8-2002

On the Construction of Linear Prewavelets over a Regular Triangulation.

Qingbo Xue
East Tennessee State University

On the Construction of Linear Prewavelets Over a Regular Triangulation

---

A thesis
presented to
the faculty of the Department of Mathematics
East Tennessee State University

In partial fulfillment
of the requirements for the degree
Master of Science in Mathematical Sciences

---

by
Qingbo Xue
August 2002

Dr. Don Hong, Chair
Dr. James Boland, Co-Chair
Dr. Robert Gardner
Dr. Debra Knisley

Keywords: Multiresolution, Prewavelets, Piecewise linear splines, Trangulations, Local support

ABSTRACT

On the Construction of Linear Prewavelets Over a Regular Triangulation

by

Qingbo Xue

In this thesis, all the possible semi-prewavelets over uniform refinements of regular triangulations have been studied. A corresponding theorem is given to ensure the linear independence of a set of different pre-wavelets obtained by summing pairs of these semi-prewavelets. This provides efficient multiresolutions of the spaces of functions over various regular triangulation domains since the bases of the orthogonal complements of the coarse spaces can be constructed very easily.

DEDICATION

This thesis is dedicated to Rui Cheng, my wife, and Haotian Xue, my son, who have supported my efforts to complete my graduate degree. Thanks for all your love and support. I love you.

ACKNOWLEDGMENTS

A special thanks to my thesis advisor, Dr. Dong Hong, who has been patient with me through the entire process. And a word of thanks to the rest of my committee who has graciously given their time to support my thesis.

Contents

ABSTRACT 2

COPYRIGHT 3

DEDICATION 4

ACKNOWLEDGMENTS 5

LIST OF FIGURES 7

1. Introduction 9

2. Basic Concepts 11

3. Multiresolution of Linear Spline Spaces over r-Triangulations 14

4. Semi-prewavelets 20

5. Examples 37

BIBLIOGRAPHY 41

VITA 43

List of Figures

1 An r-triangulation 15

2 The first refinement of the r-triangulation in Figure 1 15

3 Interior vertex a_1 and its support 21

4 Boundary vertices and their supports 21

5 Semi-prewavelet SPW(I6) 24

6 Semi-prewavelet SPW(B2) 25

7 Semi-prewavelet SPW(B3[1]), SPW(B3[2]) 27

8 Semi-prewavelet SPW(B4[1]), SPW(B4[2]) 28

9 Semi-prewavelet SPW(B5[1]), SPW(B5[2]), and SPW(B5[3]) 29

10 Semi-prewavelet SPW(B6[1]), SPW(B6[2]) and SPW(B6[3]) ($q = \frac{276}{551}$) 32

11 Pre-wavelet PW(I6,B3) 33

12 Pre-wavelet PW(B4,B2) 33

13 Pre-wavelet PW(B4,B4) 33

14 Pre-wavelet PW(I6,B4) 34

15 Pre-wavelet PW(B5,B2) 34

16 Pre-wavelet PW(B5,B3) 35

17 Pre-wavelet PW(I6,B5) 35

18 Sample r-Triangulation Domain 1 37

19 Sample r-Triangulation Domain 2 38

20 Sample r-Triangulation Domain 3 39

21 Sample r-Triangulation Domain 4 40

CHAPTER 1

Introduction

In recent years, multiresolution analysis has been intensively studied and has been used in computer graphics, differential and integral equation, manifolds, the finite element setting, and so on. Basically speaking, multiresolution is a decomposition of a function space into mutually orthogonal subspaces, each of which is endowed with a basis. The basis functions of each subspace are called wavelets if they are mutually orthogonal and prewavelets otherwise. The subspaces are called wavelet spaces and prewavelet spaces accordingly.

Piecewise linear prewavelets with small support are useful tools in approximation theory and the numerical solution of partial differential equations as applied to computer graphics and practical largescale data representation. Kotyczka and Oswald [8] constructed piecewise linear prewavelets with small support in 1995. Floater and Quak [5] published their result on piecewise linear prewavelets with small support on arbitrary triangulations in 1999. Later on, they simplified the above result by introducing the idea of semi-wavelets, which can be used to construct wavelets. These semi-wavelets and wavelets are actually semi-prewavelets and prewavelets. Some scientists omit the "pre-" part for convenience. Using this idea, Floater and Quak investigated the Type-1 triangulation in [6] and Type-2 in [5] respectively. Hong and Mu [4] have discussed the piecewise linear prewavelets with minimal support over Type-1 trangulation.

In this thesis, all the possible semi-prewavelets over uniform refinements of regular triangulations have been studied. A corresponding theorem has been given which ensures the linear independence of any set of different pre-wavelets obtained by summing pairs of these semi-prewavelets. This means that the multiresolutions of the linear function spaces over various regular triangulation domains can be done conveniently, since the bases of the orthogonal complements of the coarse spaces can be constructed very easily. Examples of multiresolutions are discussed and all the corresponding prewavelets or semi-prewavelets have been given explicitly.

CHAPTER 2

Basic Concepts

In this section we introduce some basic concepts. Most of them have commonly been used in related monographs.

Definition 2.1 A set of triangles $\mathcal{T} = \{T_1, ..., T_M\}$ is called a *triangulation* of some subset Ω of $\mathbb{R}^2$ if $\Omega = \cup_{i=1}^{M} T_i$ and

(i) $T_i \cap T_j$ is either empty or a common vertex or a common edge, $i \neq j$,

(ii) the number of boundary edges incident on a boundary vertex is two,

(iii) Ω is simply connected.

We denote by V the set of all vertices $v \in \mathbb{R}^2$ of triangles in $\mathcal{T}$ and by E the set of all edges $e = [v, w]$ of triangles in $\mathcal{T}$. By a boundary vertex or boundary edge we mean a vertex or edge contained in the boundary of Ω. All other vertices and edges will be called interior vertices and interior edges. A boundary edge belongs to only one triangle, and an interior edge to two.

For a vertex $v \in V$, the set of neighbours of v in V is

$$V_v = \{w \in V : [v, w] \in E\}.$$

Suppose next that $\mathcal{T}$ is a triangulation. Given data values $f_v \in \mathbb{R}$ for $v \in V$, there is a unique function $f : \Omega \longrightarrow \mathbb{R}$ which is linear on each triangle in $\mathcal{T}$ and interpolates the data: $f(v) = f_v,\ v \in V$. The function f is piecewise linear and the set of all such f constitute a linear space S with dimension $|V|$. For each $v \in V$, let $\phi_v : \Omega \longrightarrow \mathbb{R}$ be the unique 'hat' or nodal function in S such that for all $w \in V$,

$$\phi_v(w) = \begin{cases} 1, & w = v; \\ 0, & \text{otherwise.} \end{cases}$$

The set of functions $\Phi = \{\phi_v\}_{v \in V}$ is a basis for the space S and for any function $f \in S$,

$$f(x) = \sum_{v \in V} f(v)\, \phi_v(x), \quad x \in \Omega. \tag{2.1}$$

The support of ϕ_v is the union of all triangles which contain v:

$$M_v := \cup_{v \in T \in \mathcal{T}} T.$$

Definition 2.2 Given a triangulation $\mathcal{T}^0 = \{T_1, ..., T_M\}$. A *refined triangulation* is a triangulation $\mathcal{T}^1$ such that every triangle in $\mathcal{T}^0$ is the union of some triangles in $\mathcal{T}^1$. The result of this process is called a refinement of $\mathcal{T}^0$.

Obviously, there are various kinds of refinements. If not clearly claimed, we shall only consider the following uniform or dyadic refinement. We shall use $[u, v]$ to denote the edge incident to two vertices u, v. A triangle with vertices u, v, w will be denoted as $[u, v, w]$.

For a given triangle $T = [x_1, x_2, x_3]$, let $y_1 = (x_2 + x_3)/2, y_2 = (x_1 + x_3)/2$, and $y_3 = (x_1 + x_2)/2$ denote the midpoints of its edges. Then the set of four triangles

$$\mathcal{T}_T = \{[x_1, y_2, y_3], [y_1, x_2, y_3], [y_1, y_2, x_3], [y_1, y_2, y_3]\}$$

is a triangulation and a refinement of the coarse triangle T. The set of triangles $\mathcal{T}^1 = \cup_{T \in \mathcal{T}^0} \mathcal{T}_T$ is evidently a triangulation and a refinement of $\mathcal{T}^0$. Similarly, a whole sequence of triangulations $\mathcal{T}^j, j = 0, 1, 2, ...$, can be generated by further refinement steps.

In order to discuss some properties of $\mathcal{T}^j$ in relation to $\mathcal{T}^{j-1}$, let V^j be the set of vertices in $\mathcal{T}^j$, and define $E^j, S^j, \phi_v^j, V_v^j$, and M_v^j accordingly. A straightforward calculation shows that

$$\phi_v^{j-1} = \phi_v^j + \frac{1}{2}\sum \phi_w^j, \qquad v \in V^{j-1}, \tag{2.2}$$

and therefore we obtain a nested sequence of spaces

$$S^0 \subset S^1 \subset S^2 \subset \cdots. \tag{2.3}$$

CHAPTER 3

Multiresolution of Linear Spline Spaces over r-Triangulations

From a mathematical point of view, a computer graphic is nothing else but a function defined on a given region. On the other hand, a graphic on a domain Ω can be represented by functions in different level of function spaces such as S^j $(j = 0, 1, 2, \cdots)$ in (2.3). The difference between them is that the function from a fine space gives more detail of the original graphic than the one from a coarse space does. In an ideal situation, we can easily "witch" a function from one space into another when it is necessary. The key to choosing another function space is to use a different basis of functions. Surprisingly, the relation between the bases of these nested spaces makes it difficult to do so.

In the following we shall discuss the multiresolution of the linear spline function space defined on any r-triangulation.

Definition 3.1 $\mathcal{T}^0 = \{T_1, ..., T_M\}$ is a *regular triangulation* or simply *r-triangulation* over some domain Ω if $\mathcal{T}^0$ is a triangulation on Ω and all the elements of $\mathcal{T}^0$ are equilateral triangles.

Figure 1. gives an example of an r-triangulation over a triangle shaped region — a simply connected domain.

Clearly, a refinement $\mathcal{T}^1$ of $\mathcal{T}^0$ is still an r-triangulation, see Figure 2. Continuing the refinement process on Ω leads us to the nested sequence of spaces defined on the

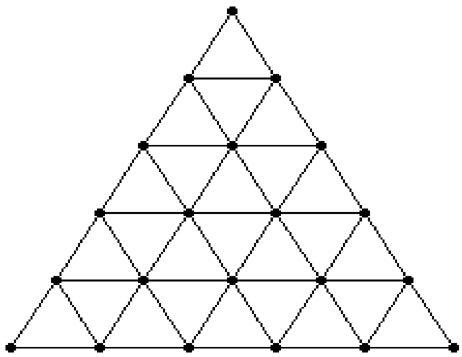

Figure 1: An r-triangulation

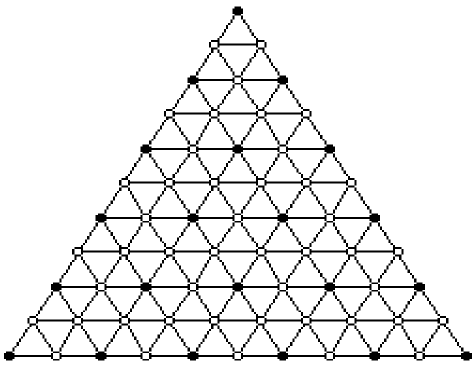

Figure 2: The first refinement of the r-triangulation in Figure 1

r-triangulation's domain Ω as we had in (2.3). All the other concepts and symbols in Section 2 can be used naturally for the r-triangulations here.

As usual, we use the following standard definition of the inner product of two continuous functions on Ω,

$$\langle f, g\rangle = \sum_{T\in\mathcal{T}^0} \frac{1}{a(T)} \int_T f(x)g(x)\,dx, \qquad f, g \in C(\Omega),$$

where $a(T)$ is the area of triangle T.

Let c be the area of any a triangle in the r-triangulation $\mathcal{T}^0$. Since all the triangles are congruent, the inner product reduces to the scaled L_2 inner product

$$\langle f, g\rangle = \frac{1}{c}\int_{\Omega} f(x)g(x)\,dx. \tag{3.1}$$

With this inner-product, the spaces S^j become inner-product spaces. Let W^{j-1} denote the relative orthogonal complement of the coarse space S^{j-1} in the fine space S^j, so that

$$S^j = S^{j-1} \oplus W^{j-1}. \tag{3.2}$$

We have the following decomposition:

$$S^n = S^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^{n-1} \tag{3.3}$$

and the dimension of W^{j-1} is $|V^j| - |V^{j-1}| = |E^{j-1}|$.

In the following, we shall try to construct a basis for the unique orthogonal complement W^{j-1} of S^{j-1} in S^j. Each of these basis functions will be called a *prewavelet* and the space W^{j-1} a *prewavelets space.* By combining prewavelet bases of the spaces W^k with the nodal bases for the spaces S^k, we obtain the framework for a multiresolution analysis (MRA). Thus any function f^n in S^n can be decomposed into its $n+1$ mutually orthogonal components:

$$f^n = f^0 \oplus g^0 \oplus g^1 \oplus \cdots \oplus g^n \tag{3.4}$$

where $f^0 \in S^0$ and $g^j \in W^j$ $(j = 0, 1, ..., n-1)$. We shall restrict our work for the construction of bases of W^k to the first refinement level since uniform refinement has been used.

Let b be any given non-zero real number, a_1 and a_2 be two neighboring vertices in V^0, and denote by $u \in V^1 \setminus V^0$ their midpoint. We define the *semiprewavelet* $\sigma_{a_1,u} \in S^1$ as the element with support contained in the support of $\phi^0_{a_1}$ and having the property that, for all $v \in V^0$,

$$\langle \phi^0_v, \sigma_{a_1,u} \rangle = \begin{cases} -b, & \text{if } v = a_1; \\ b, & \text{if } v = a_2; \\ 0, & \text{otherwise} \end{cases} \tag{3.5}$$

where

$$\sigma_{a_1,u}(x) = \sum_{v \in N^1_{a_1}} r_v \phi^1_v(x),$$

and

$$N^1_{a_1} = \{a_1\} \cup V^1_{a_1}$$

denotes the fine neighborhood of a_1. The only nontrivial inner products between $\sigma_{a_1,u}$ and coarse nodal functions ϕ^0_v occur when v belongs to the coarse neighborhood of a_1,

$$N^0_{a_1} = \{a_1\} \cup V^0_{a_1}.$$

Thus the number of coefficients and conditions are the same and, as we will subsequently establish, the element $\sigma_{a_1,u}$ is unique providing that any non-zero value of b in (3.5) is given.

Since the dimension of W^0 is equal to the number of fine vertices in V^1, i.e. $|V^1| - |V^0|$, it is natural to associate one prewavelet ψ_u per fine vertex $u \in V^1 \setminus V^0$. Since each u is the midpoint of some edge in E^0 connecting two coarse vertices a_1

and a_2 in V^0, the element of S^1,

$$\psi_u = \sigma_{a_1,u} + \sigma_{a_2,u} \tag{3.6}$$

is a prewavelet since it is orthogonal to all nodal functions $\phi_v^0,\ v \in V^0$.

The following theorem gives a sufficient condition that all the different prewavelets obtained in this way are linearly independent and hence form a basis of W^0.

Theorem 1. The set of prewavelets $\{\psi_u\}_{u\in V^1\setminus V^0}$ defined by (3.6) is a linearly independent set if

$$\psi_u(u) > \sum_{\substack{w\in V^1\setminus V^0 \\ w\neq u}} |\psi_u(w)|, \quad \forall u \in V^1 \setminus V^0$$

Proof:

Let

$$|V^0| = m, \quad |V^1 \setminus V^0| = n,$$

and

$$u_1, u_2, \cdots, u_m, u_{m+1}, \cdots, u_{m+n}$$

be any permutation of all the vertices in V^1 such that $u_j \in V^0$ $(1 \leq j \leq m)$ and $u_j \in V^1 \setminus V^0$ $(m+1 \leq j \leq m+n)$. The corresponding nodal functions are

$$\phi_j(x) = \phi_{u_j}(x), \quad j = 1, 2, \cdots, m+n.$$

Then the prewavelets in $\{\psi_u\}_{u\in V^1\setminus V^0}$ can be written as

$$\psi_i(x) = \psi_{u_{m+i}}(x) = \sum_{j=1}^{m+n} r_{ij}\phi_j(x), (1 \leq i \leq n)$$

where $r_{ij} = \psi_i(u_j)$ $(1 \leq i \leq n, 1 \leq j \leq m+n)$.

Consider the matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1,m+n} \\ r_{21} & r_{22} & \cdots & r_{2,m+n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{n,m+n} \end{bmatrix}$$

and its sub matrix

$$R_1 = \begin{bmatrix} r_{1,m+1} & r_{1,m+2} & \cdots & r_{1,m+n} \\ r_{2,m+1} & r_{2,m+2} & \cdots & r_{2,m+n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n,m+1} & r_{n,m+2} & \cdots & r_{n,m+n} \end{bmatrix}.$$

We claim that R_1 is diagonally dominant. Actually, keeping in mind that $r_{i,m+j} = \psi_i(u_{m+j}) = \psi_{u_{m+i}}(u_{m+j})$ $(1 \leq i, j \leq n)$, we know that

$$r_{i,m+i} > \sum_{\substack{1 \leq j \leq n \\ j \neq i}} |r_{i,m+j}|, \ (1 \leq i \leq n)$$

is equivalent to

$$\psi_{u_{m+i}}(u_{m+i}) > \sum_{\substack{1 \leq j \leq n \\ j \neq i}} |\psi_{u_{m+i}}(u_{m+j})|, \ (1 \leq i \leq n)$$

or

$$\psi_u(u) > \sum_{\substack{w \in V^1 \setminus V^0 \\ w \neq u}} |\psi_u(w)|, \quad \forall u \in V^1 \setminus V^0.$$

CHAPTER 4

Semi-prewavelets

We are going to establish the uniqueness of the semi-prewavelets for W^0 with regard to (3.1) and to find their coefficients. To simplify our calculation, Floater and Quak's result on inner products of nodal functions [5] , which we state here as a Lemma, will be used.

Lemma 1 Let $t(e)$ denote the number of triangles (one or two) in $\mathcal{T}^0$ containing the edge $e \in E^0$ and $t(v)$ the number of triangles (at least one) containing the vertex $v \in V^0$. If $v \in V^0$ and $w \in V^1$ are contained in the same triangle in $\mathcal{T}^0$ then

$$96\langle \phi_v^0, \phi_w^1 \rangle = \begin{cases} 6t(v), & if \ v = w; \\ 10t(e), & if \ w \text{ is the midpoint of } e; \\ t(e), & if \ e = [v, w]; \\ 4, & if \text{ otherwise.} \end{cases} \quad (4.1)$$

Let $\sigma_{a_1,u}$ be a semi-prewavelet where the fine vertex u is the midpoint of a_1 and another coarse vertex a_2. We call a_1 the *center* (vertex) of the semi-prewavelet. The *degree* of a vertex in a triangulation is the number of neighbor vertices of the vertex in that triangulation. Trivially, every coarse vertex is also a vertex in the fine triangulation and it has the same degree in both coarse and fine r-triangulations. Let $k = |V_{a_1}^0| = |V_{a_1}^1|$ be the degree of a_1. If a_1 is an interior vertex then $k = 6$. If a_1 is a boundary vertex then the value of k could range from 2 to 6. Hence there are six possible topological structures of the support of $\sigma_{a_1,u}$, which are identical to the

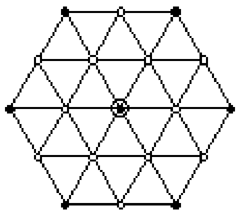

Figure 3: Interior vertex a_1 and its support

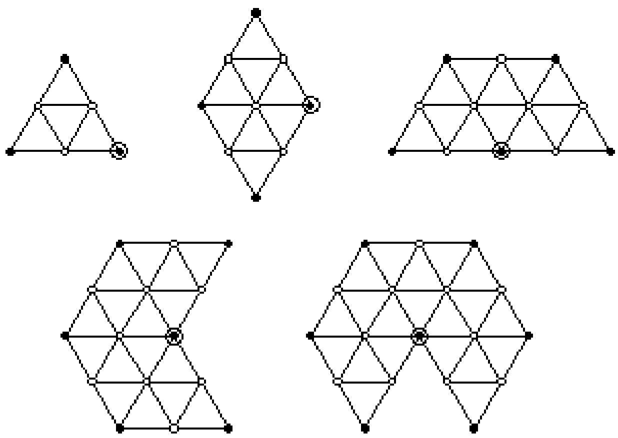

Figure 4: Boundary vertices and their supports

support $M^0_{a_1}$ of $\phi^0_{a_1}$.

For our convenience, in the later steps to construct semi-prewavelets, we shall use the following permutations of the vertices in the coarse neighborhood $N^0_{a_1}$ and the fine neighborhood $N^1_{a_1}$ of a_1:

$N^0_{a_1}$: $\quad v_1 = a_1, v_2, v_3, \cdots, v_k,$

$N^1_{a_1}$: $\quad u_1 = a_1, u_2, u_3, \cdots, u_k,$

where v_2 through v_k are all the neighbor vertices of a_1 in the coarse space labeled consecutively in the counterclockwise order, and u_j is the midpoint of the edge $[a_1, v_j]$ $(j = 2, 3, ...k)$.

Let us simply rewrite $\phi^l_{u_j}(x)$ as $\phi^l_j(x)$, where $j = 1, 2, ..., k$ and $l = 0, 1$. Thus,

$$\sigma_{a_1,u}(x) = \sum_{j=1}^{k} r_j \phi^1_j(x). \tag{4.2}$$

Let $A = (a_{ij})$ be the $k \times k$ matrix such that $a_{ij} = 96\langle \phi^0_i, \phi^1_j \rangle$. Then

$$\begin{aligned}
\langle \phi^0_i, \sigma_{v_1,u} \rangle &= \left\langle \phi^0_i, \sum_{j=1}^{k} r_j \phi^1_j \right\rangle \\
&= \sum_{j=1}^{k} r_j \langle \phi^0_i, \phi^1_j \rangle \\
&= \sum_{j=1}^{k} \frac{1}{96} a_{ij} r_j \\
&= \frac{1}{96} [a_{i1}, a_{i2}, \cdots, a_{ik}] \vec{r}
\end{aligned}$$

where

$$\vec{r} = [r_1, r_2, \cdots, r_k]^T.$$

Therefore, the "semi-orthogonal condition" (3.5) is equivalent to

$$A\vec{r} = \vec{b} \tag{4.3}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix},$$

$$\vec{r} = [r_1, r_2, \cdots, r_k]^T,$$

and

$$\vec{b} = [b_1 = -b, 0, ..., 0, b_j = b, 0, ...0]^T \text{ (here } j \text{ satisfying } u = u_j).$$

Thus, if A is invertible then (4.3) has a unique solution of the coefficients.

Figure 3 and Figure 4 give all the cases of the supports of possible semi-prewavelets up to a symmetric permutation.

Using **Lemma 1**, we can verify that A is invertible in every case. We choose the value of b in (3.5) as 66240 so that we can get integer coefficients r_j for all the semi-prewavelets, except the boundary one with the center vertex of degree 6.

Case 1 a_1 is an interior vertex, SPW(I6)

We obtain that

$$A = \begin{bmatrix} 36 & 20 & 20 & 20 & 20 & 20 & 20 \\ 2 & 20 & 4 & 0 & 0 & 0 & 4 \\ 2 & 4 & 20 & 4 & 0 & 0 & 0 \\ 2 & 0 & 4 & 20 & 4 & 0 & 0 \\ 2 & 0 & 0 & 4 & 20 & 4 & 0 \\ 2 & 0 & 0 & 0 & 4 & 20 & 4 \\ 2 & 4 & 0 & 0 & 0 & 4 & 20 \end{bmatrix}.$$

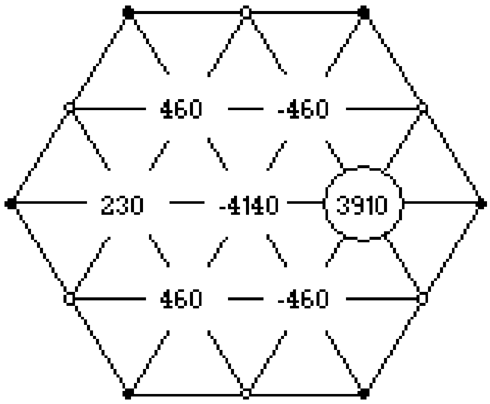

Figure 5: Semi-prewavelet SPW(I6)

A is non-singular with the inverse

$$A^{-1} = \frac{1}{1152}\begin{bmatrix} 42 & -30 & -30 & -30 & -30 & -30 & -30 \\ -3 & 65 & -11 & 5 & 1 & 5 & -11 \\ -3 & -11 & 65 & -11 & 5 & 1 & 5 \\ -3 & 5 & -11 & 65 & -11 & 5 & 1 \\ -3 & 1 & 5 & -11 & 65 & -11 & 5 \\ -3 & 5 & 1 & 5 & -11 & 65 & -11 \\ -3 & -11 & 5 & 1 & 5 & -11 & 65 \end{bmatrix}.$$

Let $\vec{b} = \begin{bmatrix} -b & b & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. Then

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -4140 & 3910 & -460 & 460 & 230 & 460 & -460 \end{bmatrix}^T.$$

Thus, a semi-prewavelet with its center a_1 as an interior vertex has been uniquely determined (Figure 5). Simply, turn its figure (Figure 5) around its center in the counterclockwise direction and step-by-step we can get all the other symmetric semi-prewavelets which share the same center a_1. We shall see this effect in the following

Figure 6: Semi-prewavelet SPW(B2)

case from another point of view.

Case 2 a_1 is a boundary vertex with degree 2, SPW(B2)

$$A = \begin{bmatrix} 6 & 10 & 10 \\ 1 & 10 & 4 \\ 1 & 4 & 10 \end{bmatrix}.$$

A is non-singular with the inverse

$$A^{-1} = \frac{1}{192} \begin{bmatrix} 42 & -30 & -30 \\ -3 & 25 & -7 \\ -3 & -7 & 25 \end{bmatrix}.$$

$$\vec{b} = \begin{bmatrix} -b & b & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -24840 & 9660 & -1380 \end{bmatrix}^T.$$

$$\vec{b} = \begin{bmatrix} -b & 0 & b \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -24840 & -1380 & 9660 \end{bmatrix}^T.$$

As we can see in Figure 5, the two subcases of SPW(B2) are symmetric. Note they are the same up to symmetry. In the following cases of other boundary semi-prewavelets we shall not mention this again:

Case 3 a_1 is a boundary vertex with degree 3, SPW(B3)

$$A = \begin{bmatrix} 12 & 10 & 20 & 10 \\ 1 & 10 & 4 & 0 \\ 2 & 4 & 20 & 4 \\ 1 & 0 & 4 & 10 \end{bmatrix}.$$

A is non-singular with the inverse

$$A^{-1} = \frac{1}{1920}\begin{bmatrix} 210 & -150 & -150 & -150 \\ -15 & 221 & -35 & 29 \\ -15 & -35 & 125 & -35 \\ -15 & 29 & -35 & 221 \end{bmatrix}.$$

$$\vec{b} = \begin{bmatrix} -b & b & 0 & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -12420 & 8142 & -690 & 1518 \end{bmatrix}^T.$$

$$\vec{b} = \begin{bmatrix} -b & 0 & b & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -12420 & -690 & 4830 & -690 \end{bmatrix}^T.$$

Case 4 a_1 is a boundary vertex with degree 4, SPW(B4)

$$A = \begin{bmatrix} 18 & 10 & 20 & 20 & 10 \\ 1 & 10 & 4 & 0 & 0 \\ 2 & 4 & 20 & 4 & 0 \\ 2 & 0 & 4 & 20 & 4 \\ 1 & 0 & 0 & 4 & 10 \end{bmatrix}.$$

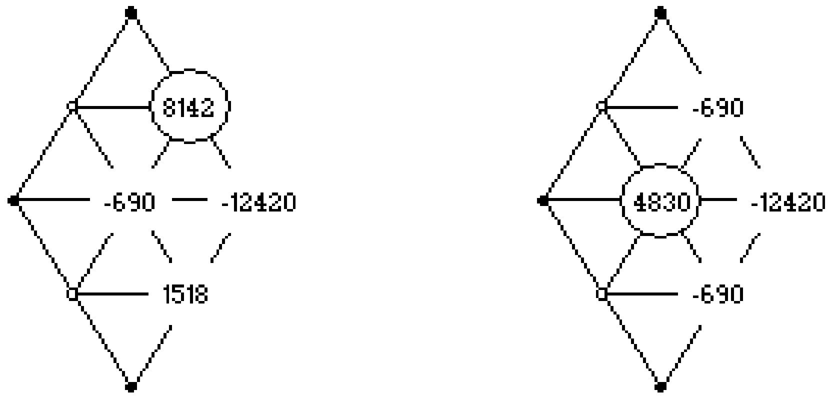

Figure 7: Semi-prewavelet SPW(B3[1]), SPW(B3[2])

A is non-singular with the inverse

$$A^{-1} = \frac{1}{576} \begin{bmatrix} 42 & -30 & -30 & -30 & -30 \\ -3 & 65 & -11 & 5 & 1 \\ -3 & -11 & 35 & -5 & 5 \\ -3 & 5 & -5 & 35 & -11 \\ -3 & 1 & 5 & -11 & 65 \end{bmatrix}.$$

$$\vec{b} = \begin{bmatrix} -b & b & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -8280 \\ 7820 \\ -920 \\ 920 \\ 460 \end{bmatrix}.$$

$$\vec{b} = \begin{bmatrix} -b & 0 & b & 0 & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -8280 \\ -920 \\ 4370 \\ -230 \\ 920 \end{bmatrix}.$$

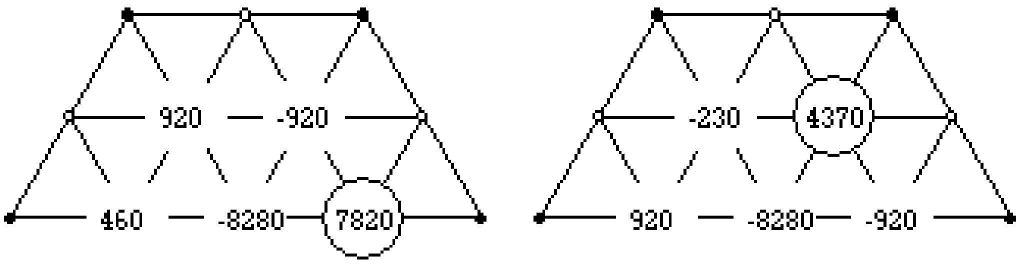

Figure 8: Semi-prewavelet SPW(B4[1]), SPW(B4[2])

Case 5 a_1 is a boundary vertex with degree 5, SPW(B5)

$$A = \begin{bmatrix} 24 & 10 & 20 & 20 & 20 & 10 \\ 1 & 10 & 4 & 0 & 0 & 0 \\ 2 & 4 & 20 & 4 & 0 & 0 \\ 2 & 0 & 4 & 20 & 4 & 0 \\ 2 & 0 & 0 & 4 & 20 & 4 \\ 1 & 0 & 0 & 0 & 4 & 10 \end{bmatrix}.$$

A is non-singular with the inverse

$$A^{-1} = \frac{1}{88320} \begin{bmatrix} 4830 & -3450 & -3450 & -3450 & -3450 & -3450 \\ -345 & 9883 & -1765 & 667 & 155 & 283 \\ -345 & -1765 & 5275 & -805 & 475 & 155 \\ -345 & 667 & -805 & 5083 & -805 & 667 \\ -345 & 155 & 475 & -805 & 5275 & -1765 \\ -345 & 283 & 155 & 667 & -1765 & 9883 \end{bmatrix}.$$

$$\vec{b} = \begin{bmatrix} -b & b & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -6210 & 7671 & -1065 & 759 & 375 & 471 \end{bmatrix}^T.$$

$$\vec{b} = \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \end{bmatrix}^T,$$

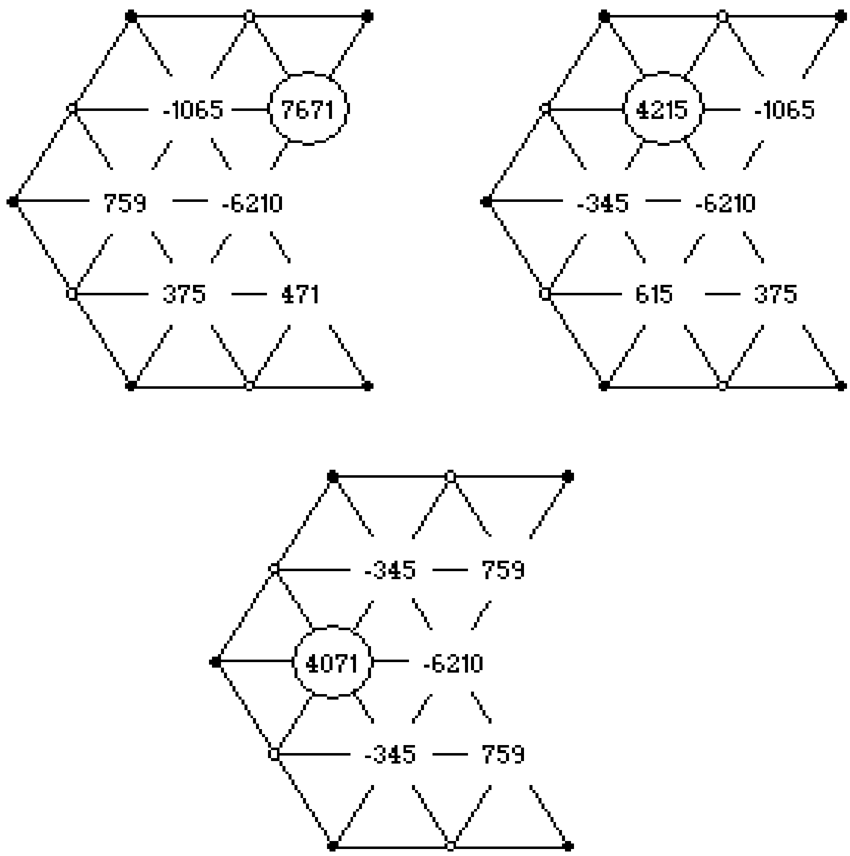

Figure 9: Semi-prewavelet SPW(B5[1]), SPW(B5[2]), and SPW(B5[3])

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -6210 & -1065 & 4215 & -345 & 615 & 375 \end{bmatrix}^T.$$

$$\vec{b} = \begin{bmatrix} -b & 0 & 0 & b & 0 & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \begin{bmatrix} -6210 & 759 & -345 & 4071 & -345 & 759 \end{bmatrix}^T.$$

Case 6 a_1 is a boundary vertex with degree 6, SPW(B6)

$$A = \begin{bmatrix} 30 & 10 & 20 & 20 & 20 & 20 & 10 \\ 1 & 10 & 4 & 0 & 0 & 0 & 0 \\ 2 & 4 & 20 & 4 & 0 & 0 & 0 \\ 2 & 0 & 4 & 20 & 4 & 0 & 0 \\ 2 & 0 & 0 & 4 & 20 & 4 & 0 \\ 2 & 0 & 0 & 0 & 4 & 20 & 4 \\ 1 & 0 & 0 & 0 & 0 & 4 & 10 \end{bmatrix}.$$

A is non-singular with the inverse

$$A^{-1} = \frac{1}{192} \begin{bmatrix} \frac{7}{160} & \frac{-1}{32} & \frac{-1}{32} & \frac{-1}{32} & \frac{-1}{32} & \frac{-1}{32} & \frac{-1}{32} \\ \frac{-1}{320} & \frac{11779}{105792} & \frac{-2173}{105792} & \frac{739}{105792} & \frac{131}{105792} & \frac{259}{105792} & \frac{227}{105792} \\ \frac{-1}{320} & \frac{-2173}{105792} & \frac{6259}{105792} & \frac{-1021}{105792} & \frac{499}{105792} & \frac{179}{105792} & \frac{259}{105792} \\ \frac{-1}{320} & \frac{739}{105792} & \frac{-1021}{105792} & \frac{6019}{105792} & \frac{-973}{105792} & \frac{499}{105792} & \frac{131}{105792} \\ \frac{-1}{320} & \frac{131}{105792} & \frac{499}{105792} & \frac{-973}{105792} & \frac{6019}{105792} & \frac{-1021}{105792} & \frac{739}{105792} \\ \frac{-1}{320} & \frac{259}{105792} & \frac{179}{105792} & \frac{499}{105792} & \frac{-1021}{105792} & \frac{6259}{105792} & \frac{-2173}{105792} \\ \frac{-1}{320} & \frac{227}{105792} & \frac{259}{105792} & \frac{131}{105792} & \frac{739}{105792} & \frac{-2173}{105792} & \frac{11779}{105792} \end{bmatrix}.$$

$$\vec{b} = \begin{bmatrix} -b & b & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \frac{276}{551} \begin{bmatrix} -9918 & 15137 & -2303 & 1337 & 577 & 737 & 697 \end{bmatrix}^T.$$

$$\vec{b} = \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \frac{276}{551} \begin{bmatrix} -9918 & -2303 & 8237 & -863 & 1037 & 637 & 737 \end{bmatrix}^T.$$

$$\vec{b} = \begin{bmatrix} -b & 0 & 0 & b & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\vec{r} = A^{-1}\vec{b} = \frac{276}{551} \left[\begin{array}{ccccccc} -9918 & 1337 & -863 & 7937 & -803 & 1037 & 577 \end{array} \right]^T.$$

Now we can get any possible prewavelets of an r-triangulation, since the above semi-prewavelets included all the possible semi-prewavelets with the exception of symmetric cases. By (3.5), to get a prewavelet, we need only to "sum" two semi-prewavelets together in such a way that the fine vertex u (which has been circled in each figure of the semi-prewavelet) is the midpoint of the centers of these two semi-prewavelets. Some prewavelets which could be often used are given by Figure 11 through Figure 17. In these figures we denote the prewavelet obtained by summing an interior semi-prewavelet, SPW(I6), and a boundary one, say SPW(Bj), by PW(I6, Bj). We denote the prewavelet obtained by summing two boundary semi-prewavelets, SPW(Bi) and SPW(Bj), by PW(Bi, Bj).

With the above results on semi-prewavelets, we are now ready to state our main result, a very useful theorem on r-triangulations.

Theorem 2. For any level of the refinements of any an r-triangulation, all the possible prewavelets can be constructed by simply summing up the two semi-prewavelets illustrated in Figure 5 through Figure 10.

Proof: Each semi-prewavelet $\sigma_{a_1,u}(x)$ illustrated in Figure 5 through Figure 10, which are all the possible cases of semi-prewavelets in r-triangulations, satisfies

$$\sigma_{a_1,u}(u) > \sum_{\substack{w \in V^1 \setminus V^0 \\ w \neq u}} |\sigma_{a_1,u}(w)|, \quad \forall u \in V^1 \setminus V^0.$$

A prewavelet is the sum of two semi-prewavelets, see (3.6). The sum can be done

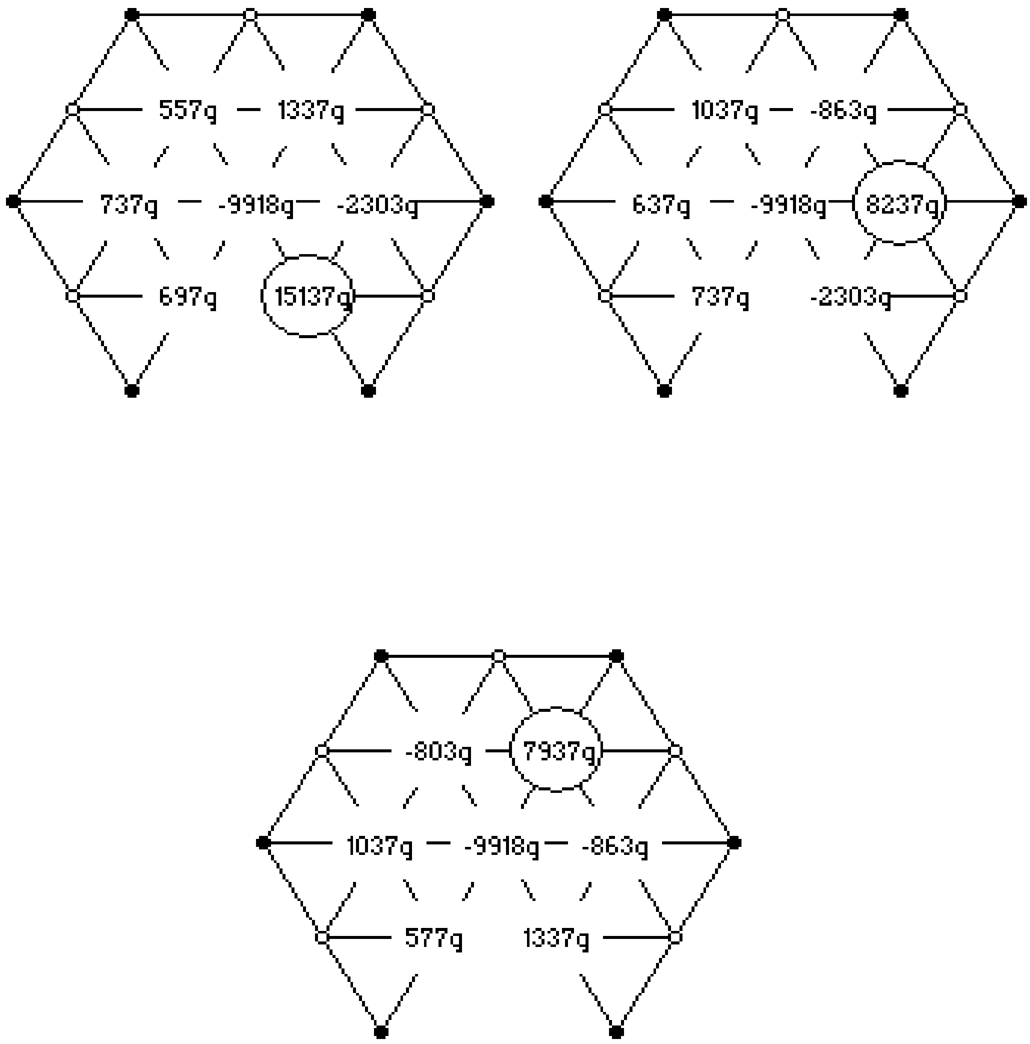

Figure 10: Semi-prewavelet SPW(B6[1]), SPW(B6[2]) and SPW(B6[3]) ($q = \frac{276}{551}$)

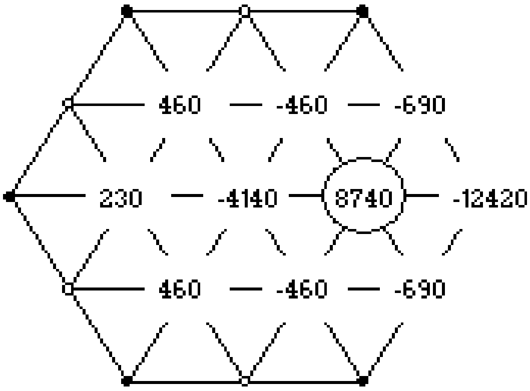

Figure 11: Pre-wavelet PW(I6,B3)

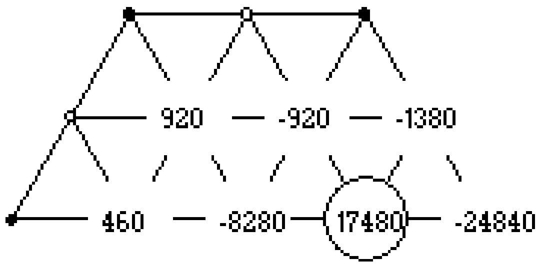

Figure 12: Pre-wavelet PW(B4,B2)

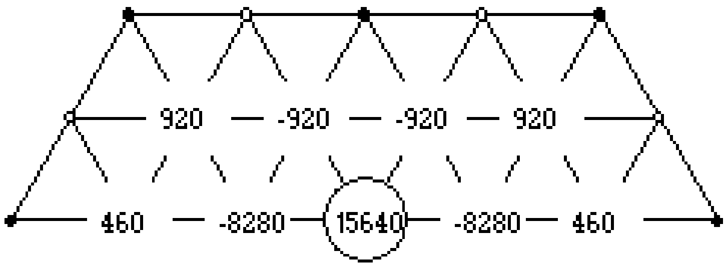

Figure 13: Pre-wavelet PW(B4,B4)

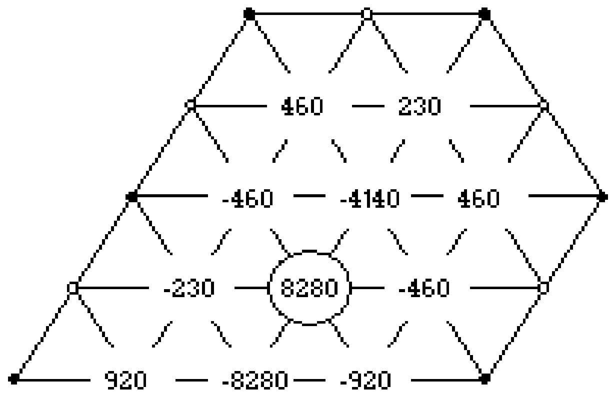

Figure 14: Pre-wavelet PW(I6,B4)

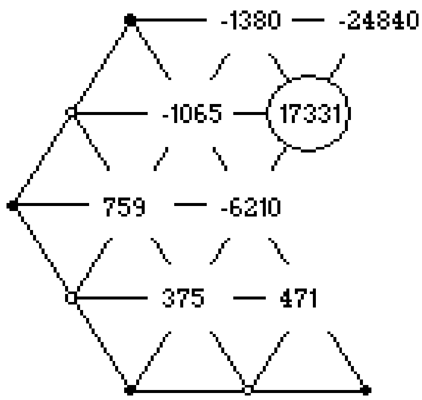

Figure 15: Pre-wavelet PW(B5,B2)

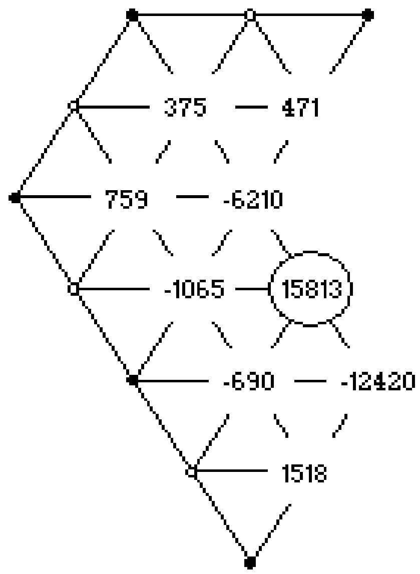

Figure 16: Pre-wavelet PW(B5,B3)

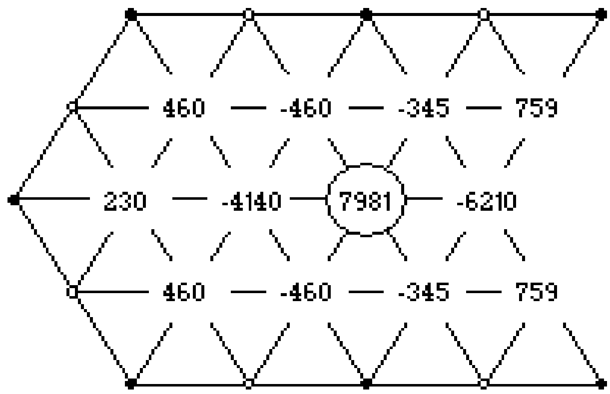

Figure 17: Pre-wavelet PW(I6,B5)

in such a way that the fine vertex u (which has been circled in each figure of semi-prewavelets) is the midpoint of the centers, a_1 and a_2, of these two semi-prewavelets in (3.6). Since the intersection of $N^1_{a_1}$ and $N^1_{a_2}$ is the single element set $\{u\}$, the only overlapped values are the values of the two semi-prewavelets functions at vertex u. Therefore, the condition of Theorem 1 in Chapter 3 is satisfied and this completes our proof.

CHAPTER 5

Examples

In this chapter, we would like to demonstrate that our Theorem 2 at the end of the last chapter can be used on various shaped domains. We can also see that all the prewavelets can be found easily by using this result.

Example 1. In the refined r-triangulation in Figure 18, there are only three types of semi-prewavelets, namely SWP(I6), SWP(B2), and SWP(B4), see Figure 5, Figure 6 and Figure 8, respectively. These semi-prewavelets can be found by simply checking if any figure in Figure 3 and Figure 4 is a subset of the given refined r-triangulation. Thus, prewavelets PW(I6,I6), PW(I6,B4) (see Figure 14), PW(B4,B4) (see Figure 13), and PW(B4,B2) (see Figure 12) are all the types of prewavelets of the fine space.

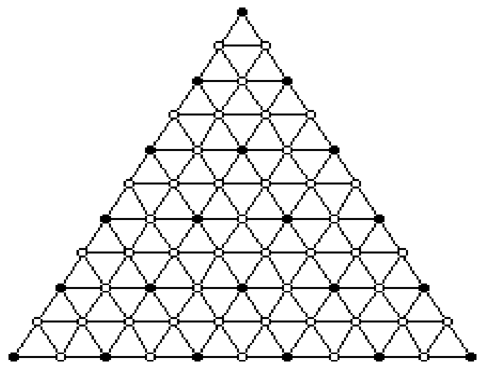

Figure 18: Sample r-Triangulation Domain 1

In the following examples we shall only point out all the types of semi-prewavelets occuring in the corresponding refinement and leave our readers to find out the types of prewavelets contained:

Example 2. In the refined r-triangulation in Figure 19, there are all the types of semi-prewavelets with exception of SPW(B6).

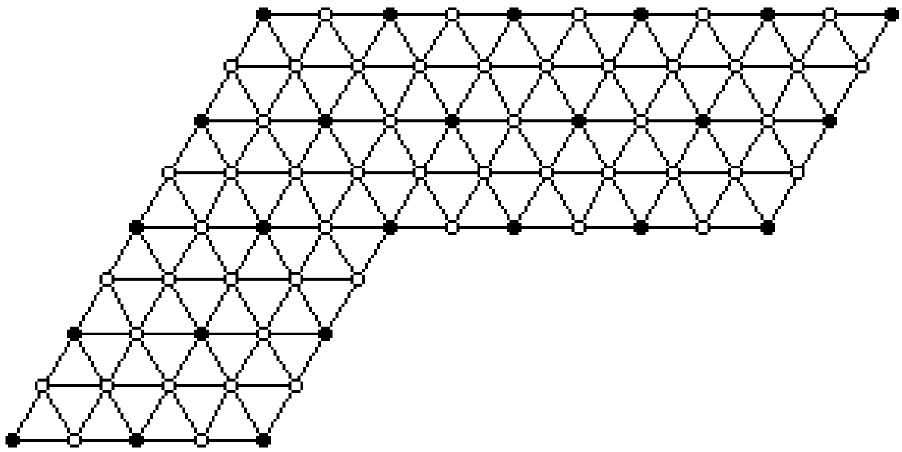

Figure 19: Sample r-Triangulation Domain 2

Example 3. In the refined r-triangulation in Figure 20, all the types of semi-prewavelets have occured. But, this does not mean that all the possible prewavelets in r-triangulations will be present in this case. For example, prewavelet type PW(B2,B2) does not occur.

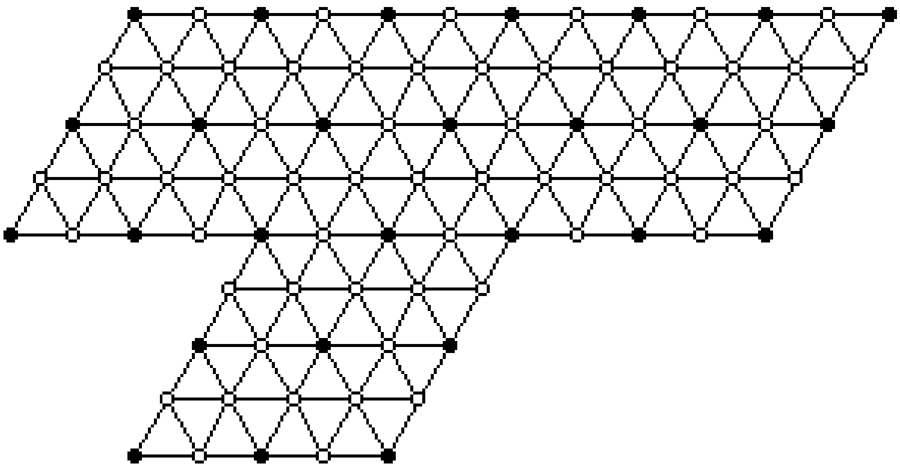

Figure 20: Sample r-Triangulation Domain 3

Although it is not necessary, rectangular shaped domain are often used for computer graphics. For this reason, one may be particularly interested in the following shaped region which can be obtained by adding some extra vertices on both the left and right sides of a rectangular area. In this way, all the refinements can be done uniformly and our theorems can be used.

Example 4. For the refined r-triangulation in Figure 21, the set of semi-prewavelets are the same as in Example 2 (Figure 19) although the two domains look so different.

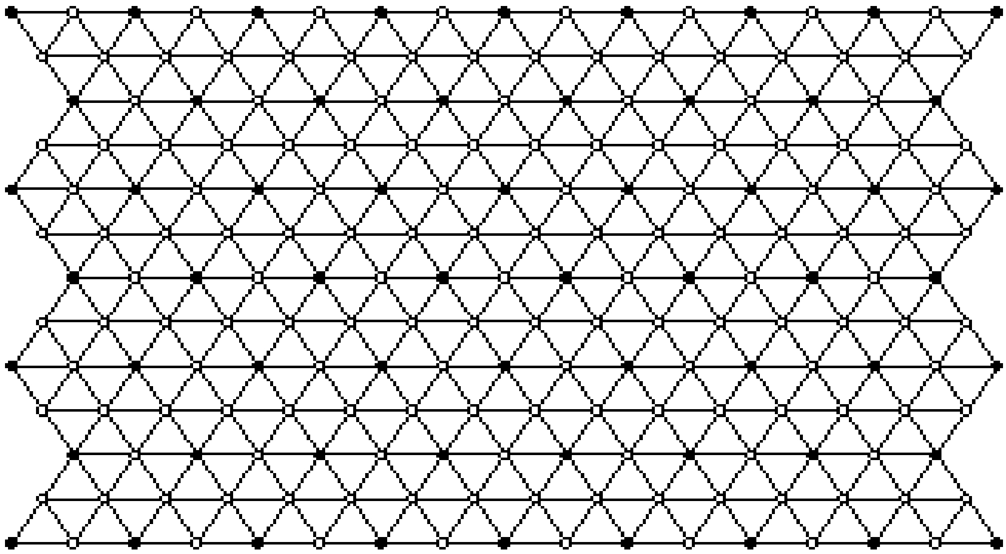

Figure 21: Sample r-Triangulation Domain 4

Remarks. Our results are valid even if the domain is not simply connected.

BIBLIOGRAPHY

[1] Chui, C. K., Multivariate Splines, *SIAM*, Philadelphia, 1988.

[2] Donovan, G.C., J.S. Geronimo, and D.P. Hardin, Compactly supported, piecewise affine scaling functions on triangulations, *Constructive Approximation* **16** (2000), 201–219.

[3] Hong, D. and Schumaker, L.L., Surface compression using hierarchical bases for bivariate C^1 cubic splines, preprint, 2002.

[4] Hong, D. and Mu, Y.A., On construction of minimum supported piecewise linear prewavelets over triangulations, In: "Wavelets and Multiresolution Methods", (T.X. He ed.), pp. 181–201, Marcel Decker Pub., New York, 2000.

[5] Floater, M. S. and Quak, E. G., Piecewise linear prewavelets on arbitrary triangula tions, *Numer. Math.* **82** (1999), 221–252.

[6] Floater, M. S. and Quak, E. G., A semiprewavelet approach to piecewise linear prewavelets on triangulations, in Approximation Theory IX, Vol 2: Computational Aspects, C. K. Chui and L. L. Schumaker (eds.), Vanderbilt University Press, Nashville, (1998), 63–70.

[7] Floater, M. S. and Quak, E. G., Linear independence and stability of piecewise linear prewavelets on arbitrary triangulations, to appear in SIAM J. Numer. Anal.

[8] Kotyczka, U. and Oswald P., Piecewise linear prewavelets of small support, in Approximation Theory VIII, Vol. 2, C. K. Chui and L. L. Schumaker, eds., World Scientific, Singapore, 1995, 235–242.

VITA

QINGBO XUE

Education: Zhongshan University, Guangzhou, China
Mathematics, B.S., 1986
East Tennessee State University, Johnson City, Tennessee
Mathematics, M.S., 2002

Professional Experience: Teacher, North China University of Technology;
Beijing, China, 1986-2000
Graduate Assistant, East Tennessee State University, College of Arts and Sciences, 2000-2002
President of Actuarial Students Association,
ETSU, Johnson City, Tennessee, 2001-2002

Publications: Xue, Qingbo (1995). "(C4, Lotus)-free Berge Graphs Are Perfect".
Analele Stiintifice Ale Universitatii "AL.I.CUZA",
Iasi, Romania. pp.65-71
Xue, Qingbo (1996). "On a Class of Square-free Graphs",
Information Processing Letters, Netherlands, 57 pp.47-48

Honors and Awards: Outstanding Scholastic Achievement Award,
East Tennessee State University, 2000-2002