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Bounds on Total Domination Subdivision Numbers.

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Bounds on Total Domination Subdivision Numbers

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by

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ABSTRACT

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The domination subdivision number of a graph is the minimum number of edges that must be subdivided in order to increase the domination number of the graph. Likewise, the total domination subdivision number is the minimum number of edges that must be subdivided in order to increase the total domination number. First, this thesis provides a complete survey of established bounds on the domination subdivision number and the total domination subdivision number. Then in Chapter 4, new results regarding bounds on the total domination subdivision number are given. Finally, a characterization of the total domination subdivision number of caterpillars is presented in Chapter 5.

DEDICATION

To my husband, for his unending love and encouragement.

ACKNOWLEDGMENTS

I would like to thank my family and friends, without whose support, this would not have been possible: Lynn Hammons, who convinced me that I could and should go further; my mom and dad and mother and father-in-law for loving my children; my children for their patience and understanding; my friends at work for tolerating me; and finally, Dr. Teresa Haynes for being an inspiring teacher and friend.

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1 INTRODUCTION

Research in graph theory has lead to many applications such as computer network designs and social and business applications. Consider the following problem. A wealthy entrepreneur buys a chain of stores currently having some problems with top-heavy management. He restructures the management of the stores so that managers of managers are eliminated and he has only one manager per store. So that no manager has complete autonomy over his/her own store, each store manager is accountable to at least one other store manager. If the owner of this franchise knows something about graph theory, he could model this situation with a graph where each store is represented by a vertex, and an edge between stores indicates accountability between managers.

Suppose, for instance, that the manager has six stores, four of which are within comfortable driving distance of one another. The owner represents these stores with vertices b, c, d, and e in graph G of Figure 1. The other two stores are somewhat geographically isolated, and are represented by vertices a and f of graph G. The edges drawn between vertices represent accountability between managers. For instance, store b is accountable to store e, and vice versa. In fact, store b is accountable to each of stores a, c, and e, and this is known as the open neighborhood of vertex b. Recognizing that store b is also accountable to itself, we know that the closed neighborhood of b consists of vertices a, b, c, and e. This leads to the following definitions.

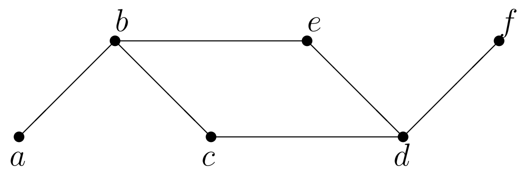

Figure 1: Graph G with order $n = 6$.

Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For instance, in Figure 1, the open and closed neighborhoods of b are as follows.

$$N(b) = \{a, c, e\}$$

$$N[b] = \{a, b, c, e\}$$

We can also easily observe accountability of groups of stores in the franchise and this leads to the following definitions. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. Refer again to Figure 1 and consider the following sets $S_1 = \{b, d\}$, $S_2 = \{b, d, e\}$, and $S_3 = \{a, b, c, d\}$. The open and closed neighborhoods of these sets are as follows.

$$N(S_1) = \{a, c, e, f\}$$

$$N[S_1] = \{a, b, c, d, e, f\}$$

$$N(S_2) = \{a, b, c, d, e, f\}$$

$$N[S_2] = \{a, b, c, d, e, f\}$$

$$N(S_3) = \{a, b, c, d, e, f\}$$

$$N[S_3] = \{a, b, c, d, e, f\}$$

In order to assess the effectiveness of his management strategy, the owner in our example plans to have a yearly meeting to discuss areas that require improvement, progress of strategies already implemented, and strategies for future growth. The owner wants every store to be represented either directly or indirectly, but because the meeting will be quite costly, he wishes to have the fewest number of managers possible involved. Now we are ready to define a dominating set of a graph G. A set S is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V - S$ has a neighbor in S. In our example, every store that does not have a manager at the meeting, ie. stores in $V - S$, has a manager who is accountable to them at the meeting. Hence, S_1, S_2 and S_3 are all dominating sets of G. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of G with cardinality $\gamma(G)$ is called a $\gamma(G)$-set. In our example, $|S_1| = 2$, $|S_2| = 3$, and $|S_3| = 4$ and since the minimum cardinality of these sets is two, we have $\gamma(G) \leq 2$. Clearly, there is no one vertex that can dominate every vertex of G, so S_1 is a $\gamma(G)$-*set.* To minimize costs, the owner should take only two managers to the meeting.

Upon reflection of the first year's meeting, the owner believes this year's meeting would be even more successful if every store had to be indirectly represented at the meeting so that no manager could exaggerate the successes of his/her own store at the meeting. The owner also believes that this strategy will improve cooperation at the meeting since each manager present will know at least one other. This leads us to a definition for a total dominating set of a graph G. A set S is a *total dominating set* if $N(S) = V$, or equivalently, every vertex in V has a neighbor in S. By this definition, S_2 and S_3 are both total dominating sets of G. The *total domination number* $\gamma_t(G)$

is the minimum cardinality of a total dominating set of G and a set S of order $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Since $|S_2| < |S_3|$, then $\gamma_t(G) \leq 3$. Note that if S is a total dominating set of G, then the subgraph $G[S]$ induced by S has no isolated vertices. Hence, any $\gamma_t(G)$-set must consists of at least two adjacent vertices, and since there clearly are no two adjacent vertices that dominate graph G, S_2 is a $\gamma_t(G)$-set.

Next, suppose the new management strategy was so successful that the owner was able to build a new store and the location was chosen to be between stores d and f. It would be more feasible for the new store to be accountable to stores f and d, and for d and f to no longer be accountable for one another. This leads us to explore the *domination subdivision number* defined by Arumugam [3]. An edge $uv \in E(G)$ is *subdivided* if the edge uv is deleted, but a new vertex x is added, along with two new edges ux and xv. The vertex x is called a *subdivision vertex*. Figure 2 shows graph G from figure 1 with edge df deleted and the subdivision vertex (new store) x added along with edges dx and xf to form the new graph G'.

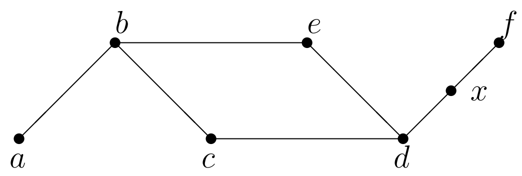

Figure 2: Graph G' formed by subdividing edge df in G with vertex x.

The owner now needs to determine if the addition of the new store will increase the number of managers he takes to his yearly meeting. Assuming the owner follows the format of the first year's meeting in which every store is either directly or indirectly represented, we define the domination subdivision number $sd_\gamma(G)$ to be the minimum number of edges that must be subdivided in order to increase the domination number. An edge can be subdivided at most once, in other words, no edge incident to a subdivision vertex can be subdivided. In this definition, we assume that every graph is of order $n \geq 3$, since the domination number of the graph K_2 does not change when its only edge is subdivided.

Referring again to Figure 2, we see that the set $S' = \{x, b\}$ is a minimum dominating set of G' and since $\gamma(G) = \gamma(G') = 2$, then subdividing df with vertex x does not increase the minimum domination number of G. Is there any one edge of G that could be subdivided to increase the minimum domination number of graph G? We will explore this by subdividing different edges of G with a subdivision vertex x.

In Figure 3, we have subdivided edge cd with a vertex x and in Figure 4, we have subdivided edge de with a vertex x. In both graphs, the set $S' = \{b, d\}$ is a minimum dominating set of G'. Since the graph is symmetrical, subdividing any one edge of ab, bc, or be will not increase the minimum domination number of G. Hence, the minimum domination subdivision number, $sd_\gamma(G) > 1$.

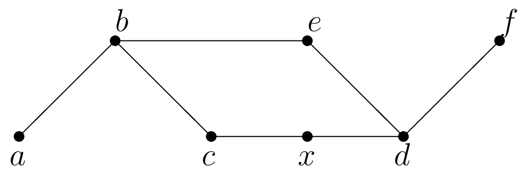

Figure 3: Graph G' formed by subdividing edge cd in G with vertex x.

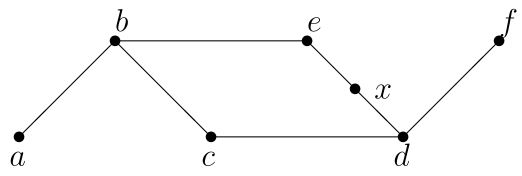

Figure 4: Graph G' formed by subdividing edge de in G with vertex x.

In fact, subdividing both ab and df with subdivision vertices x and y, respectively, (see Figure 5) results in a new graph G' in which one of a or x and one of f or y must be in every minimum dominating set S' to dominate the end-vertices a and f. However, e and c are not dominated, hence we must add another vertex, say vertex b to our dominating set. Thus, $S' = \{x, b, y\}$ and $sd_\gamma(G) = 2$.

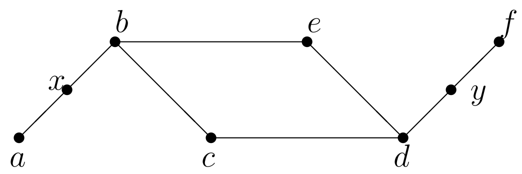

Figure 5: Graph G' formed by subdividing ab and df in G with vertices x and y.

Similarly, we define the *total domination subdivision number*, $sd_{\gamma_t}(G)$, to be the minimum number of edges that must be subdivided in order to increase the total domination number. Assume the owner follows the format of the second year's meeting in which each store is indirectly represented and recall that $\gamma_t(G) = 3$. We can establish that $sd_{\gamma_t}(G) = 1$ as follows. Subdivide df with vertex x (refer again to Figure 2). Now x and b must be in every $\gamma_t(G)$-set to dominate the end-vertices a and f, and because a total dominating set contains no isolates, there must be at least two more vertices in S' to totally dominate x and b, say c and d. Thus $S' = \{b, c, d, x\}$

and $\gamma_t(G) = 1$. With the addition of the new store, the owner must now take four managers to the meeting in order for every store to be indirectly represented.

Following is an example of a graph where the total domination subdivision number $sd_{\gamma_t}(G) = 2$. Figure 6 is a graph G where the minimum total domination number, $\gamma_t(G) = 6$. One possible total dominating set S is shown by darkened vertices.

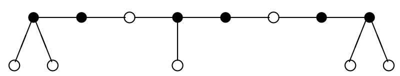

Figure 6: Graph G with $\gamma_t(G) = 6$.

To establish that the total domination subdivision number $sd_{\gamma_t}(G) \leq 2$, we must find two edges of G whose subdivision increases the total domination number of the graph. Figure 7 is a copy of graph G with two edges subdivided to form a new graph G'. The minimum total domination number of the new graph $\gamma_t(G') = 7$. Thus, we know that $sd_{\gamma_t}(G) \leq 2$.

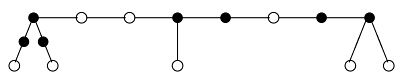

Figure 7: Graph G' with $\gamma_t(G') = 7$.

Next, we will establish that $sd_{\gamma_t}(G) \geq 2$. To accomplish this, we must subdivide one edge at a time of G and determine the minimum total domination number of each new graph G'. Figure 8 shows five instances where one edge of G is subdivided with a subdivision vertex x. In each instance, $\gamma_t(G') = 6$. Though all cases are not shown here, it is a simple task to establish that there is no one edge that can be subdivided to increase the total domination number of the graph. Therefore, $sd_{\gamma_t}(G) = 2$.

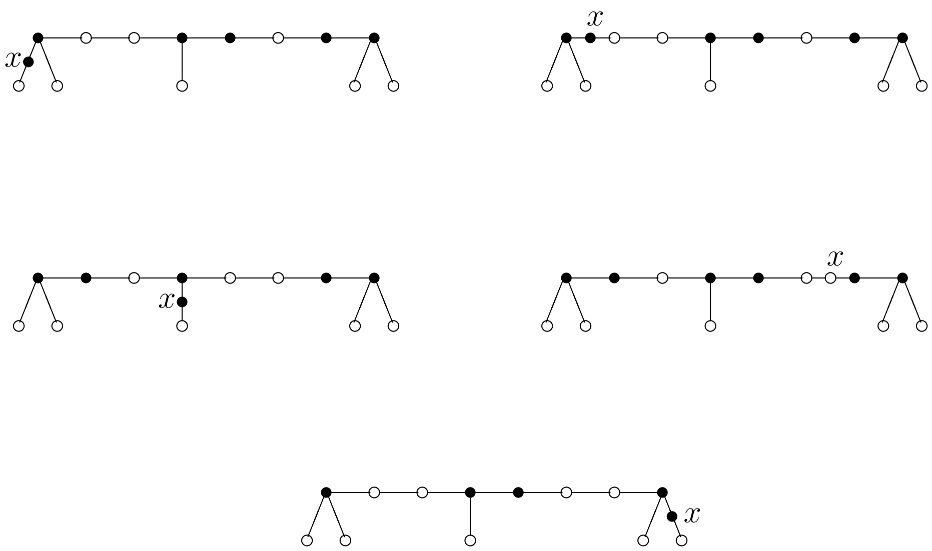

Figure 8: Five instances of graph G with one edge subdivided to form G'; $\gamma_t(G') = 6$.

Next, we present a graph where the total domination subdivision number $sd_{\gamma_t}(G) = 3$. Figure 9 is a complete graph on four vertices with $\gamma_t(K_4) = 2$. Again, one possible total dominating set of K_4 is shown with darkened vertices.

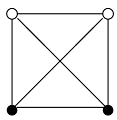

Figure 9: A complete graph K_4 where $\gamma_t(K_4) = 2$.

To establish that the total domination subdivision number $sd_{\gamma_t}(K_4) \leq 3$, we find three edges of K_4 whose subdivision increases the total domination number of the graph. Figure 10 is a copy of K_4 with three edges subdivided to form a new graph K_4'. The minimum total domination number of the new graph $\gamma_t(K_4') = 3$. Thus, we know that $sd_{\gamma_t}(K_4) \leq 3$.

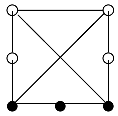

Figure 10: Graph K_4' where $\gamma_t(K_4') = 3$.

To see that $sd_{\gamma_t}(K_4) \geq 3$ we must subdivide every combination of two edges in K_4 and find the total domination number of each new graph K_4'. There are really only two distinct cases to consider; the edges subdivided are incident to a common vertex (see Figure 11a), or the edges subdivided are independent (see Figure 11b). In both cases, $\gamma_t(K_4') = 2$. Therefore, $sd_{\gamma_t}(K_4) = 3$.

(a) (b)

Figure 11: Graph K_4 with two edges subdivided to form K_4'; $\gamma_t(K_4) = 2$.

A family of graphs where the total domination subdivision number is at least 4 has been found and will be discussed in Chapter 3.

2 DOMINATION SUBDIVISION NUMBER SURVEY

Bounds on domination subdivision numbers have been studied in [3],[2], and [4]. The purpose of this chapter is to summarize the findings of this research.

It was established in [4] that $sd_\gamma(G)$ exists for all connected graphs of order $n \geq 3$. Arumugam [3], the originator of domination subdivision numbers, established the following bound on the parameter for trees.

Theorem 1 [3] *For any tree T of order $n \geq 3$, $1 \leq sd_\gamma(T) \leq 3$.*

Following this significant find, he posed this conjecture for arbitrary graphs.

Conjecture 2 [3] *For any graph G of order $n \geq 3$, $1 \leq sd_\gamma(G) \leq 3$.*

Haynes, Hedetniemi, Hedetniemi, Jacobs, Knisely, and van der Merwe [4] refuted this conjecture with the following counterexample.

Theorem 3 [4] *For any positive integer $t \geq 4$, $sd_\gamma(K_t \times K_t) = 4$.*

Thus, currently the largest known domination subdivision number is 4, so although Arumugam's upper bound does not hold, it seems probable that $sd_\gamma(G)$ could be bounded above by a constant. Nevertheless, a constant upper bound has not been established for general graphs.

On the other hand, Haynes, Hedetniemi, and Hedetniemi [3] gave the following upper bound for arbitrary graphs that can be used to show constant upper bounds for several families of graphs.

Theorem 4 [3] *For any connected graph G with adjacent vertices u and v, where $min\{deg(u), deg(v)\} \geq 2$,*

$$sd_\gamma(G) \leq deg(u) + deg(v) - 1.$$

As previously mentioned, Theorem 4 can be used to establish constant upper bounds for the domination subdivision number of many classes of graphs. For instance, every grid graph $G_{r,s}$ has $\delta(G) = 2$ as can be seen at the four corner vertices of the graph, and any neighbor of a corner vertex of a grid graph has degree three. Hence, we have the following.

Corollary 5 [3]*For any $r \times s$ grid graph $G_{r,s}$, where $2 \leq r \leq s$,*

$$1 \leq sd_\gamma(G_{r,s}) \leq 4$$

A graph G is *regular of degree k* if deg $v = k$ for each vertex v of G. Such graphs are called *k-regular* and Theorem 4 leads directly to the following.

Corollary 6 [3]*For any k-regular graph G, where $k \geq 2$,*

$$1 \leq sd_\gamma(G) \leq 2k - 1$$

Examples of Corollary 6 are 2-regular graphs C_n (*cycles*) for which $1 \leq sd_\gamma(C_n) \leq 3$ and 3-regular graphs (also called *cubic graphs*) for which $1 \leq sd_\gamma(G) \leq 5$.

2.1 Generalizations of Theorem 4

If the structure of the graph displays certain properties, we can achieve a better bound than that offered by Theorem 4. Favaron, Haynes, and Hedetniemi provided the following generalization of Theorem 4 in [2].

Theorem 7 [2] *For any graph G and edge uv, where $min\{deg(u), deg(v)\} \geq 2$,*

$$sd_{\gamma}(G) \leq |N(u) \cup N(v)| - 1 = deg(u) + deg(v) - |N(v) \cap N(u)| - 1.$$

Theorem 7 is an improvement on Theorem 4 when adjacent vertices u and v have common neighbors. For instance, refer to Graph G of Figure 12 and note that $deg(u) = deg(v) = 5$. Hence, by Theorem 4, we know that $sd_{\gamma}(G) \leq 9$. However, since $|N(u) \cap N(v)| = 2$, by Theorem 7, we can improve this upper bound to $sd_{\gamma}(G) \leq 7$.

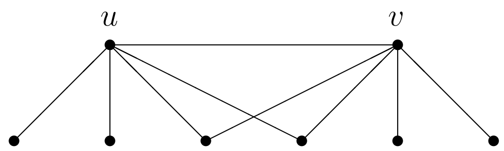

Figure 12: Graph G where $|N(v) \cap N(u)| = 2$.

Yet another generalization of Theorem 4 was established in [2] for the case of a graph with two adjacent *nonsimplicial* vertices. A vertex v in a graph G is called *simplicial* if the induced subgraph $G[N[v]]$ is a complete graph. A *clique* is defined to be a maximal complete subgraph of a graph G.

Theorem 8 [2] *Let u and v be two adjacent non-simplicial vertices of a graph G, and let r be the maximum order of a clique of $G[N(u) \cap N(v)]$. Then*

$$sd_\gamma(G) \leq deg(u) + deg(v) - 2r - 1.$$

Refer to graph G of Figure 13. By Theorem 4, we have $sd_\gamma(G) \leq 7$. A bit of improvement on this bound comes with the use of Theorem 7 where we have $sd_\gamma(G) \leq 5$. Note, however, that u and v are two adjacent nonsimplicial vertices each of degree 4, and $N(u) \cap N(v)$ contains vertices a and b that form a clique (in $N(u) \cap N(v)$) of order 2. Hence, by Theorem 8 we find $sd_\gamma(G) \leq 3$.

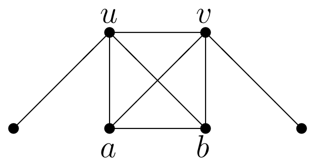

Figure 13: Graph G where $G[N(u) \cap N(v)]$ is a K_2.

For the following corollary of Theorem 8, we must first give some definitions. For a connected graph G, the *distance* $d(u, v)$ between two vertices u and v is the minimum of the length of the $u - v$ paths of G. The *line graph* $L(G)$ of a graph G is that graph whose vertices can be put in a one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. A graph and its line graph are shown in Figure 14.

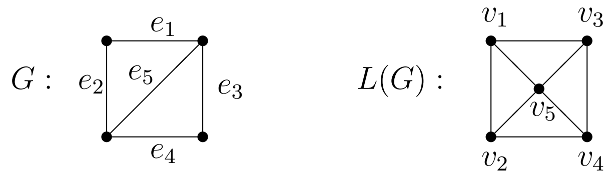

Figure 14: A graph G and its line graph $L(G)$.

Corollary 9 [2] *Let G be a graph, and let u and v be two vertices of degree at least two in G, where $d(u, v) = 2$. Then*

$$sd_\gamma(L(G)) \leq deg(u) + deg(v) - 1.$$

Referring to graph G of Figure 14, let the vertex incident with edges e_1 and e_2 be vertex u and the vertex incident with edges e_3 and e_4 be vertex v. Thus, the distance $d_G(u, v) = 2$ and $sd_\gamma(L(G)) \leq 3$.

For a graph G, the *inflated graph* G_I is formed by replacing each vertex $v_i \in V(G)$ with a clique of order $deg(v_i)$, each vertex of which is adjacent to exactly one vertex in a clique corresponding to a neighbor of v_i in G. It follows from this definition that each vertex v_i in G has the same degree as vertices within its corresponding clique in G_I. Furthermore, if $S(G)$ is the subdivision graph of G obtained by subdividing each edge of G exactly once, then the inflated graph $G_I = L(S(G))$. Following this line of reasoning, if x is a vertex of degree at least two in G, and u and v are subdivision vertices of $S(G)$ that are adjacent to x, then $deg_{S(G)}(u) = deg_{S(G)}(v) = 2$ while $d_{S(G)}(u, v) = 2$. These observations lead to the following:

Corollary 10 [2] *If G_I is the inflated graph of a graph G with $\Delta(G) \geq 2$, then*

$$1 \leq sd_\gamma(G_I) \leq 3.$$

2.2 Graphs containing a triangular vertex

The upper bound established in Theorem 4 is also improved if we know that G contains a triangular vertex. Vertex u is said to be *triangular* if every vertex $v \in N(u)$ is contained in a triangle with u. Hence, if a vertex u is triangular, then $deg(u) \geq 2$. A graph G is said to be *triangular* if it contains at least one triangular vertex and we have the following theorem.

Theorem 11 [4] *If a graph G contains a triangular vertex u, then $sd_\gamma(G) \leq deg(u)+1$.*

A graph G is *completely triangular* if every vertex in G is triangular.

Corollary 12 [4] *For every completely triangular graph G, $sd_\gamma(G) \leq \delta(G) + 1$.*

For instance, the graph in Figure 15 is completely triangular, and $\delta(G) = 3$, thus by Corollary 12 we have $sd_\gamma(G) \leq 4$.

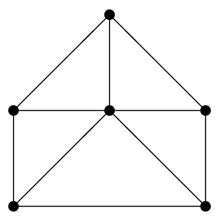

Figure 15: A completely triangular graph G.

Because every simplicial vertex of degree at least two is triangular, we have the following.

Corollary 13 [4] *If a graph G contains a simplicial vertex u of degree at least two, then $sd_\gamma(G) \leq deg(u) + 1$.*

A *k-tree* is any graph which can be obtained from a complete graph on $k + 1$ vertices by repeatedly adding a new vertex and joining it to every vertex in a complete subgraph of the existing graph of order k. Clearly, every k-tree is completely triangular.

Corollary 14 [4] *For every k-tree G, $k \geq 2$, $sd_\gamma(G) \leq k + 1$.*

The graph in Figure 16 is an example of a k-tree where $k = 2$. By Corollary 14, we have $sd_\gamma(G) \leq 3$.

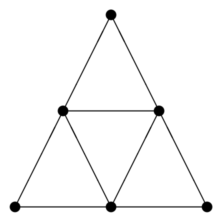

Figure 16: A 2-tree.

A *chord* is an edge between two nonconsecutive vertices of a cycle. A graph G is called *chordal* if every cycle of G of length greater than three has a chord. Every k-tree is a chordal graph. The *connectivity* or $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. A graph G is said to be *k-connected* if $\kappa(G) \geq k$ where $k \geq 1$.

Corollary 15 [4] *For every 2-connected chordal graph G, $sd_\gamma(G) \leq \delta(G) + 1$.*

Refer to Figure 17 and observe that every cycle of length at least three has a chord. Also notice that the removal of vertices u and v would result in a disconnected graph, thus $\kappa(G) \leq 2$. Clearly, there is no one vertex whose removal causes the graph to become disconnected, hence $\kappa(G) = 2$. Thus, G is a 2-connected chordal graph. Also note that this graph is completely triangular, as is every 2-connected chordal graph. Hence, because $\delta(G) = 2$, by Corollary 15 we have $sd_\gamma(G) \leq 3$.

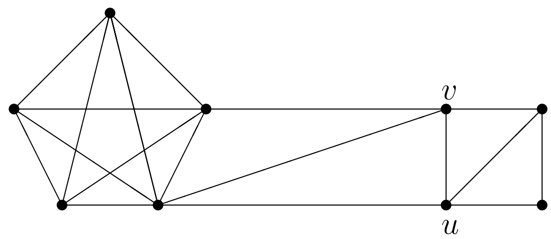

Figure 17: A 2-connected chordal graph G.

Furthermore, Theorem 11 can be used to establish constant upper bounds for $sd_\gamma(G)$ for several classes of graphs.

A *maximal outerplanar graph* is a 2-tree that is obtained from a copy of K_3 by repeatedly adding a new vertex and joining it to two adjacent vertices on the exterior face of the existing graph. Figure 16 is an example of a maximal outerplanar graph. Observe that every maximal outerplanar graph G contains at least two vertices of degree two and this is the minimum degree of any vertex in G. Also note that each vertex of degree two is a simplicial vertex and every maximal outerplanar graph is completely triangular. Hence, we have the following.

Corollary 16 [4] *For every maximal outerplanar graph G,* $sd_\gamma(G) \leq \delta(G) + 1 = 3$.

A graph G is called *maximal planar* if, for every pair u, v of nonadjacent vertices of G, the graph $G+uv$ is nonplanar. It is easy to see that every maximal planar graph is completely triangular, and because every planar graph contains a vertex of degree at most five [1], we have the following constant upper bound for maximal planar graphs.

Corollary 17 [4] *For every maximal planar graph G,* $sd_\gamma(G) \leq \delta(G) + 1 \leq 6$.

Finally, the upper bound for $sd_\gamma(G)$ holds for any graph having a vertex of degree two that is contained in a triangle because the vertex is obviously triangular.

Corollary 18 [4] *For any graph G having a vertex of degree two that forms a triangle with two other vertices,*

$$1 \leq sd_\gamma(G) \leq 3.$$

2.3 Graphs containing simplicial vertices

The bound established for $sd_\gamma(G)$ by Corollary 13 can be improved if we know more about the structure of the simplicial vertices in a graph.

Theorem 19 [4] *If G is a graph having a clique containing exactly two simplicial vertices and at least two non-simplicial vertices, then* $1 \leq sd_\gamma(G) \leq 2$.

Given three or more simplicial vertices in a clique, the domination subdivision number is even smaller.

Theorem 20 [4] *If G is a graph having three or more pairwise-adjacent simplicial vertices, then $sd_\gamma(G) = 1$.*

Figure 18 is a graph with four pairwise adjacent simplicial vertices. Thus, by Theorem 20, we have $sd_\gamma(G) = 1$.

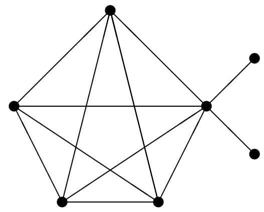

Figure 18: Graph G with four simplicial vertices.

The next theorem provides a bound on $sd_\gamma(G)$ when one of two adjacent vertices u and v is simplicial.

Theorem 21 [4] *Let u be a simplicial vertex of degree at least 2 of a graph G, and let v be a neighbor of u. Then*

$$sd_\gamma(G) \leq \min \{deg(u) + 1, deg(v) - deg(u) + 2\}.$$

Referring to Figure 19, because u is a simplicial vertex of degree 3, and v_1 is a neighbor of u of degree 6, then by Theorem 21, $sd_\gamma(G) \leq \min \{4, 5\}$. Hence $sd_\gamma(G) \leq 4$.

On the other hand, since $deg(v_2) = 4$ and v_2 is also a neighbor of u, by Theorem 21 we would have $sd_\gamma(G) \leq \min\ \{4, 3\}$. Hence, $sd_\gamma(G) \leq 3$.

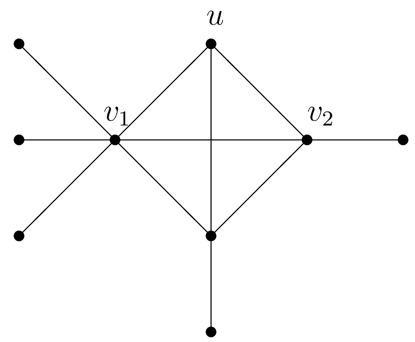

Figure 19: Graph G with simplicial vertex u.

A vertex of degree one is an *end-vertex*, and its neighbor is called a *support vertex*. A bound in terms of the degree of the neighbors of a support vertex is as follows.

Proposition 22 [2] *Let w be a vertex of degree at least two that is adjacent to a support vertex in G. Then $sd_\gamma(G) \leq deg(w) + 1 \leq \Delta(G) + 1$.*

Referring again to Figure 19, we see that u is a vertex adjacent to a support vertex v_1. Hence, by Proposition 22, $sd_\gamma(G) \leq 4$. Of course, we already have a better bound for the graph G in Figure 19 from Theorem 21.

2.4 Claw-free graphs

Graphs that do not have as an induced subgraph the star $K_{1,3}$ are referred to as *claw-free* graphs. Theorem 11 can be improved if we know that the graph is claw-free.

Theorem 23 [2] *Let u_1 and u_2 be vertices lying on a common triangle of G such that $G[N[u_1]]$ and $G[N[u_2]]$ are claw-free, and let d_1, d_2 with $d_1 \geq d_2$ denote their respective degrees. If $d_2 = 2$, then $sd_\gamma(G) \leq 3$, while if $d_2 \geq 3$, then $sd_\gamma(G) \leq$ $max\ \{d_2 + 1, d_1 - d_2 + 3\}$.*

Corollary 24 [2] *Let G be a claw-free graph, and let d_1, d_2 with $d_1 \geq d_2$ be the respective degrees of two vertices u_1 and u_2 lying on a common triangle of G. If $d_2 = 2$, then $sd_\gamma(G) \leq 3$, while if $d_2 \geq 3$, then $sd_\gamma(G) \leq max\ \{d_2 + 1, d_1 - d_2 + 3\}$.*

If G is claw-free and $\delta(G) \geq 3$, then every vertex of G lies on a triangle and we have the following.

Corollary 25 [2] *Let G be a claw-free graph with $\delta(G) \geq 3$. Let $A = \{u \in V \mid deg(u) = \delta(G)\}$, $B = \{v \in V \mid v$ has a neighbor u in A such that uv is contained in a triangle of $G\}$, and $p = min\ \{deg(v) \mid v \in B\}$. Then $sd_\gamma(G) \leq max\ \{\delta(G) + 1, p - \delta(G) + 3\}$.*

If a claw-free graph G contains a vertex u of degree $\delta(G)$ and a vertex $v \in N(u)$ of degree less than $2\delta(G) - 1$ such that uv is contained in a triangle, then $sd_\gamma(G) \leq \delta(G) + 1$.

Corollary 26 [2] *Let G be a claw-free graph with $\delta(G) \geq 3$ and maximum degree $\Delta(G)$. Then $sd_\gamma(G) \leq max\ \{\delta(G) + 1, \Delta(G) - \delta(G) + 3\}$. If, moreover, $\Delta(G) < 2\delta(G) - 1$, then $sd_\gamma(G) \leq \delta(G) + 1$.*

If G is a claw-free r-regular graph with $r \geq 2$, by Corollary 26, we have $sd_\gamma(G) \leq r + 1 = \delta(G) + 1$.

If we know that $\Delta(G) \geq 2\delta(G) - 1$, the following theorem may produce a better bound than that of Corollary 26.

Theorem 27 [2]*Let G be a claw-free graph, and let $u_1u_2 \in E(G)$ be contained in a triangle such that $deg(u_1) \geq deg(u_2)$, and under this condition, $deg(u_1) - deg(u_2)$ is minimum. Then $sd_\gamma(G) \leq deg(u_2) + 1$.*

We end our discussion of bounds on the domination subdivision number of graphs with a conjecture posed by Haynes, Hedetniemi, Hedetniemi, Jacobs, Knisely, and Van der Merwe in [4].

Conjecture 28 [4] *For every graph G with $\delta(G) \geq 2$, $sd_\gamma(G) \leq \delta(G) + 1$.*

Evidence in support of the above conjecture stems from Theorem 11, and Corollaries 12, 13, and 25.

3 TOTAL DOMINATION SUBDIVISION NUMBER SURVEY

Next, we turn our attention to what is known about the total domination subdivision number of a graph by summarizing the results of [6]. Haynes, Hedetniemi, and van der Merwe [6] defined the *total domination subdivision number* $sd_{\gamma_t}(G)$ to be the minimum number of edges that must be subdivided in order to increase the total domination number. Again an edge can be subdivided at most once, that is, no edge incident to a subdivision vertex can be subdivided. Assume that every graph is of order $n \geq 3$, because the total domination number of the graph K_2 does not change when its only edge is subdivided. It is shown in [6] that $sd_{\gamma_t}(G)$ exists for all connected graphs of order $n \geq 3$.

Paralleling Theorem 4, Haynes, Hedetniemi, and van der Merwe [6] established the following upper bound for $sd_{\gamma_t}(G)$ for arbitrary graphs.

Proposition 29 [6] *For any connected graph G of order $n \geq 3$, and for any two adjacent vertices u and v, where $min\{deg(u), deg(v)\} \geq 2$,*

$$sd_{\gamma_t}(G) \leq deg(u) + deg(v) - 1.$$

Proposition 29 can be used to obtain constant upper bounds for the total domination subdivision numbers of various classes of graphs such as the following.

Corollary 30 [6] *For any $r \times s$ grid graph $G_{r,s}$, where $2 \leq r \leq s$,*

$$1 \leq sd_{\gamma_t}(G_{r,s}) \leq 4.$$

Corollary 31 [6] *For any k-regular graph G where $k \geq 2$,*

$$1 \leq sd_{\gamma_t}(G) \leq 2k - 1.$$

To see that the lower and upper bounds of Proposition 29 are sharp, consider paths P_n and cycles C_n. It is a simple exercise to see that $sd_{\gamma_t}(P_3) = 2$. Furthermore, both paths P_n for $n \geq 4$ and cycles C_n have adjacent vertices of degree two, and it follows from Proposition 29 that $1 \leq sd_{\gamma_t}(P_n) \leq 3$ and $1 \leq sd_{\gamma_t}(C_n) \leq 3$. Hence, we have the following.

Proposition 32 [6] *For a path P_n and cycle C_n,*

$$sd_{\gamma_t}(P_n) = sd_{\gamma_t}(C_n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \text{ mod } 4, \\ 3 & \text{if } n \equiv 2 \text{ mod } 4, \\ 2 & \text{if } n \equiv 3 \text{ mod } 4. \end{cases}$$

Note that sharpness for the upper bound has only been demonstrated with graphs G having an adjacent pair of vertices of degree two. Hence, the upper bound is the constant three. It has been shown that this constant upper bound holds for any graph G where $\gamma_t(G) \in \{2, 3\}$.

Proposition 33 [6] *If G is a graph with order $n \geq 3$ and $\gamma_t(G) = 2$, then $1 \leq sd_{\gamma_t}(G) \leq 3$, and these bounds are sharp.*

Sharpness for the lower bound is achieved by a complete bipartite graph $K_{r,s}$ for $2 \leq r \leq s$ and sharpness for the upper bound is achieved by a complete graph K_n for $n \geq 4$.

Theorem 34 [6] *If G is a graph with $\gamma_t(G) = 3$, then $1 \leq sd_{\gamma_t}(G) \leq 3$.*

Turning our attention to the lower bound, it has been shown in [6] that there is no induced subgraph characterization of the graphs where the total domination subdivision number is 1. To see this, consider the *corona* $G \circ K_1$ which is defined as the graph of order $2n$ obtained from a copy of G, by adding to each vertex $v \in V(G)$, a new vertex v' and edge vv'. The graph G is obviously an induced subgraph of $G \circ K_1$ and $\gamma(G \circ K_1) = n$. If G has no isolates, then $V(G)$ is the unique total dominating set of $H = G \circ K_1$, and this leads to the following results.

Proposition 35 [6] *Every graph (of order n) with no isolates is an induced subgraph of a graph H (of order $2n$) with $sd_{\gamma_t}(H) = 1$.*

Corollary 36 [6] *There does not exist a forbidden subgraph characterization of the graphs H for which $sd_{\gamma_t}(H) = 1$.*

Yet another sufficient condition for the upper bound of three on the total domination subdivision number follows.

Proposition 37 [6] *For any graph G having a vertex of degree two which is contained in a triangle,*

$$1 \leq sd_{\gamma_t}(G) \leq 3.$$

Two direct results of Proposition 37 follow. Refer to Chapter 2 for definitions and see Figure 16 for an example of a maximal outerplanar two-tree. Clearly, every such graph is completely triangular and contains a vertex of degree two.

Corollary 38 [6] *For any two-tree G,*

$$1 \leq sd_{\gamma_t}(G) \leq 3.$$

Corollary 39 [6] *For any maximal outerplanar graph G,*

$$1 \leq sd_{\gamma_t}(G) \leq 3.$$

The evidence to this point seems to support a conjecture for total domination similar to Conjecture 2 for domination. However, Proposition 40 below shows that three is not an upper bound for $sd_{\gamma_t}(G)$ for all graphs G.

Proposition 40 *Let G be the complete bipartite graph $K_{r,r}$, for $r \geq 5$, minus a perfect matching. Then $sd_{\gamma_t}(G) \geq 4$.*

Finally, we will follow the development of bounds of the total domination subdivision number for trees. A graph T is called a *tree* if it is acyclic. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex.* A support vertex that is adjacent to more than one leaf is called a *strong support vertex.*

Leading up to an important result for the total domination subdivision number of trees, we have the following lemma for general graphs.

Lemma 41 [6] *Let G be a graph with leaves u and v. If $2 \leq dist(u,v) \leq 4$, then $sd_{\gamma_t}(G) \leq 2$.*

Corollaries 42, 43,and 44 below follow directly from Lemma 41.

Corollary 42 [6] *For any graph G with adjacent support vertices, $sd_{\gamma_t}(G) = 1$.*

Corollary 43 [6] *For any graph G with a strong support vertex, $sd_{\gamma_t}(G) \leq 2$.*

Corollary 44 [6] *Let G be a graph with two support vertices u and v. If $dist(u,v) = 2$, then $sd_{\gamma_t}(G) \leq 2$.*

Finally, we have the following result for trees.

Proposition 45 [6] *For any tree T of order $n \geq 3$,*

$$1 \leq sd_{\gamma_t}(T) \leq 3,$$

and these bounds are sharp.

Sharpness for Proposition 45 is established by Proposition 32 for paths P_n where $n \equiv 0, 1 \bmod 4$ achieves the lower bound and $n \equiv 2 \bmod 4$ achieves the upper bound.

A tree T is said to be in Class i for $i \in \{1, 2, 3\}$ if $sd_{\gamma_t}(T) = i$. Proposition 32 shows that each Class i, $1 \leq i \leq 3$, is nonempty. For additional examples of trees in each of the classes: coronas $T' \circ K_1$ for any nontrivial tree T' are in Class 1, stars $K_{1,k}$, for $k \geq 2$, are in Class 2, and the family of trees in Figure 20 is in Class 3.

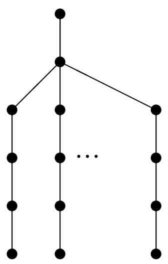

Figure 20: A family of trees T with $sd_{\gamma_t}(T) = 3$.

A *caterpillar* is a tree with the property that the removal of its leaves results in a path $u_1u_2...u_s$, referred to as the *spine* of the caterpillar. A caterpillar T is uniquely determined by the sequence of nonnegative integers $(t_1, t_2, ..., t_s)$, where t_i is the number of leaves adjacent to u_i, for $s \geq 2$. Both this sequence and its reverse sequence define T. The *code* C of the caterpillar is the larger of these two sequences. For example, the code of the caterpillar in Figure 21 is (23021).

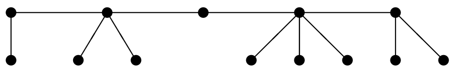

Figure 21: A caterpillar with code (23021).

In the code of caterpillar C, we consider the substrings of consecutive zeros, called *zero strings*, and label them from 1 to k. Let z_i be the number of zeros in string i, for $1 \leq i \leq k$. For example, the caterpillar with code (1001000101) has $z_1 = 2$, $z_2 = 3$, and $z_3 = 1$. The caterpillar with code (1001001001001) has $z_i = 2$, for $1 \leq i \leq 4$. Note also that the latter caterpillar is in Class 3. The Class 3 caterpillars are characterized in [6] as follows.

Theorem 46 [6] *A caterpillar T with code C is in Class 3 if and only if C has no entry greater than 1, no consecutive nonzero entries, and $z_i \equiv 2 \bmod 4$ for $1 \leq i \leq k$.*

4 NEW RESULTS FOR BOUNDS ON THE TOTAL DOMINATION SUBDIVISION NUMBER

In this chapter, we give new results that generalize the bounds given for the domination subdivision number of a graph. Refer to Chapter 2 for definitions.

A generalization of Theorem 29 for the bound on $sd_{\gamma_t}(G)$ that parallels Theorem 7 follows.

Theorem 47 *For any graph G and edge uv, where $min\{deg(u), deg(v)\} \geq 2$,*

$$sd_{\gamma_t}(G) \leq |N(u) \cup N(v)| - 1 = deg(u) + deg(v) - |N(v) \cap N(u)| - 1.$$

Proof. Let $N(v) = \{v_1, v_2, ..., v_k\}$ where $u = v_1$ and $N(u) - N[v] = \{u_1, u_2, ..., u_t\}$. (Note that $N(u) - N[v]$ may be empty.) Let G' be the graph obtained by subdividing the edge vv_i with subdivision vertex x_i, for $1 \leq i \leq k$, and the edge uu_j with subdivision vertex w_j for $1 \leq j \leq t$. Let A be the set of the subdivision vertices and S' a $\gamma_t(G')$-set such that $|S' \cap A|$ is a minimum. Clearly, no $\gamma_t(G)$-set totally dominates v in G', so $S' \cap A \neq \emptyset$.

One of u or v must be in S' to totally dominate x_1.

If both u and v are in S', then $S' - A$ is a total dominating set of G of cardinality less than $\gamma_t(G)$, a contradiction.

Assume $u \in S'$ and $v \notin S'$. Then every neighbor of v in G is in S' to totally dominate $\{x_1, x_2, ..., x_k\}$ and some x_i is in S' to dominate v. If $|S' \cap A| \geq 2$, then

$S' - A \cup \{v\}$ is a total dominating set of G of cardinality less than $\gamma_t(G)$. Thus, $|S' \cap A| = 1$. Now u is only in S' to dominate $\{w_1, w_2, ..., w_t\}$ and totally dominate $v_i \in N(u) \cap N(v)$. Then $(S' - \{u\} - A) \cup \{v\}$ is a total dominating set of G of cardinality less than $\gamma_t(G)$, a contradiction.

Assume $v \in S'$ and $u \notin S'$. Then at least one subdivision vertex is in S' to totally dominate v. If $|S' \cap A| \geq 2$, then $(S' - A) \cup \{u\}$ is a total dominating set of G of cardinality less than $\gamma_t(G)$. Therefore, assume that $|S' \cap A| = 1$. Say x_1 is the only subdivision vertex in S'. Then all neighbors of v in G except u are totally dominated by $S' - A - \{u, v\}$. Thus, $S' - \{v, x_1\} \cup \{v_i\}$ for some vertex $v_i \in N(u) \cap N(v)$ is a total dominating set of G of cardinality less than $\gamma_t(G)$. Therefore, we may assume that $x_1 \notin S'$. But then some $v_i \in N_G(v) \cap N_G(u)$ must be in S' to totally dominate u, and again $S' - A$ is a total dominating set of G of cardinality less than $\gamma_t(G)$, a contradiction. □

Surprisingly, there is a slightly stronger bound for the total domination subdivision number of a graph that contains a triangular vertex u than was established by Theorem 11 for the domination subdivision number.

Theorem 48 *If a graph G contains a triangular vertex u, then $sd_{\gamma_t}(G) \leq deg(u)$.*

Proof. Let $u \in V$ be a triangular vertex in G, and let G_u be the graph that results from subdividing every edge incident with u in G.

Now either $\gamma_t(G_u) > \gamma_t(G)$, in which case $sd_{\gamma_t}(G) \leq deg(u)$, or $\gamma_t(G_u) = \gamma_t(G)$. Assume that $\gamma_t(G_u) = \gamma_t(G)$, and let S be any $\gamma_t(G_u)$-set. Clearly, no $\gamma_t(G)$-set totally dominates u in G_u, so S must contain a subdivision vertex. Let A be the set of subdivision vertices in S.

If $u \notin S$, then every vertex of $N_G(u)$ is in S to dominate the subdivision vertices. Since u is a triangular vertex, every vertex in $N(u)$ has a neighbor in $N(u)$, implying that $S - A$ is a total dominating set of G with cardinality less than $\gamma_t(G)$, a contradiction.

Hence, we assume that $u \in S$. If $N_G(u) \cap S \neq \emptyset$, then $S - A$ is a total dominating set of G with cardinality less than $\gamma_t(G)$, a contradiction.

Therefore, we assume that $N_G(u) \cap S = \emptyset$. If $|A| \geq 2$, then $(S - A) \cup \{x\}$, where $x \in N_G(u)$, is a total dominating set of G with cardinality less than $\gamma_t(G)$, a contradiction. Thus, without loss of generality, let $A = \{x'\}$ and x the neighbor of x' in $N_G(u)$. Because u is a triangular vertex, x has a neighbor, say y, in $N_G(u)$. Now every vertex in $N_G(u) - \{x\}$ is totally dominated by vertices in $S \cap (V(G) - N_G[u])$. Hence, $(S - \{u, x'\}) \cup \{y\}$ is a total dominating set of G with cardinality less than $\gamma_t(G)$, again a contradiction. □

Theorem 48 is sharp as can be seen by the following family of graphs. Let G be the graph obtained from a complete graph K_n, $n \geq 3$, by adding a vertex u adjacent to exactly two vertices of K_n. It is easy to see that $\gamma_t(K_n) = \gamma_t(G) = 2$. We will show that $sd_{\gamma_t}(G) = deg(u) = 2$.

Because u is a triangular vertex in G, by Theorem 48, we have $sd_{\gamma_t}(G) \leq deg(u) = 2$.

To see that $sd_{\gamma_t}(G) \geq 2$, assume to the contrary that $sd_{\gamma_t}(G) = 1$. Subdivide an edge in G with a vertex w to form a new graph G' with minimum total dominating set S'. We will show that $\gamma_t(G') = \gamma_t(G) = 2$.

If w subdivides an edge incident with u, without loss of generality, say uv_1, then $S' = \{v_1, v_2\}$ is a total dominating set of G'.

If w subdivides an edge in $G - u$ incident to v_1 or v_2, without loss of generality, say v_1v_k where $k \neq 1$, then $S' = \{v_1, v_j\}$, where $j \neq k$, is a total dominating set of G'.

If w subdivides an edge in G not incident to v_1 or v_2, say v_iv_j, then $S' = \{v_1, v_i\}$ is a total dominating set of G'.

Hence, $sd_{\gamma_t}(G) = deg(u) = 2$.

Immediate consequences of Theorem 48 can be seen in the following corollaries.

Corollary 49 *For every completely triangular graph G, $sd_{\gamma_t}(G) \leq \delta(G)$.*

Corollary 50 *If a graph G contains a simplicial vertex u of degree at least two, then $sd_{\gamma_t}(G) \leq deg(u)$.*

Corollary 51 *For every k-tree G, $k \geq 2$, $sd_{\gamma_t}(G) \leq k$.*

Corollary 52 *For every 2-connected chordal graph G, $sd_{\gamma_t}(G) \leq \delta(G)$.*

Corollary 53 *For every maximal outerplanar graph G, $sd_{\gamma_t}(G) \leq \delta(G) = 2$.*

Corollary 54 *For every maximal planar graph G, $sd_{\gamma_t}(G) \leq \delta(G) \leq 5$.*

Analogous to Theorems 19 and 20 for domination subdivision numbers, we can establish a stronger bound for $sd_{\gamma_t}(G)$ than that offered by Theorem 48 if we know more about the structure of the graph. We first present a lemma.

Lemma 55 *For any complete graph K_n for $n \geq 3$, $sd_{\gamma_t}(K_n) = 2$ if $n = 3$ and $sd_{\gamma_t}(K_n) = 3$ otherwise.*

Proof. If $n = 3$, then K_n is a cycle C_n of order $n \equiv 3 \bmod 4$ and by Proposition 60, we have $sd_{\gamma_t}(C_3) = 2$.

Let S be any minimum total dominating set of K_n where $n \geq 4$. Clearly, $|S| = 2$.

First, we will show that $sd_{\gamma_t}(K_n) \geq 3$ for all $n \geq 4$. Assume to the contrary that $sd_{\gamma_t}(K_n) = 2$. Subdivide two incident edges, say uv and vw, with subdivision vertices x and y, respectively to form K'_n. Let S' be a $\gamma_t(K'_n)$-set. Then v is in S' to dominate x and y and some vertex $a \in \{V(K_n) - \{u, v, w\}\}$ is in S' to totally dominate v. But $\{a, v\}$ dominates K'_n contradicting the fact that $sd_{\gamma_t}(K_n) = 2$. Next, subdivide any two independent edges in K_n, say uv and ab with subdivision vertices x and y, respectively to form K'_n. Let S' be a $\gamma_t(K'_n)$-set. Then S' must contain one of u or v, without loss of generality, say u to dominate x, and also must contain one of a or b, without loss of generality, say a to dominate y. But then $\{u, a\}$ totally dominates K'_n. Thus $sd_{\gamma_t}(K_n) \geq 3$.

Next, we will show that $sd_{\gamma_t}(K_n) \leq 3$ for all $n \geq 4$. Let $\{u, v, w\}$ be a triangle in K_n with K'_n the graph obtained by subdividing the edges uv, vw, and uw with vertices a, b, and c, respectively. Let A be the set of subdivision vertices in K'_n. Let

B be the set of vertices in $K_n - \{u, v, w\}$. We will show that $\gamma_t(K_n) < \gamma_t(K_n')$.

Suppose not. Because $\gamma_t(K_n) = 2$, then $\gamma_t(K_n') = 2$. At least 2 of $\{u, v, w\}$ are in S' to dominate the subdivision vertices. We will assume without loss of generality that u and v are in S'. This implies that $S' \cap A = \emptyset$ and $S' \cap B = \emptyset$. But then neither u nor w is totally dominated in K_n'. □

Theorem 56 *If G is a graph having three or more pairwise-adjacent simplicial vertices, then $sd_{\gamma_t}(G) \leq 3$.*

Proof. If G is of order 3, then G is a K_3, and by Lemma 55, $sd_{\gamma_t}(G) = 2$. Hence, we will assume that G contains a clique of order at least 4. Let u, v, and w be three pairwise-adjacent simplicial vertices in graph G. We will assume that these vertices are adjacent to at least one nonsimplicial vertex, else G is a complete graph of order $n \geq 4$ and by Lemma 55, we have $sd_{\gamma_t}(G) = 3$.

Let C be the set of nonsimplicial vertices adjacent to u, v, and w, and let D be the set of simplicial vertices in the clique containing u, v, and w and the vertices in C. Let G' be the graph obtained from G by subdividing edges uv, vw, and uw with vertices a, b, and c, respectively. Let E be the set of subdivision vertices. We will show that $\gamma_t(G') > \gamma_t(G)$.

First, we will show that no $\gamma_t(G)$-set S is a total dominating set of G'.

Case 1. $S \cap C \neq \emptyset$. This implies that $|S \cap D| \leq 1$, else S is not a minimum total dominating set of G. If $u \in S$, then S does not dominate b. If $v \in S$, then S does not dominate c. If $w \in S$, then S does not dominate a.

Case 2. $S \cap C = \emptyset$. This implies that $|S \cap D| \neq \emptyset$ which in turn implies that $|S \cap D| = 2$. If $|S \cap \{u, v, w\}| = 2$, then u, v, and w are not totally dominated in G'. If $|\{u, v, w\} \cap S| \leq 1$, then at least one of a, b, or c is not dominated.

Second, we will show that no set S' containing a subdivision vertex and of cardinality $\gamma_t(G)$ is a total dominating set of G'. Let S' be a set of cardinality $\gamma_t(G)$ containing at least one subdivision vertex.

Case 1. $S' \cap C \neq \emptyset$. If $S' \cap C \geq 2$, then $S' - E$ is a total dominating set of G of cardinality less than $\gamma_t(G)$, a contradiction. If $S' \cap C = 1$, then at least one of u, v, or w is in S' to totally dominate E, but then $S' - E$ is a total dominating set of G of cardinality less than $\gamma_t(G)$.

Case 2. $S' \cap C = \emptyset$ but $S' \cap D \neq \emptyset$. If $|S' \cap D| \geq 2$, then $S' - E$ is a total dominating set of G of cardinality less than $\gamma_t(G)$, hence $|S' \cap D| = 1$. Without loss of generality, let $u \in S'$. If $|S' \cap E| \geq 2$, then $S' - E \cup \{v\}$ is a total dominating set of G of cardinality less than $\gamma_t(G)$. Hence, $|S' \cap E| = 1$. If $b \in S'$, then u is not totally dominated. If $a \in S'$, then w is not dominated. If $c \in S'$, then v is not dominated.

Case 3. $S' \cap C = \emptyset$ and $S' \cap D = \emptyset$. In this case, $S' \cap E$ is not totally dominated.

□

Following in the footsteps of Theorem 21, we have the following result.

Theorem 57 *Let u be a simplicial vertex of degree at least 2 of a graph G, and let v be a neighbor of u. Then* $sd_{\gamma_t}(G) \leq min\{deg(u), deg(v) - deg(u) + 3\}$

Proof. The first bound had been established in Theorem 48. The proof of the second bound is as follows. Let $v, w_1, ..., w_r$ be the neighbors of u, $r \geq 1$, and $v_1, ..., v_k$ the other neighbors of v (with $k = 0$ if v is also simplicial). We form a new graph G' by subdividing the edge uv with a vertex x, the edge vw_1 with a vertex y, w_1u with a vertex z, and each edge vv_i, $1 \leq i \leq k$, with a vertex a_i. Let $A = \{x, y, z, a_1 ..., a_k\}$, and let S' be a minimum total dominating set of G' such that $|S' \cap A|$ is minimum.

Clearly, if u has only v and w_1 as neighbors, then the first bound holds, thus we will assume that $r \geq 2$.

At least one of u or v must be in S' to totally dominate x. If $u \in S'$, then u dominates x and z. Hence, w_1 or v must be in S' to totally dominate y. Because $S' \cap A$ is minimum, some other w_i for $i \neq 1$ is in S' to totally dominate u. But then $S' - \{u\}$ is a total dominating set of G of cardinality less than $\gamma_t(G)$, a contradiction. Hence, $u \notin S'$. If $v \in S'$, then v dominates x and y. Hence, w_1 must be in S' to totally dominate z. Again, because $S' \cap A$ is minimum, some w_i for $i \neq 1$ is in S' to totally dominate u. But because v is only in S' to dominate $\{a_1, ..., a_k\}$ and totally dominate w_i, then $S' - \{v\}$ is a total dominating set of G of cardinality less than $\gamma_t(G)$, a contradiction. □

5 BOUNDS OF TOTAL DOMINATION SUBDIVISION NUMBERS FOR TREES

In this chapter, we characterize the caterpillars in Class i for $1 \leq i \leq 3$. We begin by restating results and definitions from Chapter 2 for easy reference in upcoming proofs.

Proposition 58 *For any graph G with adjacent support vertices, $sd_{\gamma_t}(G) = 1$.*

Concerning the total domination number of paths and cycles, we have the following.

Proposition 59 *For a path P_n or cycle C_n, $\gamma_t(P_n) = \gamma_t(C_n) = n/2$ if $n \equiv 0 \mod 4$ and $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + 1$ otherwise.*

Proposition 59 lead directly to the following classification by total domination subdivision number for paths and cycles.

Proposition 60 *For a path P_n and cycle C_n,*

$$sd_{\gamma_t}(P_n) = sd_{\gamma_t}(C_n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \mod 4, \\ 3 & \text{if } n \equiv 2 \mod 4, \\ 2 & \text{if } n \equiv 3 \mod 4. \end{cases}$$

The next theorem establishes that all trees can be classified by total domination subdivision number as Class 1, Class 2, or Class 3. However, the characterization of each of these classes remains an open problem.

Proposition 61 *For any tree T of order $n \geq 3$, $1 \leq sd_{\gamma_t}(T) \leq 3$.*

Recall in Chapter 3 we defined the code C of caterpillar T to be a sequence of nonnegative integers $(t_1, t_2, ..., t_s)$ where t_i is the number of leaves adjacent each vertex on the spine of T. In relation to that code, we labeled the zero strings from 1 to k and defined z_i as the number of zeros in string i, for $1 \leq i \leq k$.

Because all caterpillars are also trees, caterpillars can also be classified as Class 1, Class 2, or Class 3. Class 3 caterpillars were characterized as follows.

Theorem 62 *A caterpillar T with code C is in Class 3 if and only if C has no entry greater than 1, no consecutive nonzero entries, and $z_i \equiv 2 \bmod 4$ for $1 \leq i \leq k$.*

Hence, our task is to characterize the Class 1 and Class 2 caterpillars. We begin with a lemma.

Lemma 63 *For any path P_n on $n \geq 6$ vertices where $n \equiv 2, 3 \bmod 4$, there exists a minimum total dominating set that includes at least one end vertex of P_n.*

Proof Let $P_n = x_1, x_2, ..., x_n$ for $n \geq 6$ be a path with endvertices x_1 and x_n. Clearly, the Lemma holds if $n \in \{6, 7\}$, so let $n \geq 10$. Because $n \equiv 2, 3 \bmod 4$, by Proposition 59, we have $\gamma_t(P_n) = \lfloor n/2 \rfloor + 1$. We show that there exists a $\gamma_t(P_n)$-set S containing x_1. Assume $x_1 \in S$. Now $x_2 \in S$ to totally dominate x_1 and we may assume $x_{n-1} \in S$ to dominate x_n and $x_{n-2} \in S$ to totally dominate x_{n-1}. Hence, $S - \{x_1, x_2, x_{n-2}, x_{n-1}\}$ must totally dominate a path on $n - 7$ vertices.

Let $n \equiv 2 \bmod 4$. Now $n-7 \equiv 3 \bmod 4$ and we have $|S| = 4 + \lfloor (n-7)/2 \rfloor + 1 = 4 + \lfloor n/2 \rfloor - 4 + 1 = \lfloor n/2 \rfloor + 1 = \gamma_t(P_n)$. Hence, S is a $\gamma_t(P_n)$-set as desired.

If $n \equiv 3 \bmod 4$, then $n-7 \equiv 0 \bmod 4$ and by Proposition 59, we have $|S| = 4 + (n-7)/2 = 4 + \lfloor n/2 \rfloor - 3 = \lfloor n/2 \rfloor + 1$, and again S is a $\gamma_t(P_n)$-set. □

Theorem 64 *A caterpillar T with code C has $sd_{\gamma_t}(T) \geq 2$ if and only if C contains no consecutive nonzero entries and $z_i \equiv 2 \bmod 4$ or $z_i \equiv 3 \bmod 4$ for all $1 \leq i \leq k$.*

Proof If C contains consecutive nonzero entries, then by Proposition 58, $sd_{\gamma_t}(T) = 1$. Hence, we will assume that caterpillar T has code C with no consecutive nonzero entries and $z_i \equiv 2, 3 \bmod 4$ for all $1 \leq i \leq k$.

Label the vertices of zero string i as $w_1^i, w_2^i, ..., w_{z_i}^i$ and label the vertices corresponding to nonzero entries in C as $x_0, x_1, ..., x_k$ for $0 \leq i \leq k$ such that the label of the spine of C is as follows.

$$x_0, w_1^1, ..., w_{z_1}^1, x_1, w_1^2, ..., x_{i-1}, w_1^i, w_2^i, ..., w_{z_i}^i, x_i, w_1^{i+1}, ...$$

First, we will show that there exists some $\gamma_t(T)$-set S such that for an arbitrary zero string, say the jth one where $1 \leq j \leq k$, S contains x_{j-1}, w_1^j, $w_{z_j}^j$ and x_j. Clearly, all $x_i \in S$ for $0 \leq i \leq k$ since every $\gamma_t(T)$-set contains all of the support vertices. Let S be a $\gamma_t(T)$-set that contains w_1^1. Consider the $l_0 l_1$-path where l_i is a leaf adjacent to x_i. Then this path has order congruent to $2, 3 \bmod 4$. Hence, Lemma 63 implies that some minimum total dominating set S for the $l_0 l_1$-path includes x_1 and l_1, but not $w_{z_1}^1$. Now l_1 can be exchanged for w_1^2. Using the same reasoning for the $l_1 l_2$-path, there exists a total dominating set that includes x_2 and l_2. Again, l_2 can be exchanged for w_1^3. Following this pattern, there exists a minimum total dominating set that includes all w_1^i where $1 \le i \le k$ as well as a leaf l_k adjacent to x_k. Now l_k could be exchanged for $w_{z_k}^k$ and each w_1^i for $j+1 \le i \le k$ could be exchanged for $w_{z_{i-1}}^{i-1}$. Hence, we have established that there exists a minimum total dominating set which includes x_{j-1}, w_1^j, $w_{z_j}^j$ and x_j.

To see that $sd_{\gamma_t}(T) \ge 2$, assume to the contrary that $sd_{\gamma_t}(T) = 1$. Subdivide an edge uv of T forming T'. Let T_i be the subtree of T formed from

$$l_{i-1}, x_{i-1}, w_1^i, w_2^i, ..., w_{z_i}^i, x_i, l_i$$

such that uv is on the $l_{i-1}l_i$-path of T and all leaves adjacent to x_{i-1} and x_i. In T, the order of the $l_{i-1}l_i$-path is $z_i + 4$. Since $z_i + 4 \equiv 2, 3 \bmod 4$, by Proposition 60, T_i can be dominated with the same number of vertices in both T and T'. Let S and S' be a $\gamma_t(T)$-set and $\gamma_t(T')$-set, respectively. Let $S_i = S \cap V(T_i)$ and $S_i' = S' \cap V(T_i)$. Since x_{i-1} and x_i are both in S and S', it follows that $S' - V(T_i)$ and $S - V(T_i)$ both dominate $V(T) - V(T_i)$. Now $(S' - S_i') \cup S_i$ and $(S - S_i) \cup S_i'$ are dominating sets for T and T', respectively. Thus, $\gamma_t(T) = |S| \le |S' - S_i'| + |S_i| = |S' - S_i'| + |S_i'|$ and $\gamma_t(T') = |S'| \le |S - S_i| + |S_i'| = |S - S_i| + |S_i| = |S|$. Hence, $\gamma_t(T') = \gamma_t(T)$.

Conversely, assume that caterpillar T with code C has $sd_{\gamma_t}(T) \ge 2$. If C contains consecutive nonzero entries, then by Proposition 58, T is class 1. Thus, C has no consecutive nonzero entries. To see that $z_i \equiv 2, 3 \bmod 4$ for all $1 \le i \le k$, assume to the contrary that $z_i \equiv 0, 1 \bmod 4$ for some $1 \le i \le k$ and $sd_{\gamma_t}(T) \ge 2$. Let S be a minimum total dominating set of T. Then x_{i-1} and x_i are in every $\gamma_t(T)$-set and dominate all leaves adjacent to them. Hence, the ith zero string along with x_{i-1}, x_i, and each leaf adjacent to x_{i-1} and x_i can be regarded as paths

$$P_i = l_{i-1}, x_{i-1}, w_1, w_2, ..., w_{z_i}, x_i, l_i$$

where each path is of order $z_i + 4$. Since $z_i + 4 \equiv 0, 1 \bmod 4$, by Proposition 60, $sd_{\gamma_t}(P_i) = 1$ and because x_{i-1} and x_i are in every $\gamma_t(T)$-set, we have $sd_{\gamma_t}(T) = 1$. □

In conclusion, we summarize our characterization of caterpillars in each of the three classes.

Theorem 65 *A caterpillar T with code C is in*

Class 1 if and only if C contains consecutive nonzero entries or $z_i \equiv 1$ mod 4 or 0 mod 4 for some $1 \leq i \leq k$,

Class 3 if and only if C has no entry greater than 1, no consecutive nonzero entries, and $z_i \equiv 2$ mod 4 for $1 \leq i \leq k$,

Class 2 otherwise.

[1] G. Chartrand, L. Lesniak, *Graphs and digraphs*, Third Ed., Chapman and Hall, 1996.

[2] O. Favaron, T. W. Haynes, S. T. Hedetniemi, Domination subdivision numbers in graphs, To appear in *Utilitas Math.*

[3] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, Domination and independence subdivision numbers of graphs, *Discussiones Mathematicae - Graph Theory*, 20 (2000) 271 - 280.

[4] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, D. P. Jacobs, J. Knisely, and L. C. van der Merwe, Domination Subdivision Numbers, *Discussiones Mathematicae - Graph Theory*, 21 (2001) 239 - 253.

[5] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.

[6] T. W. Haynes, S. T. Hedetniemi, L. C. van der Merwe, and Total domination subdivision numbers, to appear in *J. Combin. Math. Combin. Compute.*

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