does not have Property P, we may assume without loss of generality that C_u is not very cost effective. Since π is a very cost effective partition of G, all the vertices of C_u, except possibly u, are very cost effective under π in C_u. Thus, u is not very cost effective under π in C_u, that is, u has at least as many red neighbors as blue ones in C_u. Since u is very cost effective under π in G, that is, u has more blue neighbors than red ones in G, we deduce that the two neighbors of u on C are both blue. In particular, $v \in B$, a contradiction.

Hence, we may assume that at least one of u and v, say u, is not the root of a branch. But since u and v are red under π and $\deg_G(u) = 2$, u is not very cost effective in G under π, a contradiction. Thus, the result holds.

For the converse, assume that G has Property P. To show that G is very cost effective, we proceed by induction on the number c of odd cycles in G. If G has no odd cycles, then G is bipartite and by Observation 2.11, G is very cost effective. Thus, we may assume that G has an odd cycle. If $c = 1$, then let C be the odd cycle of G. Label the vertices of C as $u_1, u_2, ..., u_k, u_1$, such that u_1 and u_k are a pair of adjacent vertices with very cost effective branches, C_{u_1} and C_{u_k}, respectively. We show that G is very cost effective by giving a very cost effective partition $\pi = \{R, B\}$ of G. Let $u_i \in R$ if i is odd and $u_i \in B$ if i is even. Then, both u_1 and u_k are red and each have exactly one blue neighbor on C. Let $pi' = \{V_1, V_2\}$ be a very cost effective partition of $H' = C_{u_1}$. If $u_1 \in V_i$, then, in H', color the vertices of V_i red and the vertices of V_{3-i} blue. Hence, u_1 is very cost effective in H', that is, u_1 has more blue neighbors than red ones in H', implying that $|N_G(u_1) \cap R| = |N_{H'}(u_1) \cap R| + 1 < |N_{H'}(u_1) \cap B| + 1 = |N_G(u_1) \cap B|$. Hence, u_1 is very cost effective in G. A similar argument shows that u_k is also very cost effective under π in G. For every u_i, $2 \le i \le k-1$, if $\deg_G(u_i) = 2$, then u_i is very cost effective in G. Assume that $\deg(u_i) \ge 3$, for some i, where $2 \le i \le k-1$, and let C_{u_i} be the branch rooted at u_i from C. Since C_{u_i} has no odd cycles, C_{u_i} is bipartite, and so, by Observation 2.11, C_{u_i} is very cost effective. Let $\pi' = \{V_1, V_2\}$ be a very cost effective partition of C_{u_i}. Relabeling the sets V_1 and V_2 if necessary, we may assume that $u_i \in V_1$ of C_{u_i}. If i is odd, then color the vertices of V_1 red and the vertices of V_2 blue. If i is even, then we color the vertices of V_1 blue and the vertices of V_2 red. Then in G, the vertices of C_{u_i} are very cost effective under $\pi = (R, B)$. Hence if $c = 1$, then G is very cost effective, establishing our base case.

Let $c > 1$, and assume that any cactus graph with Property P having fewer than c odd cycles is very cost effective. Let G be a cactus graph having $c \geq 2$ odd cycles and Property P. Begin with an odd cycle C of G. If every branch supported by C is very cost effective, then using an argument similar to the case for $c = 1$, we can show that G is very cost effective. Hence, we may assume that G has an odd cycle that supports a bad branch. Among all odd cycles with bad branches, select C to be one that minimizes the number of vertices in a bad branch supported by C. Label the vertices of C as $u_1, u_2, ..., u_k, u_1$, such that u_1 and u_k are a pair of adjacent vertices with very cost effective branches, C_{u_1} and C_{u_k}, respectively. Let C_{u_i} be a bad branch of C having the minimum number of vertices. Then, C_{u_i} has at least one odd cycle. Moreover, since C is not a cycle in C_{u_i}, C_{u_i} has fewer than c cycles. If C_{u_i} has Property P, then applying our inductive hypothesis to C_{u_i}, we have that C_{u_i} is very cost effective. This is a contradiction. Thus, we may assume that C_{u_i} does not have Property P. Since G has Property P, we conclude that u_i is a vertex on an odd cycle, say C', of C_{u_i}, and C'_{u_i} is a very cost effective branch supported by C'. Since G has Property P, there is a neighbor of u_i, say x, on C', such that C'_x is very cost effective. Coloring x the same color as u_i and proceeding as in our base case, we can show that all the vertices of C' are very cost effective. If every branch supported by C' is very cost effective, then using an argument similar to the case for $c = 1$, we can show that G is very cost effective. Thus, we may assume that for some vertex y on C', C'_y is a bad branch. But, since C'_y is a proper subgraph of C_{u_i}, C'_y has fewer vertices than C_{u_i}, contradicting our choice of C. Hence, we conclude that G is very cost effective. □

Figure 8 illustrates how a graph having property P can be partitioned into two very cost effective sets.

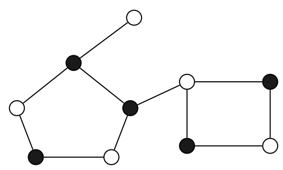

Figure 8: Graph G has property P

5 CONCLUDING REMARKS

We have characterized the very cost effective cacti graphs and showed that if a graph G or H has a very cost effective bipartition, then so does their Cartesian product $G \Box H$. We conclude with some open problems suggested by this work:

1. Characterize the very cost effective graphs.
2. We have seen that the odd-regular graphs are very cost effective. Characterize the even-regular graphs that are very cost effective.
3. Study very cost effective partitions in other families of graphs.
4. Characterize the very cost effective Cartesian products.

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