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Bicyclic Mixed Triple Systems.

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Bicyclic Mixed Triple Systems

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ABSTRACT

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In the study of triple systems, one question faced is that of finding for what order a decomposition exists. We state and prove a necessary and sufficient condition for the existence of a *bicyclic mixed triple system* based on the three possible partial orientations of the 3-cycle with twice as many arcs as edges. We also explore the existence of *rotational* and *reverse mixed triple systems.* Our principal proof technique applied is the *difference method.* Finally, this work contains a result on *packing* of *complete mixed graphs* on v vertices, M_v, with isomorphic copies of two of the mixed triples and a possible *leave* structure.

DEDICATION

I love to dedicate, as a special birthday present, this thesis to each of my beautiful youngsters: Bobga Laura (*Baby*), Bobga Dylan (*Didi*) and Bobga Bradley (*Brad*). To GOD be the glory.

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My first words of acknowledgements go to all the members of staff in the Department of Mathematics at ETSU for their words of encouragement. My special thanks to Dr Janice Huang for her inspiration and motherly advice during my studies at ETSU.
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My studies in general, and this work in particular, would not have been completed without the support of my friends, colleagues and most of all my family both here in the United States and back home in Cameroon. I very much appreciate my family's understanding during those occasions when they had to make-do without a full-time father, husband or son.
Thanks. Bobga B.B.

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1 INTRODUCTION AND BASIC DEFINITIONS

An interesting area of study in combinatorial mathematics is combinatorial design theory. The approach of modeling objects as a set of *points* (or *vertices*) and the relation between them (say) as *arcs* and/or *edges* comes in handy when studying triple systems in general and mixed triple systems in particular.

For a better understanding of this thesis, we start by giving a comprehensive list of definitions. Worthy of note is the fact that some familiar words might have a slight variation in meaning. However, the concepts remain the same.

Given a non-empty *graph* G on v vertices, we obtain a *directed graph* on v vertices (or *digraph*) D by assigning a direction to (or by *orienting*) each edge of G. D is called an *orientation* of G.

A *directed graph* on v vertices (or *digraph*) D is thus a finite non-empty set of points called vertices, together with a set of ordered pairs of distinct vertices of D, called *arcs* or *directed edges*. If $a = [x, y]$ is an arc of a digraph D, then a is said to *join* x to y and a is *incident from* x and incident *to* y, while x is incident to a and y is incident from a. We say that x and y are *adjacent* vertices.

A mixed graph on v vertices, M, can be obtained from D by introducing an edge as well as an oppositely oriented arc to each arc of D as exemplified in figure 1. We shall base our discussion on complete mixed graphs, denoted by M_v, in which every pair of vertices are connected.

In mixed graph concepts, the *out degree, od(u),* of vertex u in M_v refers to the number of vertices of M_v that are adjacent from u i.e., $od(u) = \mid N_o(u) \mid$ where the open neighborhood $N_o(u) = \{x \in V(M_v) \mid x \textit{ is adjacent from } u\}$.

The *in degree, id(u),* of vertex u in M_v refers to the number of vertices of M_v that are adjacent to u i.e.,

$id(u) = \mid N_i(u) \mid$ where $N_i(u) = \{x \in V(M_v) \mid x \text{ is adjacent to } u\}$.

The degree $d(u)$ of u in M_v is the number of edges linked to u. By the total degree of vertex u, we shall mean the sum: $od(u) + id(u) + d(u)$. It becomes clear that in M_v, the sum of its arcs and edges is always congruent to zero modulo 3.

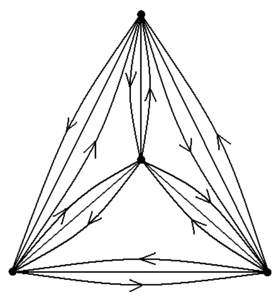

Figure 1: Complete Mixed Graph on 4 Vertices

A decomposition of a simple graph G with *isomorphic* copies of graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, and $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and the union over all the g_i's gives the graph G. The g_i's are called *blocks* of the decomposition while $V(G)$ is the vertex set of G and $E(G)$ the edge set. By replacing the *edge set* by *arc set* in the above definition, a similar definition can be obtained for a decomposition of digraphs.

A graph (digraph) decomposition into isomorphic copies of a graph (respectively directed graph) on three vertices is equivalent to a *triple system.* A K_3-decomposition of the complete graph on v vertices, K_v is called a *Steiner Triple System* of order v,

$STS(v)$, which is widely known to exist if and only if $v \equiv 1$ or 3 (mod 6).

Example: K_5 can be decomposed into two copies of the 5-cycles.

In general, whenever a complete graph, digraph, or mixed graph is decomposed into graphs (respectively digraph or mixed graph) on 3 vertices, the resulting structure is called a *triple system* and the triples are called *blocks*. There are 2 orientations of K_3, namely the 3-circuit and the transitive triple. A decomposition of the complete directed graph, denoted D_v into isomorphic copies of the 3-circuit is equivalent to a *Mendelsohn triple system* of order v, denoted $MTS(v)$ and exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [17].

A decomposition of D_v into isomorphic copies of the transitive triples is known as a *directed triple system* of order v, which is denoted $DTS(v)$ and exists if and only if $v \equiv 0$ or 1 (mod 3) [16].

Example: D_3 can be decomposed into 2 copies of the 3-circuit.

In this work, we consider 3 different triples over a complete mixed graph M_v as shown in figure 2:

T_1 in which there is a vertex with $od = 2$, $id = d = 0$, and the other two vertices each have $id = d = 1$ and $od = 0$;

T_2 in which there is a vertex with $id = 2$, $od = d = 0$, and the other two vertices each have $od = d = 1$ and $id = 0$. We may, without any loss of generality, consider T_2 as the converse of T_1;

T_3 in which there is a vertex with $od = id = 1$, a vertex with $od = d = 1$ and a third vertex with $id = d = 1$.

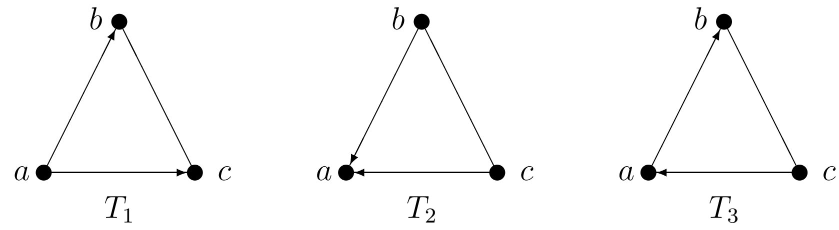

Figure 2: Orientation of the 3-Cycle

A decomposition of M_v into isomorphic copies of any of the triples T_i, for $i = 1, 2, 3$ is correspondingly called a T_i triple system of order v. It is now known that a T_i triple system of order v exists for $i \in \{1, 2, 3\}$ if and only if $v \equiv 1(\text{mod } 2)$, except for $i = 3$ and $v \in \{3, 5\}$ [11].

Let G be a graph and let $\gamma = \{g_1, g_2, \ldots, g_n\}$ be a g-decomposition of G. An *automorphism* (one-to-one and onto map from an object back into itself which preserves structures (here, blocks)) of this decomposition is a *permutation* of the vertex set $V(G)$ which fixes the set γ. That is, if g_i is a block of the triple and π is a permutation, then $\pi(g_i)$ also forms a block. Similar definitions can be given for automorphisms of directed and mixed graph decompositions.

Observe, for example, that the permutation $\pi = (0, 1, 2, 3, 4, 5, 6)$ is an automorphism of the $STS(7)$ which consist of the blocks

{(0 1 3),(1 2 4),(2 3 5),(3 4 6),(4 5 0),(5 6 1),(6 0 2)}.

Consider a permutation π on a v element set, $\{0, 1, 2, 3, \ldots, (v-1)\}$. The permutation is *cyclic* if it consists of a single cycle of length v i.e if π=$(0, 1, 2, 3, \ldots, (v-1))$. We talk of a *bicyclic* permutation π if π consists of two disjoint cycles of lengths N_1 and N_2 such that $v = N_1 + N_2$.

Calahan-Zijlstra and Gardner, 1994, proved that a *bicyclic* $STS(v)$ admitting an automorphism whose disjoint cyclic decomposition consists of two cycles of length $N_1 > 1$ and N_2 (both positive integers) exists if and only if

$N_1 \equiv 1$ or 3 (mod 6), $N_1 \neq 9$, $N_1 \mid N_2$, and $v = N_1 + N_2 \equiv 1$ or 3 (mod 6) [3].

In this work, we shall state and prove a similar result, viz.: a *bicyclic mixed triple system of order v* admitting an automorphism whose disjoint cyclic decomposition consists of two cycles of length N_1 and N_2, where $N_1 < N_2$, exists for the triples T_1 and T_2 if and only if $N_1 \equiv 1 \pmod 2$, $N_1 \mid N_2$, and $v = N_1 + N_2 \equiv 1 \pmod 2$. Bicyclic T_3 triple systems do not exist [1].

The necessary condition is established by noticing that, for a bicyclic automorphism π of a T_1 triple system with π consisting of disjoint cycles of length N_1 and N_2, π^{N_1} fixes all the points in N_1 and hence these points must form a *subsystem* of order N_1 as we shall see.

Sufficiency is established in cases using a well known method called the *difference method*, which involves direct constructions. T_2 results are then obtained as corollaries given that T_2 is the "*converse*" of T_1.

A *rotational* decomposition of the complete mixed graph M_v is a decomposition of M_v which admits an automorphism π consisting of a fixed point, denoted by ∞, and a cycle of length $(v-1)$ i.e. $\pi = (\infty)(0, 1, 2, \ldots, v-2)$.

The following result is also proved in this work:

A rotational T_i-triple system of order v

(i) *exists if and only if $v \equiv 1 \pmod 2$ when $i \in \{1, 2\}$, and*

(ii) *does not exist when $i = 3$.*

A T_i-triple system of order v is said to be *reverse* if it admits an automorphism consisting of a fixed point and $(v-1)/2$ transpositions (or simply $v/2$ transpositions when v is even). The existence of reverse T_i-triple systems follows easily from the existence of rotational T_i-triple systems.

The *orbit* of a block (triple) under an automorphism π is the image of the block under the various powers of π. A set of blocks is said to be a set of *base blocks* for a mixed triple system of order v under a permutation π if the orbits of the blocks produce a mixed triple system of order v and exactly one block occurs in each orbit.

If a decomposition of M_v does not exist, then one question to address is "can we efficiently remove isomorphic copies of a given partial orientation of a 3-cycle from M_v such that the number of arcs and edges remaining is a minimum or such that the number of arcs and edges repeated is a minimum?" These concepts are considered as *packings* and *coverings*, respectively, of complete mixed graphs on v vertices and we talk of "*the packing problem*" as well as the "*covering problem*" for mixed graphs. The remaining arcs/edges are often referred to as the *leave* of the packing while the repeated arcs/edges are called *padding* of the covering.

We shall consider the packing problem for M_v with isomorphic copies of T_1.

A *maximal packing* of a mixed graph G with isomorphic copies of a graph g is a set

$(g_1, g_2, ..., g_n)$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, and $AE(g_i) \cap AE(g_j) = \emptyset$ for $i \neq j$,

$$\bigcup_{i=1}^{n} g_i \subset G, \text{and } \mid L \mid = \mid AE(G) \backslash \bigcup_{i=1}^{n} E(g_i) \mid$$

is minimal, where $V(G)$ is the vertex set of graph G and $AE(G)$ is the *arc* and *edge* set of graph G. L represents the leave of the packing.

Packings of the complete graph on v vertices, K_v with graph g have been studied for g a 3-cycle, 4-cycle, 6-cycle and $g = K_4$. For a review of this topic, see [13].

As an example, a maximal packing of the complete graph on 5 vertices, K_5 with copies of the 3-cycle has a leave of size 4.

A minimal covering of a mixed graph G with isomorphic copies of a graph g is a set $(g_1, g_2, ..., g_n)$ where $g_i \equiv g$ and $V(g_i) \subset V(G)$ for all i,

$$G \subset \bigcup_{i=1}^{n} g_i, \text{and } \mid P \mid = \mid \bigcup_{i=1}^{n} AE(g_i) \backslash AE(G) \mid$$

is minimal (the graph $\bigcup_{i=1}^{n} g_i$ may not be mixed and $\bigcup_{i=1}^{n} AE(g_i)$ may be a multi-set). Coverings of complete graphs K_v with graph g have been studied for g being a 3-cycle, 4-cycle and 6-cycle. For a review of this topic, see [13]. A covering of K_5 with isomorphic copies of the 3-cycle has a padding $P = 2$ x K_2.

Historically speaking, the concept of mixed triple systems was initiated by Professor Robert Gardner after reading a publication by Hartman [14] entitled "The last of the triple systems". Mixed graphs (and hence mixed triples) came as a blend of both the directed and undirected graphs (respectively triples). Gardner used cyclic automorphisms to establish the existence of mixed triple systems in a paper titled "Triple Systems from Mixed Graphs" [11]. Triple systems in general were first studied and published in the *Cambridge and Dublin Mathematical Journal* in 1847 by Kirkman. In 1853, Steiner independently posed the problem for undirected graphs. Directed graph decompositions into triples later came about thanks to Mendelsohn and were studied as Mendelsohn triple systems as well as directed triple systems. There are very many fascinating publications on triple systems, but much is still to be done especially in the area of applications.

In the studies of mixed graphs, one question we try to answer is "for what v does there exist bicyclic mixed triple system of order v?" As indicated above, we have answered this question using the mixed triples T_1 and T_2 with some conditions on the sizes of the two cycles. We have further been able to show that T_3 bicyclic decompositions of M_v do not exist.

This thesis is divided into two main parts. Following this introductory chapter is the second chapter which contains the principal theorems and some proofs. It is in in this chapter that we provide a necessary and sufficient condition for the existence of a bicyclic mixed triple system of order v. A proof for the existence of both the rotational and reverse mixed triple systems of order v also appears in this chapter.

A step-by-step verification of the difference method used in the second chapter is the principal content of the third chapter. Here, we demonstrate in great detail, and with the use of some examples, how the difference method is applied in mixed triple systems. This ends the first section of our work.

The second part of this work appears in the fourth chapter and deals with some results on the packing problem raised above. We explore the leave size for a packing of a complete mixed graph on v vertices, M_v, with isomorphic copies of the triple T_1

and derive an expression for a possible leave structure. This concludes the chapter and hence the thesis.

2 BICYCLIC, ROTATIONAL AND REVERSE MIXED TRIPLES

The main content of this chapter is to appear in the Bulletin of the Institute of Combinatorics and its Applications, [1]. For clarity, we start with a brief statement of its content.

> A *mixed triple system* is a decomposition of the complete mixed graph into one of the partial orientations of a 3-cycle which consists of two arcs and one edge. An automorphism of a mixed triple system which consists of two disjoint cycles is said to be *bicyclic*. An automorphism consisting of a fixed point and a single cycle is *rotational*. An automorphism which, when applied to the vertex set of a mixed triple system listed in a particular order, reverses the order of the vertices, is said to be *reverse*. Necessary and sufficient conditions are given for the existence of bicyclic, rotational, and reverse mixed triple systems for each of the three types of mixed triple systems.

2.1 Introduction

Let K_v denote the complete graph on v vertices. For graph g, a g-*decomposition* of K_v is a set $\gamma = \{g_1, g_2, \ldots, g_n\}$ of edge disjoint subgraphs of K_v each of which is isomorphic to g and $\bigcup_{i=1}^{n} E(g_i) = E(K_v)$, where $E(G)$ is the edge set of graph G. A decomposition of the complete directed graph, D_v, is similarly defined in terms of arcs and arc sets. Throughout this thesis, we shall denote the edge between vertex

u and vertex w as the unordered pair (u, w), and the arc from vertex u to vertex w as the ordered pair $[u, w]$. An *automorphism* of a g-decomposition of K_v is a permutation of the vertex set of K_v which fixes the set γ (with automorphism of a directed graph decomposition similarly defined). An automorphism consisting of a single cycle is *cyclic*. An automorphism which consists of two disjoint cycles is *bicyclic*. An automorphism consisting of a fixed point and a single cycle is *rotational*.

An automorphism of a g-decomposition of K_v is *reverse* if, when applied to the vertices of K_v written in a particular order, that order is reversed. So a reverse automorphism consists of $v/2$ transpositions when v is even, or $(v-1)/2$ transpositions and a fixed point when v is odd.

A graph (respectively directed graph) decomposition into isomorphic copies of a graph (directed graph) on three vertices is equivalent to a triple system. A K_3-decomposition of K_v is a *Steiner triple system* of order v, which is widely known to exist if and only if $v \equiv 1$ or $3 \pmod 6$. We denote the following directed graphs as d_m and d_t;

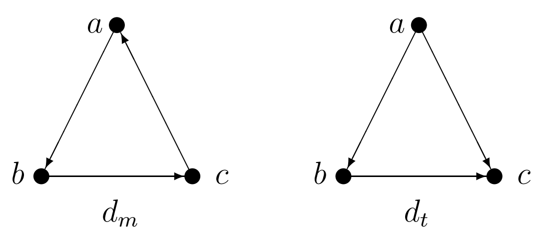

Figure 3: Directed Triples

A d_m-decomposition of D_v is a *Mendelsohn triple system* of order v and exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [17]. A d_t-decomposition of D_v is a *directed triple system* of order v and exists if and only if $v \equiv 0$ or 1 (mod 3) [16].

A cyclic Steiner triple system of order v exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [18]. A cyclic Mendelsohn triple system of order v exists if and only if $v \equiv 1$ or 3 (mod 6) [7]. A cyclic directed triple system of order v exists if and only if $v \equiv 1, 4,$ or 7 (mod 12) [8].

A bicyclic Steiner triple system of order v admitting an automorphism consisting of disjoint cycles of lengths $N_1 > 1$ and N_2 where $N_1 < N_2$ exists if and only if $N_1 \equiv 1$ or 3 (mod 6), $N_1 \neq 9$, $N_1 \mid N_2$, and $v = N_1 + N_2 \equiv 1$ or 3 (mod 6) [3]. A bicyclic directed triple system of order v admitting an automorphism consisting of disjoint cycles of lengths N_1 and N_2, where $N_1 < N_2$, exists if and only if
$N_1 \equiv 1, 4,$ or 7 (mod 12) and $N_2 = kN_1$ where $k \equiv 2$ (mod 3) [10].

Also, a rotational Steiner triple system of order v exists if and only if $v \equiv 3$ or 9 (mod 24) [19]. A rotational Mendelsohn triple system of order v exists if and only if $v \equiv 1, 3,$ or 4 (mod 6), $v \neq 10$ [5]. A rotational directed triple system of order v exists if and only if $v \equiv 0$ (mod 3) [6].

In a similar way, reverse Steiner triple systems exist if and only if $v \equiv$ 1, 3, 9, or 19 (mod 24) [9, 20, 22, 23]. A reverse directed triple system of order v exists if and only if $v \equiv$ 0, 1, 3, 4, 7, or 9 (mod 12) [4]. To the authors' knowledge, neither bicyclic nor reverse Mendelsohn triple systems have been studied.

The *complete mixed graph* on v vertices, denoted M_v, is a vertex set V of cardinality v, together with a set C of ordered and unordered pairs of V such that for all $x, y \in V$, $x \neq y$, we have $(x, y), [x, y], [y, x] \in C$. There are three partial orientations of a 3-cycle which, like M_v, contain twice as many arcs as edges. For emphasis on the structure, we repeat figure 2 below.

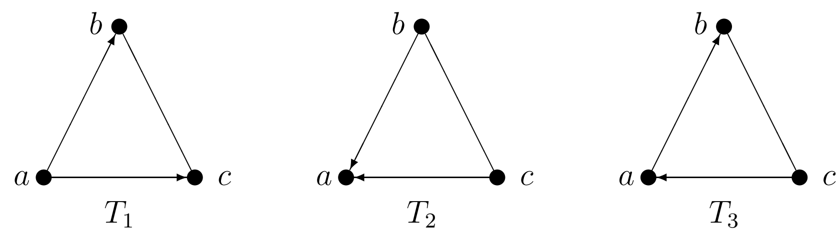

Figure 4: Orientation of the 3-Cycle

We denote T_i by the ordered triple $[a, b, c]_i$. Decompositions of M_v are defined similarly to decompositions of K_v and D_v. A T_i-decomposition of M_v is also called a T_i *triple system* of order v. For $i = 1, 2, 3$, a T_i-triple system of order v exists if and only if $v \equiv 1 \pmod 2$, except $v \in \{3, 5\}$ when $i = 3$ [11]. In fact, the constructions used in [11] take advantage of cyclic automorphisms and hence show that the existence condition is also the condition for cyclic mixed triple systems.

The purpose of this chapter is to give necessary and sufficient conditions for the existence of bicyclic, rotational, and reverse mixed triple systems.

2.2 Some Rotational STS Results

We now give some basic results to illustrate the difference method. The following is due to Peltesohn [18]: *A* $STS(v)$ *is cyclic if it admits an automorphism of type* $[0, \ldots, 0, 1]$ *and such a system exists if and only if* $v \equiv 1$ *or* 3 *(mod 6) and* $v \neq 9$.

To prove this, in the case $v \equiv 1 \pmod 6$, we note that the construction of a cyclic $STS(v)$ where $v = 6n + 1$ is equivalent to partitioning the set {1,2, …,3n } into triples such that in each triple, the sum of the two numbers is equal to the third or the sum of the three is equal to v. This is Heffter's *first difference problem* [15]. In a mixed triple system, we require that either $d_1 + e = d_2$ or $d_1 + d_2 + e \equiv 0 \pmod v$. Skolem [21] posed Heffter's first difference problem thus: partition the set $\{1, \ldots, 2n\}$ into distinct pairs (a_r, b_r) such that $b_r = a_r + r$ for $r = 1, \ldots, n$. Such a partitioning is called an *(A,n)-system*. If such a partitioning exists, then the triples (r, $a_r + n$, $b_r + n$), $r = 1, \ldots, n$ represent a solution to the above difference problem. Worthy of note is the fact that if such a partition exists, then

$$\sum b_r - \sum a_r = \frac{1}{2}n(n+1),$$

where $b_r - a_r = r$ for $r = 1, \ldots, n$.

We now present a formal proof consistent with the one given by Gardner in [12]. The necessary part follows from the fact that the total number of edges is a multiple of 3 (in a triple system) and since every vertex is of even degree, we must have v odd.

Hence $v \equiv 1$ or 3 (mod 6). Sufficiency is established in 8 cases as follows:

Case 1. Suppose $v \equiv 1 \pmod{24}$, i.e. $v = 24s + 1$. Consider the base blocks:

$(0, r, 8s - r + 1), r = 1, \ldots, 2s,$

$(0, r, 4s - r - 1), r = 1, \ldots, s - 1,$

$(0, r, 3s - r), r = 1, \ldots, s - 2,$ (omit if $s = 2$),

$(0, r, s + 1),$

$(0, r, 4s - 1),$

$(0, r, 6s).$

If $s = 1$, take the triples (0, 1, 6)(0, 2, 11)(0, 3, 10)(0, 4, 12).

Case 2. Suppose $v \equiv 7 \pmod{24}$, i.e. $v = 24s + 7$. Consider the base blocks:

$(0, r, 8s - r + 3), r = 1, \ldots, 2s,$

$(0, r, 4s - r + 1), r = 1, \ldots, s,$

$(0, r, 3s - r + 1), r = 1, \ldots, s - 2,$ (omit if $s = 2$),

$(0, r, s + 2),$

$(0, r, 6s + 2),$

$(0, r, 4s + 1).$

If $s = 0$, take the triple (0, 1, 3) and if $s = 1$, take the triples

(0, 1, 8)(0, 2, 15)(0, 3, 12)(0, 4, 14)(0, 5, 11).

Case 3. Suppose $v \equiv 13 \pmod{24}$, i.e. $v = 24s + 13$. Consider the base blocks:

$(0, r, 4s - r + 2), r = 1, \ldots, 2s,$

$(0, r, 8s - r + 4), r = 1, \ldots, s - 1,$ (omit if $s = 1$),

$(0, r, 7s - r + 3), r = 1, \ldots, s - 1$, (omit if $s = 1$),

$(0, r, 6s + 2)$,

$(0, r, 6s + 3)$,

$(0, r, 8s + 5)$,

$(0, r, 7s + 4)$.

If $s = 0$, take the triples (0, 1, 4)(0, 2, 7).

Case 4. Suppose $v \equiv 19 \pmod{24}$, i.e. $v = 24s + 19$. Consider the base blocks:

$(0, r, 8s - r - 2), r = 1, \ldots, 2s - 2$,

$(0, r, 4s - r - 1), r = 1, \ldots, s - 1$,

$(0, r, 3s - r), r = 1, \ldots, s - 2$, (omit if $s = 2$),

$(0, r, s + 1)$,

$(0, r, 4s - 1)$,

$(0, r, 6s - 1)$,

$(0, r, 8s - 1)$.

If $s = 1$, take the triples (0, 1, 5)(0, 2, 8)(0, 3, 10).

Case 5. Suppose $v \equiv 3 \pmod{24}$, i.e. $v = 24s + 3$. Consider the base blocks:

$(0, r, 4s - r + 1), r = 1, \ldots, s - 1$, (omit if $s = 1$),

$(0, r, 3s - r), r = 1, \ldots, s - 1$, (omit if $s = 1$),

$(0, r, 8s - r + 1), r = 1, \ldots, s - 1$, (omit if $s = 1$),

$(0, r, 7s - r + 1), r = 1, \ldots, s - 1$, (omit if $s = 1$),

$(0, r, 2s)$,

$(0, r, 5s+1)$,

$(0, r, 7s+1)$

$(0, r, 8s+1)$.

Case 6. Suppose $v \equiv 21 \pmod{24}$, i.e. $v = 24s + 21$. Consider the base blocks:

$(0, r, 4s-r), r = 1, \ldots, 2s-1$,

$(0, r, 8s-r), r = 1, \ldots, s-2$, (omit if $s = 2$),

$(0, r, 7s-r-1), r = 1, \ldots, s-2$, (omit if $s = 2$),

$(0, r, 6s-1)$,

$(0, r, 7s+1)$,

$(0, r, 6s)$,

$(0, r, 7s)$.

If $s = 1$, take the triples (0, 1, 5)(0, 2, 10)(0, 3, 9).

Case 7. Suppose $v \equiv 9 \pmod{24}$, i.e. $v = 24s + 9$, where $s \geq 2$. Consider the base blocks:

$(0, r, 4s-r+2), r = 1, \ldots, 2s$,

$(0, r, 7s-r+3), r = 1, \ldots, s)$,

$(0, r, 8s-r+3), r = 1, \ldots, s-2$, (omit if $s = 2$),

$(0, r, 6s+2)$,

$(0, r, 8s+4)$,

$(0, r, 7s+4)$.

If $s = 1$, take the triples (0, 1, 10)(0, 2, 12)(0, 3, 9)(0, 4, 15)(0, 5, 17).

Case 8. Suppose $v \equiv 15 \pmod{24}$, i.e. $v = 24s + 15$, where $s \geq 2$. Consider the base blocks:

$(0, r, 4s - r + 3), r = 1, \ldots, 2s,$

$(0, r, 8s - r + 4), r = 1, \ldots, s - 1,$

$(0, r, 7s - r + 3), r = 1, \ldots, s - 2,$ (omit if $s = 2$),

$(0, r, 6s + 3),$

$(0, r, 6s + 2),$

$(0, r, 6s + 4),$

$(0, r, 7s + 4),$

$(0, r, 8s + 6).$

If $s = 0$, take the triples (0, 1, 4)(0, 2, 8).

If $s = 1$, take the triples (0, 1, 12)(0, 2, 9)(0, 3, 16)(0, 4, 15)(0, 5, 17)(0, 6, 20).

This completes the proof. □

2.3 Existence of Mixed Triple Systems

The next result [10] guarantees the existence of T_i, i=1,2,3 triple systems and the constructions also give sufficient conditions for the existence of cyclic mixed triple systems.

A T_i, for $i = 1, 2, 3$ triple system exists of order v if and only if

$v \equiv 1 \pmod 2$ except $v \in \{3,5\}$ and $i = 3$.

We shall provide the proof for T_1 (and hence T_2).

The necessary condition follows from the fact that in M_v each vertex is in $(v-1)$ edges and, since there are twice as many arcs as edges, each vertex must be in $2(v-1)$ arcs. In T_i, each vertex is an element of either one arc and one edge, or two arcs. Hence, we expect that $2 \mid 3(v-1)$; i.e., $3(v-1) \equiv 0 \pmod 2$. Hence, v must be odd since $(v-1)$ is even.

We shall prove, using two cases, that if $v \equiv 1 \pmod 2$, then a T_1 triple system exists.

Case 1. Suppose $v \equiv 1 \pmod 4$, say $v = 4t+1$. Consider the blocks:

$(j, 2t-i+j, 2t+1+i+j)$, for $i = 0, 1, \ldots, t-1$, and $j = 0, 1, \ldots, 4t$;

and

$(j, 4t-i+j, 1+i+j)$, for $i = 0, 1, \ldots, t-1$, and $j = 0, 1, \ldots, 4t$.

Case 2. Suppose $v \equiv 3 \pmod 4$, say $v = 4t+3$. Consider the blocks:

$(j, 2t+1-i+j, 2t+2+i+j)$, for $i = 0, 1, \ldots, t$, and $j = 0, 1, \ldots, 4t+2$;

and $(j, 4t+2-i+j, 1+i+j)$, for $i = 0, 1, \ldots, t-1$, and $j = 0, 1, \ldots, 4t$.

In both cases, the given blocks form a T_1 - triple system.

This establishes the required result. $\square$

We note here that the above proof makes use of cyclic automorphisms.

2.4 Bicyclic Mixed Triples

In this section, we consider bicyclic mixed triple systems of order $v = N_1 + N_2$ and the automorphism $\pi = (0_1, 1_1, 2_1, \cdots, (N_1-1)_1)\ (0_2, 1_2, 2_2, \cdots, (N_2-1)_2)$, as shown below.

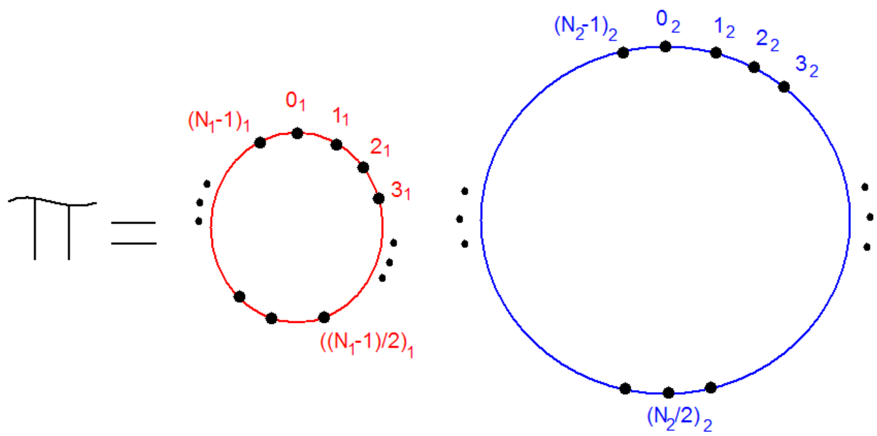

Figure 5: A Bicyclic Automorphism

Lemma 2.1 *If a bicyclic mixed triple system exists with N_1 and N_2 as described above, then $N_1 \equiv 1$ (mod 2) and $N_2 = kN_1$ for some even $k \in \mathbb{N}$.*

Proof. First, consider a mixed triple system admitting an automorphism α. Suppose α fixes vertices a and b: $\alpha(a) = a$ and $\alpha(b) = b$. Edge (a, b) is in some mixed triple, say T_i^{ab} where the vertex set of T_i^{ab} is $\{a, b, c\}$. If we apply α to T_i^{ab} we see that edge (a, b) is an edge of $\alpha(T_i^{ab})$. However, since edge (a, b) occurs in only one triple, it must be that $\alpha(T_i^{ab}) = T_i^{ab}$. From this observation follows the fact that the fixed points of α form a subsystem of the original system (that is, for any a and b fixed by α, there is a triple fixed under α which contains edge (a, b), a triple fixed under α which contains arc $[a, b]$, and a triple fixed under α which contains arc $[b, a]$). Hence the number of fixed points of an automorphism must be odd. Now, if π is the bicyclic automorphism, then π^{N_1} has N_1 fixed points and so $N_1 \equiv 1 \pmod 2$. Since $v = N_1 + N_2 \equiv 1 \pmod 2$, it follows that N_2 is even. This takes care of one part of the lemma.

To establish the divisibility which exists on the sizes of the two cycles, N_1 and N_2, let us again consider some T_i for $i = 1, 2$ as shown in figure 6 below with vertex set $\{a, b, c\}$ where the point a comes from the first cycle and b, c are points of the second cycle i.e. $a \in (0_1, 1_1, 2_1, \cdots, (N_1-1)_1)$ while $\{b, c\} \subset (0_2, 1_2, 2_2, \cdots, (N_2-1)_2)$.

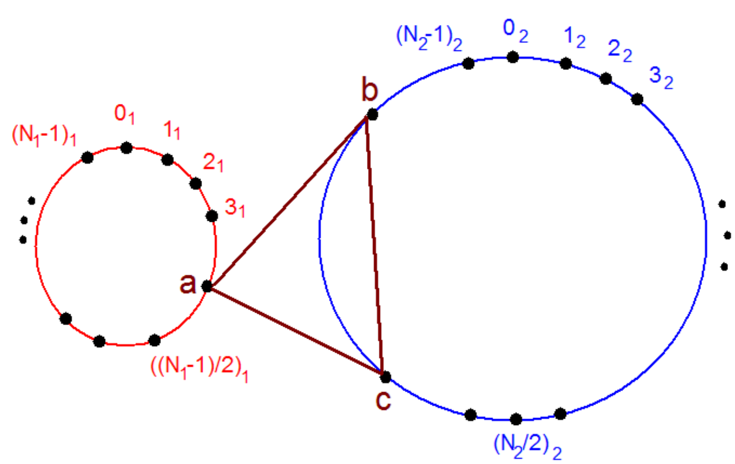

Figure 6: Decompositions

When we apply π^{N_2} to this triple, we see that $\pi^{N_2}(b) = b$ and $\pi^{N_2}(c) = c$, and hence $\pi^{N_2}(a)$ must be a. Thus, π^{N_2} fixes points $\{0_2, 1_2, \ldots, (N_2 - 1)_2\}$. However, $N_2 \equiv 0 \pmod 2$, so these cannot be the only fixed points. Therefore, π^{N_2} must fix all v points and thus N_2 is a multiple of N_1. A more elementary proof is given in [2]. □

Theorem 2.1 *A bicyclic T_1-triple system exists admitting an automorphism consisting of a cycle of length N_1 and a cycle of length N_2, where $N_1 < N_2$, if and only if $N_1 \equiv 1$ (mod 2) and $N_2 = kN_1$ for some even $k \in \mathbb{N}$.*

Proof. The necessary conditions follow from Lemma 2.1. For sufficiency, we consider cases.

Case 1. If $N_2 \equiv 0 \pmod 4$, then consider the blocks:

$$\left[0_2, \left(\frac{N_2}{4} - 1 - i\right)_2, \left(\frac{N_2}{4} + 1 + i\right)_2\right]_1 \text{ for } i = 0, 1, \ldots, \frac{N_2 - 2N_1 - 6}{4},$$
$$\left[0_2, \left(\frac{3N_2}{4} - 1 - i\right)_2, \left(\frac{3N_2}{4} + i\right)_2\right]_1 \text{ for } i = 0, 1, \ldots, \frac{N_2}{4} - 1,$$
$$\left[0_1, \left(\frac{N_1 - 3}{2} - i\right)_2, \left(\frac{N_2 - N_1 - 1}{2} + i\right)_2\right]_1 \text{ for } i = 0, 1, \ldots, \frac{N_1 - 3}{2}.$$

If $N_1 \equiv 1 \pmod 4$, then also take the blocks:

$$[0_2, i_1, (2i + 1)_2]_1 \text{ for } i = 0, 1, \ldots, \frac{N_1 - 5}{4},$$
$$\left[0_2, \left(\frac{N_1 - 1}{4} + i\right)_1, \left(\frac{N_2 - N_1 + 1}{2} + 2i\right)_2\right]_1 \text{ for } i = 0, 1, \ldots, \frac{N_1 - 5}{4},$$
$$\left[0_2, \left(\frac{N_1 + 1}{2} + i\right)_1, (2 + 2i)_2\right]_1 \text{ for } i = 0, 1, \ldots, \frac{N_1 - 5}{4},$$
$$\left[0_2, \left(\frac{3N_1 + 1}{4} + i\right)_1, \left(\frac{N_2 - N_1 + 3}{2} + 2i\right)_2\right]_1 \text{ for } i = 0, 1, \ldots, \frac{N_1 - 5}{4},$$
$$\left[0_2, \left(\frac{N_1 - 1}{2}\right)_1, \left(\frac{N_2}{4}\right)_2\right]_1 \text{ and } \left[0_1, (N_1 - 1)_2, \left(\frac{N_2 + 2N_1 - 2}{2}\right)_2\right]_1.$$

If $N_1 \equiv 3 \pmod 4$, then also take the blocks:

$[0_2, i_1, (1+2i)_2]_1$ for $= 0, 1, \ldots, \dfrac{N_1-3}{4}$,

$\left[0_2, \left(\dfrac{N_1+1}{4}+i\right)_1, \left(\dfrac{N_2-N_1+3}{2}+2i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1-7}{4}$,

$\left[0_2, \left(\dfrac{N_1+1}{2}+i\right)_1, (2+2i)_2\right]_1$ for $= 0, 1, \ldots, \dfrac{N_1-7}{4}$,

$\left[0_2, \left(\dfrac{3N_1+3}{4}+i\right)_1, \left(\dfrac{N_2-N_1+5}{2}+2i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1-7}{4}$,

$\left[0_2, \left(\dfrac{N_1-1}{2}\right)_1, \left(\dfrac{N_2}{4}\right)_2\right]_1$, $\left[0_2, \left(\dfrac{3N_1-1}{4}\right)_1, \left(\dfrac{N_2-N_1+1}{2}\right)_2\right]_1$, and

$\left[0_1, (N_1-1)_2, \left(\dfrac{N_2+2N_1-2}{2}\right)_2\right]_1$.

Case 2. If $N_2 \equiv 2 \pmod 4$, then consider the blocks:

$[0_2, i_1, (2i+1)_2]_1$ for $i = 0, 1, \ldots, \dfrac{N_1-3}{2}$,

$\left[0_2, \left(\dfrac{N_1+1}{2}+i\right)_1, (2+2i)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1-3}{2}$,

$\left[0_2, \left(\dfrac{N_2+2N_1-4}{4}-i\right)_2, \left(\dfrac{N_2+2N_1+4}{4}+i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_2-2N_1-4}{4}$,

$\left[0_2, \left(\dfrac{3N_2-2}{4}-i\right)_2, \left(\dfrac{3N_2+2}{4}+i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_2-6}{4}$,

$\left[0_1, i_2, \left(\dfrac{N_2}{2}-1-i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1-3}{2}$,

$\left[0_1, \left(\dfrac{N_1-1}{2}\right)_2, \left(\dfrac{N_2+N_1-1}{2}\right)_2\right]_1$, and $\left[0_2, \left(\dfrac{N_1-1}{2}\right)_1, \left(\dfrac{N_2+2N_1}{4}\right)_2\right]_1$.

In each case, the collection of blocks, along with their images under the permutation $(0_1, 1_1, \cdots, (N_1-1)_1)$ $(0_2, 1_2, \cdots, (N_2-1)_2)$, and the blocks for a cyclic T_1-triple system on vertices $\{0_1, 1_1, \cdots, (N_1-1)_1\}$ form a bicyclic T_1-triple system. □

Corollary 2.1 *A bicyclic T_2-triple system exists admitting an automorphism consisting of a cycle of length N_1 and a cycle of length N_2, where $N_1 < N_2$, if and only if $N_1 \equiv 1 \pmod 2$ and $N_2 = kN_1$ for some even $k \in \mathbb{N}$.*

Proof. If we reverse each of the arcs of M_v, then we get back M_v. If we reverse each of the arcs of T_1, then we get T_2, and we say that T_1 is the *converse* of T_2. Therefore, the existence of a bicyclic T_1-triple system is equivalent to the existence of a bicyclic T_2-triple system. The result then follows from Theorem 2.1. □

Theorem 2.2 *A bicyclic T_3-triple system does not exist.*

Proof. Suppose, to the contrary, that such a system does exist. By Lemma 2.1, N_2 is even. The edge $(0_2, (N_2/2)_2)$ must be in some T_3, either $T_3^a = [a_2, 0_2, (N_2/2)_2]_3$ or $T_3^b = [b_1, 0_2, (N_2/2)_2]_3$. Applying $\pi^{N_2/2}$ we fix edge $(0_2, (N_2/2)$ and get $\pi^{N_2/2}(T_3^a) = [(a + N_2/2)_2, (N_2/2)_2, 0_2)]_3 = T_3^{a\prime}$ and $\pi^{N_2/2}(T_3^b) = [b_1, (N_2/2)_2, 0_2]_3 = T_3^{b\prime}$. So we need $T_3^a = T_3^{a\prime}$ or $T_3^b = T_3^{b\prime}$, both contradictions.

2.5 Rotational Mixed Triples

We introduce a result on rotational triple systems by Phelps and Rosa [19]. A *k-rotational* Steiner triple system refers to an automorphism $[1, 0, \ldots, 0, k, 0, \ldots, 0]$, having one fixed point and k cycles of length $(v - 1)/k$ each. The following is a necessary and sufficient condition for the existence of a 1-rotational $STS(v)$.

A 1-rotational $STS(v)$ exists if and only if $v \equiv 3$ or 9 (mod 24).

Indeed, suppose there exists such a system and let us denote the automorphism of our 1-rotational $STS(v)$ by $\pi = (\infty)(0, 1, \ldots, v - 2)$ over the set $\Gamma = Z_{v-1} \cup \{\infty\}$. There are $\frac{1}{2}(v - 1)$ blocks containing ∞, each of the form $(\infty,\ i,\ i + \frac{1}{2}(v - 1))$.

All blocks of the $STS(v)$ not containing ∞ are partitioned into orbits under π of length $(v-1)$ except possibly a short orbit of length $\frac{1}{3}(v-1)$ which contains the block $(0, \frac{1}{3}(v-1)), \frac{2}{3}(v-1))$.

No 1-rotational $STS(v)$ contains blocks of the short orbit since this would require $v \equiv 1 \pmod 6$, and also, the remaining $\frac{1}{6}(v-1)(v-5)$ blocks not in the short orbit or containing ∞ must be partitioned into $\frac{1}{6}(v-5)$ orbits of length $(v-1)$. But if $v \equiv 1 \pmod 6$, then 6 will not divide $(v-5)$. Thus, the $\frac{1}{6}(v-1)(v-3)$ blocks not containing ∞ must be partitioned into $\frac{1}{6}(v-1)$ orbits of length $(v-1)$. This allows us to conclude that $v \equiv 3 \pmod 6$.

Since $(v-1)$ is even, as v is odd, the automorphism π raised to the power of $\left(\frac{v-1}{2}\right)$ is a permutation of type $[1,\frac{1}{2}(v-1),0,\ldots,0]$. The 1-rotational $STS(v)$ is also a reverse $STS(v)$ (as we shall see next) and so $v \equiv$ 1,3,9 or 19 (mod 24). Thus, we have that $v \equiv 3$ or 9 (mod 24). The converse is established from constructions of the triples.

A T_i-triple system of order v is said to be *rotational* if it admits an automorphism consisting of a fixed point and a cycle of length $(v-1)$. By taking $N_1 = 1$ in the previous section, we have necessary and sufficient conditions for the existence of a rotational T_i-triple system.

Theorem 2.3 *A rotational T_i-triple system of order v*

(i) *exists if and only if $v \equiv 1 \ (mod\ 2)$ when $i \in \{1, 2\}$, and*

(ii) *does not exist when $i = 3$.*

2.6 Reverse Mixed Triples

A T_i-triple system of order v is said to be *reverse* if it admits an automorphism consisting of a fixed point and $(v-1)/2$ transpositions. The existence of reverse T_i-triple systems follows easily from the existence of rotational T_i-triple systems.

Theorem 2.4 *A reverse T_i-triple system of order v,*

(i) *exists if and only if $v \equiv 1 \pmod 2$ when $i \in \{1, 2\}$, and*

(ii) *does not exist when $i = 3$.*

Proof. When $i \in \{1, 2\}$, for all $v \equiv 1 \pmod 2$, there exists a rotational T_i-triple system of order v admitting an automorphism π consisting of a fixed point and a cycle of length $(v-1)$. By considering $\pi^{(v-1)/2}$, we see that a reverse T_i-triple system exists for all $v \equiv 1 \pmod 2$.

When $i = 3$, we suppose a reverse T_3-triple system exists admitting the automorphism $\pi = (\infty)(0_1, 1_1)(0_2, 1_2)\cdots(0_{(v-1)/2}, 1_{(v-1)/2})$. The edge $(0_1, 1_1)$ must be in some triple, say $T_3^1 = [x, 0_1, 1_1]_3$. Now $\pi(T_3^1) = [\pi(x), 1_1, 0_1]_3$ contains edge $(0_1, 1_1)$ and so must also contain arcs $[x, 0_1]$ and $[1_1, x]$. However, $\pi(T_3^1)$ does not contain these arcs and this contradiction shows that no such T_3-triple system exists. □

3 VERIFICATION OF RESULTS AND SOME EXAMPLES

3.1 Verifications

In this chapter, we shall explore the *difference method* in detail to show how we came about with the results in chapter 2. We shall conclude the chapter with some examples. We outline the following basic notation to be used in this chapter.

For the given sets $V_1 = \{0_1, 1_1, 2_1, \cdots, (N_1 - 1)_1\}$, $V_2 = \{0_2, 1_2, 2_2, \cdots, (N_2 - 1)_2\}$, with associated permutation $\pi = (0_1, 1_1, 2_1, \cdots, (N_1 - 1)_1)(0_2, 1_2, 2_2, \cdots, (N_2 - 1)_2)$, let e_2 denote a type of difference associated with an edge of the form $(x_2\ y_2)$ which is defined to be the $\min\{(x_2 - y_2)(\text{mod } N_2),\ (y_2 - x_2)(\text{mod } N_2)\}$, and called a *pure type 2 edge difference*; let e_{12} denote a type of difference associated with an edge of the form $(x_1\ y_2)$ which is $(y - x)(\text{mod } N_1)$, and called a *mixed type 12 edge difference*; let a_2 denote the *pure arc difference* within V_2 of *type 2*; let a_{12} denote the *mixed arc difference* for an arc from V_1 into V_2; and let a_{21} denotes the *mixed arc difference* for an arc from V_2 into V_1. We note that *block* $[x_2,\ y_2,\ z_2]_1$ has associated differences of $e_2 : z_2$ - y_2; $a_2 : z_2$ - x_2; $a_2 : y_2$ - x_2. Also, for the *block* $[x_1,\ y_2,\ z_2]_1$ we have $e_2 : z_2$ - y_2 and $a_{12} : (z_2$ - $x_1)$ (mod N_1) as well as $a_{12} : (y_2$ - $x_1)$ (mod N_1). Finally, for the *block* $[x_2,\ y_1,\ z_2]_1$ we have $e_{12} : (z_2$ - $y_1)$ (mod N_1) and $a_2 : z_2$ - x_2 while $a_{21} : (y_1$ - $x_2)$ (mod N_1). In the section that follows, superscripts denote sets and serve as counters.

Case 1. If $N_2 \equiv 0 \pmod 4$, then from the block: $\left[0_2, \left(\frac{N_2}{4} - 1 - i\right)_2, \left(\frac{N_2}{4} + 1 + i\right)_2\right]_1$

for $i = 0, 1, \ldots, \dfrac{N_2 - 2N_1 - 6}{4}$,

we have the following differences:

$$e_2^1 := \left\{2, 4, 6, \ldots, \frac{N_2 - 2N_1 - 6}{2}, \frac{N_2 - 2N_1 - 2}{2}\right\}$$
$$a_2^2 := \left\{(\frac{N_2}{4} + 1), (\frac{N_2}{4} + 2), \ldots, \frac{N_2 - N_1 - 3}{2}, \frac{N_2 - N_1 - 1}{2}\right\}$$
$$a_2^1 := \left\{\frac{N_1 + 1}{2}, \frac{N_1 + 3}{2}, \ldots, (\frac{N_2}{4} - 3), (\frac{N_2}{4} - 2), (\frac{N_2}{4} - 1)\right\}.$$

For the block: $\left[0_2, \left(\dfrac{3N_2}{4} - 1 - i\right)_2, \left(\dfrac{3N_2}{4} + i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_2}{4} - 1$,

we have the following differences:

$$e_2^2 := \left\{1, 3, 5, \ldots, (\frac{N_2}{2} - N_1), \ldots, (\frac{N_2}{2} - 3), (\frac{N_2}{2} - 1)\right\}$$
$$a_2^4 := \left\{\frac{3N_2}{4}, (\frac{3N_2}{4} + 1), \ldots, (N_2 - 2), (N_2 - 1)\right\}$$
$$a_2^3 := \left\{\frac{N_2}{2}, (\frac{N_2}{2} + 1), \ldots, (\frac{3N_2}{4} - 2), (\frac{3N_2}{4} - 1)\right\}.$$

For the block: $\left[0_1, \left(\dfrac{N_1 - 3}{2} - i\right)_2, \left(\dfrac{N_2 - N_1 - 1}{2} + i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 3}{2}$,

we have the following differences:

$$e_2^3 := \left\{\frac{N_2 - 2N_1 + 2}{2}, \frac{N_2 - 2N_1 + 6}{2}, \ldots, \frac{N_2 - 8}{2}, \frac{N_2 - 4}{2}\right\}$$
$$a_{12}^2 := \left\{\frac{N_1 - 1}{2}, (\frac{N_1 - 1}{2} + 1), \ldots, (N_1 - 3)(N_1 - 2)\right\}$$
$$a_{12}^1 := \left\{0, 1, 2, \ldots, \frac{N_1 - 5}{2}, \frac{N_1 - 3}{2}\right\}.$$

For the block: $[0_2, i_1, (2i + 1)_2]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 5}{4}$,

we have the following differences:

$$e_{12}^1 := \left\{1, 2, 3, \ldots, \frac{N_1 - 5}{4}, \frac{N_1 - 1}{4}\right\}$$
$$a_{21}^1 := \left\{0, 1, 2, \ldots, \frac{N_1 - 9}{4}, \frac{N_1 - 5}{4}\right\}$$
$$a_2^5 := \left\{1, 3, 5, \ldots, \frac{N_1 - 7}{2}, \frac{N_1 - 3}{2}\right\}.$$

For the block: $\left[0_2, \left(\frac{N_1-1}{4}+i\right)_1, \left(\frac{N_2-N_1+1}{2}+2i\right)_2\right]_1$ for $i=0,1,\ldots,\frac{N_1-5}{4}$,

we have the following differences:

$$e_{12}^2 := \left\{\frac{N_1+3}{4}, \frac{N_1+7}{4}, \ldots, \frac{N_1-3}{2}, \frac{N_1-1}{2}\right\}$$
$$a_{21}^2 := \left\{\frac{N_1-1}{4}, \frac{N_1+3}{4}, \ldots, \frac{N_1-5}{2}, \frac{N_1-3}{2}\right\}$$
$$a_2^6 := \left\{\frac{N_2-N_1+1}{2}, \frac{N_2-N_1+5}{2}, \ldots, \frac{N_2-8}{2}, \frac{N_2-4}{2}\right\}.$$

For the block: $\left[0_2, \left(\frac{N_1+1}{2}+i\right)_1, (2+2i)_2\right]_1$ for $i=0,1,\ldots,\frac{N_1-5}{4}$,

we have the following differences:

$$e_{12}^3 := \left\{\frac{N_1+3}{2}, \frac{N_1+5}{2}, \ldots, \frac{3N_1-3}{4}, \frac{3N_1+1}{4}\right\}$$
$$a_{21}^3 := \left\{\frac{N_1+1}{2}, \frac{N_1+3}{2}, \ldots, \frac{3N_1-7}{4}, \frac{3N_1-3}{4}\right\}$$
$$a_2^7 := \left\{2, 4, \ldots, \frac{N_1-5}{2}, \frac{N_1-1}{2}\right\}.$$

For the block: $\left[0_2, \left(\frac{3N_1+1}{4}+i\right)_1, \left(\frac{N_2-N_1+3}{2}+2i\right)_2\right]_1$ for $i=0,1,\ldots,\frac{N_1-5}{4}$,

we have the following differences:

$$e_{12}^4 := \left\{\frac{3N_1+5}{4}, \frac{3N_1+9}{4}, \ldots, (N_1-1), (N_1)\right\}$$
$$a_{21}^4 := \left\{\frac{3N_1+1}{4}, \frac{3N_1+5}{4}, \ldots, (N_1-2), (N_1-1)\right\}$$
$$a_2^8 := \left\{\frac{N_2-N_1+3}{2}, \frac{N_2-N_1+7}{2}, \ldots, \frac{N_2-6}{2}, \frac{N_2-2}{2}\right\}.$$

For the block: $\left[0_2, \left(\frac{N_1-1}{2}\right)_1, \left(\frac{N_2}{4}\right)_2\right]_1$

we have the following differences:

$$e_{12}^5 := \left\{\frac{N_2-2N_1+2}{4}\right\} \equiv \left\{\frac{N_1+1}{2}\right\} (\text{mod } N_1)$$
$$a_{21}^5 := \left\{\frac{N_1-1}{2}\right\}$$
$$a_2^9 := \left\{\frac{N_2}{4}\right\}.$$

Finally for the block: $\left[0_1, (N_1-1)_2, \left(\frac{N_2+2N_1-2}{2}\right)_2\right]_1$, we have the following differences:

$e_2^4 := \left\{\frac{N_2}{2}\right\}$

$a_{12}^3 := \{(N_1-1)\}$

$a_{12}^4 := \left\{\frac{N_2+N_1-2}{2}\right\} \equiv \{(N_1-1)\}$ as $\frac{N_2}{2} \equiv 0(\text{mod } N_1)$.

We therefore have $a_{12}^3 \equiv a_{12}^4$. With all the base blocks taken care of, we can now count to make sure that all the arc and edge differences have been taken care of as well.

Starting with the pure type 2 arc differences, we see that if we consider a_2^5 and a_2^7 (alternately), then we shall have the differences:

$$\left\{1, 2, 3, \ldots \frac{N_1-7}{2}, \frac{N_1-5}{2}, \frac{N_1-3}{2}, \frac{N_1-1}{2}\right\}.$$

Next in the series, we consider a_2^1, a_2^9, a_2^2 respectively, to get

$$\left\{\frac{N_1+1}{2}, \frac{N_1+3}{2}, \ldots, (\frac{N_2}{4}-2), (\frac{N_2}{4}-1), (\frac{N_2}{4})\right\} \bigcup$$
$$\left\{(\frac{N_2}{4}+1), (\frac{N_2}{4}+2), \ldots, \frac{N_2-N_1-3}{2}, \frac{N_2-N_1-1}{2}\right\}.$$

We next alternate between a_2^6 and a_2^8 to obtain the differences:

$$\left\{\frac{N_2-N_1+1}{2}, \frac{N_2-N_1+3}{2}, \frac{N_2-N_1+5}{2}\right\} \bigcup$$
$$\left\{\frac{N_2-N_1+7}{2}, \ldots, \frac{N_2-8}{2}, \frac{N_2-6}{2}, \frac{N_2-4}{2}, \frac{N_2-2}{2}\right\}.$$

Finally, taking respectively a_2^3 and a_2^4, we obtain the following differences:

$$\left\{\frac{N_2}{2}, (\frac{N_2}{2}+1), \ldots, (\frac{3N_2}{4}-2), (\frac{3N_2}{4}-1), (\frac{3N_2}{4}), (\frac{3N_2}{4}+1), \ldots, (N_2-2), (N_2-1)\right\}.$$

This exhausts all the possible arc differences ((N_2-1) in number) that exist between the vertices of the big cycle N_2. Worthy of note is the fact that there are no 0 arcs differences since loops are not allowed in our M_v. An arc is between two non-identical vertices.

For verification purposes, consider for example, $N_1 = 41$ and $N_2 = 164$. We obtain the following sets:

$a_2^5 := \{1, 3, 5, \ldots, 17, 19\}$

$a_2^7 := \{2, 4, \ldots, 18, 20\}$

$a_2^1 := \{21, 22, \ldots, 39, 40\}$

$a_2^9 := \{41\}$

$a_2^2 := \{42, 43, \ldots, 60, 61\}$

$a_2^6 := \{62, 64, \ldots, 78, 80\}$

$a_2^8 := \{63, 65, \ldots, 79, 81\}$

$a_2^3 := \{82, 83, \ldots, 121, 122\}$

$a_2^4 := \{123, 124, \ldots, 162, 163\}$.

When put together, we obtain the arc differences from 1 to 163 as expected.

A similar approach can be applied to the rest of the results. For the *pure type 2* edge differences, we proceed as follows:

We first alternate between e_2^2 and e_2^1 to obtain the edge differences:

$$\left\{1, 2, 3, \ldots, (\frac{N_2}{2} - (N_1 + 2)), (\frac{N_2}{2} - (N_1 + 1))\right\}.$$

Next, we alternate e_2^2 and e_2^3 to obtain the edge differences:

$$\left\{(\frac{N_2}{2} - N_1), (\frac{N_2}{2} - (N_1 - 1)), (\frac{N_2}{2} - (N_1 - 2)), \ldots, (\frac{N_2}{2} - 4), (\frac{N_2}{2} - 3), (\frac{N_2}{2} - 2), (\frac{N_2}{2} - 1)\right\}.$$

Finally, we have $\frac{N_2}{2}$, to complete our list in total of $\frac{N_2}{2}$ *pure type 2* edge differences.

As an illustration, we see that $N_2 = 164$ and $N_1 = 41$ will give us the numbers

$\{1, 2, 3, \ldots, 39, 40\}$ followed by the numbers $\{41, 42, 43, \ldots, 78, 79, 80, 81\}$ and finally the *pure type 2* edge difference of 82. We now count mixed differences. For the arc differences, we first consider a_{12}^1, followed by a_{12}^2 to obtain the the following difference set $\left\{0, 1, 2, 3, \ldots, \frac{N_1 - 3}{2}, \frac{N_1 - 1}{2}, \ldots, (N_1 - 3), (N_1 - 2)\right\}$ and from the fact that $a_{12}^3 \equiv a_{12}^4$, we have $(N_1 - 1)$. This takes care of all the N_1 total mixed arc differences between the two cycles. Considering $a_{21}^1, a_{21}^2, a_{21}^5, a_{21}^3$ respectively, followed by a_{21}^4, we obtain the following arc differences for arcs from cycle 2 to cycle 1:

$$\left\{0, 1, 2, 3, \ldots, \frac{N_1 - 9}{4}, \frac{N_1 - 5}{4}\right\} \bigcup$$
$$\left\{\frac{N_1 - 1}{4}, \frac{N_1 + 3}{4}, \ldots, \frac{N_1 - 5}{2}, \frac{N_1 - 3}{2}, \frac{N_1 - 1}{2}\right\} \bigcup$$
$$\left\{\frac{N_1 + 1}{2}, \frac{N_1 + 3}{2}, \ldots, \frac{3N_1 - 7}{4}, \frac{3N_1 - 3}{4}\right\} \bigcup$$
$$\left\{\frac{3N_1 + 1}{4}, \frac{3N_1 + 5}{4}, \ldots, (N_1 - 3), (N_1 - 2), (N_1 - 1)\right\}.$$

This exhausts all the N_1 possible arc differences that exists for arcs from the big cycle to the small cycle. We now count the edge differences between the two cycles. Taking the sets $e_{12}^1, e_{12}^2, e_{12}^5, e_{12}^3$ in that order, followed by e_{21}^4, as we had it for the $2 \to 1$ arcs, we obtain the following arc differences:

$$\left\{1, 2, 3, \ldots, \frac{N_1 - 9}{4}, \frac{N_1 - 5}{4}, \frac{N_1 - 1}{4}\right\} \bigcup$$
$$\left\{\frac{N_1 + 3}{4}, \frac{N_1 + 7}{4}, \ldots, \frac{N_1 - 5}{2}, \frac{N_1 - 3}{2}, \frac{N_1 - 1}{2}\right\} \bigcup$$
$$\left\{\frac{N_1 + 1}{2}, \frac{N_1 + 3}{2}, \frac{N_1 + 5}{2}, \ldots, \frac{3N_1 - 3}{4}, \frac{3N_1 + 1}{4}\right\} \bigcup$$
$$\left\{\frac{3N_1 + 5}{4}, \frac{3N_1 + 9}{4}, \ldots, (N_1 - 3), (N_1 - 2), (N_1 - 1), N_1\right\}.$$

This exhausts all the N_1 edge differences between the two cycles.

We have thus counted all the pure and mixed arc and edge differences within the bigger cycle, N_2 and between the two cycles with all the differences covered and no repetition observed. This verifies our result for the case $N_2 \equiv 0 \pmod 4$ and $N_1 \equiv 1 \pmod 4$.

If $N_1 \equiv 3 \pmod 4$, (recall that $N_2 \equiv 0 \pmod 4$), then (also) take the blocks: $[0_2, i_1, (1+2i)_2]_1$ for $= 0, 1, \ldots, \frac{N_1-3}{4}$,

and have the following differences:

$a_2^1 := \left\{1, 3, 5, \ldots, \frac{N_1-5}{2}, \frac{N_1-1}{2}\right\}$.

For the block: $\left[0_2, \left(\frac{N_1+1}{4}+i\right)_1, \left(\frac{N_2-N_1+3}{2}+2i\right)_2\right]_1$ for $i = 0, 1, \ldots, \frac{N_1-7}{4}$,

we have the following differences:

$a_2^2 := \left\{2, 4, \ldots, \frac{N_1-7}{2}, \frac{N_1-3}{2}\right\}$.

For the block: $\left[0_2, \left(\frac{N_1+1}{2}+i\right)_1, (2+2i)_2\right]_1$ for $= 0, 1, \ldots, \frac{N_1-7}{4}$,

we have the following differences:

$a_2^3 := \left\{\frac{N_2-N_1+3}{2}, \frac{N_2-N_1+7}{2}, \ldots, \frac{N_2-8}{2}, \frac{N_2-4}{2}\right\}$.

For the block: $\left[0_2, \left(\frac{3N_1+3}{4}+i\right)_1, \left(\frac{N_2-N_1+5}{2}+2i\right)_2\right]_1$ for $i = 0, 1, \ldots, \frac{N_1-7}{4}$,

we have the following differences:

$a_2^4 := \left\{\frac{N_2-N_1+5}{2}, \frac{N_2-N_1+9}{2}, \ldots, \frac{N_2-10}{2}, \frac{N_2-6}{2}, \frac{N_2-2}{2}\right\}$.

For the block: $\left[0_2, \left(\frac{N_1-1}{2}\right)_1, \left(\frac{N_2}{4}\right)_2\right]_1$,

we have the difference:

$a_2^5 := \left\{\frac{N_2}{4}\right\}$

Finally, for the block: $\left[0_2, \left(\frac{3N_1-1}{4}\right)_1, \left(\frac{N_2-N_1+1}{2}\right)_2\right]_1$,

we have the difference:

$a_2^6 := \left\{ \dfrac{N_2 - N_1 + 1}{2} \right\}.$

As was with the case $N_1 \equiv 1 \pmod 4$, we see that the following four additional differences are obtained:

$$a_2^7 := \left\{ (\frac{N_2}{4} + 1), (\frac{N_2}{4} + 2), \ldots, \frac{N_2 - N_1 - 3}{2}, \frac{N_2 - N_1 - 1}{2} \right\},$$
$$a_2^8 := \left\{ \frac{N_1 + 1}{2}, \frac{N_1 + 3}{2}, \ldots, (\frac{N_2}{4} - 3), (\frac{N_2}{4} - 2), (\frac{N_2}{4} - 1) \right\},$$
$$a_2^9 := \left\{ \frac{3N_2}{4}, (\frac{3N_2}{4} + 1), \ldots, (N_2 - 2), (N_2 - 1) \right\}, \text{ and}$$
$$a_2^3 := \left\{ \frac{N_2}{2}, (\frac{N_2}{2} + 1), \ldots, (\frac{3N_2}{4} - 2), (\frac{3N_2}{4} - 1) \right\}.$$

To come up with an orderly count, we consider the difference sets. First, alternate the elements of a_2^1 and a_2^2, followed by the elements of the sets a_2^8, a_2^5, and then a_2^6 respectively. Next, alternate between a_2^3 and a_2^4. Finally, take a_2^{10} and then a_2^9 respectively to exhaust the list of $(N_2 - 1)$ pure type 2 arc differences.

As an illustration, using $N_2 = 2kN_1$ for the case $N_1 = 103$, $k = 2$ and $N_2 = 412$, we have the following difference sets:

$a_2^1 := \{1, 3, 5, \ldots, 49, 51\}$

$a_2^2 := \{2, 4, \ldots, 48, 50\}$

$a_2^8 := \{52, 53, \ldots, 101, 102\}$

$a_2^5 := \{103\}$

$a_2^7 := \{104, 105, \ldots, 153, 154\}$

$a_2^6 := \{155\}$

$a_2^3 := \{156, 158, \ldots, 202, 204\}$

$a_2^4 := \{157, 159, \ldots, 201, 203, 205\}$

$a_2^{10} := \{206, 207, \ldots, 307, 308\}$

$a_2^{9} := \{309, 310, \ldots, 410, 411\}$.

This gives a complete count of all the 411 arc differences as expected. We note here that the block $\left[0_1, (N_1-1)_2, \left(\frac{N_2+2N_1-2}{2}\right)_2\right]_1$ is the *short orbit block* and does not have a pure type 2 arc difference. We note that the differences e_2^1 and a_{12}^1 comes from the case $N_1 \equiv 1 \pmod 4$ and $N_2 \equiv 0 \pmod 4$.

A similar approach can be applied to the rest of the results. For the *mixed 12* edge differences, we proceed thus:

$$e_{12}^1 := \left\{1, 2, 3, \ldots, \frac{N_1-3}{4}, \frac{N_1+1}{4}\right\},$$
$$e_{12}^2 := \left\{\frac{N_1+3}{2}, \frac{N_1+5}{2}, \ldots, \frac{3N_1-5}{4}, \frac{3N_1-1}{4}\right\},$$
$$e_{12}^3 := \left\{\frac{N_1+5}{4}, \frac{N_1+9}{4}, \ldots, \frac{N_1-3}{2}, \frac{N_1-1}{2}\right\},$$
$$e_{12}^4 := \left\{\frac{3N_1+7}{4}, \frac{3N_1+11}{4}, \ldots, (N_1-2), (N_1-1), N_1\right\},$$
$$e_{12}^5 := \left\{\frac{N_1+1}{2}\right\},$$
$$e_{12}^6 := \left\{\frac{3N_1+3}{4}\right\}.$$

We thus exhaust our count by considering the respective difference sets: e_{12}^1, e_{12}^3, e_{12}^5, e_{12}^2, e_{12}^6 and finally e_{12}^4 . This will give a total of N_1 differences.

As an example, consider the entries $N_1 = 103$, $k = 2$ and $N_2 = 412$.

$e_{12}^1 := \{1, 2, 3, \ldots, 25, 26\}$,

$e_{12}^3 := \{27, 28, \ldots, 50, 51\}$,

$e_{12}^5 := \{52\}$,

$e_{12}^2 := \{53, 54, \ldots, 76, 77\}$,

$e_{12}^6 := \{78\}$,

$e_{12}^4 := \{79, 80, \ldots, 101, 102, 103\}$.

The *type 21 mixed* arc differences are obtained using the sets:

$$a_{21}^1 := \left\{0, 1, 2, \ldots, \frac{N_1-7}{4}, \frac{N_1-3}{4}\right\},$$
$$a_{21}^2 := \left\{\frac{N_1+1}{2}, \frac{N_1+3}{2}, \ldots, \frac{3N_1-9}{4}, \frac{3N_1-5}{4}\right\},$$
$$a_{21}^3 := \left\{\frac{N_1+1}{4}, \frac{N_1+5}{4}, \ldots, \frac{N_1-5}{2}, \frac{N_1-3}{2}\right\},$$
$$a_{21}^4 := \left\{\frac{3N_1+3}{4}, \frac{3N_1+7}{4}, \ldots, (N_1-3), (N_1-2), (N_1-1)\right\},$$
$$a_{21}^5 := \left\{\frac{N_1-1}{2}\right\},$$
$$a_{21}^6 := \left\{\frac{3N_1-1}{4}\right\}.$$

We exhaust our count by considering the respective difference sets a_{21}^1, a_{21}^3, a_{21}^5, a_{21}^2, a_{21}^6 and finally a_{21}^4 .

As an example, consider $N_1 = 103$, $k = 2$ and $N_2 = 412$, and get the difference sets:

$a_{21}^1 := \{0, 1, 2, \ldots, 24, 25\}$,

$a_{21}^3 := \{26, 27, \ldots, 49, 50\}$,

$a_{21}^5 := \{51\}$,

$a_{21}^2 := \{52, 53, \ldots, 75, 76\}$,

$a_{21}^6 := \{77\}$,

$a_{21}^4 := \{78, 79, \ldots, 100, 101, 102\}$.

We therefore have all 103 mixed arc differences for arcs from N_2 to N_1 as expected.

In all, we have counted all the pure and mixed arc and edge differences within the bigger cycle, N_2, and between the two cycles, with all the differences covered and no repetition observed. This similarly verifies the result for the case $N_2 \equiv 0 \pmod 4$ and $N_1 \equiv 3 \pmod 4$.

Case 2. If $N_2 \equiv 2 \pmod 4$, then from the blocks: $[0_2, i_1, (2i+1)_2]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 3}{2}$,

we obtain the difference: $a_2^1 := \{1, 3, 5, \ldots, (N_1 - 4), (N_1 - 2)\}$.

For the blocks: $\left[0_2, \left(\dfrac{N_1+1}{2} + i\right)_1, (2+2i)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 3}{2}$,

we obtain the differences: $a_2^2 := \{2, 4, 6, \ldots, (N_1 - 3), (N_1 - 1)\}$.

For the blocks: $\left[0_2, \left(\dfrac{N_2 + 2N_1 - 4}{4} - i\right)_2, \left(\dfrac{N_2 + 2N_1 + 4}{4} + i\right)_2\right]_1$
for $i = 0, 1, \ldots, \dfrac{N_2 - 2N_1 - 4}{4}$,

we have the following set of differences:

$$a_2^3 := \left\{N_1, (N_1+1), \ldots, \frac{N_2 + 2N_1 - 8}{4}, \frac{N_2 + 2N_1 - 4}{4}\right\}, \text{ and}$$
$$a_2^4 := \left\{\frac{N_2 + 2N_1 + 4}{4}, \frac{N_2 + 2N_1 + 8}{4}, \ldots, \frac{N_2 - 3}{2}, \frac{N_2}{2}\right\}.$$

For the blocks: $\left[0_2, \left(\dfrac{3N_2 - 2}{4} - i\right)_2, \left(\dfrac{3N_2 + 2}{4} + i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_2 - 6}{4}$,

we have the following set of differences:

$$a_2^5 := \left\{\frac{N_2 + 2}{2}, \frac{N_2 + 4}{2}, \ldots, \frac{3N_2 - 6}{4}, \frac{3N_2 - 2}{4}\right\}, \text{ and}$$
$$a_2^6 := \left\{\frac{3N_2 + 2}{4}, \frac{3N_2 + 6}{4}, \ldots, (N_2 - 2), (N_2 - 1)\right\}.$$

Finally, for the blocks: $\left[0_2, \left(\dfrac{N_1 - 1}{2}\right)_1, \left(\dfrac{N_2 + 2N_1}{4}\right)_2\right]_1$, we have the difference:
$$a_2^7 := \left\{\frac{N_2 + 2N_1}{4}\right\}.$$

To have a complete count, we start by alternating elements of a_2^1 and a_2^2. We then consider a_2^3, next a_2^7, a_2^4, a_2^5 and finally a_2^6 in that order. This will give us all our

$(N_2 - 1)$ *pure type* 2 arc differences.

Example: Take N_1=55 and N_2=330. Then we obtain the sets:

$a_2^1 := \{1, 3, 5, \ldots, 51, 53\}$,

$a_2^2 := \{2, 4, 6, \ldots, 52, 54\}$,

$a_2^3 := \{55, 56, \ldots, 108, 109\}$,

$a_2^7 := \{110\}$,

$a_2^4 := \{111, 112, \ldots, 164, 165\}$,

$a_2^5 := \{166, 167, \ldots, 246, 247\}$,

$a_2^6 := \{248, 249, \ldots, 328, 329\}$.

The *pure type* 2 edge differences are obtained from the following sets:

$$e_2^1 := \left\{2, 4, 6, \ldots, \frac{N_2 - 2N_1 - 4}{2}, \frac{N_2 - 2N_1}{2}\right\},$$
$$e_2^2 := \left\{1, 3, 5, \ldots, \frac{N_2 - 2N_1 - 2}{2}, \frac{N_2 - 2N_1 + 2}{2}\right\},$$
$$e_2^3 := \left\{\frac{N_2 - 2N_1 + 6}{2}, \frac{N_2 - 2N_1 + 10}{2}, \ldots, \frac{N_2 - 8}{2}, \frac{N_2 - 4}{2}\right\},$$
$$e_2^4 := \left\{\frac{N_2 - 2N_1 + 4}{2}, \frac{N_2 - 2N_1 + 8}{2}, \ldots, \frac{N_2 - 6}{2}, \frac{N_2 - 2}{2}\right\},$$
$$e_2^5 := \left\{\frac{N_2}{2}\right\}.$$

We alternate between e_2^2 and e_2^1. We next take alternate elements of e_2^4 and e_2^3 and lastly, e_2^5.

Consider, for example, $N_2 = 330$ for $N_1 = 55$ as above. Then we have the sets:

$e_2^2 := \{1, 3, 5, \ldots, 109, 111\}$,

$e_2^1 := \{2, 4, 6, \ldots, 108, 110\}$,

$e_2^4 := \{112, 114, \ldots, 162, 164\}$,

$e_2^3 := \{113, 115, \ldots, 161, 163\}$,

$e_2^5 := \{165\}$.

We now present a similar discussion for mixed differences.

First the *type 21 mixed* arc differences are the following:

$$a_{21}^1 := \left\{0, 1, 2, \ldots, \frac{N_1 - 5}{2}, \frac{N_1 - 3}{2}\right\},$$
$$a_{21}^2 := \left\{\frac{N_1 + 1}{2}, \frac{N_1 + 3}{2}, \ldots, (N_1 - 2), (N_1 - 1)\right\},$$
$$a_{21}^3 := \left\{\frac{N_1 - 1}{2}\right\}.$$

We exhaust our count by considering the respective difference sets: a_{21}^1, followed by a_{21}^3 and finally a_{21}^2.

As an example, consider the entries: $N_1 = 55$ and $N_2 = 330$,

and get the difference set:

$a_{21}^1 := \{0, 1, 2, \ldots, 25, 26\}$,

$a_{21}^3 := \{27\}$,

$a_{21}^2 := \{28, 29, \ldots, 53, 54\}$.

We thus have all 55 mixed arc differences for arcs from N_2 to N_1 as expected.

Next, the *mixed 12* edge differences are obtained thus:

$$e_{12}^1 := \left\{1, 2, 3, \ldots, \frac{N_1 - 3}{2}, \frac{N_1 - 1}{2}\right\},$$
$$e_{12}^2 := \left\{\frac{N_1 + 3}{2}, \frac{N_1 + 5}{2}, \ldots, (N_1 - 1), N_1\right\},$$
$$e_{12}^3 := \left\{\frac{N_1 + 1}{2}\right\}.$$

By similarly taking e_{12}^1, followed by e_{12}^3 and finally e_{12}^2, we get all the required edge differences.

As an example, consider: $N_1 = 55$ and $N_2 = 330$,

and get the difference set:

$e_{12}^1 := \{1, 2, 3, \ldots, 26, 27\}$,

$e_{12}^3 := \{28\}$,

$e_{12}^2 := \{29, 30 \ldots, 54, 55\}$.

Finally, the *12 mixed* arc differences are:

$$a_{12}^1 := \left\{0, 1, 2, \ldots, \frac{N_1 - 5}{2}, \frac{N_1 - 3}{2}\right\},$$
$$a_{12}^2 := \left\{\frac{N_1 + 1}{2}, \frac{N_1 + 3}{2}, \ldots, (N_1 - 2), (N_1 - 1)\right\},$$
$$a_{12}^3 := \left\{\frac{N_1 - 1}{2}\right\} \equiv a_{12}^4.$$

The counting is done by considering, respectively, the difference sets:

a_{12}^1, followed by a_{12}^3 and finally a_{12}^2.

As an example, consider the entries: $N_1 = 55$ and $N_2 = 330$,

and get the difference set:

$a_{12}^1 := \{0, 1, 2, \ldots, 25, 26\}$,

$a_{12}^3 := \{27\}$,

$a_{12}^2 := \{28, 29, \ldots, 53, 54\}$.

We therefore have all 55 mixed arc differences for arcs from N_1 to N_2 as expected.

We have thus counted all the pure and mixed arc and edge differences within the bigger cycle N_2, and between the two cycles, with all the differences covered and no repetition observed as well. This similarly verifies the result for the case $N_2 \equiv 2 \pmod 4$ and $N_1 \equiv 3 \pmod 4$ and also concludes our verification page.

3.2 Basic Examples

We now give some examples to further illustrate the *difference method*. We shall have values for N_1 and N_2 with the understanding that k values are immediate. We shall also construct the different blocks.

Example 1. A *bicyclic* T_1 *decomposition* of the *complete mixed graph* on 9 vertices, M_9, with cycles of length $N_1 = 3$ and $N_2 = 6$ (i.e. $N_2 \equiv 2 \pmod 4$ and $N_1 \equiv 3 \pmod 4$)) has the following 7 *base blocks*:

$[0_2\ 0_1\ 1_2]$, $[0_2\ 1_1\ 3_2]$, $[0_2\ 2_1\ 2_2]$, $[0_1\ 0_2\ 2_2]$, $[0_2\ 4_2\ 5_2]$, $[0_1\ 1_2\ 4_2]$ and $[0_1\ 1_1\ 2_1]$.

This gives a total of (5 x 6)+3+3 = 36 blocks with 36 edges and 72 arcs since the arcs are twice as many as the edges. We recall that, in M_v, there are $\binom{v}{2}$ edges and $2\binom{v}{2}$ arcs.

We have the following blocks corresponding to each base block:

1. For the base block: $[0_2\ 0_1\ 1_2]$, we have:

$[0_2\ 0_1\ 1_2]$, $[1_2\ 1_1\ 2_2]$, $[2_2\ 2_1\ 3_2]$, $[3_2\ 0_1\ 4_2]$, $[4_2\ 1_1\ 5_2]$ and $[5_2\ 2_1\ 0_2]$.

2. For the base block: $[0_2\ 1_1\ 3_2]$, we have:

$[0_2\ 1_1\ 3_2]$, $[1_2\ 2_1\ 4_2]$, $[2_2\ 0_1\ 5_2]$, $[3_2\ 1_1\ 0_2]$, $[4_2\ 2_1\ 1_2]$ and $[5_2\ 0_1\ 2_2]$.

3. For the base block: $[0_2\ 2_1\ 2_2]$, we have:

$[0_2\ 2_1\ 2_2]$, $[1_2\ 0_1\ 3_2]$, $[2_2\ 1_1\ 4_2]$, $[3_2\ 2_1\ 5_2]$, $[4_2\ 0_1\ 0_2]$ and $[5_2\ 1_1\ 1_2]$.

4. For the block: $[0_1\ 0_2\ 2_2]$, we have:

$[0_1\ 0_2\ 2_2]$, $[1_1\ 1_2\ 3_2]$, $[2_1\ 2_2\ 4_2]$, $[0_1\ 3_2\ 5_2]$, $[1_1\ 4_2\ 0_2]$, $[2_1\ 5_2\ 1_2]$.

5. For the block: $[0_2\ 4_2\ 5_2]$, we have:

$[0_2\ 4_2\ 5_2]$, $[1_2\ 5_2\ 0_2]$, $[2_2\ 0_2\ 1_2]$, $[3_2\ 1_2\ 2_2]$, $[4_2\ 2_2\ 3_2]$, $[5_2\ 3_2\ 4_2]$.

6. The short orbit block: $[0_1\ 1_2\ 4_2]$ includes

$[0_1\ 1_2\ 4_2]$, $[1_1\ 2_2\ 5_2]$, $[2_1\ 3_2\ 0_2]$.

7. Finally, the N_1 base block: $[0_1\ 1_1\ 2_1]$ contains the following blocks:

$[0_1\ 1_1\ 2_1]$, $[1_1\ 2_1\ 0_1]$, $[2_1\ 0_1\ 1_1]$.

In the next set of examples, we shall list only the resulting base blocks.

Example 2. A *bicyclic* T_1 *decomposition* of the *complete mixed graph* on 55 vertices, M_{55}, with cycles of length $N_1 = 11$ and $N_2 = 44$ (i.e. $N_2 \equiv 0 \pmod 4$ and $N_1 \equiv 3 \pmod 4$) has 33 *base blocks*:

$[0_2\ 10_2\ 12_2]$, $[0_2\ 9_2\ 13_2]$, $[0_2\ 8_2\ 14_2]$, $[0_2\ 7_2\ 15_2]$, $[0_2\ 6_2\ 16_2]$,

$[0_2\ 32_2\ 33_2]$, $[0_2\ 31_2\ 34_2]$, $[0_2\ 30_2\ 35_2]$, $[0_2\ 29_2\ 36_2]$, $[0_2\ 28_2\ 37_2]$,

$[0_2\ 27_2\ 38_2]$, $[0_2\ 26_2\ 39_2]$, $[0_2\ 25_2\ 40_2]$, $[0_2\ 24_2\ 41_2]$, $[0_2\ 23_2\ 42_2]$,

$[0_2\ 22_2\ 43_2]$, $[0_1\ 4_2\ 16_2]$, $[0_1\ 3_2\ 18_2]$, $[0_1\ 2_2\ 20_2]$, $[0_1\ 1_2\ 22_2]$,

$[0_1\ 0_2\ 24_2]$, $[0_2\ 0_1\ 1_2]$, $[0_2\ 1_1\ 3_2]$, $[0_2\ 2_1\ 5_2]$, $[0_2\ 3_1\ 18_2]$,

$[0_2\ 4_1\ 20_2]$, $[0_2\ 6_1\ 2_2]$, $[0_2\ 7_1\ 4_2]$, $[0_2\ 9_1\ 19_2]$, $[0_2\ 10_1\ 21_2]$,

$[0_2\ 5_1\ 11_2]$, $[0_2\ 8_1\ 17_2]$, $[0_1\ 10_2\ 32_2]$.

Example 3. A *bicyclic* T_1 *decomposition* of the *complete mixed graph* on 45 vertices, M_{45}, with cycles of length $N_1 = 5$ and $N_2 = 40$ (i.e. $N_2 \equiv 0 \pmod 4$ and $N_1 \equiv 1 \pmod 4$) has 25 *base blocks*:

$[0_2\ 9_2\ 11_2]$, $[0_2\ 8_2\ 12_2]$, $[0_2\ 7_2\ 13_2]$, $[0_2\ 6_2\ 14_2]$, $[0_2\ 5_2\ 15_2]$,

$[0_2\ 4_2\ 16_2]$, $[0_2\ 3_2\ 17_2]$, $[0_2\ 29_2\ 30_2]$, $[0_2\ 28_2\ 31_2]$, $[0_2\ 27_2\ 32_2]$,

$[0_2\ 26_2\ 33_2]$, $[0_2\ 25_2\ 34_2]$, $[0_2\ 24_2\ 35_2]$, $[0_2\ 23_2\ 36_2]$, $[0_2\ 22_2\ 37_2]$,

$[0_2\ 21_2\ 38_2]$, $[0_2\ 20_2\ 39_2]$, $[0_1\ 1_2\ 17_2]$, $[0_1\ 0_2\ 19_2]$, $[0_2\ 0_1\ 1_2]$,

$[0_2\ 1_1\ 18_2]$, $[0_2\ 3_1\ 2_2]$, $[0_2\ 4_1\ 19_2]$, $[0_2\ 2_1\ 10_2]$, $[0_1\ 4_2\ 24_2]$.

Example 4. A *bicyclic* T_1 *decomposition* of the *complete mixed graph* on 63 vertices, M_{63}, with cycles of length $N_1 = 9$ and $N_2 = 54$ (i.e. $N_2 \equiv 2 \pmod 4$ and $N_1 \equiv 1 \pmod 4$) has 36 *base blocks*:

$[0_2\ 0_1\ 1_2]$, $[0_2\ 1_1\ 3_2]$, $[0_2\ 2_1\ 5_2]$, $[0_2\ 3_1\ 7_2]$, $[0_2\ 5_1\ 2_2]$,

$[0_2\ 6_1\ 4_2]$, $[0_2\ 7_1\ 6_2]$, $[0_2\ 8_1\ 8_2]$, $[0_2\ 17_2\ 19_2]$, $[0_2\ 16_2\ 20_2]$,

$[0_2\ 15_2\ 21_2]$, $[0_2\ 14_2\ 22_2]$, $[0_2\ 13_2\ 23_2]$, $[0_2\ 12_2\ 24_2]$, $[0_2\ 11_2\ 25_2]$,

$[0_2\ 10_2\ 26_2]$, $[0_2\ 9_2\ 27_2]$, $[0_2\ 40_2\ 41_2]$, $[0_2\ 39_2\ 42_2]$, $[0_2\ 38_2\ 43_2]$,

$[0_2\ 37_2\ 44_2]$, $[0_2\ 36_2\ 45_2]$, $[0_2\ 35_2\ 46_2]$, $[0_2\ 34_2\ 47_2]$, $[0_2\ 33_2\ 48_2]$,

$[0_2\ 32_2\ 49_2]$, $[0_2\ 31_2\ 50_2]$, $[0_2\ 30_2\ 51_2]$, $[0_2\ 29_2\ 52_2]$, $[0_2\ 28_2\ 53_2]$,

$[0_1\ 0_2\ 26_2]$, $[0_1\ 1_2\ 25_2]$, $[0_1\ 2_2\ 24_2]$, $[0_1\ 3_2\ 23_2]$, $[0_1\ 4_2\ 31_2]$, $[0_2\ 4_1\ 18_2]$.

The figures below gives an idea of how we can obtain some *pure arc2, mixed arc21* and *edge12* differences. Here, the blocks are of the form $[0_2\ a_1\ b_2]$, where b : *arcs* $1 \leftarrow 2$ are 1 3 5 7 0 2 4 6 8 obtained from $a + e = b \pmod{N_1}$. The *pure arc2* and *edge2* links are shown in figure 7 and figure 8.

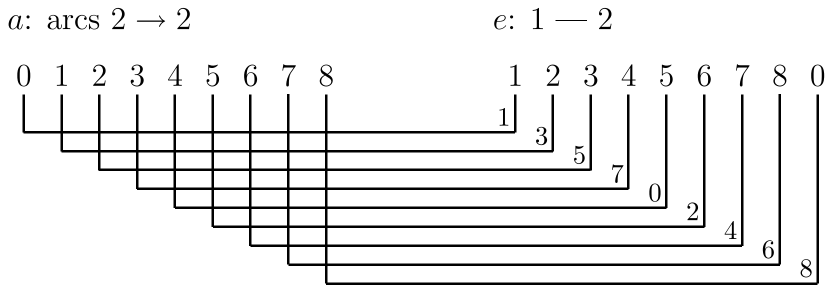

Figure 7: Pure and Mixed Differences

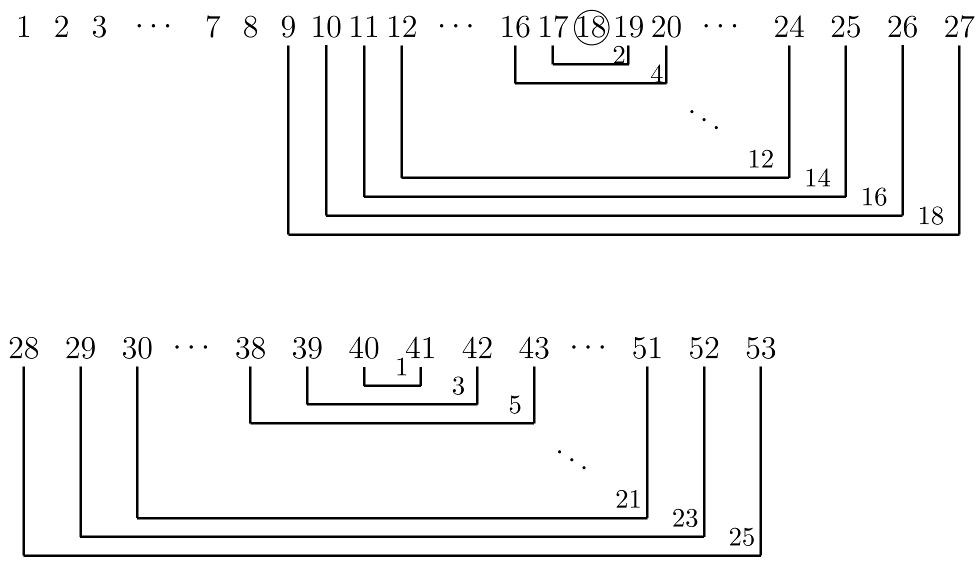

Figure 8: Pure Links

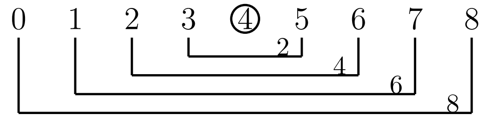

Figure 9: Modulo N_1 Differences

Some *arc*1→2 and *edge2* modulo N_1 differences are shown in figure 9. Worthy of note is the fact that since the edge2 difference is in modulo 9, we have 20≡2, etc. Also, the *short orbit block* is [0_1 4_2 31_2] since 31≡4 (mod 9).

4 THE PACKING PROBLEM

4.1 Introduction

A *maximal packing* of a mixed graph G with isomorphic copies of a graph g is a set $(g_1, g_2, ..., g_n)$ where $g_i \equiv g$ and $V(g_i) \subset V(G)$ for all i, and $AE(g_i) \cap AE(g_j) = \emptyset$ for $i \neq j$,

$$\bigcup_{i=1}^{n} g_i \subset G, \text{and} \mid AE(G) \setminus \bigcup_{i=1}^{n} E(g_i) \mid$$

is minimal, where $V(G)$ is the vertex set of graph G and $AE(G)$ is the *arc* and *edge* set of graph G. Packings of the complete graph on v vertices, K_v , with graph g have been studied for g a 3-cycle, 4-cycle, 6-cycle and $g = K_4$. For a review of this topic, see [13]

A minimal covering of a mixed graph G with isomorphic copies of a graph g is a set $(g_1, g_2, ..., g_n)$ where $g_i \equiv g$ and $V(g_i) \subset V(G)$ for all i,

$$G \subset \bigcup_{i=1}^{n} g_i, \text{and} \mid \bigcup_{i=1}^{n} AE(g_i) \setminus AE(G) \mid$$

is minimal (the graph $\bigcup_{i=1}^{n} g_i$ may not be mixed and $\bigcup_{i=1}^{n} AE(g_i)$ may be a multiset). Coverings of complete graphs K_v with graph g have been studied for g being a 3-cycle, 4-cycle and 6-cycle. For a review of this topic, see [13].

A similar definition for *maximal packings* and *minimal coverings* can be given for both directed and undirected complete graphs.

There are two orientations of the 3-cycle: the 3-circuit, C_3, denoted by any cyclic shift of $[x, y, z]_C$ and the transitive triple T, denoted by $[x, y, z]_T$ as in figure 10.

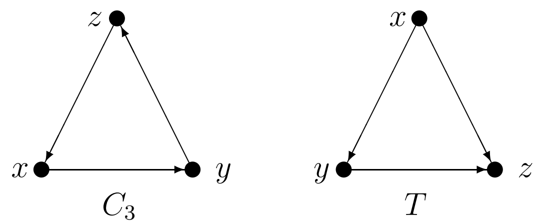

Figure 10: Three Circuit and Transitive Triple

4.2 Packing of Complete Mixed Graphs

Let D_v denote the complete directed graph on v vertices. If $(d_1, d_2, ..., d_n)$ is a packing of D_v with copies of d, then we define the directed graph $L = D_v - \bigcup_{i=1}^{n} d_i$ as the *leave* of the packing. That is, the arc set of L is $A(L) = A(D_v) \setminus \bigcup_{i=1}^{n} A(d_i)$ and the vertex set of L is induced by $A(L)$ (therefore L has no isolated vertices). A maximal packing of D_v with copies of d will therefore make |A(L)| minimal. In the event that $|\text{A(L)}| = 0$, it is said that D_v can be *decomposed* into copies of d.

The following two results are due to N. Mendelsohn [16, 17]:

Fact 1. A decomposition of D_v into copies of C_3 exists if and only if

$v \equiv 0$ or 1 (mod 3), $v \neq 6$.

Fact 2. A decomposition of D_v into copies of T exists if and only if

$v \equiv 0$ or 1 (mod 3).

The decomposition in Fact 1 is called a *Mendelsohn Triple System*, denoted by

$MTS(v)$ while that in Fact 2 is called a *Directed Triple System*, and denoted by $DTS(v)$. Therefore, discussions on packing of D_v with the 3-circuit C_3, or with the transitive triple T, is valid only when $v \equiv 2 \pmod 3$ (and when $v = 6$). The following two results were proven in [13].

Fact 3. A maximal packing of D_v with copies of the Transitive Triple T and leave L satisfies:

1. $|$ A(L)$| = 0$ if $v \equiv 0$ or $1 \pmod 3$, or

2. $|$ A(L)$| = 2$ and $L = C_2$ if $v \equiv 2 \pmod 3$.

Proof: With $v \equiv 2 \pmod 3$, $|$ A(L)$| = v(v-1) \equiv 2 \pmod 3$ and if we can demonstrate a packing where $|$ A(L)$| = 2$, then it certainly must be maximal.

<u>Case 1.</u> If $v \equiv 2 \pmod{12}$, say $v = 12t + 2$, then consider the set of triples:

$\{[0, 3t-i, 3t+1+i]_T \mid i = 0, 1, \ldots, t-1\} \bigcup$

$\{[0, 5t-i, 5t+2+i]_T \mid i = 0, 1, \ldots, t-1\} \bigcup$

$\{[0, 7t+2+i, 7t-i]_T \mid i = 0, 1, \ldots, t-2\} \bigcup$

$\{[0, 9t+1+i, 9t-i]_T \mid i = 0, 1, \ldots, t-1\} \bigcup$

$\{[0, x, 5t+1]_T, [0, y, 7t+1]_T\}$.

<u>Case 2.</u> If $v \equiv 5 \pmod{12}$, say $v = 12t + 5$, then consider the set of triples:

$\{[0, 3t-i, 3t+1+i]_T \mid i = 0, 1, \ldots, t-1\} \bigcup$

$\{[0, 5t-i, 5t+2+i]_T \mid i = 0, 1, \ldots, t-1\} \bigcup$

$\{[0, 7t+3+i, 7t+1-i]_T \mid i = 0, 1, \ldots, t-1\} \bigcup$

$\{[0, 9t + 3 + i, 9t + 2 - i]_T \mid i = 0, 1, \ldots, t - 1\} \bigcup$

$\{[0, x, 5t + 1]_T, [0, y, 7t + 2]_T\}$.

Case 3. If $v \equiv 8 \pmod{12}$, say $v = 12t + 8$, then consider the set of triples:

$\{[0, 3t + 2 - i, 3t + 3 + i]_T \mid i = 0, 1, \ldots, t\} \bigcup$

$\{[0, 5t + 3 - i, 5t + 5 + i]_T \mid i = 0, 1, \ldots, t - 1\} \bigcup$

$\{[0, 7t + 6 + i, 7t + 4 - i]_T \mid i = 0, 1, \ldots, t - 1\} \bigcup$

$\{[0, 9t + 6 + i, 9t + 5 - i]_T \mid i = 0, 1, \ldots, t - 1\} \bigcup$

$\{[0, x, 5t + 4]_T, [0, y, 7t + 5]_T\}$.

Case 4. If $v \equiv 11 \pmod{12}$, say $v = 12t + 11$, then consider the set of triples:

$\{[0, 3t + 2 - i, 3t + 3 + i]_T \mid i = 0, 1, \ldots, t\} \bigcup$

$\{[0, 5t + 3 - i, 5t + 5 + i]_T \mid i = 0, 1, \ldots, t - 1\} \bigcup$

$\{[0, 7t + 6 + i, 7t + 4 - i]_T \mid i = 0, 1, \ldots, t - 1\} \bigcup$

$\{[0, 9t + 7 + i, 9t + 6 - i]_T \mid i = 0, 1, \ldots, t\} \bigcup$

$\{[0, x, 5t + 4]_T, [0, y, 7t + 5]_T\}$.

In each case, the given set of triples along with their images under the various powers of the permutation $(x)(y)(0, 1, ..., (v - 3))$ form a packing of graph D_v where $V(D_v) = \{x, y, 0, 1, ..., v - 3\}$ with copies of C_3 and leave $L = C_2$ with arc set given by $A(L) = \{(x, y), (y, x)\}$.

Finally, we note that the total-degree (i.e. the in-degree plus the out-degree) of each vertex of D_v is $2(v - 1)$ and the total degree of each vertex of T is 2. So any packing of D_v with copies of T will have a leave L with each vertex of even total-degree. Therefore an optimal packing must have $L = C_2$.

For packing of D_v with copies of C_3, we note as above that each vertex of D_v has in-degree equal to out-degree, hence it must be that each vertex of a leave also has this property.

The following result was also established;

Fact 4. *A maximal packing of D_v where $v \neq 6$, with copies of the 3-circuit C_3, and leave L satisfies:*

1. $|$ A(L)$| = 0$ if $v \equiv 0$ or 1 (mod 3), $v \neq 6$, or

2. $|$ A(L)$| = 2$ and $L = C_2$ if $v \equiv 2$ (mod 3).

Proof: With $v \equiv 2$ (mod 3), $|$ A(L)$| = v(v-1) \equiv 2$ (mod 3), if we can demonstrate a packing where $|$ A(L)$| = 2$, i.e. $L = C_2$, then it certainly must be maximal.

<u>Case 1.</u> If $v \equiv 2$ (mod 6), say $v = 6t + 2$, then consider the set of triples:

$\{[0, 3t + i, 6t - 1 - i]_C \mid i = 0, 1, \ldots, t - 1\} \bigcup$

$\{[0, 4t + 1 + i, t - 1 - i]_C \mid i = 0, 1, \ldots, t - 2\} \bigcup$

$\{[x, 0, 4t]_C, [y, 0, 5t]_C\}$.

<u>Case 2.</u> If $v \equiv 5$ (mod 6), say $v = 6t + 5$, then consider the set of triples:

$\{[0, 2 + i, 3t + 2 - i]_C \mid i = 0, 1, \ldots, t - 1\} \bigcup$

$\{[0, 4t + 4 + i, t - i]_C \mid i = 0, 1, \ldots, t - 2\} \bigcup$

$\{[x, 0, 1]_C, [y, 0, 6t + 2]_C\} \bigcup$

$\{[0, 4t + 1, 2t]_C$, (omit if t=0)$\}$.

In each case, the given set of triples along with their images under the powers of the permutation $(x)(y)(0, 1, ..., v-3)$ form a maximal packing of D_v, $v \neq 6$, where $V(D_v) = \{x, y, 0, 1, ..., v-3\}$ with copies of C_3 and leave $L = C_2$ where $A(L) = \{(x, y), (y, x)\}$.

We recall that a *mixed graph* on v vertices is an ordered pair (V, C) where V is a set of vertices, |V| $= v$, and C is a set of ordered and unordered pairs, denoted $[x, y]$ and (x, y) respectively, of elements of V. An ordered pair $[x, y] \in C$ is called an *arc* of (V, C) while the unordered pair $(x, y) \in C$ is called an *edge* of (V, C).

Similarly, the *complete mixed graph* on v vertices, denoted M_v, is the mixed graph (V, C) where, for every pair of distinct vertices $v_1, v_2 \in V$, we have
$\{\ [v_1, v_2],\ [v_2, v_1],\ (v_1, v_2)\ \} \subset C$.

The following important results [10, 11], established in the second chapter, are necessary;

Fact 5. *A T_1-triple system of order v exists if and only if* $v \equiv 1 (\text{mod } 2)$.

We again recall that the *converse* of a mixed graph (V, C), is the mixed graph (V, C') where $C' = \{[v_2, v_1] |\ [v_1, v_2] \in C\} \bigcup \{(v_1, v_2) | (v_1, v_2) \in C\}$.
It is easy to see that the converse of M_v is M_v. Thus the existence of T_1-triple system implies the existence of a T_2-triple system. Gardner [11] thus had the following as a consequence:

Fact 6. *A T_2-triple system of order v exists if and only if* $v \equiv 1 (\text{mod } 2)$.

From the above two results, we see that packings of M_v with T_1 or T_2 will make sense only for the case when v is even.

We now state the basic results in this chapter. We recall first that in T_1, we have *total out degree* 2 and in M_v, we have *out degree* $(v-1)$ at each vertex. So, in the *leave* of a packing of M_v with T_1, each vertex will have *out degree* at least 1. This gives at least v arcs and $\frac{v}{2}$ edges for even v.

Hence, $|A(L)| \geq v$, $\quad |E(L)| \geq \frac{v}{2} \quad \Rightarrow |L| = \frac{3v}{2}$. This establishes the necessary condition of the following theorem.

Theorem: A maximal packing of M_v with copies of T_1 exists and the leave size satisfies $|L| = \frac{3v}{2}$. Furthermore, an example structure of the leave will be of the form:

1. For $v \equiv 0 \pmod 4$,

$$L = L_e \bigcup L_a, \text{where}, L_e = \{\ (i, \frac{v}{2} + i\ ) : i = 0, 1, \ldots, (\frac{v}{2} - 1)\},$$

$$L_a = \{\ [i, \frac{3v}{4} + i\ ] : i = 0, 1, \ldots, (v-1)\}.$$

2. For $v \equiv 2 \pmod 4$,

$$L = L_e \bigcup L_a, \text{where}, L_e = \{\ (i, \frac{v}{2} + i\ ) : i = 0, 1, \ldots, \frac{v}{2}\},$$

$$L_a = \{\ [i, \frac{3(v-2)}{4} + 1 + i\ ] : i = 0, 1, \ldots, (v-1)\}.$$

Indeed, we construct the resulting *base blocks* and show that the leave has the predicted structure for each case.

We shall have two cases: $v = 4k$ and $v = 4k + 2$ with the parity of the edge difference, e being a deciding factor. In each case, we will give a diagram to help us see how the difference triples are formed as well as help in isolating the leave. We shall also have tables that display the various entries for each edge and arc difference.

Case 1: $v = 4k$.

Let us in general consider e odd.

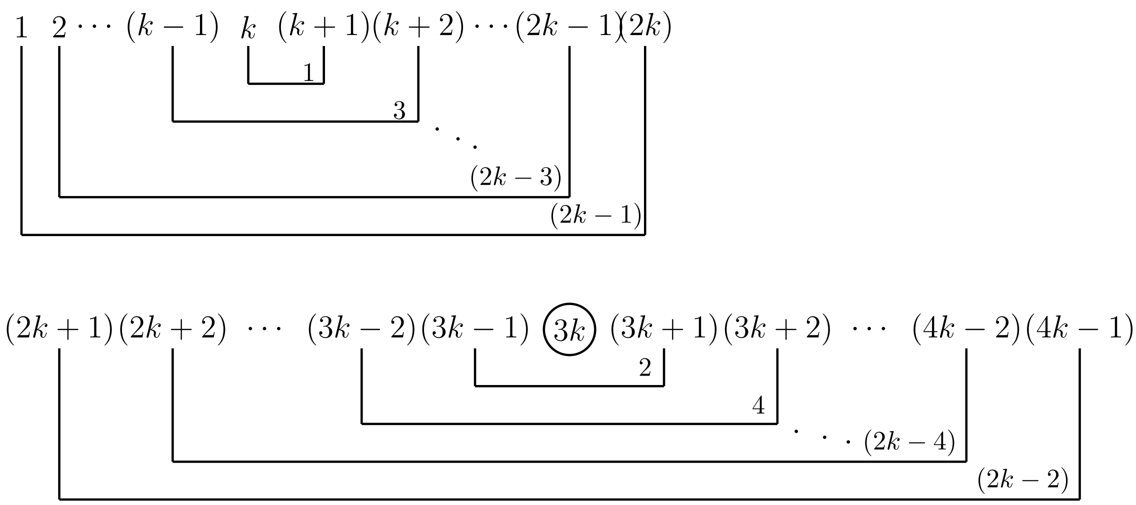

Figure 11: Associated Differences for $v = 4k$

Table 1: $v = 4k$ and e Odd

e	a_1	$a_2 = e + a_1$
1	k	$k+1$
3	$k-1$	$k+2$
5	$k-2$	$k+3$
$\vdots$	$\vdots$	$\vdots$
$2k-3$	2	$2k-1$
$2k-1$	1	$2k$

$$e : 2r+1, \qquad a_1 : k-r, \qquad a_2 : k+1+r, \qquad r = 0, 1, \ldots, (k-1).$$

We take these differences and associate them with T_1's to get the blocks:

$[0, k-r, k+r+1]$, r = 0, 1, ..., $k-1$.

Observe that this is just for one from a total of v vertices. Thus, we are missing all the arcs $(0, k+1), (1, k+2)$, etc. Therefore we need to introduce a counter, i, to count from 0 to $(v-1)$. Hence, the base block for the packing of M_v, $v = 4k$, e odd, with isomorphic copies of T_1 becomes:

$[i, k-r+i, k+r+1+i]$, r = 0, 1, ..., k - 1, i = 0, 1, ..., ($4k$ - 1)

We shall use the $2k$ edge difference and the $3k$ arc difference in the leaves.

Similarly, for e even, we have 2 with the entries:

$$e : 2r+2, \qquad a_1 : 3k-1-r, \qquad a_2 : 3k+1+r, \qquad r = 0, 1, \ldots, k-2.$$

We take these differences and associate them with T_1's to get the blocks:

$[0, 3k-1-r, 3k+1+r]$, r = 0, 1, ..., $(k-2)$.

Table 2: $v = 4k$ and e Even

e	a_1	$a_2 = e + a_1$
2	$3k-1$	$3k+1$
4	$3k-2$	$3k+2$
6	$3k-3$	$3k+3$
$\vdots$	$\vdots$	$\vdots$
$2k-4$	$2k+2$	$4k-2$
$2k-4$	$2k+1$	$4k-1$

Thus we similarly introduce a counter, i, ranging from 0 to $(v-1)$. Hence, the base block for the packing of M_v, $v = 4k$, e even with isomorphic copies of T_1 becomes:

$$[i, 3k-1-r+i, 3k+1+r+i\,], \qquad r = 0, 1, \ldots, k-2\,, \qquad \mathrm{i} = 0,\ 1,\ \ldots,\ (4k-1)$$

We now use the $2k$ edge difference and the $3k$ arc difference in the leaves as follows:

edge difference of $2k : (0, 2k), (1, 2k+1), \ldots, (2k-1, 4k-1)$

arcs difference of $3k : [0, 3k], [1, 3k+1], \ldots, [4k-1, 3k-1]$

Hence, we have

$$L = L_e \bigcup L_a, \text{where}, L_e = \{(i, \frac{v}{2} + i) : i = 0, 1, \ldots, (\frac{v}{2} - 1)\};$$

$$L_a = \{[i, \frac{3v}{4} + i] : i = 0, 1, \ldots, (v-1)\},$$

and, therefore, our leave size is $(v + \frac{v}{2}) = \frac{3v}{2}$. This completes the proof of case 1.

Case 2: $v = 4k + 2$.

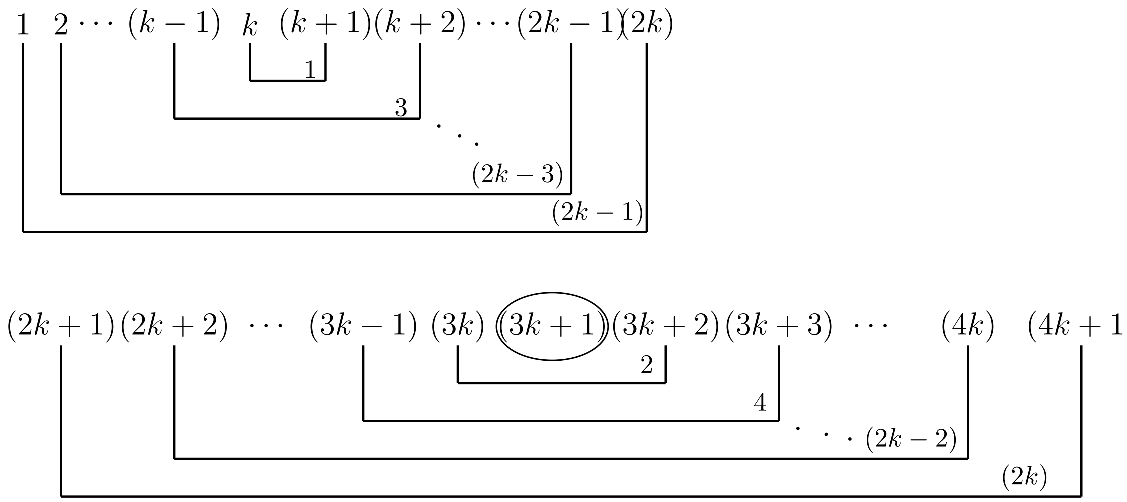

Figure 12: Associated Differences for $v = 4k + 2$

Table 3: $v = 4k + 2$ and e Odd

e	a_1	$a_2 = e + a_1$
1	k	$k+1$
3	$k-1$	$k+2$
5	$k-2$	$k+3$
$\vdots$	$\vdots$	$\vdots$
$2k-3$	2	$2k-1$
$2k-1$	1	$2k$

$$e : 2r+1, \qquad a_1 : k-r, \qquad a_2 : k+1+r, \qquad r = 0, 1, \ldots, k-1.$$

We take these differences and associate them with T_1's to get the blocks:

$$[0, k-r, k+r+1], \qquad r = 0,\ 1,\ \ldots,\ (k-1).$$

Observe again that this is just for one vertex. Thus we are missing all the arcs $(0, k+1), (1, k+2)$, etc. Therefore we introduce a counter, i, ranging from 0 to $(v-1)$. Hence the base blocks for the packing of M_v, $v = 4k+2$, e odd, with isomorphic copies of T_1 becomes:

$$[i, k-r+i, k+r+1+i], \qquad r = 0, 1, \ldots, (k-1), \qquad i = 0, 1, \ldots, (4k-1).$$

We shall use the $2k$ edge difference and the $3k$ arc difference in the leaves.

Similarly, for e even, we have table 4

Table 4: $v = 4k+2$ and e Even

e	a_1	$a_2 = e + a_1$
2	$3k$	$3k+2$
4	$3k-1$	$3k+3$
6	$3k-2$	$3k+4$
$\vdots$	$\vdots$	$\vdots$
$2k-2$	$2k+2$	$4k$
$2k$	$2k+1$	$4k+1$

$$e : 2r+2, \qquad a_1 : 3k-1-r, \qquad a_2 : 3k+1+r, \qquad r = 0, 1, \ldots, (k-2).$$

We take these differences and associate them with T_1's to get the blocks:

$$[0, 3k-r, 3k+2+r], \qquad r = 0, 1, \ldots, (k-1).$$

Thus, we introduce a counter, i, ranging from 0 to $(v-1)$. Hence, the base blocks for the packing of M_v, $v = 4k+2$, e even with isomorphic copies of T_1 become:

$$[i, 3k-r+i, 3k+2+r+i], \qquad r = 0, 1, \ldots, (k-2), \qquad i = 0, 1, \ldots, (4k+1).$$

We now use the $2k+1$ edge difference and the $3k+1$ arc difference in the leaves as follows: edge difference of $2k : (0, 2k+1), (1, 2k+2), \ldots, (2k, 4k+1)$, and arcs difference of $3k : [0, 3k+1], [1, 3k+2], \ldots, [4k+1, 3k]$

Hence, we have

$$L = L_e \bigcup L_a, \text{where}, L_e = \{(i, \frac{v}{2} + i) : i = 0, 1, \ldots, (\frac{v}{2} - 1)\},$$

$$L_a = \{[i, \frac{3(v-2)}{4} + 1 + i] : i = 0, 1, \ldots, (v-1)\},$$

and, therefore, our leave size is $(v + \frac{v}{2}) = \frac{3v}{2}$. This completes the proof of case 2 and hence the theorem.

From the observation that T_2 is the converse of T_1, we immediately have the following:

Corollary: A maximal packing of M_v with copies of T_2 exists and the leave size satisfies $|L| = \frac{3v}{2}$. Furthermore, an example structure of the leave will be of the form:

1. For $v \equiv 0 \pmod 4$,

$$L = L_e \bigcup L_a, \text{where}, L_e = \{(i, \frac{v}{2} + i) : i = 0, 1, \ldots, (\frac{v}{2} - 1)\},$$

$$L_a = \{[i, \frac{3v}{4} + i] : i = 0, 1, \ldots, (v-1)\}.$$

2. For $v \equiv 2 \pmod 4$,

$$L = L_e \bigcup L_a, \text{where}, L_e = \{(i, \frac{v}{2} + i) : i = 0, 1, \ldots, \frac{v}{2}\},$$

$$L_a = \{[i, \frac{3(v-2)}{4} + 1 + i] : i = 0, 1, \ldots, (v-1)\}.$$

This concludes our research.

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