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Extensions of the Cayley-Hamilton Theorem with Applications to Elliptic Operators and Frames.

Alberto Mokak Teguia
East Tennessee State University

Extensions of the Cayley-Hamilton Theorem with Applications to Elliptic Operators and Frames

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by

Alberto Mokak Teguia

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Jeff Knisley, Ph.D., Chair

Anant Godbole, Ph.D.

Robert Gardner, Ph.D

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ABSTRACT

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The Cayley-Hamilton Theorem is an important result in the study of linear transformations over finite dimensional vector spaces. In this thesis, we show that the Cayley-Hamilton theorem can be extended to self-adjoint trace-class operators and to closed self-adjoint operators with trace-class resolvent over a separable Hilbert space. Applications of these results include calculating operator resolvents and finding the inverse of a frame operator.

DEDICATION

I dedicate this thesis to my family, to Dr. Thomas and Janice Huang, and to Lakeisha R. Brown.

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1 INTRODUCTION

The Cayley-Hamilton Theorem (CHT) can be stated as follows:

Theorem 1 *If A is a linear transformation of a finite-dimensional vector space into itself and if $p(z) = \det(A - zI)$ where I is the identity transformation, then*

$$p(A) = 0.$$

This important result has many applications throughout linear algebra, including algorithms to calculate inverse and exponential of linear transformations. Because the characteristic polynomial of A is defined by

$$p(x) = \det(A - xI)$$

where I is the identity on V, it also follows that if $\lambda \in \sigma_p(A)$, the point spectrum of A, then $p(\lambda) = 0$. Thus, a generalization of CHT is that if A is a linear transformation over a finite-dimensional vector space V, then there exists a polynomial $p(x)$ which vanishes on $\sigma_p(A)$ for which $p(A) = 0$.

Several attempts have been made to generalize the CHT in recent literature. In [23], CHT has been extended to the case of supermatrices over finite dimensional spaces, which has many applications in physics. In [13], an extension of the CHT for a special type of algebra is introduced. Also, Ponge [17] recently attempted to extend a version of CHT both to compact operators and to operators with a compact resolvent over a Hilbert space H. Although the paper has now been withdrawn due to an error, he had many good insights into how to generalize CHT to an algebraic sum of finite-dimensional subspaces that generalize the eigenspaces of a compact operator.

There are no straightforward means of defining the determinant (and so, the characteristic polynomial) for linear transformations over infinite-dimensional vector spaces. Instead, there are many extensions of the determinant concept, each one designed to work in a given situation. The majority of these definitions are derived from the equality

$$\ln(\det(T)) = \operatorname{tr}(\ln(T)).$$

Here, we will make use of this relation and of zeta functions and analytic continuation since it allows the determinant to be defined for a large class of elliptic operators (see [20, 15, 11] for details). This suggests that an extension of CHT is possible in at least some special classes of operators.

In this thesis, we derive such an extension for self-adjoint trace-class operators and for closed self-adjoint operators with trace-class resolvent over a Hilbert space, H having a trace-class resolvent. In particular, one of our results is, if L is an elliptic operator with trace-class resolvent, then there is a function $p\,(x, y)$ which is meromorphic in both x and y for which

$$p(L, y) = p(0, y)P_{\ker(L)}$$

where $P_{\ker(L)}$ is the projection onto $\ker\,(L)$. In chapter 2, we present some of the preliminary ideas needed for these results. In chapter 3, we present previous research related to our work. In chapter 4, we present extensions of CHT and some corollaries. In chapter 5, we present some applications of our results and in chapter 6 we present our future objectives.

2 PRELIMINARIES

2.1 Cayley-Hamilton Theorem

2.1.1 Statement and Proof of the Cayley-Hamilton Theorem

The materials of this section can be found in any undergraduate linear algebra book ([3, 5]) The Cayley-Hamilton Theorem (CHT) states that (in a finite dimensional space), every operator (or square matrix) is annihilated by its characteristic polynomial. Let's define the concept of a characteristic polynomial and then prove the CHT.

Definition 1 *Let T be a linear operator on a finite dimensional vector space V and let A be the matrix associated with T. The characteristic polynomial of T is defined by*

$$p(z) = det(A - zI).$$

The characteristic polynomial is a $dim(V)$ degree polynomial, whose zeros are the eigenvalues of the operator T. The following lemma is closely related to the CHT and plays an important role in its proof.

Lemma 2 *Let $P(z)$ and $Q(z)$ be polynomials (of finite degrees) with coefficients that are linear transforms over a finite vector space V (or $n \times n$ matrices). Let T be a linear transform over V.*

$$\begin{aligned} If \quad P(z) &= Q(z)(T - zI), then \\ P(T) &= 0. \end{aligned}$$

Proof. Let $Q(z) = B_0 + B_1 z + ... + B_n z^n$. Then

$$\begin{aligned} P(z) &= (B_0 + B_1 z + ... + B_n z^n)(T - zI) \\ &= B_0 T + B_1 T z + ... + B_n T z^n - (B_0 z + B_1 z^2 + ... + B_n z^{n+1}) \\ &= B_0 T + (B_1 T - B_0) z + ... + (B_n T - B_{n-1}) z^n - B_n z^{n+1} \end{aligned}$$

Hence, the polynomial applied to the operator T yields

$$\begin{aligned} P(T) &= B_0 T + (B_1 T - B_0) T + ... + (B_n T - B_{n-1}) T^n - B_n T^{n+1} \\ &= 0 \end{aligned}$$

■

For a square matrix S (which can be used to represent a linear transform over V), we have

$$(\mathrm{adj} S) S = (\det S) I.$$

Thus, for any square matrix A,

$$adj(zI - A).(zI - A) = \det(zI - A).I.$$

But $\mathrm{adj}(zI - A)$ can be written as polynomial in z with square matrices coefficients, thus, by lemma 2

$$p(A) = 0 \quad \text{with} \quad p(z) = \det(zI - A).$$

This proves the Cayley-Hamilton Theorem:

Theorem 3 *If T is a linear operator on a finite dimensional vector space V, then T satisfies its own characteristic equation, that is,*

$$p(T) = 0.$$

This is a beautiful theorem mathematically. It gives a direct relation between an operator, its spectrum (to be defined later), and its trace. But the importance of this theorem goes far beyond its beauty. It has many applications, both in pure and applied mathematics. We present some of these applications in the next section.

2.1.2 Inverse of a Square Matrix

Many problems arising in science require finding the inverse of a matrix. For square matrices, there are algorithms for calculating inverses using the co-factor expansion. But as the order of the matrix gets large, these algorithms become inefficient.

Theorem 4 *Let A be an $n\times n$ matrix, with characteristic polynomial $p(z) = \sum_{i=0}^{n} a_i z^i$. If $a_0 \neq 0$, then A^{-1} exists and can be written as*

$$A^{-1} = -\frac{1}{a_0}\sum_{i=1}^{n} a_i A^{i-1}.$$

Proof. Recall that, by definition, A is invertible if there exists a matrix B such that

$$AB = BA = I,$$

in which case $A^{-1} = B$. This is true if and only if 0 is not an eigenvalue of A. By the CHT, $p(A) = 0$. That is,

$$\begin{aligned} a_0 I &= -\sum_{i=1}^{n} a_i A^i \\ \text{i.e.}\quad I &= A\left(-\frac{1}{a_0}\sum_{i=1}^{n} a_i A^{i-1}\right) \quad \text{and} \\ I &= \left(-\frac{1}{a_0}\sum_{i=1}^{n} a_i A^{i-1}\right) A \end{aligned}$$

The result then follows. ■

This theorem gives us an efficient method for computing the inverse of a square matrix using only matrix addition and substraction. Similarly, A^k for k, an integer, can be expressed as a linear combination of A^{n-1}, A^{n-2}, ...,A^1, I.

2.1.3 CHT and the Functional Calculus

Let A be a square matrix of order n, a polynomial $q(x)$ of degree r, such that $r > n$ and $p(x)$ the characteristic polynomial of A. By Euclid's algorithm, there exist unique polynomials $t(x)$ and $R(x)$ such that

$$q(z) = t(z) \cdot p(z) + r(z).$$

Thus, $q(A) = t(A)p(A) + r(A)$. Applying the CHT, we get:

Lemma 5 *Let A be a square matrix of order n, a polynomial $q(x)$ of degree r, such that $r > n$ and $p(x)$ the characteristic polynomial of A. Then*

$$q(A) = r(A)$$

where r is the (unique) remainder polynomial when $q(z)$ is divided by p.

This lemma gives a nice way to compute the image of a square matrix under a polynomial of finite order. We have a similar result for analytic functions.

2.2 Infinite Products

Because we will be using infinite products in this thesis, let us begin by stating some well-established properties of infinite products. These have been extracted from [2, 16]. We begin with the definition of the convergence of an infinite product.

Definition 2 *[2, 16] Let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers and let*

$$P_N = \prod_{n=1}^{N} p_n$$

If $\{P_n\}_{n=1}^{\infty}$ *converges to a number* $L \neq 0$*, we say that the infinite product converges and we write*

$$\prod_{n=1}^{\infty} p_n = L.$$

Proposition 6 *Let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence such that the sequence*

$$\left\{ P_N = \prod_{n=1}^{N} p_n \right\}_{N \in \mathbb{N}}$$

converges. Then

$$p_n \longrightarrow 1$$

as $n \longrightarrow \infty$.

Proof. Well,

$$\begin{aligned} p_N &= \frac{P_N}{P_{N-1}} \\ \text{thus} \quad \lim_{N\to\infty} p_N &= \lim_{N\to\infty} \frac{P_N}{P_{N-1}} \\ &= \frac{\lim_{N\to\infty} P_N}{\lim_{N\to\infty} P_{N-1}} \quad \text{since both limits exist} \\ &= 1 \end{aligned}$$

■

Some properties of infinite products are derived using infinite series of logarithms. For example, we have the following propositions.

Proposition 7 *[16] Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers. Then*

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges if and only if

$$\sum_{n=1}^{\infty}\ln(1+a_n)$$

converges, where we consider the principal branch of the logarithm.

Proof. Let's suppose that $\sum_{n=1}^{\infty}\ln(1+a_n)$ converges. We have

$$\prod_{n=1}^{N}(1+a_n) = e^{\sum_{n=1}^{N}\ln(1+a_n)}.$$

Then, since the exponential function is entire and does not take on the value zero, the infinite product is convergent.

Now suppose that $\prod_{n=1}^{\infty}(1+a_n)$ converges, say to L. We need to show that $\log P_n$ converges to the same branch of $\log L$. Clearly, $\log\left(\frac{P_n}{L}\right) \to 0$ as $n \to \infty$.

For every n, there is an integer h_n such that

$$\log\left(\frac{P_n}{L}\right) = S_n - \log(P) + h_n(2\pi i)$$

where S_n is the nth partial sum of the above series. We then have

$$\left(h_{n+1}-h_n\right)(2\pi i) = \log\left(\frac{P_{n+1}}{L}\right) - \log\left(\frac{P_n}{L}\right) - log(1+a_n)$$

and taking the argument we get

$$\left(h_{n+1}-h_n\right)(2\pi) = arg\left(\frac{P_{n+1}}{L}\right) - arg\left(\frac{P_n}{L}\right) - arg(1+a_n)$$

This implies that for n sufficiently large, we have $h_{n+1} = h_n$. The result then follows. ■

Proposition 8 *[16] Let $\{a_n\}$ be a sequence of complex numbers. Then $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if and only if $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

Proof. We know that

$$\begin{aligned} \ln(1+x) &\leq e^x \ \ \forall\, x\ \in \mathbb{R}^+, \quad \text{then} \\ \sum_{n=1}^{N} \ln|1+a_n| \leq \sum_{n=1}^{N} \ln(1+|a_n|) &\leq e^{\sum_{n=1}^{N}|a_n|} \quad \forall\, N \in\ \mathbb{N}. \end{aligned}$$

Thus, if $\sum_{n=1}^{\infty}|a_n|$ converges, so does $\sum_{n=1}^{\infty}\ln|1+a_n|$, and hence $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely by proposition 7. Let's suppose that $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely. Clearly,

$$\begin{aligned} \ln(x) &\leq x+1 \ \ \forall\, x\ \in \mathbb{R}^+, \quad \text{then} \\ \sum_{n=1}^{N}|a_n| &= e^{\ln\left(\sum_{n=1}^{N}|a_n|\right)} \\ &= e^{\prod_{n=1}^{N}(\ln|a_n|)} \\ &\leq e^{\prod_{n=1}^{N}(|1+a_n|)} \ \ \forall\, N \in\ \mathbb{N}. \end{aligned}$$

Thus $\sum_{n=1}^{\infty}|a_n|$ is also convergent. ∎

Proposition 9 *[16] An absolutely convergent infinite product of complex numbers is convergent.*

Proof. Let $\prod_{n=1}^{\infty} a_n$ be an absolute convergent infinite product. Then its sequence of partial products (P_n) is also absolutely convergent. Then, both the real and the imaginary parts of this sequence are absolutely convergent, and thus are convergent.

This implies that the the sequence is convergent, hence the infinite product is convergent. ■

Combining the two previous propositions gives us:

Proposition 10 *[2]*

Let $\{a_n\}$ be a sequence of complex numbers. Then $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges .

Proposition 11 *[16] Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on a region G. The infinite product $\prod_{n=1}^{\infty}(1+f_n(z))$ converges (uniformly) in G if $\sum_{n=1}^{\infty}(|f_n(z)|)$ converges (uniformly) in G.*

Proof. Suppose that $\sum_{n=1}^{\infty}(|f_n(z)|)$ converges . By proposition 10, $\prod_{n=1}^{N}(1+|f_n(z)|)$ is also convergent. But

$$\prod_{n=1}^{N}|1+f_n(z)| \leq \prod_{n=1}^{N}(1+|f_n(z)|),$$

so $\prod_{n=1}^{\infty}(1+f_n(z))$ converges absolutely, and by the proposition 9, it converges. The case of uniform convergence follows similarly. ■

2.3 Elementary Operator Theory

Let's review some basic operator theory. The following was extracted from [1] and [18].

Definition 3 *A vector space V is said to be metrizable if there exists a function*

$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ *satisfying:*

$$\begin{aligned}\langle g, f\rangle &= \overline{\langle f, g\rangle} \\ \langle a_1 f_1 + a_2 f_2, g\rangle &= a_1\langle f_1, g\rangle + \ a_2\langle f_2, g\rangle \\ \langle f, f\rangle &\geq 0, \quad \textit{with equality only for} \ \ f = 0\end{aligned}$$

where g, f, f_1, f_2 *are arbitrary vectors in* V *and* a_1, a_2 *are arbitrary complex numbers.* $\langle f, g\rangle$ *is called the inner product of* f *and* g.

Definition 4 *A Hilbert space is an infinite dimensional inner product space which is a complete metric space with respect to the metric generated by the inner product.*

For the rest of this thesis, we use H to denote an arbitrary Hilbert space. An important example of a Hilbert space is l^2.

Definition 5 *Define*

$$l^2 = \{\{x_n\}_1^\infty \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty\}.$$

The inner product defined on l^2 *is*

$$<,>: \{x_n\}_1^\infty, \{y_n\}_1^\infty \longmapsto \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

On a subset of a Hilbert space H, we can define the same inner product as on H, but not all subsets of H are complete. If a subset of H is closed, then it is complete since H is complete. Thus, it is again a Hilbert space. These types of subsets of H play an important role in operator theory.

Definition 6 *A non-empty closed subset of a Hilbert space* H *is called a subspace.*

Definition 7 *A Hilbert space H is said to be separable if there exists a countable set $S \subseteq H$ such that*

$$\overline{S} = H,$$

that is, such that S is dense in H.

As we mentioned above, l^2 is an important Hilbert space. This is because all separable Hilbert spaces are isomorphic to l^2 ([1]). Another Hilbert space often studied is $L^2(a, b)$.

Definition 8 *Let (a, b) be an interval. $L^2(a, b)$ is the set of all complex valued Lebesque measurable functions f defined on (a, b) such that $|f|^2$ is Lebesque integrable on (a, b). The inner product defined on this space is:*

$$<,>: f, g \longmapsto \int_a^b f(u)\overline{g(u)}du.$$

Let's define a type of mapping acting on Hilbert spaces.

Definition 9 *On a Hilbert space H , an operator T is a function that maps elements f of a non-empty subspace D of H into an element of H. T is said to be linear if*

$$T(af + bg) = aTf + bTg,$$

$\forall f, g$ in the domain of T (the collection of all $f \in H$ for which Tf exists in H) and for arbitrary $a, b \in \mathbb{C}$.

Definition 10 *An operator T defined on a Hilbert space H is continuous at a point $f_0 \in D_T$ if*

$$\lim_{f \to f_0} Tf = Tf_0.$$

A norm can be associated with every operator over a Hilbert space ([1]).

Definition 11 *Let T be an operator defined over a Hilbert space H, with domain D. The norm of T is defined as*

$$\|T\| = \sup_{f \in D} \frac{\|Tf\|}{\|f\|} = \sup_{f \in D, \|f\|=1} \|Tf\| .$$

An operator is said to be bounded if its norm is finite.

Proposition 12 *[1] the following are true:*

1)A bounded linear operator is continuous on its domain.

2)If a linear operator is continuous at one point, then it is bounded.

Definition 12 *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of vectors in H. We say that this sequence converges weakly to the vector f and we write $f_k \underset{\rightarrow}{w} f$ if*

$$\lim_{k \to \infty} < f_k, h >=< f, h >$$

$\forall h \in H$.

We say that this sequence converges (strongly) to the vector f and we write $f_k \to f$ if

$$\lim_{k \to \infty} \|f_k - f\| = 0.$$

Proposition 13 *[1] If the sequence of vectors $(f_k)_{k \in \mathbb{N}}$ converges weakly to the vector f and if*

$$\lim_{k \to \infty} \|f_k\| = \|f\| ,$$

then

$$\lim_{k \to \infty} \|f_k - f\| = 0,$$

that is $(f_k)_{k \in \mathbb{N}}$ converges strongly to f.

Definition 13 *Let* $\{A_n\}_{n=1}^{\infty}$ *be a sequence of bounded linear operators defined everywhere in* H*. Suppose this sequence converges to an operator* A*. We say that the convergence is:*

1) weak, and we write $A_n \underset{\rightarrow}{w} A$ *if*

$$A_n f \underset{\rightarrow}{w} Af$$

$\forall f \in H.$

2) strong, and we write $A_n \rightarrow A$ *if*

$$A_n f \rightarrow Af$$

$\forall f \in H.$

3) uniform, and we write $A_n \Longrightarrow A$ *if*

$$\|A_n - A\| \rightarrow 0$$

Definition 14 *An operator* T *is closed if for every sequence* $\{f_k\}_{k \in \mathbb{N}}$ *in the domain of* T*, the relations*

$$\lim_{k \to \infty} f_k = f \quad \textit{and} \quad \lim_{k \to \infty} T f_k = g$$

imply that

$$f \in D_T \quad \textit{and} \quad Tf = g.$$

For the rest of this section, let T be a closed linear operator over H, with domain denoted by D_T.

Definition 15 *A complex number* λ *is called an eigenvalue of a linear operator* T *if there exists a non-zero vector* $f \in$ *such that*

$$Tf = \lambda f. \tag{1}$$

Such a vector is called an eigenvector (associated with λ).

For each eigenvalue λ of T, the set of all eigenvectors associated with λ is a subspace of H, called the eigenspace associated with λ. The multiplicity of λ is the dimension of its corresponding eigenspace.

Definition 16 *A subspace $H_1 \subseteq H$ is called an invariant subspace of T if*

$$\forall f \in D_T \cap H_1$$

we have

$$Tf \in H_1.$$

We can define an operator T_1 on $D_T \cap H_1$ by

$$T_1 f = Tf \quad \forall f \in D_T \cap H_1.$$

T_1 is the restriction of T to H_1.

Definition 17 *Let $H_1 \subseteq H$. H_1 reduces the operator T if H_1 and $H_1^\perp$ are invariants subspaces of T and $P_{H_1} D_T \subset D_T$.*

Theorem 14 *[1] Let S be a subspace of H. If T is reduced by S, then*

$$Tf = T_1 f_1 + T_2 f_2$$

where $T_1 = T\mid_S, T_2 = T\mid_{S^\perp}, f_1 = P_S, f_2 = P_{S^\perp}$

Definition 18 *Let T be an operator on H. If D_T is dense in H, then there exists an operator T^* defined on*

$$D_{T^*} = \{g \mid \exists \ \ g^* \in H \ \textit{and} \ f \in D_T \ \textit{with} \ \langle Tf, g\rangle = \langle f, g^*\rangle\}$$

such that

$$T^*g = g^*,$$

with g^* *as defined in* D_{T^*}. T^* *is called the adjoint of* T.

Definition 19 *Let* T *be an operator on* H. T *is a self-adjoint operator if*

$$T = T^*,$$

that is ,if

$$\langle Tf, g\rangle = \langle f, Tg\rangle$$

$\forall f, g \in H$.

Proposition 15 *Let* T *be a self-adjoint operator on* H. *If* λ *is an eigenvalue of* T, *then* λ *is real.*

Proof. Let f be a non-zero eigenvector associated with λ. Then

$$\begin{aligned} \lambda\langle f, f\rangle &= \langle Tf, f\rangle \\ &= \langle f, Tf\rangle \\ &= \overline{\lambda}\langle f, f\rangle. \end{aligned}$$

Thus, $\lambda = \overline{\lambda}$ ■

Definition 20 *Let* T *be an operator on* H. T *is a symmetric operator if* D_T *is dense in* H *and* $\forall f, g \in D_T$,

$$\langle Tf, g\rangle = \langle f, Tg\rangle$$

Clearly, if T is symmetric operator, $\langle Tf, f\rangle$ is real. If it has the additional property that, $\forall f \in D_T$,

$$\langle Tf, f\rangle \geq 0,$$

then T is said to be positive. A negative symmetric operator is defined similarly.

Theorem 16 *[1] A symmetric operator whose range is H is self-adjoint.*

Corollary 17 *[1] If a bounded self-adjoint operator T on H has an inverse, then the inverse is self-adjoint.*

Definition 21 *An operator T defined over a Banach space B is compact if it maps the unit ball to a set of compact closure.*

In a Hilbert space, compact operators are characterized as follows:

Theorem 18 *An operator T defined over a Hilbert space H is compact if and only if it is the strong limit of a sequence of finite rank operators.*

For the remainder of this section, let T be a bounded linear operator defined on H.

Definition 22 *The spectrum of T, denoted by $\sigma(T)$ is the set*

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid (T - \lambda I)^{-1} \ \ \textit{does not exist or is unbounded}\}.$$

The point spectrum of T, denoted by $\sigma_p(T)$ is the set

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \lambda \ \ \textit{is an eigenvalue of } T\}.$$

Clearly,

$$\sigma_p(T) \subset \sigma(T).$$

Definition 23 *Let $\lambda \in \mathbb{C} \setminus \sigma(T)$. The resolvent of T (associated with λ) is the operator*

$$R_\lambda = (T - \lambda I)^{-1}.$$

Theorem 19 *[1]* ***Hilbert relation*** *We have*

$$R_\lambda - R_\mu = (\lambda - \mu) R_\mu R_\lambda.$$

where $\lambda, \mu \in \mathbb{C} \setminus \sigma(T)$.

Proof. Since $\lambda, \mu \in \mathbb{C} \setminus \sigma(T)$, R_λ, R_μ are well-defined and invertible,

$$\begin{aligned} R_\lambda &= R_\mu (T - \mu I) R_\lambda \\ R_\mu &= R_\mu (T - \lambda I) R_\lambda \\ R_\lambda - R_\mu &= R_\mu (T - \mu I - T + \lambda I) R_\lambda \\ &= (\mu - \lambda) R_\mu R_\lambda. \end{aligned}$$

■

Proposition 20 *[8] Let T be a self-adjoint operator on H. Then there exists a sequence of units vectors $\{h_n\}_{n \in \mathbb{N}}$ such that*

$$\langle T h_n, h_n \rangle \to \|T\|$$

as $n \to \infty$.

Lemma 21 *Let T be a compact operator on H. If there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ and a real number λ such that*

$$\|(T-\lambda I)h_n\| \to 0,$$

then λ is an eigenvalue of T.

Proof. Since T is compact, there exists a subsequence $\{h_{n_i}\}_{n\in\mathbb{N}}$ of $\{h_n\}_{n\in\mathbb{N}}$ such that

$$\|Th_{n_i} - h\| \to 0,$$

for some $h \in H$. We have

$$\begin{aligned}
\lambda h_{n_i} &= (\lambda I - T)h_{n_i} + Bh_{n_i} \\
&\to 0 + h = h. \\
\|T(\lambda h_{n_i} - h)\| &= \left\| T\left(\frac{\lambda h_{n_i} - h}{\|\lambda h_{n_i} - h)\|}\right)\right\| \|\lambda h_{n_i} - h)\| \\
&\leq \|B\| \, \|\lambda h_{n_i} - h)\| \\
&\to 0 \text{ as } i \to \infty, \\
\text{and then } \left\|Th_{n_i} - \lambda^{-1}Th)\right\| &\to 0, \\
\text{so } h &= \lambda^{-1}Th.
\end{aligned}$$

That is, λ is an eigenvalue of T ■

Theorem 22 *[1] If T is non zero compact self-adjoint operator, then one of $\|T\|, -\|T\|$ is an eigenvalue of T.*

Proof. We know (proposition 20) there is a sequence $\{h_n\}_{n\in\mathbb{N}}$ in H such that $\langle Th_n, h_n\rangle \to a$, where $a = \|T\|$ or $a = -\|T\|$. We have

$$\begin{aligned} 0 \leq \|T(h_n - a)h_n\|^2 &= \|Th_n\| - 2a < Th_n, h_n > +a^2\|h_n\|^2 \\ &\leq 2a^2 - 2a < Th_n, h_n > \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Thus $\|T(h_n - a)h_n\| \to 0$ as $n \to \infty$. This implies that a is an eigenvalue of T. ■

Theorem 23 *[18]**(Schur)** Let T be a non-zero compact self-adjoint operator over H. Then there exists a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of eigenvalues of T such that*

1)$|\lambda_i| \geq |\lambda_{i+1}|$ and if the sequence is not finite, it converges to zero.

2) $\sigma(T) = \sigma_p(T)\bigcup\{0\}$

3) $\ker(T - \lambda_i I)$ has finite dimension $\forall i \in \mathbb{N}$ and if P_i is the projection of H onto $\ker(T - \lambda_i I)$ then

$$P_n P_m = \delta_{nm}.$$

4) $T = \sum_{i\in\mathbb{N}} \lambda_i P_i$.

Proof. We will just present a sketch of the proof. The complete proof can be found in [18] . Let $H_0 = H$. We know that $\|T\|$ or $-\|T\|$ is an eigenvalue of T. Let λ_0 be this eigenvalue and h_0 an eigenvector associated to it. Let $H_1 = (\text{span}\{h_0\})^{\perp}$ and $T_1 = T\backslash_{H_1}$. If T_1 is non-zero, we can prove that T_1 is reduced by H_1 is a compact self-adjoint operator over H_1. Then T_1 has an eigenvalue $\lambda_1 = \|T_1\|$ or $\lambda_1 = -\|T_1\|$. Clearly, $\|T\| \geq \|T_1\|$.

Now let h_1 be a non-zero eigenvector associated to λ_1 , $H_2 = (\text{span}\{h_0, h_1\})^{\perp}$ and T_2 the restriction of T_1 on H_2. Similarly, if T_2 is non-zero, it can be shown that T_2 is a compact operator over H_2. It then has an eigenvalue $\lambda_2 = \|T_2\|$ or $\lambda_2 = -\|T_2\|$.

We can repeat this procedure (infinitely many times or until we obtain a zero operator). All the results of this theorem are derived from this construction. ■

Theorem 24 *Let T be a compact normal operator on a separable Hilbert space. Then T diagonalizes in an orthogonal basis, that is*

$$H = \bigoplus_{\lambda \in \sigma_p(T)} \ker(T - \lambda I).$$

Proof. By 3) in theorem 23, $\{\ker(T - \lambda_i I)\}_{\sigma_p(T)}$ is a orthogonal family of subspaces of H. Clearly,

$$\bigoplus_{\lambda \in \sigma_p(T) \setminus 0} \ker(T - \lambda I) \subseteq H.$$

Let h be a non-zero element of H such that h does not belongs to $\bigoplus_{\lambda \in \sigma_p \setminus 0(T)} \ker(T - \lambda I)$. Then, 4) in theorem 23 implies that

$$Th = 0,$$

that is, zero is an eigenvalue of T and

$$h \in \ker(T).$$

Hence, $\forall h \in H$,

$$h \in \bigoplus_{\lambda \in \sigma_p(T)} \ker(T - \lambda I).$$

■

We complete this section with the following definitions:

Definition 24 *Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for H. If A is a linear operator over H for which*

$$\sum_{j=1}^{\infty} \|Ae_j\|^2 < \infty$$

then A is said to be a Hilbert-Schmidt *operator.*

Products of Hilbert-Schmidt operators are of special interest in operator theory.

Definition 25 *An operator T is said to be a trace-class operator if it is the product of Hilbert-Schmidt operators .We define its trace by*

$$tr(A) = \sum_{j=1}^{\infty} \langle Ae_j, e_j \rangle$$

The set of all trace-class operators is a 2-sided ideal in the space of bounded linear operators $L(H)$ over H.

2.4 Elementary Frame Theory

In this section, we present some elementary frame theory. The following was extracted from [9, 10, 14].

Definition 26 *[9] Let H be a separable Hilbert space , and $< \cdot, \cdot >$ be the inner product defined on H. A family of elements $\{f_i\}_{i=1}^{\infty} \subset H$ is a frame if $\exists A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\rangle f, f_i \langle|^2 \leq B\|f\|^2 \tag{2}$$

for any arbitrary $f \in H$.

The constants A, B are called lower and upper frame bounds respectively, and are not unique. The optimal frame bounds are the largest and the smallest possible value of A and B respectively. If it is possible to have $A = B$, then the frame is said to be tight. It is said to be exact if it is no longer a frame when an arbitrary element is removed.

Proposition 25 *[9] Every finite collection of H is a frame for its span.*

Proof. Let $\{f_i\}_{i=1}^n$ be a finite collection of H. Since the collection is finite there exist A, B, C, D, E such that

$$\begin{aligned}
C &= max\{\|f\|^2, f \in Span\{f_i\}_{i=1}^n\}. \\
D &= min\{\sum_{i=1}^n |\rangle f, f_i\langle|^2, f \in Span\{f_i\}_{i=1}^n\}. \\
F &= max\{\sum_{i=1}^n |\rangle f, f_i\langle|^2, f \in Span\{f_i\}_{i=1}^n\}. \\
E &= min\{\|f\|^2, f \in Span\{f_i\}_{i=1}^n\}. \\
A &= \frac{D}{C}. \\
B &= \frac{F}{E}.
\end{aligned}$$

Clearly, we have

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\rangle f, f_i\langle|^2 \leq B\|f\|^2$$

$\forall\ f \in Span\{\{f_i\}_{i=1}^n\}$. ■

Now we define the frame operator.

Definition 27 *[9] The frame operator $S : H \to H$ is defined as*

$$Sf = \sum_{i=1}^{\infty} \rangle f, f_i \langle f_i.$$

The following theorem gives us some properties of the frame operator.

Theorem 26 *[14] The frame operator S, as in definition 27, is a bounded, self-adjoint, positive and invertible operator.*

Proof. This theorem has been proved in [14]. We will just prove that S is self-adjoint and positive. The remainer is proven using (2). First note that $\mathrm{Dom}(S) = H$. let $f, g \in H$.

$$\begin{aligned}
\langle Sf, g\rangle &= \left\langle \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i, g \right\rangle \\
&= \sum_{i=1}^{\infty} \langle f, f_i\rangle \langle f_i, g\rangle. \\
\langle f, Sg\rangle &= \left\langle f, \sum_{i=1}^{\infty} \langle g, f_i\rangle f_i \right\rangle \\
&= \sum_{i=1}^{\infty} \overline{\langle g, f_i\rangle} \langle f, f_i\rangle \\
&= \sum_{i=1}^{\infty} \langle f_i, g\rangle \langle f, f_i\rangle \\
\text{then,}\ \langle Sf, g\rangle &= \langle f, Sg\rangle.
\end{aligned}$$

Thus, S is self-adjoint.

For the positivity of S we have

$$\begin{aligned}
\langle Sf, f\rangle &= \left\langle \sum_{i=1}^{\infty}\langle f, f_i\rangle f_i, f \right\rangle \\
&= \sum_{i=1}^{\infty}\langle f, f_i\rangle\langle f_i, f\rangle \\
&= \sum_{i=1}^{\infty}\langle f, f_i\rangle\overline{\langle f, f_i\rangle} \\
&= \sum_{i=1}^{\infty}\|\langle f, f_i\rangle\|^2 \\
&\geq 0.
\end{aligned}$$

This complete the proof. ■

Note that the frame operator reduces to the identity operator if the frame forms an orthonormal basis of H. Using the invertibility property of S, we have:

Theorem 27 *[10] All $f \in H$ can be represented as an infinite linear sum of the elements in the frame family.*

Proof. Since S is invertible, we have

$$f = SS^{-1}f = \sum_{i=1}^{\infty} < f, S^{-1}f_i > f_i.$$

This is called the frame decomposition. ■

Let's define the related notion of a Riesz basis of a Hilbert Space H.

Definition 28 *[9] $\{\varphi_k\}$ is a Riesz basis for H if it is complete and*

$$\exists A, B > 0 : A\sum |c_k|^2 \leq \left\|\sum c_k\varphi_k\right\|^2 \leq B\sum |c_k|^2$$

for all finite sequences $\{c_k\}$ of complex scalars.

A Riesz basis is a frame, and numbers A, B in the above definition coincide with the frame bounds.

Another example of frames is nonharmonic Fourier Series. We know that the family of elements $\{\frac{1}{\sqrt{2\pi}}e^{imx}\}_{m\in\mathbb{Z}}$ form an orthonormal basis for $L^2(-\pi,\pi)$. In general, we have:

Definition 29 *[9] Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers. A linearly independent subset of $L^2(-\pi,\pi)$ of the form $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is called a nonharmonic basis, and the expansion*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n x}$$

is called a nonharmonic Fourier series.

Among frames, nonharmonic Fourier Series have been intensively studied, and some nice theorems about them have been proven. One of these results is **Kadec's 1/4 Theorem** [9], stating that if $\{\lambda_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ and

$$\sup{}_{m\in\mathbb{Z}}\{|\lambda_n - n|\} < \frac{1}{4},$$

then $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$. A generalization of this theorem is:

Theorem 28 *[9] Let $\{\lambda_n\}_{n\in\mathbb{Z}}$, and $\{\mu_n\}_{n\in\mathbb{Z}}$ be sequences of real numbers. suppose that $\{e^{i\mu_n x}\}_{n\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$ with bounds A, B. If there exists a constant $L < \frac{1}{4}$ such that*

$$|\mu_n - \lambda_n| \leq L, and \quad 1 - \cos\pi L + \sin\pi L < \sqrt{\frac{A}{B}},$$

then $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$ with bounds

$$A\left(1 - \sqrt{\frac{B}{A}}\left(1 - \cos\pi L + \sin\pi L\right)\right)^2, \quad B\left(2 - \cos\pi L + \sin\pi L\right)^2.$$

The main feature of a frame is the frame decomposition. The coefficients $\{\langle f, S^{-1}f_i\rangle\}$ are called frame coefficients. Theorem 27 says that, if $\{f_i\}_{i=1}^{\infty}$ is a frame for H, then every $f \in H$ can be represented as a linear combination of the f_i. Moreover, as it has been proven in [9], this decomposition is not unique. Some operators are complicated to study. But if we know how they act on some building blocks (frames) of the Hilbert space, we can find how they act on the rest of the space by finding the frame coefficients. This is why, in many applications, it is important to approximate the frame coefficients. There are several approaches to this, one of which is the so called *projection method*. Let P_n be the projection on $H_n = \text{span}\{f_i\}_1^n$. Clearly,

$$P_n f = \sum_{i=1}^{n} < f, S_n^{-1} f_i > f_i,$$

where

$$S_n f = \sum_{i=1}^{n} < f, f_i > f_i,$$

is defined from H_n to H_n. We do not have an expression for S^{-1} . But we can compute S_n^{-1} (using the CHT). The following question then arises naturally:

$$\text{Is it true that} \qquad < f, S_n^{-1} f_i > \to < f, S^{-1} f_i >? \tag{3}$$

The answer is “yes” for a Riesz basis, but “no” in general [9]. Several attempts have been made to provide a sequence of coefficients converging to the frame coefficients. The main idea is to find a sequence of finite collections of H for which a convergence similar to (3) can be obtained.

In [7] the authors present this approach:

Theorem 29 *Let $\{\varphi_k\}$ be a frame. Given $n \in \mathbb{N}$, let A_n denote a lower frame bound* *for* $\{\varphi_k\}_{k\in I_n}$ *and choose a finite set* J_n *containing* I_n *such that*

$$\sum_{k\notin J_n} |< \varphi_j, \varphi_k >| \leq \frac{A_n}{n|I_n|}, \quad \forall \;\; j \in \;\; I_n.$$

Let $V_n : H_n \to H_n$ *denote the frame operator for finite family* $\{P_n\varphi_k\}_{k\in J_n}$*. Then*

$$V^{-1}P_nf \to S^{-1}f \quad as \quad n \to \infty, \quad \forall \;\; f \in H.$$

This result is most important to use for the case of exponential frames. [10] gives a more general result, where the following lemma plays an important role:

Lemma 30 *[10] Given* $n \in \mathbb{N}$*, there exists a number* $m(n)$ *such that*

$$\frac{A}{2}\|f\|^2 \leq \sum_{i=1}^{n+m(n)} |< f, f_i >|^2, \forall f \in H_n.$$

The following theorem can then be proven.

Theorem 31 *[10] Let* $\{f_i\}_{i=1}^{\infty}$ *be a frame with bounds* A, B*. For* $n \in \mathbb{N}$*, choose* $m(n)$ *as in the above lemma. Then*

$$\left(P_nS_{n+m(n)}\right)^{-1}P_nf \to S^{-1}f \quad for \quad n \to \infty, \quad \forall \;\; f \in H.$$

The two preceding theorems provide an answer to the problem, but their implementation is only efficient in some specific cases.

3 A REVIEW OF CURRENT RESULTS

3.1 Extensions of the Cayley-Hamilton Theorem

Several attempts have been made to generalize the CHT in recent literature. Here, we will discuss some of these attempts.

3.1.1 Extension of the CHT to Super-Matrices

This is a summary of [23]. Let's start by the definition of a super-matrix:

Definition 30 *[23] A $(p+q)\times(p+q)$ super-matrix is a block matrix of the form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are $p\times p, p\times q, q\times p, q\times q$ matrices.

The super-trace and the super-determinant are defined respectively as follows:

$$\begin{aligned} \mathrm{Str}(M) &= \mathrm{tr}(A) - \mathrm{tr}(D) \\ \mathrm{Sdet} &= \exp(\mathrm{Str}(\ln(M))) \end{aligned}$$

We can obtain from these definitions the following identities:

$$\begin{aligned} \mathrm{Str}(M_1M_2) &= \mathrm{Str}(M_1M_2) \\ \mathrm{Sdet} &= \frac{\det(A - BD^{-1}C)}{\det(D)} \\ &= \frac{\det(A)}{\det(D - CA^{-1}B)} \\ \mathrm{Sdet}(M_1M_2) &= \mathrm{Sdet}(M_1M_2) \end{aligned}$$

with $\det(A), \det(D)$ non-zero. In [23], using these properties, the authors define polynomials having the property that they are annihilated by the super-matrix .

Theorem 32 *(Extended Cayley-Hamilton Theorem) Let M and $(xI - M)$ be $(p+q)\times(p+q)$ super-matrices, $x \in \bigwedge_0$, with Sdet$(xI - M) = \widetilde{F}/\widetilde{G} = F/G$. Then, for* *any common factor R such that $\widetilde{F} = R\overline{f}, \widetilde{G} = R\overline{g}$ and S such that $F = Sf, G = Sg$, where $\widetilde{f}/\widetilde{g} = f/g$, the polynomial $P(x) = \overline{f}(x)g(x) = f(x)\overline{g}(x)$ annihilates M. That is*

$$P(M) = 0.$$

Note: Terms used in this theorem are defined in the paper.

3.1.2 Cayley-Hamilton Decomposition

Here we discuss the approach of the paper by Ponge [17]. Although the paper has been withdrawn, there are many useful ideas and results in the manuscript.

Definition 31 *Let H be a separable Hilbert space and T be a compact operator defined on H. Given an eigenvalue $\lambda \in \sigma_p(T)\backslash\{0\}$, the characteristic space $E_\lambda(T)$ and the characteristic projector $\prod_\lambda(T)$ are given by the formulas*

$$\begin{aligned} E_\lambda(T) &= \bigcup_{k\geq 1} ker(T - \lambda I)^k \\ \prod_\lambda(T) &= \frac{-1}{2i\pi}\int_{\Gamma_{(\lambda)}} (T - uI)^{-1}\, du \end{aligned}$$

where $\Gamma_{(\lambda)}$ is a small circle about λ which isolates λ from the rest of the spectrum.

This is a generalization of the eigenspace and the projection associated to an eigenvalue. With only the compactness condition, it is not true that

$$H = \overline{+_{\lambda\in\sigma_p(T)} E_\lambda(T)},$$

where $+$ represents the algebraic sum. However, there are some well-known properties of the characteristic spaces and the characteristic projectors.

Theorem 33 *[19] The family* $(\prod_\lambda(T))_{\lambda\in\sigma_p(T)\backslash\{0\}}$ *is an orthogonal family of projectors, that is*

$$\prod_\lambda(T)\prod_\mu(T) = \delta_{\lambda\mu} \textit{ for any } \lambda,\mu \in\sigma_p(T)\backslash 0.$$

Moreover for any $\lambda \in \sigma_p(T)\backslash 0$, *the projector* $\prod_\lambda(T)$ *has finite rank.*

Theorem 34 *[12] Let* $\lambda \in \sigma_p(T)\backslash 0$. *Then:*

1) $E_\lambda(T)$ *has finite dimension and there is an integer* $N \geq 1$ *such that* $E_\lambda(T) = \ker(T-\lambda I)^N$.

2) The subspace $E_{\overline{\lambda}}(T*)^\perp$ *is globally invariant by* T *and we have* $H = E_\lambda(T) + E_{\overline{\lambda}}(T*)^\perp$.

3) The projector $\prod_\lambda(T)$ *projects onto* $E_\lambda(T)$ *and along* $E_{\overline{\lambda}}(T*)^\perp$.

In [17], the author proposes an extension of this concept to $0 \in \sigma_P(T)$ as follows:

$$\prod_0(T) = \lim_{r\to 0^+}\frac{-1}{2i\pi}\int_{|u|=r}(T-uI)^{-1}\,du \quad \text{and} \quad E_0(T) = \mathrm{im}\prod_0(T).$$

In the manuscript, the author claims the following conjecture is true, although the proof is flawed.

Conjecture 35 *Let* $(\prod_n)_{n\geq 1}$ *be an orthogonal sequence of projectors. Then:*

1) The series $\sum_{n\geq 1}\prod_n$ *converges in* $\Omega(H)$, *the space of bounded linear operators defined on* H, *to the projector onto* $\overline{+_{n\geq 1} im \prod_n}$ *and along the subspace* $\bigcap_{n\geq 1} ker \prod_n$.

2) Let $\prod_0 = I - \sum_{n\geq 1}\prod_n$. *Then* $(\prod_n)_{n\geq 0}$ *is an orthogonal sequence of projector and we have*

$$H = \overline{+_{n\geq 0} im \prod_n} \quad \textit{and} \quad \sum_{n\geq 0}\prod_n = I.$$

3.2 Determinant of an Operator in Infinite Dimensional Space

There are various ways to calculate the determinant of a linear transformation over a finite dimensional vector space. This can be done using the co-factor expansion or the CHT. However, in the infinite dimensional case, calculating the determinant of an operator is often quite difficult.

The first difficulty is deciding how to define the determinant of an operator. We cannot use the same definition as in the finite case. This question does not have a unique answer, and none of the answers proposed so far have been generally accepted. Also, it is known that the different available definitions generally produce different results. However, we note that most of these definitions are derived from the identity

$$\log \det(T) = \operatorname{tr} \log(T),$$

where T is a linear operator defined on a Hilbert space.

Among all the definitions of the determinant of a linear transformation over an infinite dimensional space, the one having the most mathematical acceptance is the one using the derivative of the zeta function associated with the operator and was first introduced by Ray and Singer.

Definition 32 *Let T be an operator with a trace-class inverse defined on a separable Hilbert space H. The zeta-function associated with T is*

$$\zeta(s) = \operatorname{tr}\left((T)^{-s}\right).$$

The determinant of T, when it exists, is defined to be

$$\det(T) = e^{-\zeta'(0)}.$$

The importance of this definition comes from the fact the it can be applied to elliptic operators, often encountered in physics, chemistry, biology.

Definition 33 *[21] A partial differential D operator of order K that acts on $\mathbb{C}^m$-valued C^∞ functions defined on $\mathbb{R}^n$ can be written as*

$$D = \left(\sum_{|\alpha| \leq k} a_\alpha^{ij} D^\alpha \right)_{i,j=1,\cdots,m}$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ *It is said to be of elliptic type if for every* $\epsilon = \{\epsilon_1, \cdots, \epsilon_n\} \neq 0 \in \mathbb{R}^n$,

$$\sigma(D)(\epsilon) = \left(\sum_{|\alpha|=k} a_\alpha^{ij}(x) \epsilon^\alpha \right)$$

is non-singular.

Unfortunately, as illustrated in [11], there are some anomalies arising from this definition. A major problem arising from this definition is that $\operatorname{tr}\left((T)^{-s}\right)$ exists only for $\Re(s) > a$ for some $a > 0$. Thus, the determinant requires that the zeta function be analytically continued to a region including the origin ([15, 22]). However, it is easy to verify that this definition reduces to the normal definition of determinant in finite dimension.

3.3 Analytic and Meromorphic Continuations

Definition 34 *A function is said to be analytic on an open region R of the complex plane if it is complex differentiable at every point in R.*

Note: The term *holomorphic* is a synonym for analytic.

Often, we need to know the behavior of a function outside of its domain of analyticity. To obtain this information, we "extend" the function to a bigger domain, preserving analyticity. This technique is called *analytic continuation.*

Definition 35 *Suppose D_1 and D_2 are connected, open subsets of $\mathbb{C}$, and that $D_1 \subset D_2$. Suppose*

$$f : D_1 \to \mathbb{C} \quad \textit{and} \quad g : D_2 \to \mathbb{C}$$

are analytic on their respective domains. If

$$g(z) = f(z) \;\; \forall \;\; z \in D_1,$$

then g is the analytic continuation of f on D_2.

Example: Let

$$f(z) = \sum_{k=0}^{\infty} z^k.$$

The ratio test shows that $f(z)$ is analytic only on the region $R = |z| < 1$. Consider

$$g(z) = \frac{1}{1-z}$$

$z \in \mathbb{C} \setminus \{1\}$. Clearly, g is analytic on its domain $D_g = \mathbb{C} \setminus \{1\}$. The Maclaurin expansion of g is $\sum_{k=0}^{\infty} z^k$ and is equal to $g(z)$ on R. thus,

$$g = f \;\; \text{on R.}$$

Then g is the analytic continuation of f on D_g.

A natural question is whether or not the analytic continuation is unique. Suppose g_1, g_2 are two analytic continuations of a function $f : D_1 \to \mathbb{C}$ analytic on D_1 (a connected open subset of the complex plane) to a region D_2 of the complex plane.

Then D_1 is a subset of $D = \{z \in \mathbb{C} / g_1(z) = g_2(z)\}$. Thus D contains a limit point. Thus, g_1, g_2 are equal everywhere on D_2. We just proved the following theorem:

Theorem 36 *The analytic continuation of a function f in an open connected subset of the complex plane containing $Dom f$ is unique.*

A concept similar to the one of analytic functions is the one of meromorphic functions.

Definition 36 *A meromorphic function is a function that is holomorphic everywhere on the complex plane except on a set of isolated poles, which are not essential singularities.*

Every meromorphic function can be expressed as the ratio between two holomorphic functions (with the denominator not identically 0). The poles then occur at the zeroes of the denominator.

The meromorphic continuation of a function is defined similarly as the definition of the analytic continuation.

Finding analytic and meromorphic continuations of different classes of operators is an active research focus. In [15], meromorphic continuation of the zeta functions associated with elliptic operators is derived. This will be of great important in our work.

4 EXTENSIONS OF THE CAYLEY-HAMILTON THEOREM

A single generalized extension of the CHT to linear operators over an infinite-dimensional separable Hilbert space is unlikely to exist. Restrictions are needed, which will provide us with operators having some properties useful in obtaining extensions. We start by presenting some of these properties for the classes of operators we will use.

Let's state the *Spectral theorem* for self-adjoint operators [4].

Theorem 37 *If A is a (possibly unbounded) self-adjoint operator over a separable Hilbert space H and if A is either compact or has a compact resolvent, then the spectrum of A is*

$$\sigma(A) = \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}$$

and there exists a sequence of commuting, finite-rank, self-adjoint projections $P_1, P_2, \ldots$ such that

$$I = \sum_{n=1}^{\infty} P_n \quad \textit{and} \quad A = \sum_{n=1}^{\infty} \lambda_n P_n.$$

Proposition 38 *Let P be a projection operator defined on H. If $P(H)$ is finite, then P is trace-class and*

$$tr(P) = \dim(P(H)).$$

Proof. Let $\{e_j\}_1^{\infty}$ be an orthonormal basis of H.

$$\begin{aligned}
\operatorname{tr}(P) &= \sum_{j=1}^{\infty} \langle Pe_j, e_j \rangle \\
&= \sum_{e_j \in \operatorname{im}(P)} \langle Pe_j, e_j \rangle \\
&= \sum_{e_j \in \operatorname{im}(P)} 1 \\
&= \dim(\operatorname{im}(P)).
\end{aligned}$$

This completes the proof. ■

Proposition 39 *A trace-class operator defined on H is compact.*

Proof. Let ${e_j}_{n=1}^{\infty}$ be an orthonormal basis of H and A a trace-class operator on H. Let

$$A_n = \begin{cases} A \text{ on } \operatorname{span}\{e_1, \cdots, e_n\}, \\ \\ 0, \text{ on } H \setminus \operatorname{span}\{e_1, \cdots, e_n\} \end{cases}.$$

Let $\epsilon > 0$ be given. Since T is trace-class, $\sum_{n=N+1}^{\infty} \|Ae_j\|^2 < \epsilon$ for some $N \in \mathbb{N}$. Let $f = \sum_{j=1}^{\infty} c_j e_j$ be arbitrary in H. Now

$$\begin{aligned}
\|(A - A_N) f\|^2 &= \langle (A - A_N) f, (A - A_N) f \rangle \\
&= \left\langle \sum_{j=N+1}^{\infty} c_j A e_j, \sum_{i=N+1}^{\infty} c_i A e_i \right\rangle \\
&= \sum_{j=N+1}^{\infty} \sum_{i=N+1}^{\infty} c_j \overline{c_i} \langle Ae_j, Ae_i \rangle \\
&\leq \sum_{j=N+1}^{\infty} \sum_{i=N+1}^{\infty} |c_j||\overline{c_i}| \|Ae_j\| \|Ae_i\| \\
&\leq \left(\sum_{j=N+1}^{\infty} |c_j| \|Ae_j\| \right) \left(\sum_{i=N+1}^{\infty} |\overline{c_i}| \|Ae_i\| \right) \\
&\leq \left[\left(\sum_{j=N+1}^{\infty} |c_j|^2 \right) \left(\sum_{j=N+1}^{\infty} \|Ae_j\|^2 \right) \right]^2 \leq \epsilon^2 \|f\|^2 .
\end{aligned}$$

Thus $\|(A - A_N)\| \to 0$ as $N \to \infty$. But the A_N's have finite rank. Thus the result follows by theorem 18. ■

4.1 CHT for Invertible Self-Adjoint Trace-Class Operators

Here, we let T be an invertible self-adjoint trace-class operator.

Proposition 40 *Let T be a compact normal operator on a separable Hilbert space. Then, for every $n \in \mathbb{N}$,*

$$\lambda \in \sigma_p(T) \quad \textit{iff} \quad \lambda^n \in \sigma_p(T^n)$$

and

$$ker(T^n - \lambda^n I) = ker(T - \lambda I) \quad \forall \quad \lambda \in \sigma_p(T)$$

Proof. Clearly, one direction of both statements is trivial. Now since T is a compact normal operator, then so is T^n. Let $A = \{\lambda^n \mid \lambda \in \sigma_p(T)\}$ and $B = \sigma_p(T^n)\backslash A$. Theorem 24 implies that

$$\begin{aligned} H &= \bigoplus_{\lambda\in\sigma_p(T)} \ker(T-\lambda I) \\ H &= \bigoplus_{\lambda\in\sigma_p(T^n)} \ker(T^n-\lambda^n I) \\ &= \bigoplus_{\lambda\in A} \ker(T^n-\lambda^n I) \bigoplus_{\lambda\in B} \ker(T^n-\lambda^n I). \\ \text{But} \quad \ker(T-\lambda I) &\subseteq \ker(T^n-\lambda^n I). \quad \text{Thus} \\ H &= \bigoplus_{\lambda\in\sigma_p(T)} ker(T-\lambda I) \subseteq \bigoplus_{A} ker(T^n-\lambda I). \end{aligned}$$

This implies that $B = \emptyset$, and completes the proof of the corollary. ■

Lemma 41 *Let T be a non-zero trace-class (hence compact) self adjoint operator and $z, y \in H$ such that*

$$|z| > \|T\| .$$

Then

$$tr[\ln(zI - T)] = \sum_{\lambda\in S_p(T)} n_\lambda \ln(z - \lambda),$$

where we define

$$\ln(zI - T) = \ln(z)I - \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{z^n}T^n.$$

and n_λ is the order of the generalized (or characteristic) eigenspace associated with λ.

In the lemma, we defined $\ln(zI - T) = \ln(z)I - \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{z^n}T^n$. We now present the motivation behind this definition.

Theorem 42 ***C. Neumann expansion:****[18] If T is a bounded linear operator, and $\|T\| < 1$ then $I - T$ is invertible and*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Moreover,

$$\left\|(I - T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}$$

Proof. Let Ω be the collection of bounded linear operators defined on H. We know that Ω is complete. $T \in \Omega$ and $\|T\| < 1$. Then $\sum_{n=0}^{\infty} \|T\|^n$ is a convergent geometric series. Let $T_N = \sum_{n=0}^{N} T^n$, $N, P \in \mathbb{N}$, P arbitrary. Then

$$\begin{aligned}
\lim_{N\to\infty} \|T_N - T_{N+P}\| &= \lim_{N\to\infty} \left\|\sum_{n=N+1}^{N+P} T^n\right\| \\
&\leq \lim_{N\to\infty} \sum_{n=N+1}^{N+P} \|T^n\| \\
&\leq \lim_{N\to\infty} \sum_{n=N+1}^{\infty} \|T^n\| \\
&= 0.
\end{aligned}$$

Since $\sum_{n=0}^{\infty} \|T\|^n$ converges, $\left\{\sum_{n=0}^{N} T^n\right\}_{N\in\mathbb{N}}$ is a Cauchy sequence. It then converges, say to

$$T_0 = \sum_{n=0}^{\infty} T^n,$$

since Ω is complete.

Now

$$\begin{aligned}
(I-T)T_0 &= T_0 - TT_0 \\
&= \sum_{n=0}^{N} T^n - \sum_{n=0}^{N} T^{n+1} \\
&= I. \\
\text{Similarly,}\quad T_0(I-T) &= I. \\
\text{Hence}(I-T)^{-1} &= \sum_{n=0}^{N} T^n.
\end{aligned}$$

The rest of the proof can be found in [18]. ■

Corollary 43 *[18] If T is a bounded linear operator and if $z \in \mathbb{C}$ with $|z| > \|T\|$, then $zI - T$ is invertible and*

$$(zI-T)^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{z^{n+1}}.$$

Proof. $|z| > \|T\|$ implies that $1 > \left\|\frac{T}{z}\right\|$. We have

$$\begin{aligned}
zI - T &= z(I - z^{-1}T), \\
\text{but}\quad \left\|z^{-1}T\right\| &= \frac{\|T\|}{|z|} \\
&< 1, \quad \text{thus} \quad (I - z^{-1}T) \quad \text{is invertible and} \\
(I - z^{-1}T)^{-1} &= \sum_{n=0}^{\infty} \frac{T^n}{z^n},
\end{aligned}$$

$$\begin{aligned} \text{hence} \quad (zI - T)^{-1} &= z^{-1}(I - z^{-1}T)^{-1} \\ &= \sum_{n=0}^{\infty} \frac{T^n}{z^{n+1}}. \end{aligned}$$

This complete the proof. ■

Using the above corollary, we obtain the following definition:

Definition 37 *Let T be a bounded linear operator on H, and $z \in \mathbb{C}$ such that $|z| > \|T\|$. Then,*

$$\ln (zI - T) = \ln(z)I - \sum_{n=1}^{\infty} \frac{1}{n} \frac{T^n}{z^n}.$$

Proposition 44 *Let T be a self-adjoint compact operator. Then*

$$T^n = \sum_{\lambda \in \sigma_p(T)} \lambda^n P_\lambda.$$

Proof. Since T is a self-adjoint compact operator, so is T^n. We can then apply the spectral theorem stated earlier to T^n, together with proposition 24 and obtain the result. ■

Now consider the R.H.S. of the equality in the lemma. We have

$$\operatorname{tr}[\ln (zI - T)] = \operatorname{tr}\left[\ln(z)I - \sum_{n=1}^{\infty} \frac{1}{n} \frac{T^n}{z^n}\right].$$

Also, since $|z| > \|T\|$, then $|z| > \lambda, \forall \ \lambda \in \sigma_p(T)$. Then,

$$\begin{aligned} (z - \lambda)^{-1} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{z^{n+1}} \\ \text{thus} \quad \ln(z - \lambda) &= \ln z - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\lambda^n}{z^n}. \end{aligned}$$

The L.H.S. becomes

$$\begin{aligned}
\sum_{\lambda\in S_p(T)} n_\lambda \ln(z-\lambda) &= \sum_{\lambda\in S_p(T)} \left(n_\lambda \ln z - \sum_{n=1}^{\infty} n_\lambda \frac{1}{n}\frac{\lambda^n}{z^n} \right) \\
&= \sum_{\lambda\in S_p(T)} \left(\operatorname{tr}(P_\lambda) \ln z - \sum_{n=1}^{\infty} \operatorname{tr}(P_\lambda) \frac{1}{n}\frac{\lambda^n}{z^n} \right) \\
&= \operatorname{tr}\left[\sum_{\lambda\in S_p(T)} \left(P_\lambda \ln z - \sum_{n=1}^{\infty} P_\lambda \frac{1}{n}\frac{\lambda^n}{z^n} \right) \right] \\
&= \operatorname{tr}\left[\ln(z) I - \sum_{\lambda\in S_p(T)} \sum_{n=1}^{\infty} P_\lambda \frac{1}{n}\frac{\lambda^n}{z^n} \right] \\
&= \operatorname{tr}\left[\ln(z) I - \sum_{n=1}^{\infty} \frac{1}{n}\frac{T^n}{z^n} \right].
\end{aligned}$$

This completes the proof of the lemma. Moreover, the lemma and its proof motivated us to make the following definition

Definition 38 *The characteristic function of T is defined to be the meromorphic continuation in z of*

$$g(z,y) = e^{\sum_{\lambda\in S_p(T)} n_\lambda \left[\ln(1-\frac{\lambda}{z}) - \ln(1-\frac{\lambda}{y})\right]}$$

from the domain $D = \{|z| > \|T\|\}$ to the punctured plane $\mathbb{C} - \{0\}$.

The main result for $g(z,y)$ is given below.

Theorem 45

$$g(z,y) = \prod_{\lambda\in S_p(T)} \left(1 - \lambda \frac{z^{-1} - y^{-1}}{1 - \lambda y^{-1}}\right)^{n_\lambda}$$

where g is defined on $\mathbb{C} \times \rho(T)$.

Proof. We have

$$\begin{aligned} e^{\sum_{\lambda\in S_p(T)} n_\lambda\left[\ln(1-\frac{\lambda}{z})-\ln(1-\frac{\lambda}{y})\right]} &= e^{\sum_{\lambda\in S_p(T)} n_\lambda\left[\ln(1-\frac{\lambda}{z})(1-\frac{\lambda}{y})^{-1}\right]} \\ &= e^{\ln\left[\prod_{\lambda\in S_p(T)}\left(1-\lambda\frac{z^{-1}-y^{-1}}{1-\lambda y^{-1}}\right)^{n_\lambda}\right]} \\ &= \prod_{\lambda\in S_p(T)}\left(1-\lambda\frac{z^{-1}-y^{-1}}{1-\lambda y^{-1}}\right)^{n_\lambda}. \end{aligned}$$

This complete the proof. ■

Corollary 46 *$g(\lambda_n, y) = 0$ for all $\lambda_n \in \sigma_p(T)$.*

Notice that in a finite dimensional space, $g(z, y)$ reduces to the characteristic polynomial. Consider the function

$$p(z,y) = \prod_{\lambda\in S_p(T)}\left(1-\lambda\frac{z^{-1}-y^{-1}}{1-\lambda y^{-1}}\right) = \prod_{n=1}^{\infty}\left(1-\lambda_n\frac{z^{-1}-y^{-1}}{1-\lambda_n y^{-1}}\right),$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ are the eigenvalues of T. Note that this is related to the concept of minimal polynomial in the finite case.

Proposition 47 *There exists an orthonormal basis of H whose vectors are eigenvectors of T.*

Proof. This is a direct consequence of theorem 24 ■

Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of H formed by eigenvectors of T. Then we have the following lemma for $y \in \rho(T)$:

Lemma 48 *$\forall f \in dom(T)$, the sequence*

$$\left\{\prod_{n=1}^{N}\left(I-\lambda_n\frac{T^{-1}-y^{-1}I}{1-\lambda_n y^{-1}}\right)\right\}$$

converges strongly to 0.

Proof. For each $N \in Z^{+}$, let

$$p_N(T, y) = \prod_{n=1}^{N} \left(I - \lambda_n \frac{T^{-1} - y^{-1} I}{1 - \lambda_n y^{-1}} \right).$$

Given any $f \in \mathrm{Dom}(T)$ with $f = \sum_{j=1}^{\infty} c_j e_j$, we let $h_N = p_N(T, y) f$. Since the product p_N is finite, we have

$$\begin{aligned} h_N &= \sum_{j=1}^{\infty} c_j \prod_{n=1}^{N} \left(I - \lambda_n \frac{T^{-1} - y^{-1} I}{1 - \lambda_n y^{-1}} \right) e_j \\ &= \sum_{j=1}^{\infty} c_j \prod_{n=1}^{N} \left(1 - \lambda_n \frac{\lambda_j^{-1} - y^{-1}}{1 - \lambda_n y^{-1}} \right) e_j \\ &= \sum_{j=N+1}^{\infty} c_j \prod_{n=1}^{N} \left(1 - \lambda_n \frac{\lambda_j^{-1} - y^{-1}}{1 - \lambda_n y^{-1}} \right) e_j. \end{aligned}$$

If $g \in H$ is given by $g = \sum_{j=1}^{\infty} d_j e_j$, then

$$\langle h_N, g \rangle = \sum_{j=N+1}^{\infty} \overline{d_j}\, c_j \prod_{n=1}^{N} \left(1 - \lambda_n \frac{\lambda_j^{-1} - y^{-1}}{1 - \lambda_n y^{-1}} \right),$$

so that $\langle h_N, g \rangle \to 0$ for all $g \in H$, which implies that $p_N(T, y)$ converges weakly to 0 on $\mathrm{Dom}(T)$. The same argument can be used to show that

$$\langle h_N, h_N \rangle \to 0.$$

Hence the convergence is strong by theorem 13. ■

We define the functions $B_j(y)$ by

$$p(z, y) = 1 - \sum_{j=1}^{\infty} B_j(y)(z^{-1} - y^{-1})^j.$$

We now use the above results to develop an extension of CHT.

Theorem 49 *Extended Cayley-Hamilton Theorem:*

Let T be an invertible self-adjoint trace-class operator. $p(z, y)$ annihilates T , that is,

$$p(T, y) f = 0, \forall f \in H.$$

Proof. This is a direct consequence of lemma 48. $\blacksquare$

Corollary 50 $I = \sum_{j=1}^{\infty} B_j(y)(T^{-1} - y^{-1}I)^j$.

4.2 CHT for Closed Symmetric Operators with Trace-Class Resolvent

In this section, let T be a closed self-adjoint invertible operator with a trace-class resolvent having a finite number of negative eigenvalues and domain $\mathrm{Dom}(T) \subset H$, dense in H (we will refer to this as (*)), and let

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

denote the eigenvalues of T with each repeated up to multiplicity (they have finite multiplicity since T^{-1} is compact).

Note: $\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$ converges since $0 \in \rho(T)$.

For any $z \in \mathbb{C}$, there exists $N \in \mathbb{N}$, $\epsilon \in \mathbb{R}$ such that $\mathrm{Re}(z) < \lambda_N + \epsilon$ (since T is unbounded) with, $\lambda_N + \epsilon \in \rho(T)$, thus

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n - z|}$$

converges uniformly over $\rho(T)$. By proposition 11, this implies that $\prod_{n=1}^{\infty} \left(1 - \frac{z-y}{\lambda_n - y}\right)$ and $\prod_{n=1}^{\infty} \left(1 - \frac{1}{\lambda_n - z}\right)$ converge uniformly in $\mathbb{C} \times \rho(T)$ and $\rho(T)$ respectively. Then $\sum_{n=1}^{\infty} \ln \left(1 - \frac{z-y}{\lambda_n - y}\right)$ converges uniformly for all z and y in the half-plane $\mathrm{Re}(z) < \lambda_1$ (in this region, $-\pi < arg \left(1 - \frac{z-y}{\lambda_n - y}\right) < \pi$).

Let $h(s,z) = tr(T - zI)^{-s}$. On the region where this function is analytic, we have

$$\begin{aligned}
h(s,z) &= tr(T - zI)^{-s} \\
&= h(s,y) + \int_y^z h_u(s,u)du \\
&= h(s,y) + s\int_y^z tr(T - uI)^{-s-1}du \\
h_s(s,z) &= h_s(s,y) + \int_y^z tr(T - uI)^{-s-1} + s\int_y^z \frac{d}{ds}\left(tr(T - uI)^{-s-1}\right).
\end{aligned}$$

Clearly $tr(T - zI)^{-s-1}$ is analytic on a region containing 0. We can extend T to an elliptic operator E defined on $H^{'} = H \bigoplus \ker_E$. The zeta function associated with E has a meromorphic continuation on a region containing 0 ([15]). Thus, so does $E - zI$. Then, $h(s,z)$ has a meromorphic continuation, say $k(s,z)$, on a region containing zero. Let R be the intersection of these two regions. We then have

$$\begin{aligned}
k_s(s,z) &= k_s(s,y) + \int_y^z tr(T - uI)^{-s-1}du + s\int_y^z \frac{d}{ds} tr(T - uI)^{-s-1}du \\
k_s(0,z) &= k_s(0,y) + \int_y^z tr(T - uI)^{-1}du.
\end{aligned}$$

This allows us to define the following characteristic function.

Definition 39 *The characteristic function of T is defined to be the analytic continuation of*

$$g(z,y) = e^{-k_s(0,y) - \int_y^z tr(T-uI)^{-1}du}$$

to $\mathbb{C} \times \rho(T)$.

The main result for $g(z,y)$ is given below.

Theorem 51 *If $p(z,y) = e^{k_s(0,y)} g(z,y)$, then*

$$p(z,y) = \prod_{n=1}^{\infty}\left(1 - \frac{z-y}{\lambda_n - y}\right),$$

where p is defined on $\mathbb{C} \times \rho(T)$.

Proof. Because $\operatorname{tr}(T - zI)^{-1} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - z}$ is uniformly convergent on $\rho(T)$, we have

$$\begin{aligned} \int_y^z \operatorname{tr}(T - uI)^{-1} du &= \int_y^z \sum_{n=1}^{\infty} \frac{1}{\lambda_n - u} du \\ &= \sum_{n=1}^{\infty} \int_y^z \frac{1}{\lambda_n - u} du \\ &= -\sum_{n=1}^{\infty} \ln\left(1 - \frac{z-y}{\lambda_n - y}\right). \end{aligned}$$

Thus, $g(z, y) = e^{-k_s(0,y)} \prod_{n=1}^{\infty} \left(1 - \frac{z-y}{\lambda_n - y}\right)$. ■

Corollary 52 *$p(\lambda_n, y) = 0$ for all $\lambda_n \in \sigma_p(T)$.*

Notice that if T is finite dimensional, then $g(z, y)$ reduces to the normal definition of the characteristic polynomial and

$$\det(T) = e^{-k_s(0,0)}.$$

Proposition 53 *[1] If S is an invertible self-adjoint linear operator defined on a separable Hilbert space, then S^{-1} is also self-adjoint.*

Proof. S is invertible and self-adjoint implies that S^* and $(S^{-1})^*$ both exist and satisfy

$$(S^*)^{-1} = (S^{-1})^*.$$

But $S^* = S$ since S is self-adjoint. The result then follows. ■

This proposition allows us to derive:

Lemma 54 $H = \bigoplus_{\lambda\in\sigma_p(T)} \ker(T-\lambda I)$

Proof. T^{-1} is a compact normal operator. Thus, by theorem 24, it diagonalizes in an orthonormal basis. Then $H = \bigoplus_{\lambda\in\sigma_p(T^{-1})} \ker(T^{-1}-\lambda I)$. Also,

$$\ker(T^{-1}-\lambda I) = \ker(T - \frac{1}{\lambda} I), \forall \lambda \in \sigma_p(T^{-1})$$

The result then follows for T. ■

Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of H formed by eigenvectors of T. Then we have the following lemma for $y \in \rho(T)$:

Lemma 55 *$\forall f \in Dom(T)$, the sequence*

$$\left\{ \prod_{n=1}^{N} \left(I - \frac{T-yI}{\lambda_n - y} \right) f \right\}_{N=1}^{\infty}$$

converges strongly to 0.

Proof. For each $N \in Z^+$, let

$$p_N(T,y) = \prod_{n=1}^{N} \left(I - \frac{T-yI}{\lambda_n - y} \right).$$

Given any $f \in \mathrm{Dom}(T)$ with $f = \sum_{j=1}^{\infty} c_j e_j$, we let $h_N = p_N(T,y) f$. Since the product p_N is finite, we have

$$\begin{aligned} h_N &= \sum_{j=1}^{\infty} c_j \prod_{n=1}^{N} \left(I - \frac{T-yI}{\lambda_n - y} \right) e_j \\ &= \sum_{j=1}^{\infty} c_j \prod_{n=1}^{N} \left(1 - \frac{\lambda_j - y}{\lambda_n - y} \right) e_j \\ &= \sum_{j=N+1}^{\infty} c_j \prod_{n=1}^{N} \left(1 - \frac{\lambda_j - y}{\lambda_n - y} \right) e_j. \end{aligned}$$

If $g \in H$ is given by $g = \sum_{j=1}^{\infty} d_j e_j$, then

$$\langle h_N, g \rangle = \sum_{j=N+1}^{\infty} \overline{d_j}\ c_j \prod_{n=1}^{N} \left(1 - \frac{\lambda_j - y}{\lambda_n - y}\right)$$

so that $\langle h_N, g \rangle \to 0$ for all $g \in H$, which implies that $p_N(T, y)$ converges weakly to 0 on $\text{Dom}(T)$. The strong convergence follows similarly. ■

We define the functions $B_j(y)$ by

$$p(z, y) = 1 - \sum_{j=1}^{\infty} B_j(y)(z - y)^j.$$

Then we immediately obtain directly the following :

Theorem 56 *(Extended Cayley-Hamilton Theorem)*

Let T be a closed self-adjoint invertible operator with a trace-class resolvent having a finite number of negative eigenvalues and domain $Dom(T) \subset H$, dense in H. Then

$$p(T, y)f = 0,$$

$\forall\ f\ \in\ Dom(T)$.

Corollary 57 $I = \sum_{j=1}^{\infty} B_j(y)(T - yI)^j$.

Corollary 58 $(T - yI)^{-1} = \sum_{j=1}^{\infty} B_j(y)(T - yI)^{j-1}$.

From this corollary we obtain directly an expression of T^{-1}. We now use the above results to develop another extension of CHT. Let A be an operator with a trace-class resolvent and let

$$H_A = H \ominus \ker(A)$$

such that A/H_A satisfies the conditions (*) where $H = H_A \bigoplus ker(A)$. Then the results of the last section imply the following:

Theorem 59 *Let $P_{\ker(A)}$ denote the orthogonal projection on* $\ker(A)$. *Then*

$$p(A,y) = p(0,y)P_{\ker(A)},$$

where $p(z,y)$ is as defined in theorem 51 for the operator A/H_A on H_A

Proof. First note that since $\ker(A)$ reduces A, it also reduces $p(A,y)$. Let $f \in H$. We can write $f = f_0 + f_1$ with $f_0 \in \ker(A)$, $f_1 \in H_A$.

$$\begin{aligned} p(A,y)f &= p(A,y)f_0 + p(A,y)f_1 \\ &= f_0 - \sum_{j=1}^{\infty} B_{yj}(A-yI)^j f_0 + f_1 - \sum_{j=1}^{\infty} B_{yj}(A-yI)^j f_1 \\ &= \left(I - \sum_{j=1}^{\infty} B_{yj}(A-yI)^j\right) f_0 \\ &= \left(1 - \sum_{j=1}^{\infty} B_{yj}(-y)^j\right) f_0. \end{aligned}$$

Then,

$$p(A,y)f = p(0,y)P_{\ker(A)}f.$$

■

We can deduce directly from this theorem that $p(A,0) = P_{ker(A)}$. We also have the following corollaries.

Corollary 60 $\sum_{j=1}^{\infty} B_j(0)A^j + P_{\ker(A)} = I$.

This implies that $A\left(\sum_{j=1}^{\infty} B_j(0)A^{j-1}\right) = P_{H_A}$.

Corollary 61 $(A-yI)^{-1} = \sum_{j=1}^{\infty} B_j(y)A^{j-1} + p(0,y)\,(A-yI)^{-1}\,P_{\ker(A)}$.

Now we consider the case of a positive elliptic operator L of order greater than one on H.

Theorem 62 *Let L be a positive elliptic operator of order greater than one defined on H. We have*

$$p(L,y) = p(0,y)P_{\ker(L)},$$

where $p(z,y)$ is as defined in theorem 51 for the operator L/H_L on H_L

Proof. If $\lambda \in \sigma_p(L)$ then $\lambda = 0(n^2)$ for some $n \in \mathbb{N}$ since L has order greater than two. Moreover, positive elliptic operators are closed and self-adjoint. Thus L meets the conditions of theorem 15. ■

To illustrate this theorem, consider the following example.

EXAMPLE1: Let $f \in D = C^\infty(\mathbb{R}) \cap L_p^2[-\pi,\pi]$, where $L_p^2[-\pi,\pi]$ is the set of 2π periodic functions that are square integrable on $[-\pi,\pi]$. Consider the operator $L = -\frac{d^2}{dx^2}$ defined on H. It is easy to verify that this operator is a positive elliptic operator of order 2. Clearly, $H_L = \{f \in H \mid \int_{-\pi}^{\pi} f(x)dx = 0\}$. Thus, if $\int_{-\pi}^{\pi} f(x)dx = 0$, we have (with $p(z,y)$ and B_{yj} as defined in theorem 59)

$$f(x) = \sum_{j=1}^{\infty} B_{0j}\left(L^j f\right)(x).$$

In general we have

$$p(L,y)f = \frac{1}{2\pi}p(0,y)\int_{-\pi}^{\pi} f(x)dx.$$

We illustrate this example on Maple (see code “Example 1”).

Note: Other values of B_{0j} can be chosen with the results remaining valid. Indeed, for this example, the theorem remains true for $B_j = \frac{(-1)^{j+1}.\pi^{2j}}{(2j+1)!}$. This hopefully will facilitate eventual numerical applications of these results.

Remark: Since L annihilates its kernel, it follows that

$$Lp(L,y)=0$$

on the domain of L.

Theorem 59 gives an expression of the projection on the kernel of L. A similar expression can also be derived for projections on eigenspaces of L.

Theorem 63 *If E_k is the eigenspace corresponding to λ_k and $p_k(z,y)$ is the polynomial corresponding to the operator $L-\lambda_k I$ in theorem 15, then*

$$p_k\left(L,y+\lambda_k\right)=p_k(0,y)P_{E_k}.$$

Proof. By theorem 59, $p_k\left(L-\lambda_k I,y\right)=p_k(0,y)P_{ker(L-\lambda_k)}=p_k(0,y)P_{E_k}$. But $p_k\left(L-\lambda_k I,y\right)=p_k\left(L,y+\lambda_k\right)$. ∎

We define the functions $B_{j,n}(y)$ by

$$p_n(z,y)=1-\sum_{j=1}^{\infty}B_{j,n}(y)(z-y)^j.$$

We can then combine theorem 63 with the spectral theorem and obtain the following corollary:

Corollary 64 *We have*

$$\begin{aligned}
I &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{-1}{p_n(0,y)}B_{j,n}(y)\left(L-yI\right)^j, \\
L &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{-\lambda_n}{p_n(0,y)}B_{j,n}(y)\left(L-yI\right)^j,
\end{aligned}$$

with $B_{0n}(y)^n=-1\ \forall\ y$.

5 APPLICATIONS

Here, we consider applications of the extension of CHT to closed self-adjoint operator with trace-class resolvent (having simple eigenvalues) L on H. To apply these results, it is important to be able to estimate the $B_j(y)$'s (or $B_{j,n}(y)$'s). If all the eigenvalues of the operator are known, this is an easy task since

$$p(z,y) = \prod_{j=1}^{\infty}\left(1 - \frac{z-y}{\lambda_j - y}\right) = 1 - \sum_{n=1}^{\infty} B_j(y)\,(z-\lambda_j)^j\,.$$

Unfortunately, in general we do not know the eigenvalues of the operator and estimating them is often a hard problem.

By corollary 58, on $H_L = H \ominus \ker(L)$, we have

$$L^{-1}_{|H_L} = q(L,0),$$

where

$$q(z,y) \quad = \quad \frac{1-p(z,y)}{z} \quad \text{and}$$

and $p(z,y)$ is defined as the extension of

$$g(z,y) \quad = \quad e^{-\int_y^z \operatorname{tr}(T-uI)^{-1}du}.$$

defined on $\rho(L)\times\rho(L)$ to $\mathbb{C}\times\rho(L)$ (since L has simple eigenvalues). This extension is given by

$$p(z,y) \quad = \quad 1 - \sum_{j=1}^{\infty} B_j(y)(z-y)^j.$$

$$q(z,y) \quad = \quad \frac{1-p(z,y)}{z} \quad \text{and}$$

It is easy to show that

$$B_i(y) = -\frac{1}{i!}\lim_{z\to 0} p_z^{(i)}(z,y).$$

But we have

$$\begin{aligned}
\frac{g_z^{(1)}(z,y)}{g(z,y)} &= -\mathrm{tr}(T-zI)^{-1} \\
\frac{g_z^{(2)}(z,y)}{g(z,y)} &= -\mathrm{tr}(T-zI)^{-2} + \left(\mathrm{tr}(T-zI)^{-1}\right)^2 \\
\frac{g_z^{(3)}(z,y)}{g(z,y)} &= -2\mathrm{tr}(T-zI)^{-3} + 3\mathrm{tr}(T-zI)^{-2}\mathrm{tr}(T-zI)^{-1} \\
&\quad - \left(\mathrm{tr}(T-zI)\right)^3 \\
&\vdots \quad .
\end{aligned}$$

The R.H.S. of the above equalities are analytic on $\mathbb{C}$. We then have:

$$\begin{aligned}
\frac{p_z^{(1)}(z,y)}{p(z,y)} &= -\mathrm{tr}(T-zI)^{-1} \\
\frac{p_z^{(2)}(z,y)}{p(z,y)} &= -\mathrm{tr}(T-zI)^{-2} + \left(\mathrm{tr}(T-zI)^{-1}\right)^2 \\
\frac{p_z^{(3)}(z,y)}{p(z,y)} &= -2\mathrm{tr}(T-zI)^{-3} + 3\mathrm{tr}(T-zI)^{-2}\mathrm{tr}(T-zI)^{-1} \\
&\quad - \left(\mathrm{tr}(T-zI)\right)^3 \\
&\vdots \quad .
\end{aligned}$$

Also, $p(0,0) = 1$. Thus all the coefficients $B_i(0)$ can be expressed as functions of $\mathrm{tr}(L^{-1}), \mathrm{tr}(L^{-2}), \cdots, \mathrm{tr}(L^{-i}), \quad \forall i \in \mathbb{N}$. Thus finding the $B_i(0)$'s is equivalent to finding $\mathrm{tr}(L^{-N})$. There are several ways to calculate this trace. We can either do it by direct computation or by using a trace formula. Here, a useful trace formula is the Krein's trace formula:

Theorem 65 *[6] Let A and B be two self-adjoint operators with $B - A \in S_1$. There exists a function $\xi \in L^1(\mathbb{R})$ such that*

(a) for every $f \in K = \{f \mid fx = \int_{\mathbb{R}} \frac{e^{tsx}-1}{s} d\mu(s), \quad where \quad \mu \quad is\ a\ finite$ measure on $\mathbb{R}\}$,

$$tr[f(B) - f(A)] = \int_{\mathbb{R}} f^{'}(x)\xi(x)dx.$$

In particular,

$$tr[B - A] = \int_{\mathbb{R}} (x)\xi(x)dx.$$

(b) $\|\xi\|_1 \le \|B - A\|_1$

(c) If $A \le B$, then $0 \le \xi$ a.e.

(d) $\xi(x) = 0$ outside of any interval containing $\sigma(A) \cup \sigma(B)$

Since L is self-adjoint operator with trace class resolvent, $(L - \lambda I)^{-N}$ (with $\lambda \in \sigma(L)$) is a self-ajoint trace class operator ($\forall\ N \ \in \mathbb{N}^{+}$). Thus, we can find its trace by letting $A = 0$ and $B = (L - \lambda I)^{-N}$ in the above theorem.

5.1 Differential Equation

Let H be a Hilbert space and L a differential operator defined on H. Let $g \in H$ and consider the equation

$$Lf - \lambda f = g \quad f \in H,$$

with $\lambda \in \rho(L)$. In general, we cannot find an exact solution to this type of differential equation. It becomes important to find some numerical methods to approximate the solutions of this type of equation.

If the operator L is a closed, self-adjoint operator with with trace-class resolvent, $g \in ker(L)^{\perp}$ and if $\lambda \in \rho(L)$, then, by corollary 61, we have

$$f = (L - \lambda I)^{-1} g = \sum_{j=1}^{\infty} B_j(\lambda)(L - \lambda I)^{j-1} g.$$

Since we can approximate the $B_j(\lambda)$, this provides us with a method to approximate f numerically.

We illustrate this on Maple, and the codes of an implementation of this are given in appendix "Differential equation".

5.2 Inverse Frame Operator

Let the family of elements $\{f_i\}_{i=1}^{\infty}$ be a frame of the Hilbert space H, with S the frame operator associated with this family, defined by

$$S : H \to H,\ Sf = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i.$$

We know that S is a bounded positive self-adjoint operator. Suppose there exists an operator T defined on H such

$$L = TST$$

is a closed, unbounded, self-adjoint operator with trace-class resolvent and $\ker(T) = \ker(L)$ (with preferably finite dimension). Let $H = \ker(L) \bigoplus H_L$. By theorem 59, on H_L, we have

$$L^{-1} = q(L, 0),$$

where

$$q(z,y) \quad = \quad \frac{1-p(z,y)}{z}.$$

Theorem 66 *$\forall\ f\ \in H_L = H \ominus \ker(L)$,*

$$S^{-1} = TL^{-1}T.$$

This theorem provides us with a method to numerically approximate the inverse frame operator. Now, let's show how this theorem can be applied to a special class of frames, the non-harmonic Fourier Series.

Example 3: Let $C = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $|\lambda_n - n| < \frac{1}{4}\ \ \forall \lambda_n \in C$. By theorem 28 , $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a frame for $H = L^2(-\pi,\pi)$.

Let S be the frame operator associated with this family and T be the second differential operator defined on H. We know that T is a closed unbounded operator, and $ker(T) = \{f \in H \mid \text{f is constant}\}$ has dimension one.

Theorem 67

$$L = TST : f \to \sum_{n=1}^{\infty} \lambda_n^4 < f, e^{i\lambda_n} > e^{i\lambda_n z}$$

is a closed, unbounded, self-adjoint operator with trace-class resolvent with, $ker(L) = ker(T)$.

Proof. Since $|\lambda_n - n| < \frac{1}{4}$, it follows that L is unbounded, with a trace-class resolvent. Let $f, g \in H$. We have

$$\begin{aligned} < Lf, g > &= \sum_{n=1}^{\infty} \lambda_n^4 < f, e^{i\lambda_n} >< e^{i\lambda_n z}, g > \\ &= \sum_{n=1}^{\infty} \lambda_n^4 \overline{< e^{i\lambda_n z}, g >} < f, e^{i\lambda_n} > \\ &= \left\langle f, \sum_{n=1}^{\infty} \lambda_n^4 < g, e^{i\lambda_n} > e^{i\lambda_n z} \right\rangle \\ &= < f, Lg > \end{aligned}$$

thus L is self-adjoint. Since S is bounded, then so is ST (implying that ST is continuous). Suppose there exists a sequence $\{f_i\}_{i=1}^{\infty}$ such that

$$\begin{aligned} \lim_{n\to\infty} f_i &= f \text{ and } \lim_{n\to\infty} Lf_i = g \\ \text{then } \lim_{n\to\infty} STf_i &= g_0 \text{ we know the limit exists by continuity of } ST \\ \text{and } \lim_{n\to\infty} T(STf_i) &= g. \end{aligned}$$

Thus, since T is closed, we have $Lf = T(STf) = g$. ■

We can then apply theorem 66, and get

$$S^{-1}f = \sum_{j=1}^{\infty} B_j T(L)^{j-1} Tf,$$

$\forall f \in H \ominus \ker(L)$ where B_j can be approximated using the jth derivative of $p(z, y)$.

The codes of an implementation (illustration) of this example in Maple are in appendix “Inverse Frame Operator”.

6 FUTURE RESEARCH

6.1 Extension without the Self-Adjoint Restriction.

To extend the CHT, we restricted ourselves to trace-class self-adjoint operators and closed self-adjoint operators with trace-class resolvent. The approach used in [17], although not right, suggests that it is possible to drop the self-adjoint restriction.

Note that we have

$$\dim(E_\lambda) = n_\lambda$$

$\forall \lambda \in \sigma_p(T)$, with E_λ and n_λ as defined earlier. Without the self-adjoint restriction, our results will be of the form:

Theorem 68 *Let T be a trace-class operator (respectively, a closed operator with trace-class resolvent). Then there exists a polynomial $p(z,y)$ such that*

$$\begin{aligned} p(\lambda, y) &= 0 \quad \forall \lambda \in \sigma_p(T) \quad \textit{and} \\ p(T, y) &= p(0, y) P_E, \quad \textit{with} \\ E &= H \ominus \left(\oplus_{\lambda \in \sigma_p(T)} E_\lambda \right), \end{aligned}$$

where the E_λ's are the characteristic spaces as defined earlier.

This is a nice result but a limited result. It is limited because we do not know the dimension of E, and it can be infinite. In such a case, this theorem is useless.

The questions are then:

1) How can we associate a pseudo-zero eigenvalue to T?

2) How can we express E as the sum of pseudo-characteristic spaces with finite dimension?

An insight to answer the first question can be found in the definition:

$$\prod_0(T) = \lim_{r \to 0^+} \frac{-1}{2i\pi} \int_{|u|=r} (T - uI)^{-1}\, du \quad \text{and} \quad E_0(T) = \operatorname{im}\left(\prod_0(T)\right)$$

given in [17].

If we can successfully answer any of these questions, we will be able to obtain a new CHT for a larger class of operators with many possible applications.

6.2 Commutators.

Definition 40 *The commutator operator, denoted by* $[.,.]$ *is an operator defined over the cross-product of collection of linear operators over* H Λ *by:*

$$[.,.] : \Lambda \times \Lambda \to T, L \longmapsto [T, L] = TL - LT.$$

The commutator operator is considered a generalization of differentials operators.

But our second extension of the CHT is applicable to some differential operators. A natural question is then under which conditions (say(**)) on T and L do we have:

Conjecture 69 *Let* T *and* L *be operators over* H *such that (**) is true. Then there exists a polynomial* $p(z, y)$ *such that*

$$p([T, L], y) = p(0, y)P_E.$$

where E *is defined as depending only on* T *and* L.

(**) can be conditions implying that $[T, L]$ meets the requirements of our previous results. In that regard, suppose we know the properties of T, and we know that $[T, L]$ meets the requirements of our previous results. Can we recover information or derive results about L from theorem 69?

6.3 Other Possible Applications

In addition, we have some preliminary work in the following areas.

- Coherent States
- Number Operator and its Properties
- Green's Functions in Neuroscience

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.1 Example 1

```
> restart:with(plots):
> evl:=1-sin(Pi*sqrt(x))/sqrt(x)/Pi;

> nterms:=10;
> An:=[seq(coeftayl(evl, x=0,

> i),i=1..nterms)];

> f:=-cos(x);
```

Must check to see that C0 = 0. This sets it to be 0 automatically.

```
> int(f,x=-Pi..Pi)/2/Pi;

> f:=f-%;

> fa:=sum(An[i]*(-1)^(i)*diff(f,x$(2*i)),i=1..nterms):
> #Approximation in red to curve in blue.

> #If you only blue, approximation is visually same as actual
> p1:=plot(f,x=-2*Pi..2*Pi,color=blue,thickness=2):

> p2:=plot(fa,x=-2*Pi..2*Pi,color=red):

> display(p1,p2,view=[-2*Pi..2*Pi,-3..3]);


> fapp[1]:=-An[1]*f:

> for j from 2 to nterms

> do

> fapp[j]:=-An[1]*f+sum(An[i]*(-1)^(i)*diff(f,x$(2*i-2)),i=2..j):

> end do:

> ani:=display(seq(plot(fa,fapp[j],x=-2*Pi..2*Pi,color=[blue,red]) ,

j=1..nterms),insequence=true):

> display(ani,view=[-2*Pi..2*Pi,-3..3],insequence=true);
```

```
> # Select image and play animation to see how approximations converge

to actual
> with(Maplets):

> with(Elements):

> with(Tools):
> s1m:=Maplet([

> "This is for n=17 and for f(x) = -cos(x)",

> [Plotter[Q](plots[display](p1,p2,view=[-2*Pi..2*Pi,-3..3])),

> Plotter[P](plots[display](ani,view=[-2*Pi..2*Pi,-3..3],insequence=true),

continuous=false)],

> [ Button("play",SetOption(P('play')=true)),

> Button("stop",SetOption(P('`stop`')=true)),

> Button("pause",SetOption(P('pause')=true)) ],

> [ "delay",

> Slider[DELAY](100..500, 200,

> onchange=SetOption(target=P, `option`='delay', Argument(DELAY))) ],

> Button("ok",Shutdown()) ]):

> Maplets[Display](s1m);
```

```
> restart:with(plots):
> evl:=1-sin(Pi*sqrt(x))/sqrt(x)/Pi;

> nterms:=10;
> An:=[seq(coeftayl(evl, x=0,

> i),i=1..nterms)];

> f:=-cos(x);
```

Must check to see that C0 = 0. This sets it to be 0 automatically.

```
> int(f,x=-Pi..Pi)/2/Pi;

> f:=f-%;

> fa:=-An[1]*f +sum(An[i]*(-1)^(i)*diff(f,x$(2*i-2)),i=2..nterms):
> #Approximation in red to curve in blue.

> #If you only blue, approximation is visually same as actual
> p1:=plot(f,x=-2*Pi..2*Pi,color=blue,thickness=2):

> p2:=plot(fa,x=-2*Pi..2*Pi,color=red):

> display(p1,p2,view=[-2*Pi..2*Pi,-3..3]);



> fapp[1]:=-An[1]*f:
> for j from 2 to nterms do

> fapp[j]:=-An[1]*f+sum(An[i]*(-1)^(i)*diff(f,x$(2*i-2)),i=2..j):

> end do:

> ani:=display(seq(plot(fa,fapp[j],x=-2*Pi..2*Pi,color=[blue,red]),

j=1..nterms),     insequence=true):

> display(ani,view=[-2*Pi..2*Pi,-3..3],insequence=true);

> # Select image and play animation to see how approximations converge

to actual
```

```
> with(Maplets):

> with(Elements):

> with(Tools):
> s1m:=Maplet([

> "This is for n=17 and for f(x) = -cos(x)",

> [Plotter[Q](plots[display](p1,p2,view=[-2*Pi..2*Pi,-3..3])),

> Plotter[P](plots[display](ani,view=[-2*Pi..2*Pi,-3..3],insequence=true),
continuous=false)],

> [ Button("play",SetOption(P('play')=true)),

> Button("stop",SetOption(P('`stop`')=true)),

> Button("pause",SetOption(P('pause')=true)) ],

> [ "delay",

> Slider[DELAY](100..500, 200,

> onchange=SetOption(target=P, `option`='delay', Argument(DELAY))) ],

> Button("ok",Shutdown()) ]):

> Maplets[Display](s1m);
```

```
> restart:with(LinearAlgebra):with(plots):interface(displayprecision=-1):

> dlen:=25; omega1:=5:  omega2:=-2:

> omega3:=Pi:

> coeff1:=2:

> coeff2:=3:

> coeff3:=5:

> forig:=coeff1*cos(omega1*x)+coeff2*cos(omega2*x)+coeff3*cos(omega3*x);

> h:=evalf(2*Pi/(dlen-1),15); #Sampling period

> data1:=[seq(evalf(coeff1*cos(omega1*(-Pi+(i-1)*h))

> +coeff2*cos(omega2*(-Pi+(i-1)*h))

> +coeff3*cos(omega3*(-Pi+(i-1)*h)),15),i=-10..dlen+10)]:
> datalen1:=dlen+20;

> datalen:=dlen;

> plots[listplot]([seq([-Pi+(i-1)*h,data1[i+10]],i=1..dlen)]);

> y[0]:=data1:  #Actual data is y0

> MM:=10:  #Max Degree of CH polynomial

> Numdata:= datalen1:  #Upper bound on the  length of the array

> Numdata1:= datalen1:
> LL:=1:  #Lower bound on length of array

> for j from 1 to MM do

> Numdata:=Numdata-1:

> LL:=LL+1:

> Numdata1:=Numdata1-2:
```

```
> y[j]:= array(LL..Numdata):      # create an empty 1 by Numdata-2 array
for y[j]#
> for i from LL to Numdata do    #generate the approximation for y[j]#
> y[j][i]:= (y[j-1][i+1]-2*y[j-1][i]+y[j-1][i-1])/(h^2):
> end do:
> end do:
> if type( Numdata-1 , even ) = true then
> N:= Numdata-1:
> else
> N:= Numdata-2:
> end if:
> EstError:=1:
> NN:=1:kkk:=2:
> while(abs(evalf(EstError,20)) > 0.1 and NN<10) do
> NN:=NN+1:
> AAA[kkk]:= Matrix(1..NN,1..1):
> BBB[kkk]:=Matrix(1..NN,1..NN):
> for i from 0 to NN-1 do
> for j from LL+1 to N-1 do
> if type( j , even ) = true then
> f[j]:=2*y[NN][j]*y[i][j]:
> else
> f[j]:=4*y[NN][j]*y[i][j]:
> end if:
> end do:
```

```
> f[LL]:=y[NN][LL]*y[i][LL]:
> f[N]:=y[NN][N]*y[i][N]:
> c:=0:
> for k from LL to N do
> c:=c+f[k]:
> end do:
> AAA[kkk][i+1,1]:=(h*c)/3:
> end do:
> for p from 0 to NN-1 do
> for q from 0 to NN-1 do
> for j from LL+1 to N-1 do
> if type( j , even ) = true then
> f[j]:=2*y[p][j]*y[q][j]:
> else
> f[j]:=4*y[p][j]*y[q][j]:
> end if:
> end do:
> f[LL]:=y[p][LL]*y[q][LL]:
> f[N]:=y[p][N]*y[q][N]:
> c:=0:
> for k from LL to N do
> c:=c+f[k]:
> end do:
```

```
> BBB[kkk][p+1,q+1]:=(h*c)/3:
> end do:
> end do:
> if (Rank( BBB[kkk] )= NN ) then
> CCC[kkk]:=evalf(MatrixMatrixMultiply(MatrixInverse(BBB[kkk]),AAA[kkk])
,20):
> ee[NN]:=0:
> for r from LL to  Numdata do
> Esterror1:= y[NN][r]:
> for m from 1 to NN do
> Esterror1:=Esterror1 - CCC[kkk][m,1]*y[m-1][r]:
> end do:
> ee[NN]:=ee[NN]+ abs(evalf(Esterror1,20)):
> end do:
> EstError:=(ee[NN])/(Numdata-LL+1):
> eee[NN]:=EstError:
> for j from LL+1 to N-1 do
> if type( j , even ) = true then
> f[j]:=2*y[NN][j]*y[NN][j]:
> else
> f[j]:=4*y[NN][j]*y[NN][j]:
> end if:
> end do:
```

```
> f[LL]:=y[NN][LL]*y[NN][LL]:
> f[N]:=y[NN][N]*y[NN][N]:
> c:=0:
> for k from LL to N do
> c:=c+f[k]:
> end do:
> cc:=(h*c)/3:
> for q from 0 to NN-1 do
> for j from LL+1 to N-1 do
> if type( j , even ) = true then
> f[j]:=2*y[NN][j]*y[q][j]:
> else
> f[j]:=4*y[NN][j]*y[q][j]:
> end if:
> end do:
> f[LL]:=y[NN][LL]*y[q][LL]:
> f[N]:=y[NN][N]*y[q][N]:
> c:=0:
> for k from LL to N do
> c:=c+f[k]:
> end do:
> ccc[q]:=(h*c)/3:
> end do:
```

```
> eeee[NN]:=cc:

> for i from 0 to NN-1 do

> eeee[NN]:=eeee[NN]-CCC[kkk][i+1,1]*ccc[i]:

> end do:

> kkk:=kkk+1:

> end if:

> end do:

> evalf(EstError,20);

> NN;


> dlen:=250;

> h:=evalf(2*Pi/(dlen-1),15); #Sampling period

> data1:=[seq(evalf(coeff1*cos(omega1*(-Pi+(i-1)*h))

> +coeff2*cos(omega2*(-Pi+(i-1)*h))

> +coeff3*cos(omega3*(-Pi+(i-1)*h)),15),i=-10..dlen+10)]:
> datalen1:=dlen+20;

> datalen:=dlen;

> plots[listplot]([seq([-Pi+(i-1)*h,data1[i+10]],i=1..dlen)]);

> y[0]:=data1:  #Actual data is y0

> MM:=10:  #Max Degree of CH polynomial
> Numdata:=datalen1:#upper bound on the

> length of the array

> Numdata1:=datalen1:

> LL:=1:  #Lower bound on length of array
```

```
> for j from 1 to MM do
> Numdata:=Numdata-1:
> LL:=LL+1:
> Numdata1:=Numdata1-2:
> y[j]:= array(LL..Numdata):      # create an empty 1 by Numdata-2 array
for y[j]#
> for i from LL to Numdata do    #generate the approximation for y[j]#
> y[j][i]:= (y[j-1][i+1]-2*y[j-1][i]+y[j-1][i-1])/(h^2):
> end do:
> end do:
> if type( Numdata-1 , even ) = true then
> N:= Numdata-1:
> else
> N:= Numdata-2:
> end if:
> AAA[NN]:= Matrix(1..NN,1..1):
> BBB[NN]:=Matrix(1..NN,1..NN):
> for i from 0 to NN-1 do
> for j from LL+1 to N-1 do
> if type( j , even ) = true then
> f[j]:=2*y[NN][j]*y[i][j]:
> else
> f[j]:=4*y[NN][j]*y[i][j]:
> end if:
> end do:
```

```
> f[LL]:=y[NN][LL]*y[i][LL]:
> f[N]:=y[NN][N]*y[i][N]:
> c:=0:
> for k from LL to N do
> c:=c+f[k]:
> end do:
> AAA[NN][i+1,1]:=(h*c)/3:
> end do:
> for p from 0 to NN-1 do
> for q from 0 to NN-1 do
> for j from LL+1 to N-1 do
> if type( j , even ) = true then
> f[j]:=2*y[p][j]*y[q][j]:
> else
> f[j]:=4*y[p][j]*y[q][j]:
> end if:
> end do:
> f[LL]:=y[p][LL]*y[q][LL]:
> f[N]:=y[p][N]*y[q][N]:
> c:=0:
> for k from LL to N do
> c:=c+f[k]:
> end do:
> BBB[NN][p+1,q+1]:=(h*c)/3:
> end do:
> end do:
```

```
> if (Rank( BBB[NN] )= NN ) then
> CCC[NN]:= evalf(MatrixMatrixMultiply(MatrixInverse(BBB[NN]),AAA[NN]),20):
> end if :

> for i from 1 to NN do
> aa[i]:=CCC[NN][i,1]:
> end do:
> Est:=[solve(sum(aa[jj]*x^(jj-1),jj=1..NN)=x^NN,x)]:
> for i from 1 to NN do
> alpha[i]:=evalf(sqrt(abs(Est[i])),15);
> end do;
> #alpha[2]:=-alpha[2];
> A1:=Matrix(1..NN,1..1):
> B1:=Matrix(1..NN,1..NN):
> C1:=Matrix(1..NN,1..1):
> for j from 1 to NN do
> x[j]:=array(LL..Numdata):
> for i from LL to Numdata do
> x[j][i]:=evalf(cos(alpha[j]*(-Pi+h*(i-LL))),10):
> end do:
> end do:
```

```
> for i from 0 to NN-1 do
> for j from LL+1 to N-1 do
> if type( j , even ) = true then
> f[j]:=2*y[0][j]*x[i+1][j]:
> else
> f[j]:=4*y[0][j]*x[i+1][j]:
> end if:
> end do:
> f[LL]:=y[0][LL]*x[i+1][LL]:
> f[N]:=y[0][N]*x[i+1][N]:
> c:=0:
> for k from LL to N do
> c:=c+f[k]:
> end do:
> A1[i+1,1]:=(h*c)/3:
> end do:
> for p from 0 to NN-1 do
> for q from 0 to NN-1 do
> for j from LL+1 to N-1 do
> if type( j , even ) = true then
> f[j]:=2*x[p+1][j]*x[q+1][j]:
> else
> f[j]:=4*x[p+1][j]*x[q+1][j]:
> end if:
> end do:
```

```
> f[LL]:=x[p+1][LL]*x[q+1][LL]:

> f[N]:=x[p+1][N]*x[q+1][N]:

> c:=0:

> for k from LL to N do

> c:=c+f[k]:

> end do:

> B1[p+1,q+1]:=(h*c)/3:

> end do:

> end do:
> C1:=

> evalf(MatrixMatrixMultiply(MatrixInverse(B1),A1),10);
> for i from 1 to NN do

> aaa[i]:=C1[i,1]:

> end do:
> f:=sum(aaa[jjj]*cos(alpha[jjj]*t),jjj=1..NN);

> fr:=sum(aaa[jjj]*cos(alpha[jjj]*t) ,jjj=1..NN);
> plots[listplot]([seq([-Pi+(i-1)*h,data1[i+11]],i=1..dlen)]):

> plots[listplot]([seq([-Pi+(i-1)*h,

> subs(t=-Pi+(i-1)*h,f)],i=1..dlen)],color=red,thickness=2):

> rr:=display(%,%%):display(rr);

> sqrt(sum((subs(t=-Pi+(ii-1)*h,f)-data1[ii+10])^2,ii=1..dlen)/dlen);
> err:=sqrt(sum((subs(t=-Pi+(ii-1)*h,f)-data1[ii+10])^2,ii=1..dlen)/dlen);

> for j from LL+1 to N-1 do

> if type( j , even ) = true then

> f[j]:=2*y[0][j]*y[0][j]:
```

```
> else

> f[j]:=4*y[0][j]*y[0][j]:

> end if:

> end do:

> f[LL]:=y[0][LL]*y[0][LL]:

> f[N]:=y[0][N]*y[0][N]:

> normf:=sqrt(add(f[jj],jj=LL..N)*h/3);:

> err/normf;
> with(Maplets):

> with(Elements):

> with(Tools):
> s1m:=Maplet([

> [[[Label("The original function is",'font'= Font("helvetica", 16))],

> [MathMLViewer('value' = MathML[Export](forig))],

> [Label("The approximation of the original is",'font'= Font("helvetica",
16))],

> [MathMLViewer('value' = MathML[Export](fr))],

> [Label("The Approximation(in red) Versus the Original(in black)",'font'=
Font("helvetica", 16))]],

> Plotter[Q](plots[display](rr))],

> Button("ok",Shutdown()) ]):

> Maplets[Display](s1m);
```

VITA

Alberto Mokak Teguia

119 West Pine Street #2

Johnson City, TN 37601

almotek@yahoo.fr

Education

- East Tennessee State University (ETSU), Johnson City, 37614, TN.

  Masters of Science in Mathematics (August 2005).

- University of Buea, Buea , Cameroon.

  Bachelors of Science in Mathematics (December 2003).

Submitted Paper

- A.P. Godbole and A.M. Teguia: *A Sierpienski Graph and Some of its Properties.* Submitted for publication to the Australasian Journal of Combinatorics.

Work in Progress

- J. Knisley and A.M. Teguia: *An Extension of the Cayley-Hamilton Theorem to a class of Elliptic Operators.*

- J. Gardner, A.M. Teguia, A. Vuong, N. Watson and C. Yerger: *Domination Cover Pebbling.*

- A.P. Godbole and A.M. Teguia: *Coincidences of Multinomial Coefficients.*

- A.M. Teguia: *Domination k-Cover Pebbling.*

Experience

- Research Experience for Undergraduates (REU) , ETSU, Summer 2002.

  Participant.

- ETSU, January 2004-May 2005.

  Graduate Assistant.

- REU, ETSU, Summer 2004.

  Graduate Assistant.

Conferences

- University of Central Florida, Orlando, FL.

  Applied Actuarial Research Conference, Spring 2004.

- Institute of Pure and Applied Mathematics, Los Angeles, CA.

  Blackwell-Tapia Prize Presentation Conference. Fall 2004

  Poster: *An Extension of the Cayley-Hamilton Theorem to a class of Elliptic Operators.*

- ETSU.

  Department of Mathematics Seminar. Fall 2004

  Talk:*Domination Cover Pebbling.*

  Talk:*An Extension of the Cayley-Hamilton Theorem to a class of Elliptic Operators.*

- Atlanta, GA.

  Joint AMS, MAA, SIAM Annual Meeting. January 2005.

  Talk:*An Extension of the Cayley-Hamilton Theorem to a class of Elliptic Operators.*

- Florida Atlantic University. Boca Raton,FL.

  Thirty-sixth Southeastern International Conference on Combinatorics

  Graph Theory, and Computing. Spring 2005.

  Talk: *A Sierpienski Graph and Some of its Properties.*

- Virginia Polytechnic Institute and State University, Blacksburg, VA.

  SIAM Student Chapter Seminar. Spring 2005.

  Invited Presentation: *Extensions of the Cayley-Hamilton Theorem with Applications to Elliptic Operators and Frames.*

Activities, Organizations and Honors

- Actuarial Student Association, ETSU.

  Member.

- University of Buea, Buea Cameroon.

  Outstanding Mathematics Senior , Fall 2004.

- Society Of Actuaries.

  SOA Exam 1.