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Unifications of Pythagorean Triple Schema

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Unifications of Pythagorean Triple Schema

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Abstract

Euclid's Method of finding Pythagorean triples is a commonly accepted and applied technique. This study focuses on a myriad of other methods behind finding such Pythagorean triples. Specifically, we discover whether or not other ways of finding triples are special cases of Euclid's Method.

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1 Overview

What is a Pythagorean triple? In short, a Pythagorean triple is any set of three positive integers a, b and c for which the Pythagorean Theorem holds. The most common Pythagorean triple is $(3, 4, 5)$ where the calculation

$$\begin{aligned}
 3^2 + 4^2 &= 5^2 \\
 9 + 16 &= 25 \\
 25 &= 25\checkmark
 \end{aligned}$$

shows those integers withstand the Pythagorean Theorem. Pythagorean triples have been known and developed since ancient times with the oldest record dating back to 1900 BC [1]. There are fifty Pythagorean triples where the hypotenuse, c , is less than 100, but there are infinitely many triples in general. Thus, when it comes to discovering what values of a, b and c create a triple, a method for finding Pythagorean triples becomes useful and necessary. As might be expected, the amount and depth of research and resulting methods that have been discovered from 1800 BC - Present is great; this paper researches only eight such methods within a significant range in time. Note that the following methods discussed each possess a correlating proof which will not be written in this paper yet could easily be found if desired.

2 Methods for Generating Pythagorean Triples

2.1 Classical Methods

The three classical methods of generating Pythagorean triples are Pythagoras' Method, Euclid's Formula, and Plato's Formula. Although these three are prevalent methods in both the old and present world of mathematics, they are merely the beginning of where Pythagorean Triple research has gone. The following is a brief introduction to each method including the history, the creator(s), and how it all works.

2.1.1 Pythagoras' Method, c. 540 BC [1]

Pythagoras was a Greek mathematician whose name is well-known in most math classrooms. His method of generating Pythagorean triples is the following: Suppose there is a given odd integer; this odd integer is the lesser of the two sides of the triangle of which contain the right angle. Then, the odd integer is squared, and we subtract one from that number, and divide that by two in order to get the greater of the sides about the right angle. Adding one to the length of the greater side retrieves the remaining side. Here's an example:

- $a = \mathbf{3}$ (this is the odd integer to begin with)
- $3^2 = 9$ (squared the odd integer given)
- $9 - 1 = 8$ (subtract one from digit above)
- $8 \div 2 = 4$ (divide by two—gives the second side)
- $4 + 1 = \mathbf{5}$ (add one to get remaining side)

The example above displays the Pythagoras' method of solving for the (3, 4, 5) Pythagorean triple. A simple way to look at this method would be the formula

$$\text{side } a : \text{side } b = \frac{a^2-1}{2} : \text{side } c = \frac{a^2+1}{2}$$

where $a \geq 3$ is an odd integer.

2.1.2 Plato's Method, c. 380 BC [1] [2]

Plato's formula for generating Pythagorean triples is similar to that of Pythagoras'. The main change is that the starting side (a) is even rather than odd. The formula is the following:

$$\text{side } a : \text{side } b = \left(\frac{a}{2}\right)^2 - 1 : \text{side } c = \left(\frac{a}{2}\right)^2 + 1$$

where $a \geq 4$ is an even integer. The same triple is formed using either Pythagoras' or Plato's method. Let us show this by starting with the even integer four

$$\text{side } a = \mathbf{4} : \text{side } b = \left(\frac{4}{2}\right)^2 - 1 = \mathbf{3} : \text{side } c = \left(\frac{4}{2}\right)^2 + 1 = \mathbf{5}$$

which indeed yields the (3, 4, 5) Pythagorean triple.

2.1.3 Euclid's Method, c. 300BC [3] [4]

Euclid, the father of geometry, was also a Greek mathematician. Euclid's Formula for generating Pythagorean triples is one of the most used and well-known methods. Given two arbitrary integers m and n where $m > n > 0$, this formula states that

$$a = m^2 - n^2, b = 2mn, c = m^2 + n^2$$

form a Pythagorean triple. In order for the triple generated by Euclid's formula to be primitive, both m and n must be coprime and not both odd. Euclid's formula generates all primitive triples, but it does not produce all of the triples that exist. For example, the non-primitive triple (9, 12, 15) cannot be obtained via Euclid's method since 15 cannot be expressed as the sum of two squares. In order to retrieve those remaining triples there must be a parameter k implemented into the formula. Then, the following uniquely generates all of the Pythagorean triples:

$$a = k \cdot (m^2 - n^2), b = k \cdot (2mn), c = k \cdot (m^2 + n^2)$$

where m , n , and k are positive integers, $m > n$, and m and n coprime and not both odd.

2.2 Post-Classical Methods

2.2.1 Fibonacci's Method, c.1170 - c.1250 [5]

Fibonacci's method of generating Pythagorean triples is the following:

$$\begin{aligned} a &= 2F_n F_{n+1} \\ b &= (F_{n+1})^2 - (F_n)^2 \\ c &= (F_n)^2 + (F_{n+1})^2 \end{aligned}$$

where $n \geq 5$ is odd. Let's look at an example. Suppose $n = 5$. Then, we have that $F_n = 5$ and $F_{n+1} = 8$. Thus,

$$\begin{aligned} a &= 2F_n F_{n+1} \\ &= 2(5)(8) \\ &= 80 \\ b &= (F_{n+1})^2 - (F_n)^2 \\ &= 8^2 - 5^2 \\ &= 39 \\ c &= (5)^2 + (8)^2 \\ &= 89 \end{aligned}$$

which gives the Pythagorean triple (80,39,89).

2.2.2 Stifel's Method, 1544 [6]

Michael Stifel was a German mathematician and professor. Stifel's method of generating Pythagorean triples is the following: We start with the mixed fraction $x\frac{x}{2x+1}$ which can be expressed as an improper fraction as

$$\begin{aligned}x\frac{x}{2x+1} &= \frac{x(2x+1)+x}{2x+1} \\ &= \frac{2x^2+2x}{2x+1} \\ &= \frac{2x(x+1)}{2x+1}.\end{aligned}$$

Then, we use the numerator and denominator of the mixed fraction as the a and b of the Pythagorean triple. For an example let us set $x = 2$. Then,

$$\begin{aligned}2\frac{2}{2(2)+1} &= \frac{2(2(2)+1)+2}{2(2)+1} \\ &= \frac{8+4}{5} \\ &= \frac{12}{5}\end{aligned}$$

which gives us that $a = 5$ and $b = 12$. Thus, $c = 13$ by the Pythagorean Theorem and we have the triple $(5, 12, 13)$.

2.2.3 Ozanam's Method, 1694 [7]

Jacques Ozanam was a French mathematician who republished Stifel's method and added his version of a similar sequence. Ozanam's method of generating Pythagorean triples is the following: We start with the mixed fraction $x\frac{4x+3}{4x+4}$ which can be expressed as an improper fraction as

$$x\frac{4x+3}{4x+4} = \frac{x(4x+4)+(4x+3)}{4x+4}.$$

Then, we use the numerator and denominator of the mixed fraction as the a and b of the Pythagorean triple. Plug this a and b into the Pythagorean theorem and then after performing some basic algebra it is found that $c = 4x^2 + 8x + 5$. Let us suppose that $x = 1$. Then,

$$\begin{aligned}1\frac{4(1)+3}{4(1)+4} &= \frac{(4+4)+(4+3)}{4+4} \\ &= \frac{15}{8}\end{aligned}$$

so that $a = 15$ and $b = 8$. The Pythagorean Theorem gives us that $c = 17$ and thus we have the $(8, 15, 17)$ triple. Note that Stifel and Ozanam's method combined produce all primitive triples of Plato and Pythagoras.

2.3 Modern Methods

2.3.1 Portia's Method, n.d.

We are not sure of the origins of Portia's method. Further, we simply named this method after Dr. Holly Portia Anthony of Tennessee Technology University after she informed us that the method was brought to her attention by a colleague who, in turn, had seen it in an old Algebra text. Portia's method of generating Pythagorean triples is the following: Given two fractions, where

$$\frac{a}{b} \cdot \frac{2b}{a} = 2,$$

then adding two to each fraction,

$$2\frac{a}{b} \text{ and } 2\frac{2b}{a},$$

making those improper fractions,

$$\frac{2b+a}{b} \text{ and } \frac{2a+2b}{a},$$

and setting them equal to cross multiply,

$$\begin{aligned} \frac{2b+a}{b} &= \frac{2a+2b}{a} \\ a(2b+a) &= b(2a+2b), \end{aligned}$$

gives you that

$$m = a(2b+a) \text{ and } n = 2b(a+b).$$

2.3.2 Dickson's Method, 1920 [8] [9]

Leonard Eugene Dickson is the namesake behind this method. He was an American Mathematician who focused his research mainly on abstract algebra and number theory. His method for finding Pythagorean triples is published in a Carnegie Institute of Washington journal titled *History of the Theory of Numbers*.

Dickson's Method states that the integers

$$\begin{aligned} a &= r + s \\ b &= r + t \\ c &= r + s + t \end{aligned}$$

where r, s, t are positive integers form a Pythagorean triple (a, b, c) with the condition that $r^2 = 2st$. Thus, r is any even integer and s and t are factors of $\frac{r^2}{2}$. In the case that s and t are relatively prime, the triple will be primitive; however, all Pythagorean triples can be generated by use of this method.

3 Objectives

For the sake of having the objectives of this study explicitly stated, our two main goals are

(a) to determine if there exists, for each method listed previously, a unique m and n such that Euclid's Method maps to the other method selected.

(b) to identify whether or not Euclid's Method encompasses all other given methods.

4 Theorems and Proofs

4.1 Theorem 1

Theorem. Plato's method of generating Pythagorean triples is a special case of Euclid's method. Setting $m = \frac{a}{2}$ and $n = 1$, where $a \geq 4$ is even, gives the Plato triples.

Proof. The Pythagorean Theorem is

$$a^2 + b^2 = c^2,$$

Euclid's Formula is

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2,$$

and Plato's formula is

$$(a)^2 + \left(\left(\frac{a}{2}\right)^2 - 1\right)^2 = \left(\left(\frac{a}{2}\right)^2 + 1\right)^2.$$

Then, plugging in $m = \frac{a}{2}$ and $n = 1$ we have that

$$\begin{aligned} a &= 2mn \\ &= 2\left(\frac{a}{2}\right)(1) \\ &= \left(\frac{2a}{2}\right) \\ &= a \\ b &= m^2 - n^2 \\ &= \left(\frac{a}{2}\right)^2 - (1)^2 \\ &= \left(\frac{a}{2}\right)^2 - 1 \\ c &= m^2 + n^2 \\ &= \left(\frac{a}{2}\right)^2 + (1)^2 \\ &= \left(\frac{a}{2}\right)^2 + 1 \end{aligned}$$

Thus, setting $m = \frac{a}{2}$ and $n = 1$ where $a \geq 4$ is even into Euclid's Formula results in Plato's method, as desired.

4.2 Theorem 2

Theorem The Pythagoras method for generating Pythagorean triples is a special case of Euclid's method. When $m = \frac{a+1}{2}$ and $n = \frac{a-1}{2}$, where $a \geq 3$ is odd, the Pythagoras' triples are obtained.

Proof. The Pythagorean Theorem is

$$a^2 + b^2 = c^2,$$

Euclid's Formula is

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2,$$

and Pythagoras' formula is

$$(a)^2 + \left(\frac{a^2-1}{2}\right)^2 = \left(\frac{a^2+1}{2}\right)^2.$$

Then, plugging in $m = \frac{a+1}{2}$ and $n = \frac{a-1}{2}$ we have that

$$\begin{aligned} a &= m^2 - n^2 \\ &= \left(\left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2\right) \\ &= \left(\frac{a^2+2a+1}{4} - \frac{a^2-2a+1}{4}\right) \\ &= \frac{4a}{4} \\ &= a \end{aligned}$$

and

$$\begin{aligned} b &= 2mn \\ &= \left(2\left(\frac{a+1}{2}\right)\left(\frac{a-1}{2}\right)\right) \\ &= \left(2\left(\frac{a^2-1}{4}\right)\right) \\ &= \frac{2(a^2-1)}{4} \\ &= \frac{a^2-1}{2} \end{aligned}$$

and

$$\begin{aligned}
c &= m^2 + n^2 \\
&= \left(\left(\frac{a+1}{2}\right)^2 + \left(\frac{a-1}{2}\right)^2\right) \\
&= \left(\frac{a^2+2a+1}{4} + \frac{a^2-2a+1}{4}\right) \\
&= \frac{2a^2+2}{4} \\
&= \frac{a^2+1}{2}.
\end{aligned}$$

Thus, setting $m = \frac{a+1}{2}$ and $n = \frac{a-1}{2}$ where $a \geq 3$ is odd into Euclid's Formula results in Pythagoras' method, as desired.

4.3 Theorem 3

Theorem. Stifel's case of generating Pythagorean triples is a special case of Euclid's method. Specifically, we get the Stifel triples by setting $m = x + 1$ and $n = x$.

Proof. Stifel's method of generating Pythagorean triples is the following: We start with the mixed fraction $x \frac{x}{2x+1}$ which can be expressed as an improper fraction as

$$\begin{aligned}
x \frac{x}{2x+1} &= \frac{x(2x+1) + x}{2x+1} \\
&= \frac{2x^2 + 2x}{2x+1} \\
&= \frac{2x(x+1)}{2x+1}
\end{aligned}$$

Then, we use the numerator and denominator of the mixed fraction as the a and b of the Pythagorean triple. When we do this we get

$$\begin{aligned}
a^2 + b^2 &= (2x(x+1))^2 + (2x+1)^2 \\
&= 4x^2(x^2 + 2x + 1) + 4x^2 + 4x + 1 \\
&= 4x^4 + 8x^3 + 8x^2 + 4x + 1 \\
&= (2x^2 + 2x + 1)^2
\end{aligned}$$

where the final step gives us $c = 2x^2 + 2x + 1$. Continuing, show that the Pythagorean triple $(2x + 1)$, $(2x^2 + 2)$, and $(2x^2 + 2x + 1)$ can be obtained by a specific choice of m and n in Euclid's method. First, we try $m = x + 1$ and $n = x$. This gives us

$$\begin{aligned}
m^2 + n^2 &= (x + 1)^2 + x^2 \\
&= 2x^2 + 2x + 1 \\
&= c \\
m^2 - n^2 &= (x + 1)^2 - x^2 \\
&= 2x + 1 \\
&= a \\
2mn &= 2(x + 1)x \\
&= 2x^2 + 2 \\
&= b
\end{aligned}$$

as claimed.

4.4 Theorem 4

Theorem. Ozanam's case of generating Pythagorean triples is a special case of Euclid's method. To get the Ozanam triples by using Euclid's method one must set $m = 2x + 2$ and $n = 1$.

Proof. Ozanam's method of generating Pythagorean triples is the following: We start with the mixed fraction $x \frac{4x+3}{4x+4}$ which can be expressed as an improper fraction as

$$x \frac{4x+3}{4x+4} = \frac{x(4x+4) + (4x+3)}{4x+4}.$$

Then, we use the numerator and denominator of the mixed fraction as the a and b of the Pythagorean triple. When we do this we get

$$\begin{aligned}
a^2 + b^2 &= (4x + 4)^2 + (x(4x + 4) + (4x + 3))^2 \\
&= (4x + 4)^2 + (4x^2 + 8x + 3)^2 \\
&= 16x^4 + 64x^3 + 104x^2 + 80x + 25 \\
&= (4x^2 + 8x + 5)^2
\end{aligned}$$

where the final step gives us $c = 4x^2 + 8x + 5$. Continuing, show that the Pythagorean triple $(4x + 4)$, $(4x^2 + 8x + 3)$, and $(4x^2 + 8x + 5)$ can be obtained by a specific choice of m and n in Euclid's method. First, we try $m = 2x + 2$ and $n = 1$. This gives us

$$\begin{aligned}
m^2 + n^2 &= (2x + 2)^2 + 1^2 \\
&= 4x^2 + 8x + 5 \\
&= c \\
m^2 - n^2 &= (2x + 2)^2 - 1^2 \\
&= 4x^2 + 8x + 3 \\
&= b \\
2mn &= 2(2x + 2)(1) \\
&= 4x + 4 \\
&= a
\end{aligned}$$

as claimed.

4.5 Theorem 5

Theorem. The Fibonacci method for generating Pythagorean triples is a special case of Euclid's method. We get the Fibonacci triples by setting $m = F_{n+1}$ and $n = F_n$.

Proof. Fibonacci's method of generating Pythagorean triples is the following:

$$\begin{aligned}
a &= 2F_n F_{n+1} \\
b &= (F_{n+1})^2 - (F_n)^2 \\
c &= (F_{n+1})^2 + (F_n)^2
\end{aligned}$$

where $n \geq 5$ is odd. Plugging in $m = F_{n+1}$ and $n = F_n$ we have that

$$\begin{aligned}
a &= 2mn \\
&= 2(F_n)(F_{n+1}) \\
&= a \\
b &= m^2 - n^2 \\
&= (F_{n+1})^2 - (F_n)^2 \\
&= b \\
c &= m^2 + n^2 \\
&= (F_{n+1})^2 + (F_n)^2 \\
&= c.
\end{aligned}$$

as desired.

4.6 Theorem 6

Theorem. Portia's method for generating Pythagorean triples is a special case of Euclid's method. We get Portia's triples by setting $m = \frac{a+1}{2}$ and $n = \frac{a-1}{2}$.

Proof. Portia's method of generating Pythagorean triples is the following: Given two fractions, where

$$\frac{a}{b} \cdot \frac{2b}{a} = 2,$$

then adding two to each fraction,

$$2\frac{a}{b} \text{ and } 2\frac{2b}{a},$$

making those improper fractions,

$$\frac{2b+a}{b} \text{ and } \frac{2a+2b}{a},$$

and setting them equal to cross multiply,

$$\begin{aligned} \frac{2b+a}{b} &= \frac{2a+2b}{a} \\ a(2b+a) &= b(2a+2b), \end{aligned}$$

gives you that

$$m = a(2b+a) \text{ and } n = 2b(a+b).$$

Then, to map it to Euclid's method, we set $m = a+b$ and $n = b$. Thus,

$$\begin{aligned} m^2 + n^2 &= (a+b)^2 + b^2 \\ &= a^2 + 2b^2 + 2ab \\ &= a^2 + 2b(a+b) \\ m^2 - n^2 &= (a+b)^2 - b^2 \\ &= a^2 + 2ab \\ &= a(a+2b) \\ 2mn &= 2(a+b)b \\ &= 2ab + 2b^2 \\ &= 2b(a+b) \end{aligned}$$

as desired.

5 Conclusion and Applications

Although we were able to make a connection between Euclid and six other methods, it was discovered during the process of trying to map Dickson's method to Euclid's that not *all* methods out there prove to be special cases. It turns out that Euclid ends up mapping to and potentially being a special case of Dickson. So, could it be that Dickson's Method is the actual parent method? In regard to applying this newfound knowledge, how could this be used in middle or high school Math Education? Showing students multiple ways of finding Pythagorean triples and getting the same results could prove to be beneficial in the math classroom. It can give the students an option to select whichever method they are most comfortable with implementing on a test as well as give them a deeper understanding of the world of mathematics. Furthermore, Algebra students, in particular, could have a great opportunity for engagement in the classroom if they were given the task to make the connections that we did. The bulk of the calculations within the proofs are simply algebra concepts that middle or high school algebra students would be familiar with; if those students were asked to show how one method maps to another given a specific m and n , then they could show such is true through use of relatively simple algebra techniques. Advanced Algebra students could be further challenged by having to figure out the m 's and n 's followed by showing how the algebra backs up the claims made.

6 Future Work

Moving forward, the goal is to dig more into Dickson's Method and answer the question of whether or not Dickson is the parent method. This would need to involve the six methods looked at in this study in addition to other various methods out there for finding the Pythagorean triples. If Dickson's Method proves to encompass all of the other methods out there, then that would perhaps spark an increased emphasis on that method. The effects of this might be seen in Math Education somewhere down the road as teachers begin to realize the significance of a seemingly modern Pythagorean Triple solving approach.

References

- [1] J. H. Barnett, "Generating Pythagorean triples: the methods of Pythagoras and of Plato via Gnomons," 2017.
- [2] G. R. Proclus *et al.*, "A commentary on the first book of Euclid's Elements," 1970.
- [3] D. Joyce, "Book x, proposition xxix," *Euclid's Elements*, Clark University, 1997.

- [4] L. C. Bruno, *Math and Mathematicians: The History of Math Discoveries Around the World*. Math and Mathematicians, U X L, 2002.
- [5] W. Feavyour, “Pythagorean triples in the fibonacci sequence.”
- [6] F. J. Swetz *et al.*, “Mathematical treasures - Michael Stifel’s Arithmetica Integra,” 2011.
- [7] J. Ozanam, *Recreations in Mathematics and Natural Philosophy*. Math and Mathematicians, G. Kearsley; 49, 1814.
- [8] J. Rukavicka, “Dickson’s method for generating Pythagorean triples revisited.,” *European Journal of Pure & Applied Mathematics*, vol. 6, no. 3, 2013.
- [9] L. E. Dickson, *History of the theory of numbers: Diophantine Analysis*, vol. 2. Courier Corporation, 2013.