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Construction of Piecewise Linear Wavelets.

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Construction of Piecewise Linear Wavelets

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ABSTRACT

Construction of Piecewise Linear Wavelets

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It is well known that in many areas of computational mathematics, wavelet based algorithms are becoming popular for modeling and analyzing data and for providing efficient means for hierarchical data decomposition of function spaces into mutually orthogonal wavelet spaces. Wavelet construction in more than one-dimensional setting is a very challenging and important research topic. In this thesis, we first introduce the method of constructing wavelets by using semi-wavelets. Second, we construct piecewise linear wavelets with smaller support over type-2 triangulations. Then, parameterized wavelets are constructed using the orthogonality conditions.

DEDICATION

This thesis is dedicated to Lianyun Song, my wife, and XiangKun Cao, my son, who have supported my efforts to complete my graduate degree. Thanks for all your love and support.

ACKNOWLEDGEMENTS

A special thanks to my thesis advisor, Dr. Don Hong, who has been patient with me through the entire process. And a word of thanks to the rest of my committee who has graciously given their time to support my thesis. And a word of thanks to all faculty, staff and fellow graduate students in the Department of Mathematics who helped and taught me in these two years.

Contents

CHAPTER 1

Introduction

It is well known that wavelets have become an important tool of mathematical analysis with a wide range of applications in mathematical physics, computational mathematics, image compressing, detecting self-similarity in a time series and musical tones, to mention a few (see [6,16,17,19,21]).

The first mention of wavelets appeared in an appendix to the thesis of A. Haar (1909). One property of the Haar wavelet is that it has compact support, which means that it vanishes outside of a finite interval. Unfortunately, Haar wavelets are not continuously differentiable which somewhat limits their applications. In the 1930s, several groups working independently researched the representation of functions using scale-varying basis functions. Understanding the concepts of basis functions and scale-varying basis functions is key to understanding wavelets.

Between 1960 and 1980, the mathematicians Guido Weiss and Ronald R. Coifman studied the simplest elements of a function space, called atoms, with the goal of finding the atoms for a common function and the "assembly rules" that allow the reconstruction of all the elements of the function space using these atoms. In 1980, Grossman and Morlet, a physicist and an engineer, broadly defined wavelets in the context of quantum physics. These two researchers provided a way of thinking of wavelets based on physical intuition.

In 1985, Stephane Mallat gave wavelets an additional jump-start through his work in digital signal processing. He discovered some relationships between quadrature mirror filters, pyramid algorithms, and orthonormal wavelet bases. Inspired in part by these results, Y. Meyer constructed the first non-trivial wavelets. Unlike the Haar wavelets, the Meyer wavelets are continuously differentiable; however they do not have compact support. A couple of years later, Ingrid Daubechies used Mallat's work to construct a set of wavelet orthonormal basis functions that are perhaps the most elegant, and have become the cornerstone of wavelet applications today (see [5]). Almost instantaneously it became a success story with thousands of papers written by now with wide ranging applications. Charles K. Chui and Jianzhong Wang used splines to construct wavelets and opened a channel to construct smooth wavelets with short support by using spline functions (see [3]).

It is much more challenging to construct wavelets in a higher dimensional setting. In fact, even the case of continuous piecewise linear wavelet construction is unexpectedly complicated (see [7,9-12,4-15]) and the references therein. However, most real application problems are multivariate or multiparameter in nature. There creating great demand to study multivariate wavelets. In recent years, many researchers studied wavelets over triangulations (see [7–15, 25] and references therein).

In this thesis, we emphasize the construction of piecewise linear prewavelets with smaller support over type-2 triangulations. The thesis is organized as follows. In Chapter 2, we introduce the concept of multiresolution over type-2 triangulations. In Chapter 3, wavelets over type-2 triangulations are constructed by using semi-wavelets. In Chapter 4, the smaller support wavelets are constructed. In Chapter 5, parameterized wavelets are constructed by using the orthogonality conditions.

CHAPTER 2

Multiresolutions for Type-2 Triangulations

The two diagonals of each square $S_{ij} = [i, i+1] \times [j, j+1] \quad for \quad i, j \in Z$ in the plane, divide the square into four congruent triangles. Following convention, we will refer to the set of all such triangles as a type-2 triangulation. We also refer to any subtriangulation as a type-2 triangulation and we will be concerned with the bounded subtriangulation τ^0, generated by the square S_{ij} for $i = 0, 1, 2, ..., m-1$ and $j = 0, 1, ..., n-1$, for some arbitrary m, n, see Figure 2.1. Throughout this thesis we will assume, for the sake of simplicity, that $m \geq 2$ and $n \geq 2$, though wavelet constructions can be made in a similar way when either $m = 1$ or $n = 1$ (or both). We let V^0 and E^0 denote the vertices and edges respectively in τ^0, so that

$$V^0 = \{(i,j)\}_{i=0,\cdots,m,\ j=0,\cdots,n} \cup \{(i+\frac{1}{2}, j+\frac{1}{2})\}_{i=0,\cdots,m-1,\ j=0,\cdots,n-1}$$

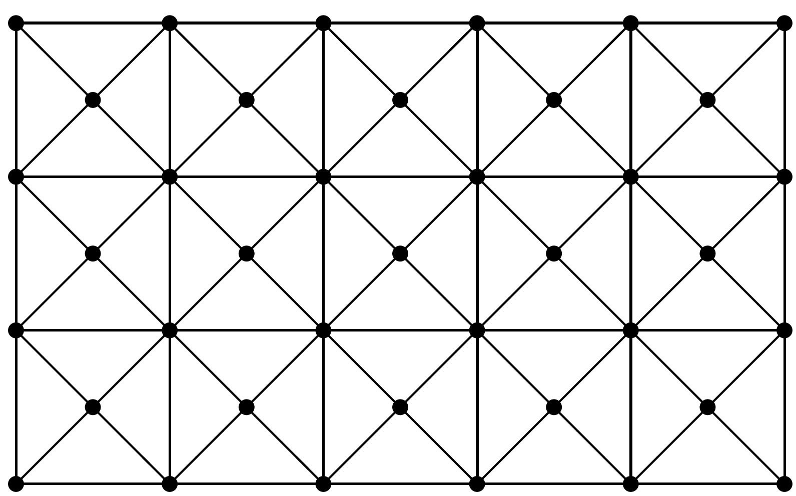


Figure 2.1 A type-2 triangulations.

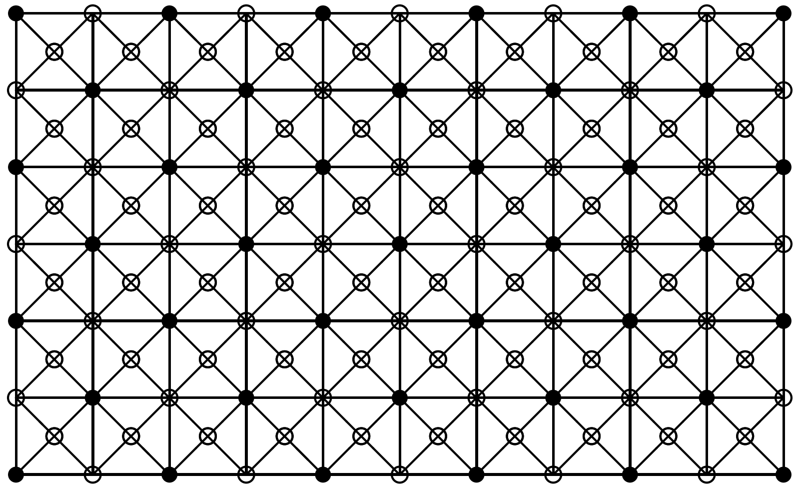


Figure 2.2 The first refinement.

Let $S^0 = S_1^0(\tau^0)$ be the linear space of continuous functions over τ_0 which are linear over every triangle. A basis for S^0 is given by the nodal functions ϕ_v^0 in S^0, for $v \in V^0$, satisfying $\phi_v^0(w) = \delta_{vw}$. The support of $\phi_{i+1/2,j+1/2}$ is the square S_{ij}, while the support of ϕ_{ij}^0 is the diamond enclosed by the polygon with vertices $(i-1,j)$,$(i,j-1)$,$(i+1,j)$,$(i,j+1)$, suitably truncated if the point (i,j) lies on the boundary of the domain $D = [0,m] \times [0,n]$.

Next consider the refined triangulation τ^1, also of type-2, formed by adding lines in the four directions halfway between each pair of existing parallel lines, as in Figure 2.2, and define V^1, E^1, the linear space S^1, and the basis ϕ_u^1, $u \in V^1$ accordingly. Then S^0 is subspace of S^1 and a refinement equation relates the coarse nodal functions ϕ_v^0 to the fine ones ϕ_v^1. In order to formulate this equation we define

$$V_v^0 = \{w \in V^0 : \ w \text{and } v \text{ are neighbours } \ in \ \ V^0\},$$

and

$$V_v^1 = \{u = (w+v)/2 \in V^1 : \ w \in V_v^0\}$$

Thus V_v^0 is the set of neighbors of v in V_v^1 is the set of midpoints between v and its coarse neighbors. For example when v is an interior vertex, there are two cases:

$$V^1_{i+1/2,j+1/2} = \{(i+1/4, j+1/4), (i+3/4, j+1/4), (i+3/4, j+3/4), (i+1/4, j+3/4)\},$$

and

$$V^1_{i,j} = \{(i+1/2, j), (i+1/4, j+1/4), (i, j+1/2), (i-1/4, j+1/4),$$
$$(i-1/2, j), (i-1/4, j-1/4), (i, j-1/2), (i+1/4, j-1/4)\}.$$

Then the refinement equation is easily seen to be

$$\phi^0_v = \phi^1_v + \frac{1}{2} \sum_{u \in V^1_v} \phi^1_u(x)$$

Let W^0 be the orthogonal complement space, then $S^1 = S^0 \oplus W^0$, treating S^0 and S^1 as Hilbert spaces equipped with the inner product

$$\langle f, g \rangle = \int_D f(x) g(x) dx, \qquad f,\ g \in L_2(D)$$

Ideally we would like a basis of functions with small support for the purpose of conveniently representing the decomposition of a given function f^1 in S^1 into its two unique components $f^0 \in S^0$ and $g^0 \in W^0$;

$$f^1 = f^0 \oplus g^0.$$

We will call any basis functions *wavelets*. Clearly the refinement of τ^0 can be continued indefinitely, generating a nested sequence

$$S^0 \subset S^1 \subset \cdots S^k \subset \cdots,$$

and if we define the wavelet space W^{k-1} to be the orthogonal complement at every refinement level k, then

$$S^k = S^{k-1} \oplus W^{k-1}.$$

We obtain the decomposition

$$S^n = S^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^{n-1},$$

for any $n \geq 1$. By combining wavelet bases for the spaces W^k with the nodal bases for the spaces S^k , we obtain the framework for a multiresolution analysis (MRA). Note that the basis elements of any W^k can simply be taken to be a dilation of the basis elements for W^0 and therefore we restrict our study purely to W^0.

We classify these seven different structure wavelets over type-2 triangulations into three types —- interior wavelet, boundary wavelet, and corner wavelet according to the position of $u \in V^1 \setminus V^0$.

Interior wavelet: If the two neighbor vertices of $u \in V^1 \setminus V^0$ in V^0 are the interior vertex (not on boundary and corner), then this wavelet is called an interior wavelet.

Boundary wavelet: If at least one of the two neighbor vertices of $u \in V^1 \setminus V^0$ in V^0 is the boundary vertices (not in the corner), then this wavelet is called a boundary wavelet.

Corner wavelet: If one of the two neighbor vertices of $u \in V^1 \setminus V^0$ in V^0 is the corner vertex, then this wavelet is called a corner wavelet.

CHAPTER 3

Semi-wavelets and Wavelets

In this chapter, we will introduce the method to construct wavelets using semi-wavelet ideas. That is, our approach to constructing wavelets for the wavelet space W^0 is to sum the pairs of semi-wavelets, elements of the finite space which have small support and are close to being in the wavelet space, that they are orthogonal to all but two of the nodal functions in the coarse space.

Letting v_1 and v_2 be two neighboring vertices in V^0, and denoting by $u \in V^1 \setminus V^0$ their midpoint, we define the $semi-wavelet$ $\sigma_{v_1,u} \in S^1$ as the element with support contained in the support of $\phi^0_{v_1}$ and having the property that, for all $v \in V^0$,

$$\langle \sigma_{v_1,u}, \phi^0_v \rangle = \begin{cases} -1 \text{ if } v = v_1; \\ 1 \text{ if } v = v_2; \\ 0 \text{ otherwise.} \end{cases} \quad (3.1)$$

Thus $\sigma_{v_1,u}$ has the form

$$\sigma_{v_1,u}(x) = \sum_{v \in N_{v_1^1}} a_v \phi^1_v(x)$$

where

$$N_{v_1^1} = \{v_1\} \cup V^1_{v_1}$$

denotes the fine neighborhood of v_1. The only non-trivial inner products between $\sigma_{v_1,u}$ and coarse nodal functions ϕ^0_v occur when v belongs to the coarse neighborhood of v_1,

$$N_{v_1^0} = \{v_1\} \cup V_{v_1}^0.$$

Thus the number of coefficients and conditions are the same and, as we will subsequently establish, the element $\sigma_{v_1,u}$ is unique.

Since the dimension of W^0 is equal to the number of fine vertices in V^1, i.e. $|V^1| - |V^0|$, it is natural to associate one wavelet ψ_u per fine vertex $u \in V^1 \setminus V^0$. Since each u is the midpoint of some edge in E^0 connecting two coarse vertices v_1 and v_2 in V^0, the element of S^1,

$$\psi_u = \sigma_{v_1,u} + \sigma_{v_2,u} \quad (3.2)$$

is a wavelet since it is orthogonal to all nodal functions $\phi_v^0,\ v \in V^0$.

First we construct the interior semi-wavelets by using the definition of semi-wavelets. Initially we consider only interior vertices v_1 and there are two cases: (1) $v_1 = (i+1/2, j+1/2)$ and (2) $v_1 = (i,j)$. Firstly, if $v_1 = (i+1/2, j+1/2)$, then $\sigma_{v_1,u}$ has support contained in S_{ij} and its fine and coarse neighborhoods are

$$N_{v_1}^1 = \{(i+1/2, j+1/2), (i+1/4, j+1/4),$$

$$(i+3/4, j+1/4), (i+3/4, j+3/4), (i+1/4, j+3/4)\}, \quad (3.3)$$

and

$$N_{v_1}^0 = \{(i+1/2, j+1/2), (i,j), (i+1,j), (i+1,j+1), (i,j+1)\} \quad (3.4)$$

Thus there are five coefficients and five constraints imposed by the definition of semi-wavelet and we must solve the linear system

$$Ax = b \quad (3.5)$$

where

$$A = (\langle \phi_v^0, \phi_w^1 \rangle)_{v \in N_{v_1}^0, w \in N_{v_1}^1}$$

$$b = \begin{cases} (-1,1,0,0,0)^T & \text{if } v_2 = (i,j); \\ (-1,0,1,0,0)^T & \text{if } v_2 = (i+1,j); \\ (-1,0,0,1,0)^T & \text{if } v_2 = (i+1,j+1); \\ (-1,0,0,0,1)^T & \text{if } v_2 = (i,j+1); \end{cases}$$

and the ordering of the vertices in $N_{v_1^0}$ and $N_{v_1^1}$ is the same in (2.1) and (2.2). Due to the symmetry, we can simply assume that $b = (-1,1,0,0,0)^T$ and the coefficients of the remaining three semi-wavelets are the same but rotated appropriately around v_1. In order to compute the entries in the 5×5 matrix A. We apply the following standard lemma.

Lemma 3.1 *Let $T = [x_1, x_2, x_3]$ be a triangle and let $f, g : T \to R$ be two linear functions. If $f_i = f(x_i)$ and $g_i = g(x_i)$ for $i = 1,2,3$, and a(T) is the area of the triangle T, then*

$$\int_T f(x)g(x)dx = \frac{a(T)}{12}(f_1g_1 + f_2g_2 + f_3g_3 + (f_1+f_2+f_3)(g_1+g_2+g_3)).$$

Using this lemma, and the fact that

$$< f, g >= \sum_{T \in \tau^1} \int_T f(x)g(x)dx$$

for any f and g in S^1, one can compute the entries $\langle \phi_v^0, \phi_w^0 \rangle$ of A and one finds that

$$A = \frac{1}{192}\begin{bmatrix} 20 & 6 & 6 & 6 & 6 \\ 3 & 8 & 1 & 0 & 1 \\ 3 & 1 & 8 & 1 & 0 \\ 3 & 0 & 1 & 8 & 1 \\ 3 & 1 & 0 & 1 & 8 \end{bmatrix}.$$

Thus the vector of coefficients of $\sigma_{v1,u}$ is given by

$$x = A^{-1}b = \frac{1}{2}(-48, 64, 8, 16, 8)^T$$

The coefficients are shown in Figure 3.1 after multiplying them by a factor of 2 (the same scaling will be applied to all later semi-wavelet coefficients). The vertex v_1 is in the center of the figure (the only coarse vertex where $\sigma_{v_1,u}$ is non-zero) and the fine vertex $u = (v_1 + v_2)/2$.

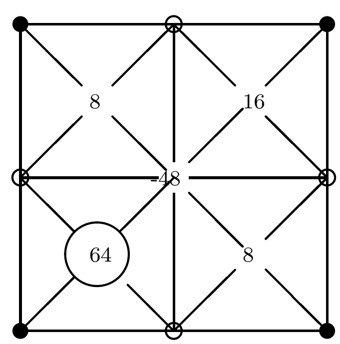


Figure 3.1 First interior semi-wavelet.

In case (2), we suppose that $v_1 = (i, j)$, whose fine neighborhood is

$$N_{v_1}^1 = \{(i,j), (i+1/2, j), (i+1/4, j+1/4), (i, j+1/2), (i-1/4, j+1/4),$$

$$(i-1/2, j), (i-1/4, j-1/4), (i, j-1/2), (i+1/4, j-1/4)\},$$

and whose coarse neighborhood is

$$N^0_{v_1} = \{(i,j),(i+1,j),(i+1/2,j+1/2),(i,j+1),(i-1/2,j+1/2),$$

$$(i-1,j),(i-1/2,j-1/2),(i,j-1),(i+1/2,j-1/2)\},$$

Thus we again solve the linear system (3.5) where A is this time a 9×9 matrix and b is either

$$(-1,1,0,0,0,0,0,0)^T \quad \text{or} \quad (-1,0,1,0,0,0,0,0)^T$$

depending on whether $v_2 = (i+1/2,j)$ or $v_2 = (i+1/4,j+1/4)$ and the remaining six possible coarse neighbors v_2 lead to the same coefficients , only rotated. Applying Lemma 3.1 we find after some calculation that

$$A = \frac{1}{192}\begin{bmatrix} 24 & 12 & 8 & 12 & 8 & 12 & 8 & 12 & 8 \\ 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 4 & 6 & 4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 6 \end{bmatrix}.$$

Thus if $b = (-1,1,0,0,0,0,0,0,0)^T$, we find that

$$x = A^{-1}b = \frac{1}{2}(-24,38,-24,4,0,2,0,4,-24)^T,$$

and if $b = (-1,0,1,0,0,0,0,0,0)^T$, we find that

$$x = A^{-1}b = \frac{1}{2}(-48,-3,76,-3,8,3,4,3,8)^T.$$

These two semi-wavelets are illustrated in Figures 3.2 and 3.3. Using the three interior semi-wavelets of Figure 3.1, 3.2 and 3.3 provides us with two wavelets ψ_u, from (3.2). The first of these, in Figure 3.4, is the sum of two semi-wavelets from 3.2 and the second, in Figure 3.4, the sum of the semi-wavelets in Figures 3.1 and 3.3. Symmetries and rotations of these two give us all interior wavelets ψ_u in the sense that v_1 and v_2 are both interior vertices of τ^0.

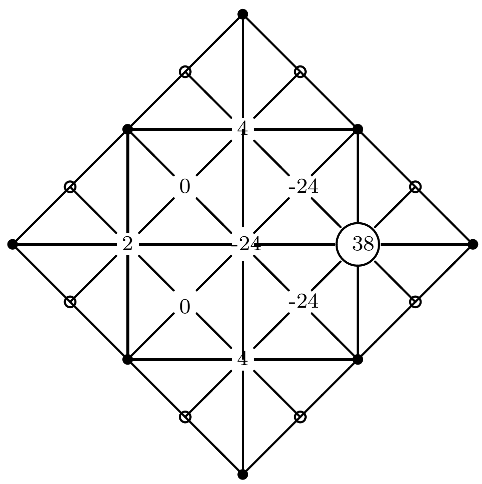


Figure 3.2. Second interior semi-wavelet.

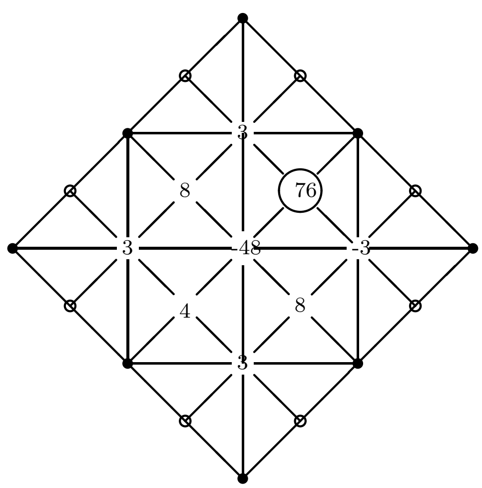


Figure 3.3. Third interior semi-wavelet.

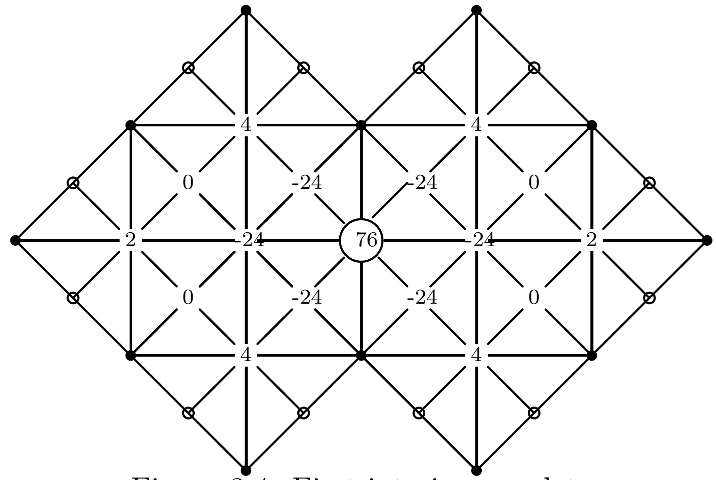


Figure 3.4. First interior wavelet.

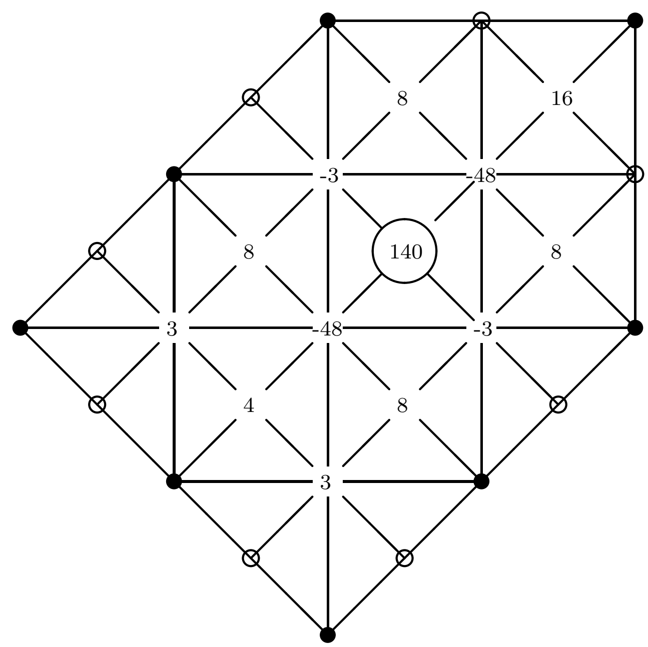


Figure 3.5. Second interior wavelet.

Now we consider the case where v_1 is a boundary vertex, which means that $v_1 = (i, j)$. Let us suppose first that v_1 lies on an edge of the domain, but not at one of the four corners, thus we assume without loss of generality that $j = 0$ and $0 < i < m$. The coarse and fine neighborhoods of v_1 are then

$$N_{v_1}^1 = \{(i,0), (i+1,0), (i+1/2,1/2), (i,1), (i-1/2,1/2), (i-1,0),$$

and

$$N_{v_1}^0 = \{(i,0),(i+1/2,0),(i+1/4,j+1/4),(i,1/2),(i-1/4,1/4),(i-1/2,0),$$

respectively, and the matrix A has dimension 6×6. From Lemma 3.1 we find that

$$A = \frac{1}{192}\begin{bmatrix} 12 & 6 & 8 & 12 & 8 & 6 \\ 1/2 & 6 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 0 & 1 & 12 & 1 & 0 \\ 1 & 0 & 0 & 4 & 6 & 4 \\ 1/2 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}.$$

If $v_2 = (i+1,j)$, then $b = (-1,1,0,0,0,0)^T$ and $x = A^{-1}b = \frac{1}{2}(-48,76,-48,8,0,4)^T$.

If $v_2 = (i+1/2,j+1/2)$, then $b = (-1,0,1,0,0,0)^T$ and $x = A^{-1}b = \frac{1}{2}(-96,-6,84,0,12,6)^T$.

If $v_2 = (i,j+1)$ then $b = (-1,0,0,1,0,0)^T$ and $x = A^{-1}b = \frac{1}{2}(-48,8,-24,40,-24,8)^T$.

These three elements are shown in Figures 3.6, 3.7, and 3.8. Summing two of the first boundary semi-wavelets gives us the boundary wavelet ψ_u shown in Figure 3.9. Summing the second boundary semi-wavelet and the first interior semi-wavelet gives us the wavelet ψ_u shown in Figure 3.10. Finally, summing the third edge semi-wavelet and the second interior semi-wavelet gives us the edge wavelet ψ_u shown in Figure 3.11. Up to rotation and symmetries these elements provide all wavelets ψ_u for which one of v_1 and v_2 is an interior vertex while the other one lies on the boundary but not at a corner.

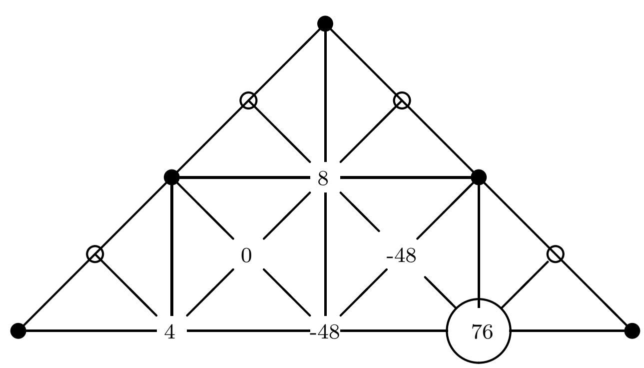


Figure 3.6. First boundary semi-wavelet.

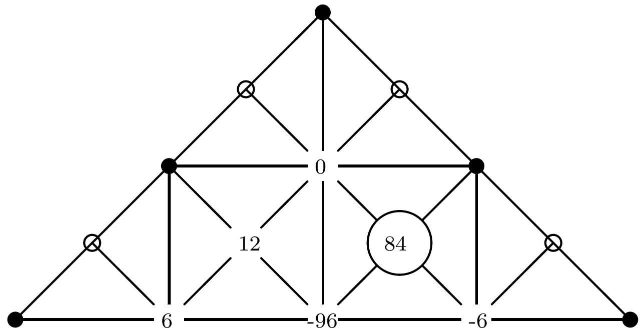


Figure 3.7. Second boundary semi-wavelet.

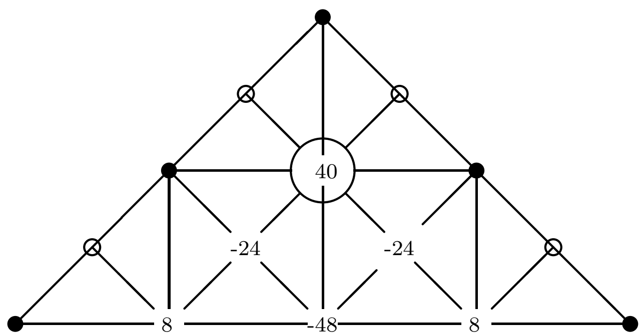


Figure 3.8. Third boundary semi-wavelet.

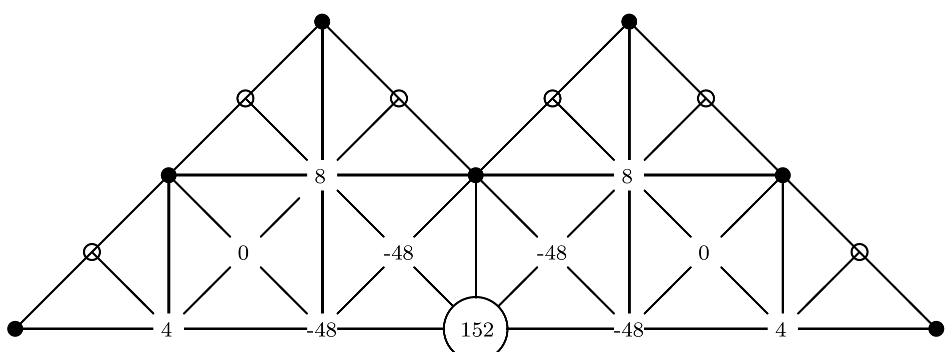


Figure 3.9. First boundary wavelet

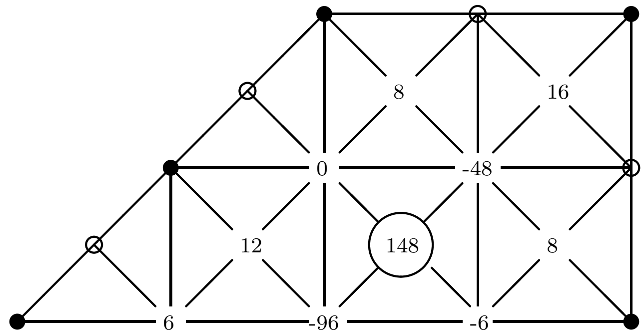


Figure 3.10. Second boundary wavelet.

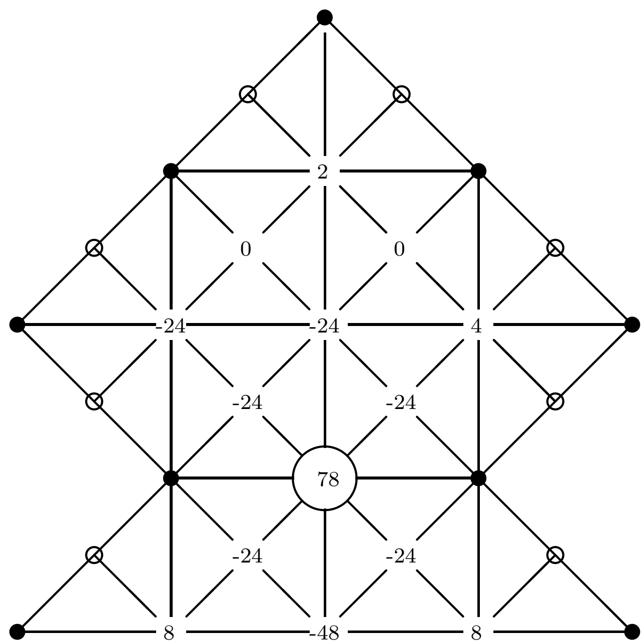


Figure 3.11. Third boundary wavelet.

In the case that v_1 is one of the four corners of the domain, we may suppose without loss of generality that $v_1 = (0, 0)$. The coarse and fine neighborhoods of v_1 are then

$$N^0_{v_1} = \{(0,0), (1,0), (1/2, 1/2), (0,1)\},$$

and

$$N_{v_1}^0 = \{(0,0),(1/2,0),(1/4,1/4),(0,1/2)\},$$

and the matrix A has dimension 4×4, specifically,

$$A = \frac{1}{192}\begin{bmatrix} 6 & 6 & 8 & 6 \\ 1/2 & 6 & 1 & 0 \\ 1 & 4 & 6 & 4 \\ 1/2 & 0 & 1 & 6 \end{bmatrix}.$$

There are only two cases, up to symmetry: if $v_2 = (1,0)$ then $b = (-1,1,0,0)^T$ and $x = A^{-1}b = \frac{1}{2}(-96,80,-48,16)^T$; while if $v_2 = (1/2,1/2)$ then $b = (-1,0,1,0)^T$ and $x = A^{-1}b = \frac{1}{2}(-192,0,96,0)^T$; the two semi-wavelets are shown in Figures 3.12, and 3.13. Summing the first corner semi-wavelet and the first boundary semi-wavelet yields the wavelet in Figure 3.14 and summing the second corner semi-wavelet and the first interior semi-wavelet yields the wavelet in Figure 3.15. Symmetries and rotations of these give us all remaining wavelets ψ_u.

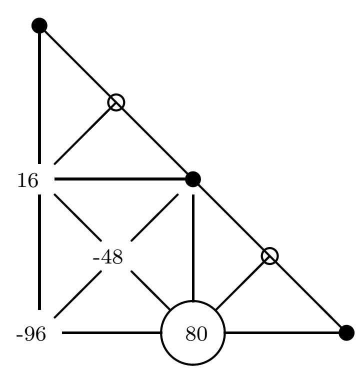


Figure 3.12. First corner semi-wavelet.

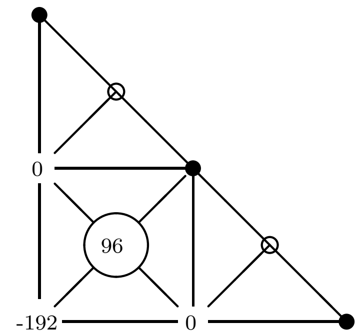


Figure 3.13. Second corner semi-wavelet.

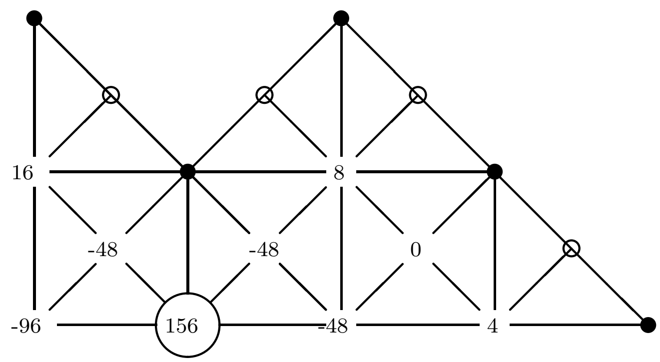


Figure 3.14. First corner wavelet.

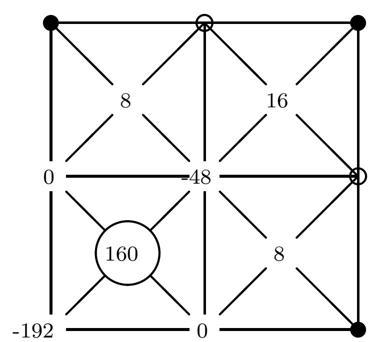


Figure 3.15. Second corner wavelet.

In order to prove that the above wavelets consist of wavelet basis, we introduce the following Lemma.

Lemma 3.2 *A set of wavelets* $\Psi = (\psi_{u_1}, \psi_{u_2}, \cdots, \psi_{u_n})$ *in* W^0 *is a basis of* W^0 *if the matrix* $Q = (q_{u_i,u_j})_{i,j}$ *is nonsingular, where* $q_{u_i,u_j} = \psi_{u_j}(u_i)$.

Proof. Given a linear combination $\sum_{j=1}^{n} c_j \psi_{u_j}$, which is identically zero, evaluation at u_i yields

$$\sum_{j=1}^{n} c_j \psi_{u_j}(u_i) = \sum_{j=1}^{n} c_j q_{u_i,u_j} = 0$$

and so $Qc = 0$ where $c = (c_1, c_2, c_3, \cdots, c_n)^T$, Therefore $c = 0$.

In the following, we will prove that the wavelets defined by (3.2) can consist of the wavelet basis.

Theorem 3.3 *The set of wavelets* $\{\phi_u\}_{u \in V^1 \setminus V^0}$ *defined by (3.2) is a basis for the wavelet space* W^0.

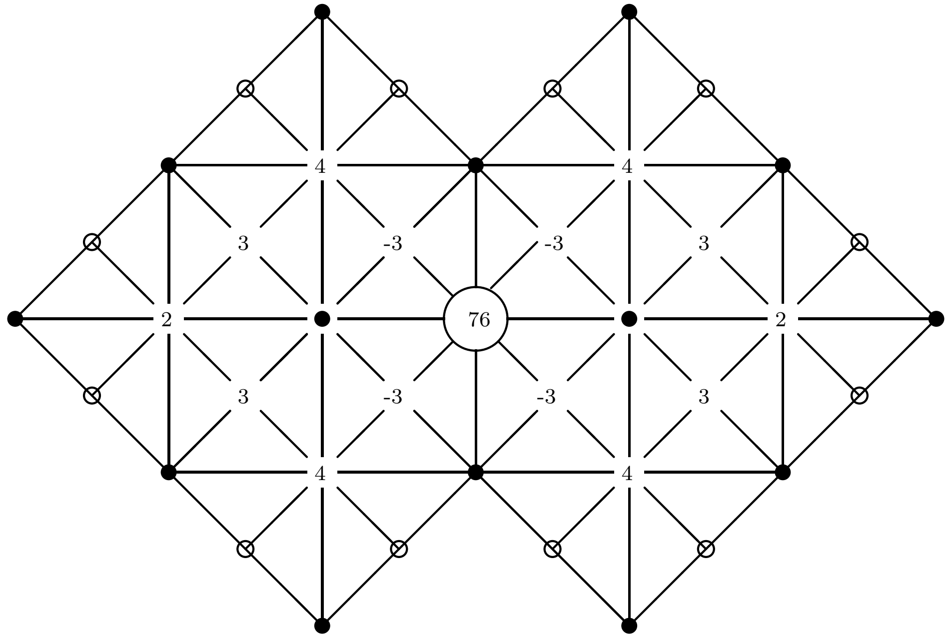
Proof. It is sufficient to show that the wavelets ϕ_u are linearly independent. We demonstrate this by showing that the square matrix

$$Q = (\phi_v(u))_{u,v \in V^1 \setminus V^0}$$

is diagonally dominant and therefore non-singular. Diagonal dominance is clearly equivalent to the condition that

$$\phi_v(v) > \sum_{u \in V^1 \setminus V^0, u \neq v} |\phi_u(v)|, \quad \text{for all } v \in V^1 \setminus V^0.$$

Thus for each v in $V^1 \setminus V^0$ we need to show the sum of the absolute values of coefficients at v of wavelets other than ϕ_v is less than the coefficient at v of ϕ_v itself. It turns out that this condition does indeed hold in every topological case. In Figure 3.16 each distinct topological case of $v \in V^1 \setminus V^0$ is illustrated by placing the value $\phi_u(v)$ at u for each relevant u. The vertex v is circled in each case. Thus the coefficients in each figure are the non-zero elements of the row v of the matrix Q.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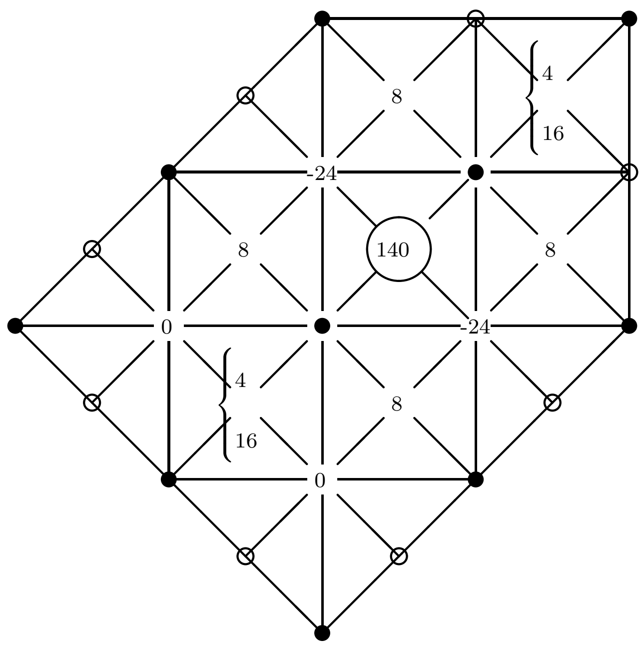
4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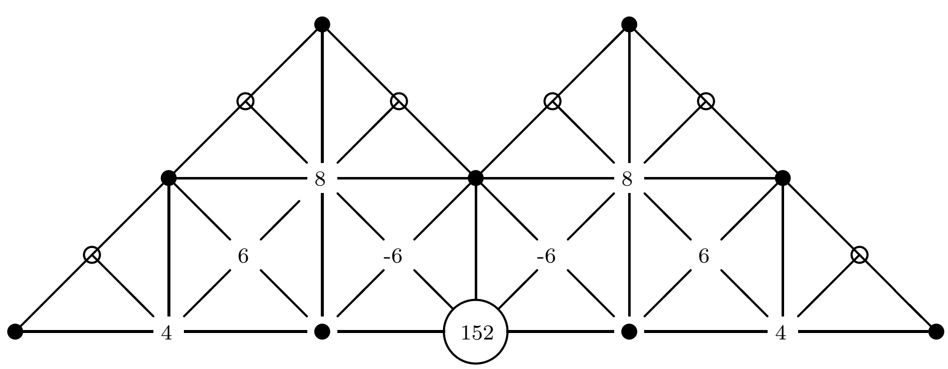
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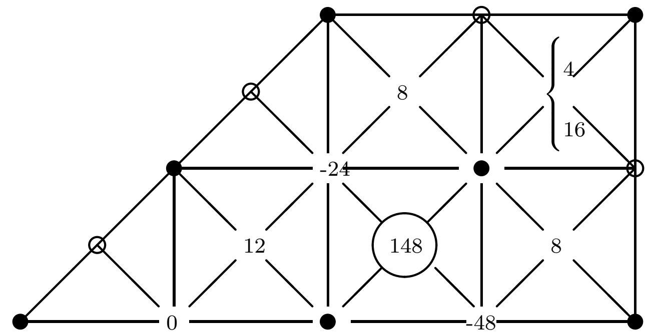
-6
6
4
152
4



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12
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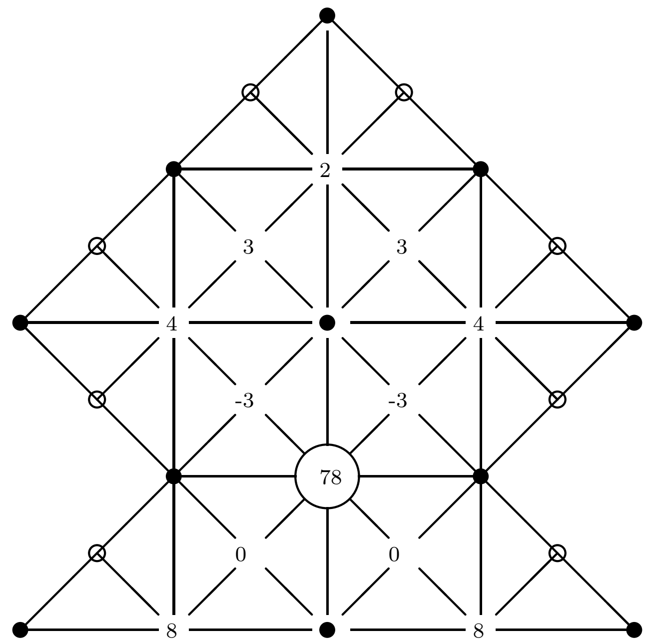
-48



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3
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78
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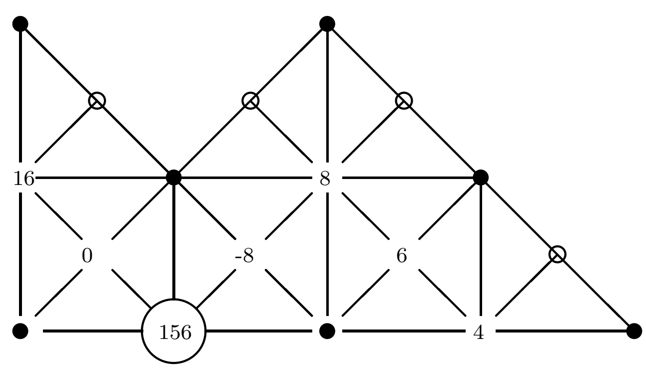
0
8
8



16
8
0
-8
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156
4

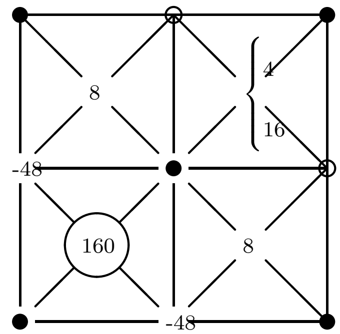


Figure 3.16. Wavelets evaluations

CHAPTER 4

The Smaller Support Wavelets

In Chapter 3, we used the semi-wavelet idea to obtain the small support and unique wavelets over type-2 triangulations. In this chapter, we will work on second interior wavelet and second boundary wavelet—the smaller support interior wavelet and boundary wavelet will be constructed.

4.1 Construction of the Smaller Support Wavelets

First, we will work on the second boundary wavelet. Support vertices are labeled in the following Figure 4.1 and P_i $i = 1, \cdots, 6$ are labeled for the old vertices. This wavelet is called the second smaller support boundary wavelet.

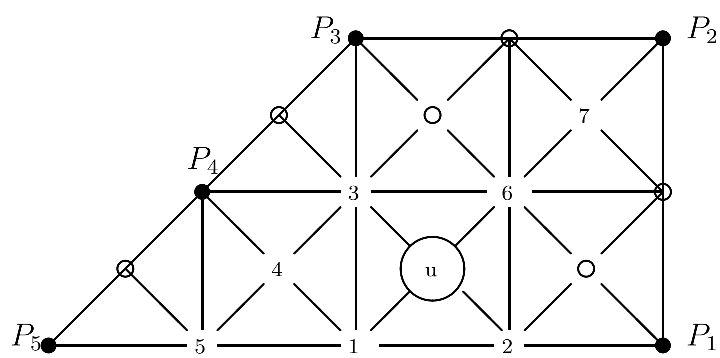


Figure 4.1. The second smaller support boundary wavelet

Let ϕ_u^0 be the wavelet function at u which has the following expression:

$$\phi_u^0 = A\phi_u^1 + B_1\phi_1^1 + B_2\phi_2^1 + B_3\phi_3^1 + B_4\phi_4^1 + B_5\phi_5^1 + B_6\phi_6^1 + B_7\phi_7^1$$

Where ϕ_u^0 is a wavelet function at vertex u in W^0, and $\phi_i^1,\ i = u, 1, \cdots, 7$ are nodal basis functions at u, $i = 1, \cdots, 7$ in S^1. By the orthogonal conditions, the following inner products must be zeros,

$$\langle \phi_u^0, \phi_1^0 \rangle = 0, \;\; \langle \phi_u^0, \phi_{p_i}^0 \rangle = 0, \; i = 1, \cdots 5, \;\; \langle \phi_u^0, \phi_6^0 \rangle = 0$$

By Lemma 3.1 and computation, we obtain the following linear equations:

$$\begin{cases} 8A + 12B_1 + 6B_2 + 12B_3 + 8B_4 + 6B_5 + 3B_6 + 0B_7 = 0, \\ 1A + \frac{1}{2}B_1 + 6B_2 + 0B_3 + 0B_4 + 0B_5 + 3B_6 + 1B_7 = 0, \\ 0A + 0B_1 + 0B_2 + 0B_3 + 0B_4 + 0B_5 + 3B_6 + 8B_7 = 0, \\ 1A + 1B_1 + 0B_2 + 12B_3 + 1B_4 + 0B_5 + 3B_6 + 1B_7 = 0, \\ 0A + 1B_1 + 0B_2 + 4B_3 + 6B_4 + 4B_5 + 0B_6 + 0B_7 = 0, \\ 0A + \frac{1}{2}B_1 + 0B_2 + 0B_3 + 1B_4 + 6B_5 + 0B_6 + 0B_7 = 0, \\ 6A + 1B_1 + 4B_2 + 4B_3 + 0B_4 + 0B_5 + 20B_6 + 6B_7 = 0, \end{cases}$$

The coefficient matrix of the above linear equations is

$$D_1 = \begin{bmatrix} 8 & 12 & 6 & 12 & 8 & 6 & 3 & 0 \\ 1 & \frac{1}{2} & 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 8 \\ 1 & 1 & 0 & 12 & 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 4 & 6 & 4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & 6 & 0 & 0 \\ 6 & 1 & 4 & 4 & 0 & 0 & 20 & 6 \end{bmatrix}.$$

Let the vector $v_1 = [A, B_1, B_2, B_3, B_4, B_5, B_6, B_7]^T$, the solutions of

$$D_1 v_1 = 0$$

be,

$$v_1 = \frac{1}{2}\left[\frac{204}{5}k, -\frac{144}{5}k, \frac{6}{5}k, \frac{8}{5}k, \frac{12}{5}k, 2k, -\frac{64}{5}k, \frac{24}{5}k\right]^T, \quad (1)$$

where k is a non-zero arbitrary real number.

From the above solutions, it is clear that the second smaller support boundary wavelet only needs 8 points of support but second boundary wavelet needs 9 points of support and is not unique, depending on the value of k.

In the following, we study the second smaller support interior wavelet based on the same structure as second interior wavelet. The support vertices are labeled and some vertices in V^0 are also labeled in the following Figure 4.2. This wavelet is called the second smaller support interior wavelet.

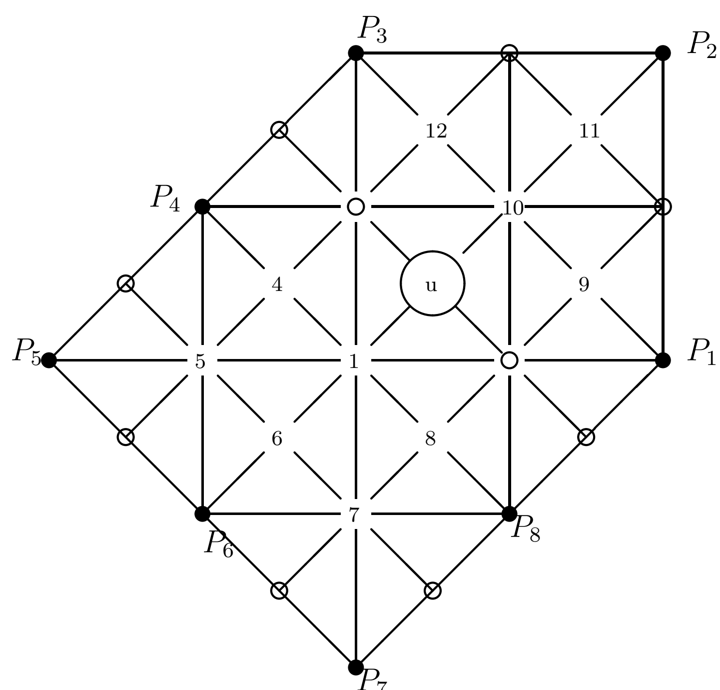


Figure 4.2. The second smaller support interior wavelet

Let's assume that the wavelet at u has the following expression:

$$\phi_u^0 = B_1\phi_1^1 + A\phi_u^1 + B_4\phi_4^1 + B_5\phi_5^1 + B_6\phi_6^1$$

$$+B_7\phi_7^1 + B_8\phi_8^1 + B_9\phi_9^1 + B_{10}\phi_{10}^1 + B_{11}\phi_{11}^1 + B_{12}\phi_{12}^1$$

Where ϕ_u^1 is a basis wavelet function in W^0, and $\phi_i^1,\ i = u, 1, \cdots, 12$ are nodal basis functions in S^1.

By the orthogonal conditions, $\langle \phi_u^0, \phi_{P_i}^0 \rangle = 0,\ i = 1, \cdots, 8$ and $\langle \phi_u^0, \phi_1^0 \rangle = 0$, $\langle \phi_u^0, \phi_{10}^0 \rangle = 0$, we will obtain the following linear equations:

$$
\begin{cases}
24B_1 + 8A + 8B_4 + 12B_5 + 8B_6 + 12B_7 + 8B_8 + B_9 + 3B_{10} + 0B_{11} + B_{12} = 0 \\
B_1 + 1A + 0B_4 + 0B_5 + 0B_6 + 0B_7 + 1B_8 + 8B_9 + 3B_{10} + 1B_{11} + 0B_{12} = 0 \\
B_1 + 6A + 0B_4 + 0B_5 + 0B_6 + 0B_7 + 0B_8 + 6B_9 + 20B_{10} + 6B_{11} + 6B_{12} = 0 \\
B_1 + 1A + 1B_4 + 0B_5 + 0B_6 + 0B_7 + 0B_8 + 0B_9 + 3B_{10} + B_{11} + 8B_{12} = 0 \\
B_1 + 0A + 6B_4 + 4B_5 + 0B_6 + 0B_7 + 0B_8 + 0B_9 + 0B_{10} + 0B_{11} + 0B_{12} = 0 \\
B_1 + 0A + B_4 + 12B_5 + 1B_6 + 0B_7 + 0B_8 + 0B_9 + 0B_{10} + 0B_{11} + 0B_{12} = 0 \\
B_1 + 0A + 0B_4 + 4B_5 + 6B_6 + 4B_7 + 0B_8 + 0B_9 + 0B_{10} + 0B_{11} + 0B_{12} = 0 \\
B_1 + 0A + 0B_4 + 0B_5 + B_6 + 12B_7 + B_8 + 0B_9 + 0B_{10} + 0B_{11} + 0B_{12} = 0 \\
B_1 + 0A + 0B_4 + 0B_5 + 0B_6 + 4B_7 + 6B_8 + 4B_9 + 0B_{10} + 0B_{11} + 0B_{12} = 0 \\
0B_1 + 0A + 0B_4 + 0B_5 + 0B_6 + 0B_7 + 0B_8 + B_9 + 3B_{10} + 8B_{11} + B_{12} = 0
\end{cases}
$$

The coefficient matrix of the above linear equations is the following:

$$D_2 = \begin{bmatrix} 24 & 12 & 8 & 12 & 8 & 12 & 8 & 12 & 8 & 1 & 3 & 0 & 1 \\ 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 6 & 20 & 6 & 6 \\ 1 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 8 \\ 1 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 8 & 1 \end{bmatrix}.$$

We solve this linear equation and obtain the following solutions

$$B_1 = -15k_1, A = \frac{253}{6}k_1, B_4 = \frac{11}{6}k_1, B_5 = k_1, B_6 = \frac{7}{6}k_1, B_7 = k_1$$

$$B_8 = \frac{11}{6}k_1, B_9 = k_1, B_{10} = -14k_1, B_{11} = 5k_1, B_{12} = k_1$$

where k_1 is a non-zero arbitrary real number.

This second smaller support interior wavelet only needs 11 points of support, but second interior wavelet needs 13 points of support.

Due to symmetry of type-2 triangulations, rotation will generate all types of wavelets which have the same structures as these two wavelets.

4.2 Wavelet Basis

Theorem 4.4 *Let $k = 5$ and $k_1 = 3$. Then the set of $\{\phi_u^0\}_{u \in V^1 \setminus V^0}$ which contains the first interior wavelet, the second smaller support interior wavelet, the first boundary wavelet, the second smaller support boundary wavelet, the third boundary wavelet, the first corner wavelet and the second corner wavelet is a basis for wavelet space W_0.*

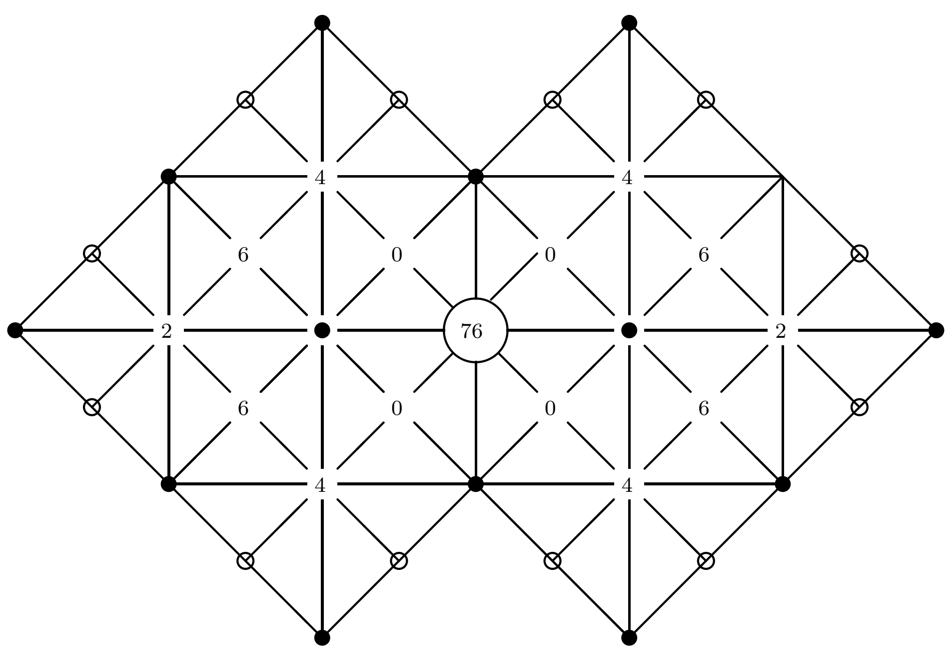
Proof. It is sufficient to show that the wavelets $\{\phi_u^0\}_{u\in V^1\setminus V^0}$ are linearly independent. We demonstrate this by showing that the following square matrix

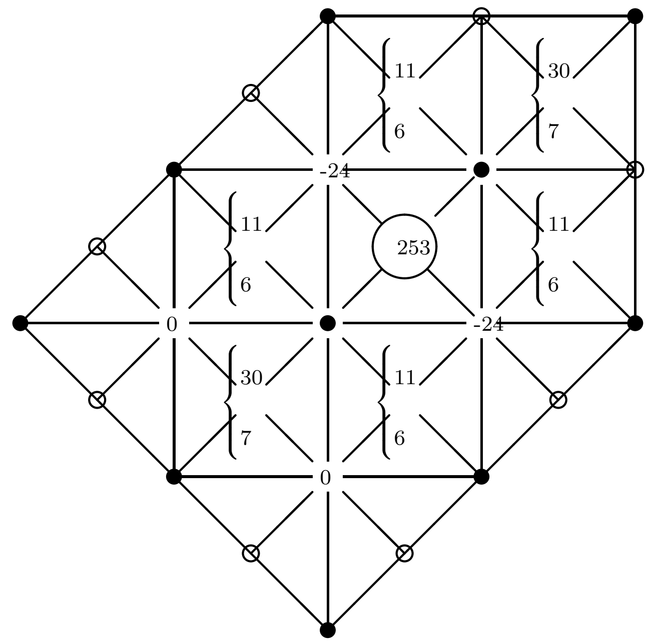
$$Q = (\phi_v^0(u))_{u,v\in V_1\setminus V_0}$$

is diagonally dominant and therefore non-singular. Diagonal dominance is clearly equivalent to the condition that

$$|\phi_v(v)| > \sum_{u\in V_1\setminus V_0, u\neq v} |\phi_u(v)| \qquad mbox for \quad \text{all} \quad v \in V_1 \setminus V_0$$

By computing the value of every wavelet at u, we can get non-zero values in each row of Q as shown in the Figure 4.3, and it is sufficient to prove that if every row is dominant in matrix Q, then Q is non-singular matrix, that is, these wavelets can consist of wavelet basis.

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11
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253
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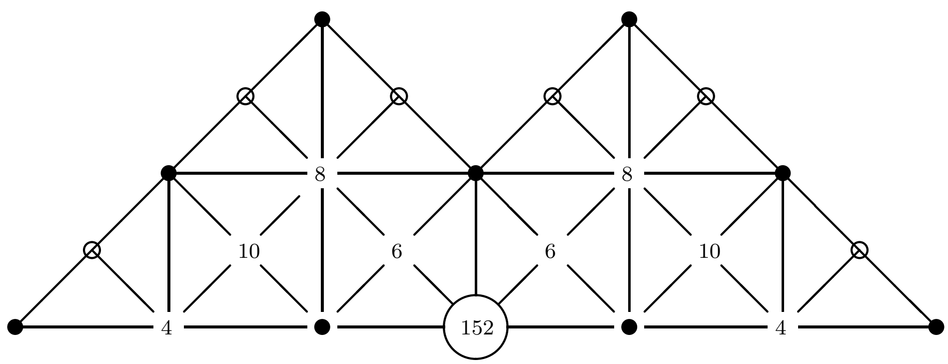
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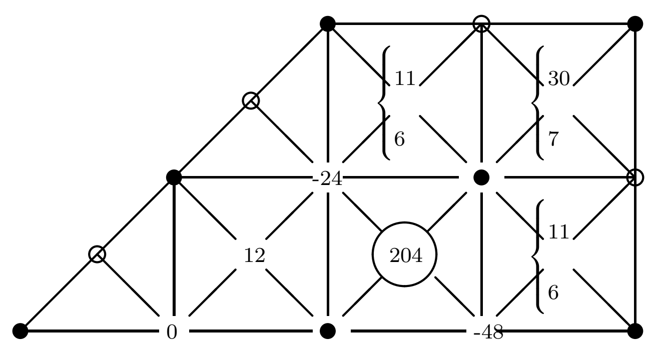
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-48

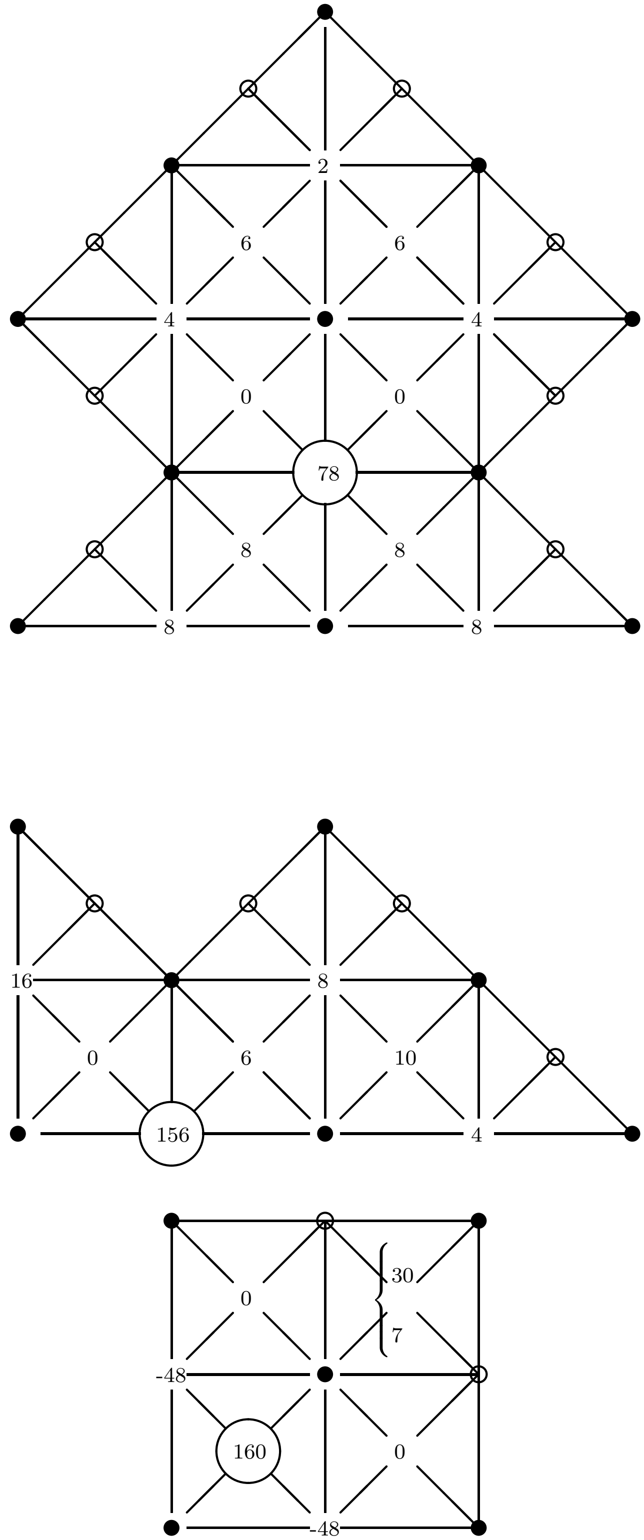


Figure 4.3. Evaluation of wavelets

From the above discussion, we know that we can choose different k,k_1 to ensure that these two smaller support wavelets combining other five structure wavelets in Chapter 3 can consist of wavelet basis for wavelet space W^0.

4.3 The Range of Parameters in the Wavelet Basis

In this section, we will give the sufficient conditions of k and k_1 to ensure that the first interior wavelet, the second smaller support interior wavelet, the first boundary wavelet, the second smaller support boundary wavelet, the third boundary wavelet, the first corner wavelet, and the second corner wavelet can consist of wavelet basis for wavelet space W^0.

Theorem 4.5 *If k and k_1 satisfy the following conditions,*

$$\frac{144}{149} < |k| < \frac{34}{5}$$

$$\frac{2645}{1192} < |k_1| < 20$$

$$4|k| + 5|k_1| < 65$$

Then the set of $\{\phi_u^0\}_{u \in V^1 \setminus V^0}$ which contains the first interior wavelet, the second smaller support interior wavelet, the first boundary wavelet,the second boundary wavelet, the third boundary wavelet, the first corner wavelet and the second corner wavelet is a basis for wavelet space W_0.

Proof. We consider the first interior wavelet, the second smaller support interior wavelet, the first boundary wavelet, the second smaller support boundary wavelet, the third boundary wavelet, the first corner wavelet and the second corner wavelet. In order to verify that these wavelets can consist of the basis, we need to prove that the following matrix is non-singular

$$Q = (\phi_u^0(v))_{v \in V^1 \setminus V^0}.$$

Figures 4.4 shows every non-zero values in every row of Q. We obtain the following dominant inequalities in every row from each figure.

$$\frac{253}{3}|k_1| > 48 + \frac{11}{3}|k_1| + \frac{22}{3}|k_1| + 10|k_1|$$

$$\frac{204}{5}|k| > 48 + 24 + \frac{12}{5}|k| + \frac{22}{6}|K| + 10|K|$$

$$76 > 8|k_1| + 8 + 8 + 4$$

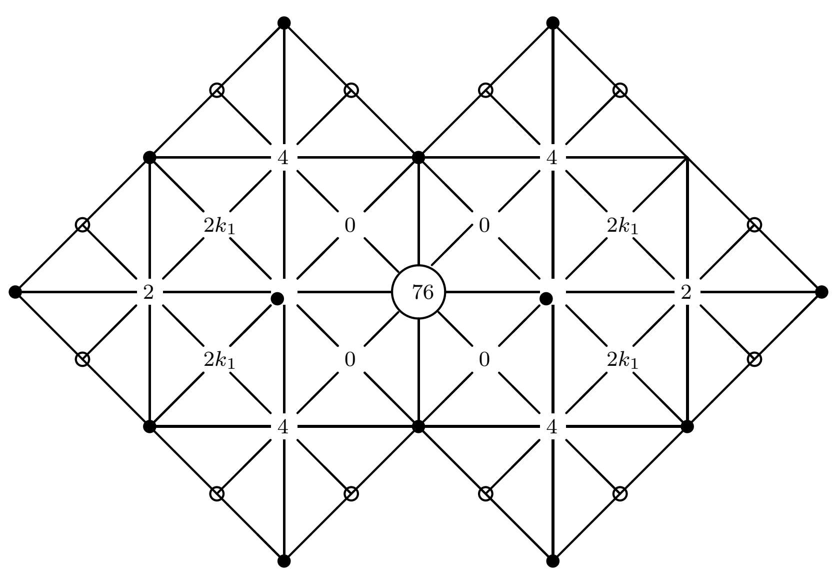
$$102 > 24 + 4|k_1| + \frac{12}{5}|k|$$

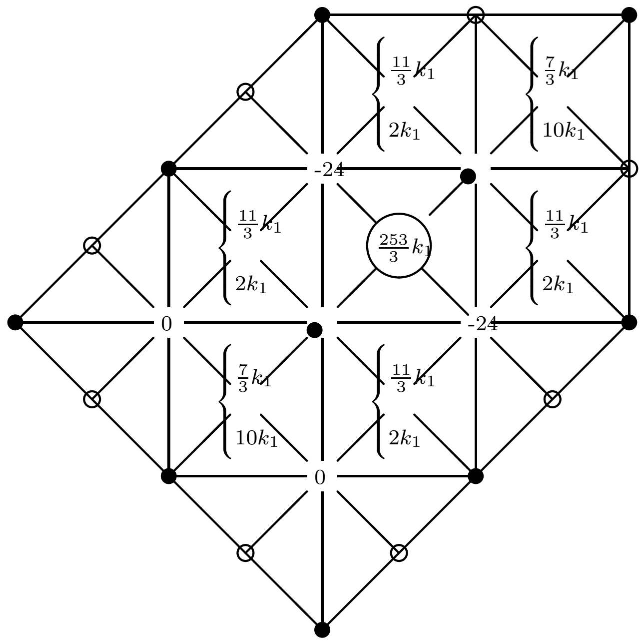
$$78 > 16 + 8 + 2 + 4|k_1| + \frac{16}{5}|k|$$

$$156 > 28 + \frac{16}{5}|k|$$

$$160 > 92 + 10|k_1|.$$

We solve the above inequalities— the range for k and k_1 in the theorem will be obtained.




$\frac{11}{3}k_1$
$2k_1$
$\frac{7}{3}k_1$
$10k_1$
-24
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$\frac{253}{3}k_1$
$\frac{11}{3}k_1$
$2k_1$
0
-24
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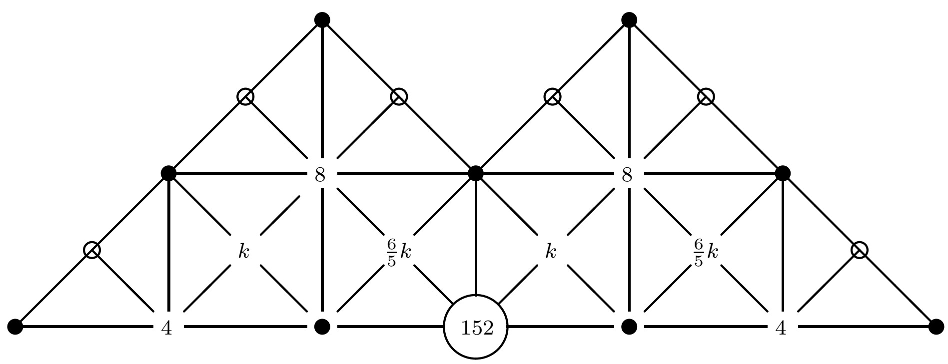
$10k_1$
$\frac{11}{3}k_1$
$2k_1$
0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k
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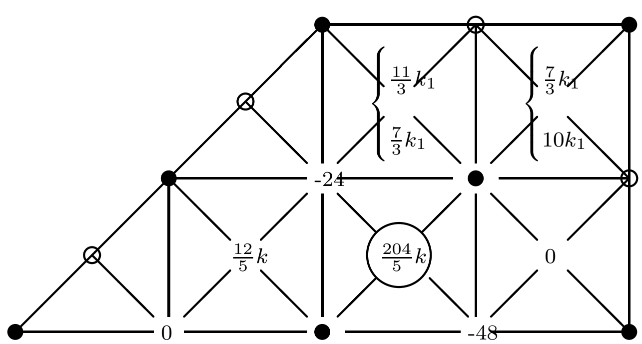
4
152
4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$\frac{7}{3}k_1$
$\frac{7}{3}k_1$
$10k_1$
-24
$\frac{12}{5}k$
$\frac{204}{5}k$
0
0
-48

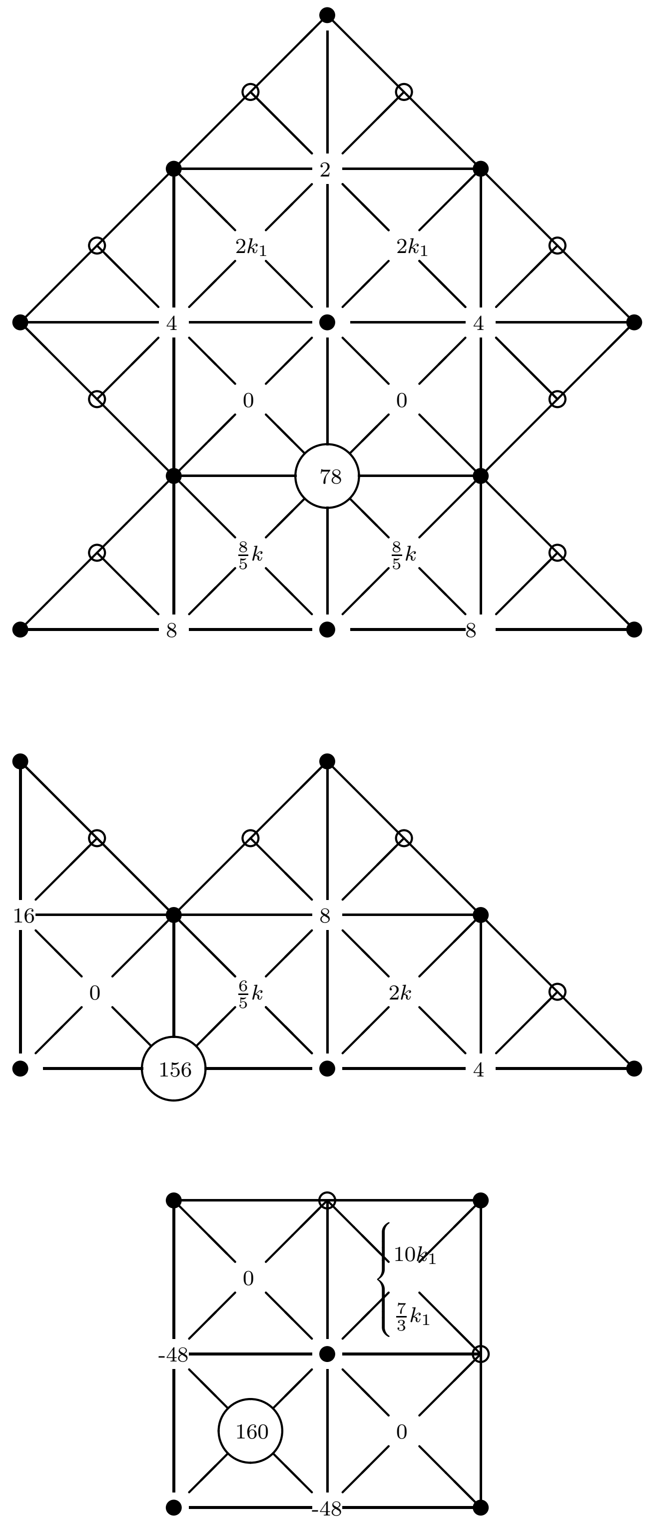


Figure 4.4. Evaluation of wavelets

CHAPTER 5

Parameterized Wavelet Basis

In this chapter, we will construct the parameterized wavelet basis over type-2 triangulations. The two smaller support wavelets are discussed in Chapter 4 (See the Figures 5.1 and 5.2). Since there are parameters in these two wavelets, we call the following wavelets parameterized wavelet 1 and parameterized wavelet 2, respectively.

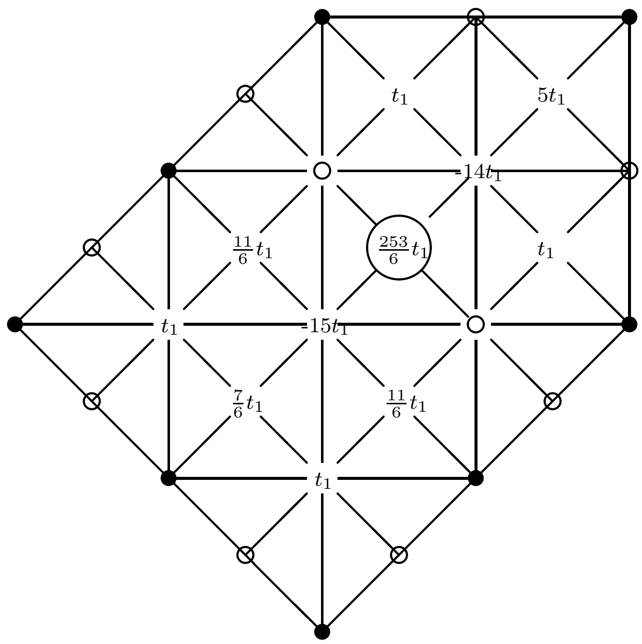


Figure 5.1. Parameterized Wavelet 1

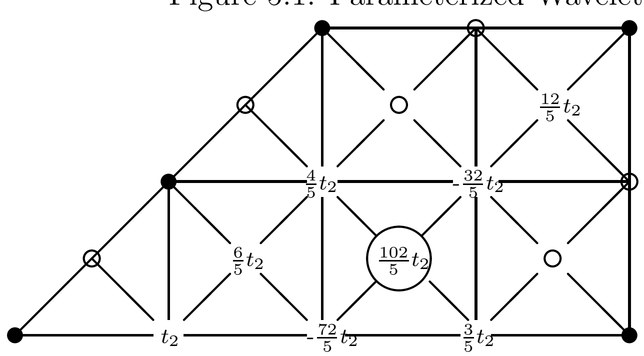


Figure 5.2. Parameterized Wavelet 2

In fact, the above wavelets are interior and boundary wavelets. According to symmetry, the rotation of the two wavelets can form the other interior and edge wavelets which have the same structure as these two parameterized wavelets.

We will construct the other five paramaterized wavelets directly. First we consider the following figure and label the vertices in the Figure 5.3.

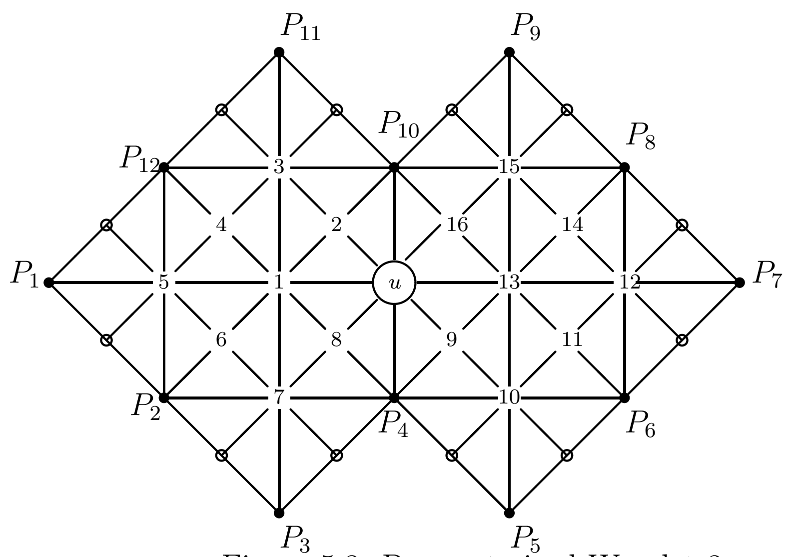


Figure 5.3. Parameterized Wavelet 3

Let σ_u be the wavelet at vertex u with the following expression:

$$\sigma_u = A\phi_u^1 + B_1\phi_1^1 + B_2\phi_2^1 + B_3\phi_3^1 + B_4\phi_4^1 + B_5\phi_5^1 + B_6\phi_6^1 + B_7\phi_7^1 + B_8\phi_8^1$$

$$+B_9\phi_9^1 + B_{10}\phi_{10}^1 + B_{11}\phi_{11}^1 + B_{12}\phi_{12}^1 + B_{13}\phi_{13}^1 + B_{14}\phi_{14}^1 + B_{15}\phi_{15}^1 + B_{16}\phi_{16}^1$$

Here A and B_i $(i = 1, \cdots, 16)$ will be determined by using the orthogonality conditions. By using the orthogonal conditions such as $\langle \sigma_u, \phi_{P_i}^0 \rangle = 0$, $i = 1, \cdots, 12$ and $\langle \sigma_u, \phi_1^0 \rangle = 0$,$\langle \sigma_u, \phi_{13}^0 \rangle = 0$, we obtain the following equations:

$$
\begin{cases}
4A + B_1 + 6B_2 + 4B_3 = 0 \\
B_1 + B_2 + 12B_3 + B_4 = 0 \\
B_1 + 4B_3 + 6B_4 + 4B_5 = 0 \\
B_1 + B_4 + 12B_5 + B_6 = 0 \\
B_1 + 4B_5 + 6B_6 + 4B_7 = 0 \\
B_1 + B_6 + 12B_7 + B_8 = 0 \\
4A + 4B_7 + 6B_8 + 6B_9 + 4B_{10} = 0 \\
B_9 + 12B_{10} + B_{11} + B_{13} = 0 \\
4B_{10} + 6B_{11} + 4B_{12} + B_{13} = 0 \\
B_{11} + 12B_{12} + B_{13} + B_{14} = 0 \\
4B_{12} + B_{13} + 6B_{14} + 4B_{15} = 0 \\
12A + B_1 + B_3 + B_8 + 8B_9 + 12B_{10} + \\
8B_{11} + 12B_{12} + 24B_{13} + 8B_{14} + 12B_{15} + 8B_{16} = 0
\end{cases}
$$

Let the vector $v_1 = [A, B_1, B_2, \cdots, B_{16}]^T$, then the coefficient matrix of the above linear equation is

$$
M_1 = \begin{bmatrix}
4 & 1 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 6 \\
0 & 1 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 6 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 6 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 12 & 1 \\
12 & 24 & 8 & 12 & 8 & 12 & 8 & 12 & 8 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
12 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 12 & 8 & 12 & 24 & 8 & 12 & 8
\end{bmatrix}.
$$

We let $B_{14} = 0$, $B_{11} = 0$,$B_6 = 0$,and $B_4 = 0$, and the coefficient matrix will be transformed into the following submatrix:

$$
M_{11} = \begin{bmatrix}
4 & 1 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 6 \\
0 & 1 & 1 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 4 & 6 & 6 & 4 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 \\
12 & 24 & 8 & 12 & 12 & 12 & 8 & 1 & 0 & 0 & 1 & 0 & 1 \\
12 & 1 & 1 & 0 & 0 & 0 & 1 & 8 & 12 & 12 & 24 & 12 & 8
\end{bmatrix}.
$$

We solve the following equation $M_1 v_1 = 0$, where 0 is a column vector.

$$v_1 = [38t_3, -12t_3, -12t_3, 2t_3, 0, t_3, 0, 2t_3, -12t_3, -12t_3, -12t_3, 0, 2t_3, t_3, 0, -12t_3, 2t_3, -12t_3]^T$$

where t_3 is an arbitrary non-zero real number.

By the similar way, we label the support vertices and compute the coefficients over the Figure 5.4.

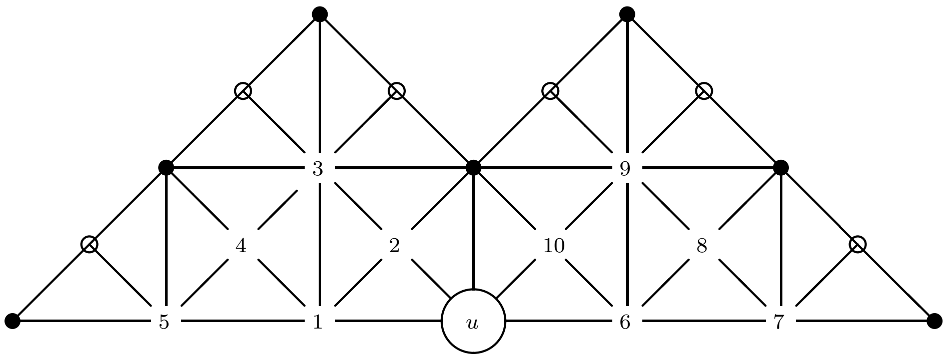


Figure 5.4. Parameterized Wavelet 4

First, we assume that the wavelet at vertex u has the following expression:

$$\sigma_u = A\phi_u^1 + B_1\phi_1^1 + B_2\phi_2^1 + B_3\phi_3^1 + B_4\phi_4^1 + B_5\phi_5^1 +$$

$$B_6\phi_6^1 + B_7\phi_7^1 + B_8\phi_8^1 + B_9\phi_9^1 + B_{10}\phi_{10}^1.$$

By the orthogonality conditions, we will obtain the matrix:

$$M_2 = \begin{bmatrix} 6 & 12 & 8 & 12 & 8 & 6 & 1/2 & 0 & 0 & 0 & 1 \\ 6 & 1/2 & 1 & 0 & 0 & 0 & 12 & 6 & 8 & 12 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 12 & 1 \\ 4 & 1 & 6 & 4 & 0 & 0 & 1 & 0 & 0 & 4 & 6 \\ 0 & 1 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1 & 6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We let $B_4 = 0$ and $B_8 = 0$, and obtain the coefficient matrix:

$$M_{22} = \begin{bmatrix} 6 & 12 & 8 & 12 & 6 & 1/2 & 0 & 0 & 1 \\ 6 & 1/2 & 1 & 0 & 0 & 12 & 6 & 12 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 12 & 1 \\ 4 & 1 & 6 & 4 & 0 & 1 & 0 & 4 & 6 \\ 0 & 1 & 1 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let the column vector $v_2 = [A, B_1, B_2, B_3, B_5, B_6, B_7, B_8, B_9, B_{10}]^T$ be the column vector. By solving the linear equation $M_{22}v_2 = 0$, where 0 is a column vector, we will obtain the solutions

$$v_2 = [38t_4, -12t_4, -12t_4, 2t_4, t_4, -12t_4, t_4, 2t_4, -12t_4]^T$$

where t_4 is an arbitrary non-zero real number.

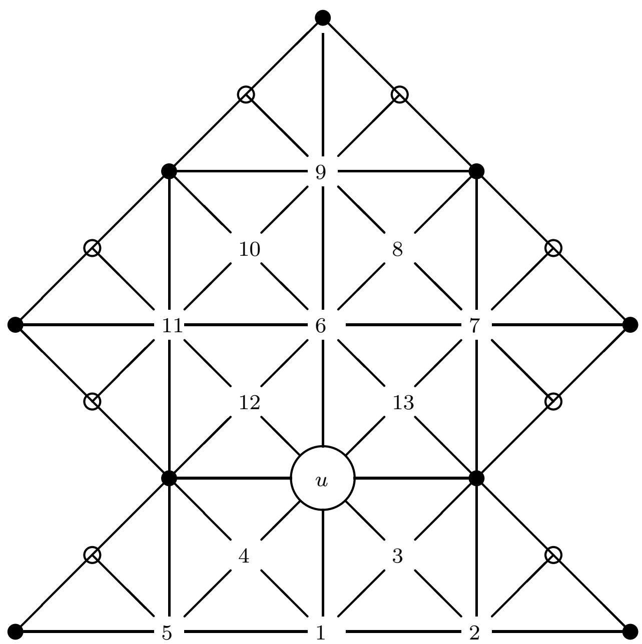


Figure 5.5. Parameterized Wavelet 5

We consider the boundary wavelet at u in the Figure 5.5 and it has the expression:

$$\sigma_u = A\phi_u^1 + B_1\phi_1^1 + B_2\phi_2^1 + B_3\phi_3^1 + B_4\phi_4^1 + B_5\phi_5^1 + B_6\phi_6^1 +$$

$$B_7\phi_7^1 + B_8\phi_8^1 + B_9\phi_9^1 + B_{10}\phi_{10}^1 + B_{11}\phi_{11}^1 + B_{12}\phi_{12}^1 + B_{13}\phi_{13}^1.$$

By the orthogonality conditions, we obtain the following linear equations:

$$\begin{cases} 12A + B_1 + B_3 + B_4 + 24B_6 + 12B_7 + 8B_8 \\ +12B_9 + 8B_{10} + 12B_{11} + 8B_{12} + 8B_{13} = 0 \\ 12A + 12B_1 + 6B_2 + 8B_3 + 8B_4 + 6B_5 + B_6 + B_{13} + B_{14} = 0 \\ \frac{1}{2}B_1 + 6B_2 + B_3 = 0 \\ 4A + B_1 + 4B_2 + 6B_3 + B_6 + 4B_7 + 6B_{13} = 0 \\ B_6 + 12B_7 + B_8 + B_{13} = 0 \\ B_6 + 4B_7 + 6B_8 + 4B_9 = 0 \\ B_6 + B_8 + 12B_9 + B_{10} = 0 \\ B_6 + 4B_9 + 6B_{10} + 4B_{11} = 0 \\ B_6 + B_{10} + 12B_{11} + B_{12} = 0 \\ 4A + B_1 + 6B_4 + 4B_5 + B_6 + 4B_{11} + 6B_{12} = 0 \\ \frac{1}{2}B_1 + B_4 + 6B_5 = 0 \end{cases}$$

By letting $B_{10} = 0$ and $B_8 = 0$, and letting the vector v_3 be the following form:

$$v_3 = [A, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_9, B_{11}, B_{12}, B_{13}]^T,$$

we solve the deleted linear equations and obtain the following solutions:

$$v_3 = [39t_5, -24t_5, 4t_5, -12t_5, -12t_5, 4t_5, -12t_5, 2t_5, 0, t_5, 0, 2t_5, -12t_5, -12t_5]^T$$

where t_5 is an arbitrary non-zero real number.

Let's consider the other structure wavelet in the Figure 5.6 and label the vertices in the Figure 5.6,

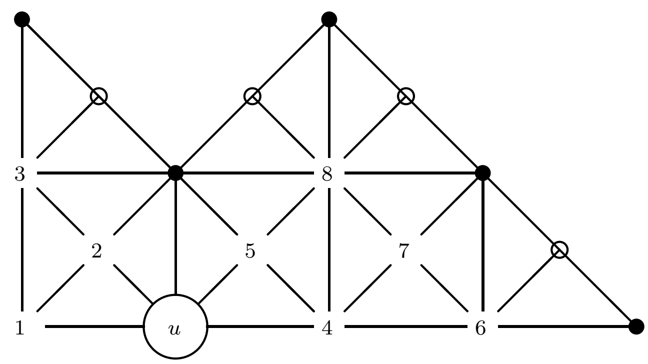


Figure 5.6. Parameterized Wavelet 6

Let σ_u be the wavelet on the vertex u and suppose it has the following expression:

$$\sigma_u = A\phi_u^1 + B_1\phi_1^1 + B_2\phi_2^1 + B_3\phi_3^1 + B_4\phi_4^1 + B_5\phi_5^1 + B_6\phi_6^1 + B_7\phi_7^1 + B_8\phi_8^1.$$

Assume that vector v_4 has the following expression:

$$v_4 = [A, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8]^T.$$

By the orthogonality conditions and letting $B_7 = 0$, we obtain the coefficient matrix M_4 of linear equation $M_4 v_4 = 0$:

$$M_4 = \begin{bmatrix} 6 & 6 & 8 & 6 & 1/2 & 1 & 0 & 0 \\ 6 & 1/2 & 1 & 0 & 12 & 8 & 6 & 12 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 12 \\ 4 & 1 & 6 & 4 & 1 & 6 & 0 & 4 \\ 0 & 1/2 & 1 & 6 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solving the linear equation $M_4 v_4 = 0$, we obtain the following solution:

$$v_5 = [39t_6, -24t_6, -12t_6, 4t_6, -12t_6, -12t_6, t_6, 0, 2t_6]^T.$$

where t_6 is an arbitrary non-zero real number.

Let's consider the seventh structure wavelet in the following figure and label the vertices on the Figure 5.7.

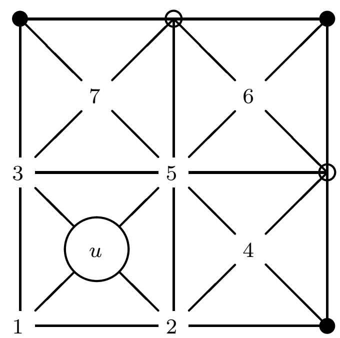


Figure 5.7. Parameterized Wavelet 7

Let σ_u be the wavelet on vertex u in the parameterized wavelet 7 and suppose it has the expression:

$$\sigma_u = A\phi_u^1 + B_1\phi_1^1 + B_2\phi_2^1 + B_3\phi_3^1 + B_4\phi_4^1 + B_5\phi_5^1 + B_6\phi_6^1 + B_7\phi_7.$$

Assume that vector v_5 has the expression:

$$v_5 = [A, B_1, B_4, B_5, B_6, B_7]^T.$$

Let $B_2 = 0$ and $B_3 = 0$,by the orthogonality conditions, we obtain the coefficient matrix M_5 of linear equation $M_5 v_5 = 0$:

$$M_5 = \begin{bmatrix} 6 & 1 & 6 & 20 & 6 & 6 \\ 8 & 6 & 1 & 3 & 0 & 1 \\ 1 & 1/2 & 8 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 8 & 1 \\ 1 & 1/2 & 0 & 3 & 1 & 8 \end{bmatrix}.$$

By solving the linear equation $M_5 v_5 = 0$, we obtain the solution:

$$v_5 = [20t_7, -24t_7, t_7, -6t_7, 2t_7, t_7]^T$$

where t_7 is an arbitrary non-zero real number.

Due to the symmetry, the rotations of the above seven parameterized wavelets can form any wavelet functions on the new vertex in $u \in V^1 \setminus V^0$. So we can obtain all wavelet functions in W^0.

When $t_3 = 1$, $t_4 = 1$, $t_5 = 2$, $t_6 = 2$, and $t_7 = 4$, the above five wavelets can be transformed into the first interior wavelet, the first boundary wavelet, the third boundary wavelet, the first corner wavelet, and the second corner wavelet in Chapter 3. These parameterized wavelets have smaller support and are not unique depending on the parameters t_i. In the following, we will give sufficient conditions of these parameters t_i, $i = 1, \cdots, 7$ to ensure that these seven wavelets can form a wavelet basis.

Theorem 5.6 *We consider the above seven parameterized wavelets* $1, \cdots 7$, *if* t_i,$i = 1, \cdots, 7$ *in the parameterized wavelets* $1, \cdots, 7$ *satisfy the following conditions,*

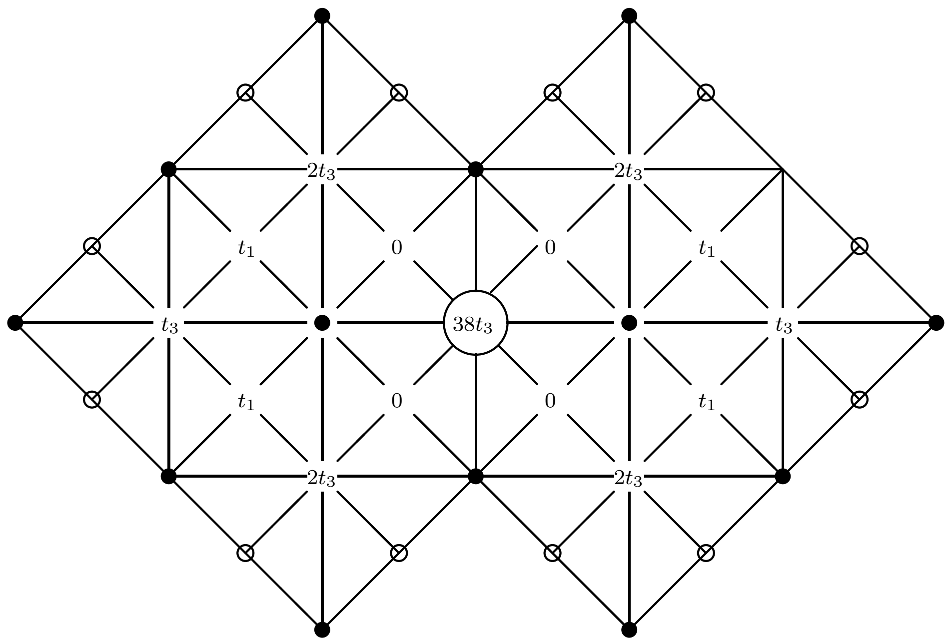
$$\frac{144}{149}|t_3| < |t_1| < \min\{7|t_3|, 5|t_7 - 6|t_6|\}$$

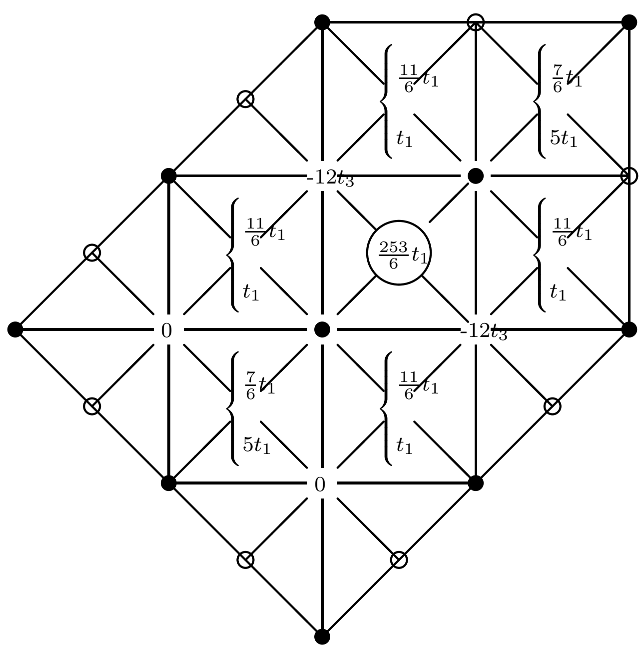
$$\frac{5}{96}(\frac{41}{6}|t_1| + 12|t_4| + 12|t_5|) < |t_2| < \min(\frac{5}{8}(18|t_4 - 4|t_5|),$$
$$\frac{5}{8}(39|t_5| - 4|t_4| - 5|t_3| - 2|t_1|, \frac{1}{2}(35|t_6| - 4|t_5| - 5|t_4|))$$

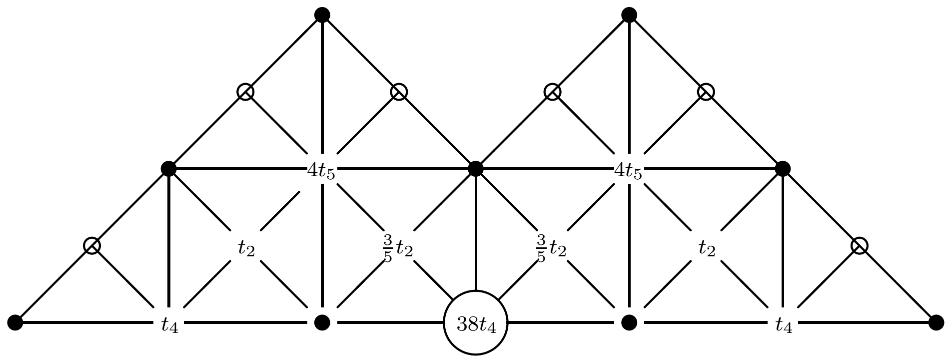
where $t_i \neq 0$.

Then these seven parameterized wavelets can consist of a wavelet basis.

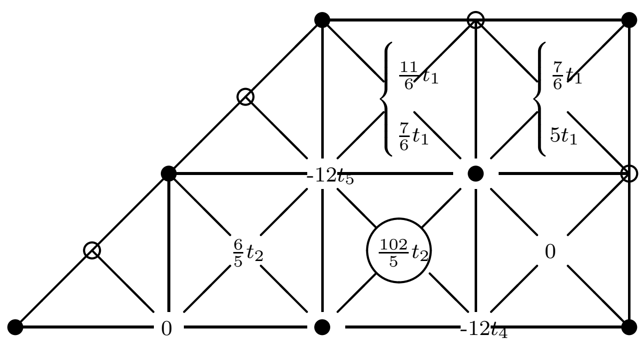
Proof. Let $Q = (\phi_u^1(v))_{v \in V^1 \setminus V^0}$ be a matrix evaluated at u by every parameterized wavelet. The following figures show that the non-zero values of rows in matrix Q.



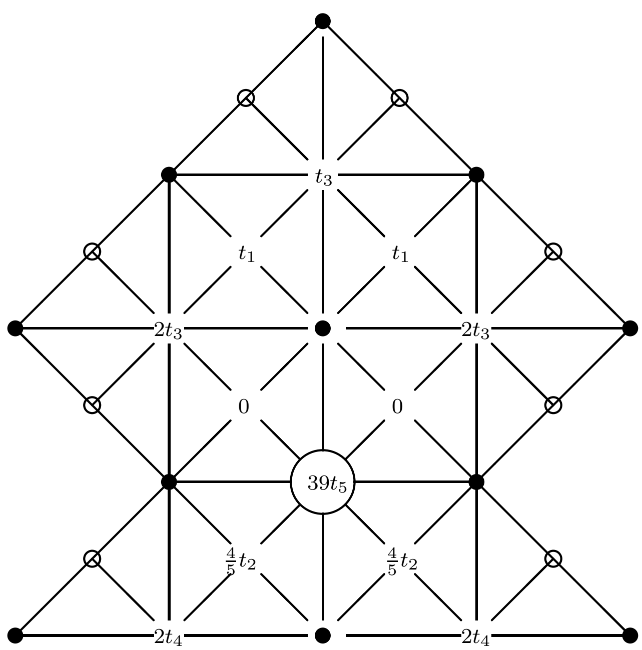




$4t_5$
$4t_5$
t_2
$\frac{3}{5}t_2$
$\frac{3}{5}t_2$
t_2
t_4
$38t_4$
t_4



$\frac{11}{6}t_1$
$\frac{7}{6}t_1$
$\frac{7}{6}t_1$
$5t_1$
$-12t_5$
$\frac{6}{5}t_2$
$\frac{102}{5}t_2$
0
0
$-12t_4$



t_3
t_1
t_1
$2t_3$
$2t_3$
0
0
$39t_5$
$\frac{4}{5}t_2$
$\frac{4}{5}t_2$
$2t_4$
$2t_4$

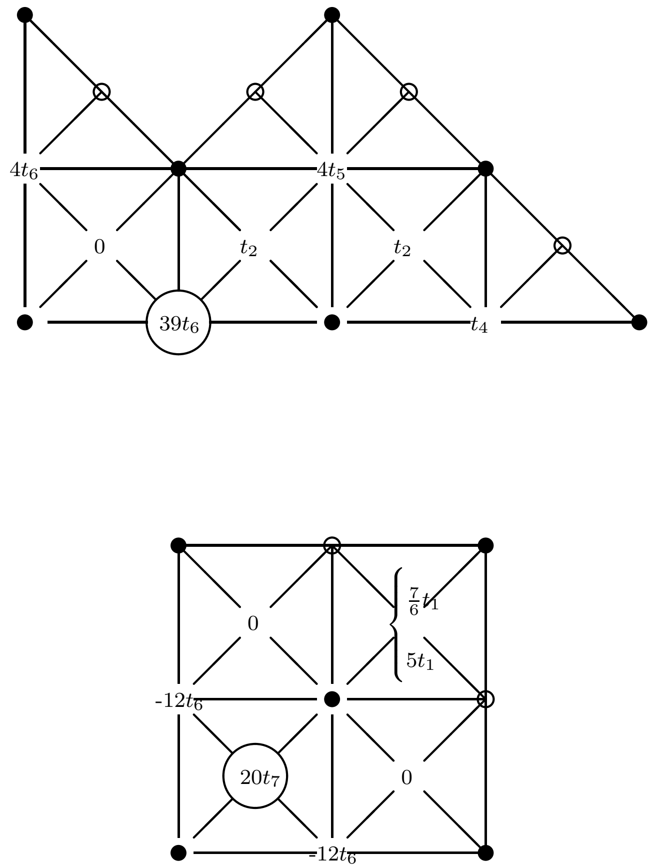


Figure 5.8 Evaluation of parameterized wavelets

It is easy to verity that if the t_i satisfy the above conditions, then matrix Q is a row dominant matrix, and Q is nonsingular, that is, these parameterized wavelets can consist of the wavelet basis for wavelet space W^0 .

It is clear that we can construct the smaller support wavelets over type-2 triangulations. These smaller support wavelets combining other wavelets can consist of the wavelet basis on the wavelet space. Parameterized wavelets are proposed and constructed by using parameters and these parameterized wavelets can form the wavelet basis when these parameters satisfy some conditions, that is, we can pick infinite t_i $i = 1, \cdots, 7$ to obtain the wavelet basis.

Remarks

1: In Chapter 4, we construct the smaller support wavelets over a type-2 triangulation. Does the smallest support linear piecewise wavelet basis exist over the type-2 triangulations?

2: Does the smallest support linear piecewise parameterized wavelet basis exist over this type-2 triangulation?

3: In Chapter 5, we prove that parameterized wavelets can consist of a wavelet basis when parameters satisfy some conditions. Is it possible to find the better bounds for t_i to ensure that these seven parameterized wavelets can form a wavelet basis?

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