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Life Annuities under Random Rates of Interest.

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Life Annuities Under Random Rates of Interest

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ABSTRACT

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We begin by examining the accumulated value functions of some annuities-certain. We then investigate the accumulated value of these annuities where the interest is a random variable under some restrictions. Calculations are derived for the expected value and the variance of these accumulated values and present values. In particular we will examine an annuity-due of k yearly payments of 1. Then we will consider an increasing annuity-due of k yearly payments of $1, 2, \cdots, k$. And finally, we examine a decreasing annuity-due of k yearly payments of $n, n-1, \cdots, n-k+1$, for $k \leq n$.

Finally we extend our analysis to include a contingent annuity. That is an annuity in which each payment is contingent on the continuance of a given status. Specifically, we examine a life annuity under which each payment is contingent on the survival of one or more specified persons. We extend our methods from the previous sections to derive the formula of the expected value for the present value of the life annuities of a future life time at a random rate of interest.

DEDICATION

I would like to dedicate this thesis in memory of my grandfather, James Emerson Roller, who graduated with a Masters of Arts Degree From East Tennessee State University in 1954. Also to my parents, Hayden and Kathy Baker, who taught me that through God all things are possible. Without their support this thesis would not have been possible. And finally to my beautiful daughter Kasey, who's love and patience were never ending, and for reminding me that one plus one is equal to two. To all of you, I am truly grateful.

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CHAPTER 1

INTRODUCTION

1.1 Definitions And Notations

A brief introduction to the theory of interest can be found in [3]. We define *interest* as the compensation that a borrower of capital pays to a lender of capital for its use [4]. The *effective rate of interest* is a measure of interest paid at the end of the period. And, the *effective rate of discount*, denoted by d, is a measure of interest paid at the beginning of the period. The theory of *compound interest* assumes that the interest earned is automatically reinvested. In this paper we assume the interest is compounded unless otherwise stated. Thus using compounded interest, if we invest 1 at an effective yearly rate of interest i, then the accumulated value at time $t \geq 0$ is given by

$$a(t) = (1 + i)^t.$$

$a(t)$ is called the *accumulation function.* Also, we define an *amount function* $A(t)$ to give the accumulated value at time $t \geq 0$ of an original investment of P. Hence we have [4]

$$A(t) = P * a(t).$$

In addition, it is sometimes desired to determine how much a person must invest initially in order to accumulate an amount of 1 at the end of t years. This is called the *present value* and is denoted by v^t. So that for $t \geq 0$

$$v^t = a^{-1}(t) = \frac{1}{(1 + \mathrm{i})^{\mathrm{t}}}.$$

Notice if t=1 we have

$$v(1+i) = 1.$$

Kellison states two rates of discount or interest are equivalent if a given amount is invested for the same length of time at each of the rates and produces the same accumulated value [4]. We may express d as a function of i to be

$$d = \frac{\mathrm{i}}{(1+\mathrm{i})}$$

so we have the equalities

$$d = iv,$$

$$v + d = 1.$$

Now we can define an annuity as a series of payments made at equal intervals of time, called payment periods [4]. Examples of annuities include house rents, mortgage payments, installment payments on automobiles, and interest payments on money. If the payments are made at the end of each payment period for n periods, the annuity is called an *annuity-immediate.* If instead, the payments are made at the beginning of each interval, the annuity is called an *annuity-due.*

An *annuity-certain* is one for which the payments begin and end at fixed dates. That is, payments are certain to be made for a fixed period of time. This fixed period, from the beginning of the first interval of payment to the end of the last interval is the *term* of the annuity-certain. For example, mortgage payments constitute an annuity-certain. [4]

1.2 Annuities Under Constant Rates of Interest

Now let us consider an n-period annuity due with yearly interest rate i and yearly payments of 1. We will assume $k \leq n$ in this paper, unless otherwise stated. We denote the accumulated value after k years as $\ddot{s}_{\overline{k}|i}$. So we have

$$\begin{aligned} \ddot{s}_{\overline{n}|i} &= (1+i)^n + (1+i)^{n-1} + \cdots + (1+i)^2 + (1+i) \\ &= \frac{(1+\mathrm{i})[(1+\mathrm{i})^{\mathrm{n}} - 1]}{(1+\mathrm{i}) - 1} \\ &= \frac{(1+\mathrm{i})^{\mathrm{n}} - 1}{\mathrm{d}}. \end{aligned}$$

We also note that the accumulated value at time k can be written as

$$\ddot{s}_{\overline{k}|i} = (1+i)(1+\ddot{s}_{\overline{k-1}|i}).$$

Next we will consider an increasing annuity-due. In particular, we will exam the accumulated value after k years of an increasing annuity-due of k yearly payments of $1, 2, ..., k$, respectively, and denote it as $(I\ddot{s})_{\overline{k}|i}$. This is a common case in interest theory and is known to be given by the formula

$$(I\ddot{s})_{\overline{k}|i} = \frac{\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{i}} - \mathrm{k}}{\mathrm{d}}.$$

However, we will note that we can also write

$$(I\ddot{s})_{\overline{k}|i} = (1+i)^k + 2(1+i)^{k-1} + \cdots + (k-1)(1+i)^2 + k(1+i)$$

In addition to the traditional increasing annuity, Zaks derives an equation for the accumulated value after k years of an increasing annuity-due of k yearly payments of $1^2, 2^2, ..., k^2$, respectively, which he denotes by $(I^2\ddot{s})_{\overline{k}|i}$ [6]. By definition we know

$$(I^2\ddot{s})_{\overline{k}|i} = (1+i)^k + 2^2(1+i)^{k-1} + \cdots + (k-1)^2(1+i)^2 + k^2(1+i).$$

Next, we note that

$$(I^2\ddot{s})_{\overline{k}|i} - v(I^2\ddot{s})_{\overline{k}|i} = (1-v)(I^2\ddot{s})_{\overline{k}|i} = d(I^2\ddot{s})_{\overline{k}|i}.$$

Since $d = 1 - \frac{1}{1+\mathrm{i}}$, we have

$$d(I^2\ddot{s})_{\overline{k}|i} = [(1+i)^2 - (1+i)^{k-1}] + 2^2[(1+i)^{k-1} - (1+i)^{k-2}]$$

$$+ \cdots + [(k-1)^2(1+i)^2 - (k-1)^2(1+i)] + [k^2(1+i) - k^2].$$

Combining like terms we obtain

$$d(I^2\ddot{s})_{\overline{k}|i} = (1+i)^k + \cdots + (2k-1)(1+i) - k^2.$$

From which we have

$$d(I^2\ddot{s})_{\overline{k}|i} = 2[(1+i)^k + \cdots + k(1+i)] - [(1+i)^k + \cdots + (1+i)] - k^2.$$

Thus,

$$(I^2\ddot{s})_{\overline{k}|i} = \frac{2(\mathrm{I\ddot{s}})_{\overline{\mathrm{k}}|\mathrm{i}} - \ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{i}} - \mathrm{k}^2}{\mathrm{d}}.$$

Similarly, it can be shown that for $m = 3, 4, ...$, a recursive formula of the above can be derived for $(I^m\ddot{s})_{\overline{k}|i}$, which is defined as the accumulation of an annuity-due of k payments of $1^m, 2^m, ...k^m$.

Thus we have now established formulae for annuities-due with constant rates of interest having level payments, having k yearly increasing payments of $1, 2, ..., k$, and having k yearly increasing payments of $1^2, 2^2, ..., k^2$.

CHAPTER 2

ANNUITIES-CERTAIN WITH RANDOM RATES OF INTEREST

Thus far we have assumed a constant rate of interest throughout the term of the annuity. Yet, in our economic world, a level rate of interest is not always the case. Thus it is necessary to consider the case of random rates of interest. Many attempts have been made to evaluate stochastic interest rate models and to investigate their impact under various scenarios: see Kellison (1991), Bowers, et al (1997), and Zaks (2001). In this chapter we will examine the accumulated value of some annuities-certain over a term in which the rate of interest is a random variable under some restrictions. We will use a method developed by Zaks in his paper, 'Annuities under random rates of interest'in which we derive the expected value and the variance of the accumulated value.

2.1 Single Payments and Level Payments

We will let i_k denote the rate of interest for year k, that is for the interval $k-1$ to k, for $k = 1, 2, \cdots, n$. And we assume that $i_1, \cdots, i_n$ are independent random variables. Now, let us first consider the case of a single payment of 1 at the start of the term. That is $c_1 = 1, c_2 = c_3 = \cdots = c_k = 0$. We will let C_k be the accumulated value after k years. Then we have

$$C_k = (1 + i_1)(1 + i_2) \cdots (1 + i_k)$$

or

$$C_k = C_{k-1}(1 + i_k) \quad \text{for } k = 2, \cdots, n.$$

Next lets consider the case where C_k is the accumulation after k years of an annuity-due of k yearly payments of 1. So $c_1 = c_2 = \cdots = c_n = 1$. In this case we have

$$C_k = (1+i_1)(1+i_2)\cdots(1+i_k)+(1+i_2)(1+i_3)\cdots(1+i_k)+\cdots+(1+i_{k-1})(1+i_k)+(1+i_k).$$

Equivalently we have, [4]

$$C_k = (C_{k-1} + k)(1 + i_k).$$

or

$$C_k = \sum_{t=1}^{t} \prod_{s=1}^{t} (1 + i_{n-s+1}).$$

Notice this is a long and tedious calculation.

However, let us suppose that for each k, we have $E(i_k) = j$ and $Var(i_k) = s^2$. Zaks [6] expresses the expected value of a payment of 1 in year k and the interest earned in the kth year as

$$E(1 + i_k) = 1 + j = \mu.$$

So,

$$E[(1 + i)^2] = E(1 + 2i_k + i_k^2)$$

$$= 1 + 2j + E(i_k^2).$$

Recall that

$$Var(i_k) = s^2 = E(i_k^2) - [E(i_k)]^2$$

$$= E(i_k^2) - j^2,$$

or equivalently we have

$$E(i_k^2) = s^2 + j^2.$$

Hence,

$$E[(1+i_k)^2] = 1 + 2j + j^2 + s^2$$

$$= (1+j)^2 + s^2.$$

If we let

$$f = 2j + j^2 + s^2$$

we obtain

$$E[(1+i_k)^2] = 1 + f =: m.$$

Thus the variance of the amount accumulated during year k can be denoted as

$$Var(1+i_k) = m - \mu^2.$$

Now let us again consider the case of a single investment at the beginning of the first year. In particular, let $c_1 = 1, c_2 = c_3 = \cdots = c_k = 0$. Recall we found

$$C_k = (1+i_1)\cdots(1+i_k) = C_{k-1}(1+i_k) \quad \text{for } k = 2, \cdots, n.$$

We define $E[C_k]$ to be μ_k. Since the i_k's are independent we have

$$\mu_k = \mu_{k-1}\mu,$$

and thus

$$\mu_k = \mu^k.$$

Notice this is our expected value for the accumulated value after k years with a single investment of 1 at the beginning. Now we wish to find the variance. Using the same reasoning as above, we can write

$$m_k = m_{k-1}m.$$

And so

$$m_k = m^k.$$

Hence we have

$$Var(C_k) = m^k - \mu^{2k}.$$

We can generalize these findings as the following theorem [6].

Theorem 2.1 *If C_k denotes the future value after k years of a single initial investment of 1. And, if the yearly rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $Var(i_k) = s^2$, and $i_1, i_2, ..., i_n$ are independent variables, then*

$$E(C_n) = (1+j)^n$$

and,

$$Var(C_n) = ((1+j)^2 + s^2)^n - (1+j)^{2n}.$$

Next, we will reconsider the case of an annuity-due with k yearly payments of 1. That is, $c_1 = \cdots = c_n = 1$. Letting C_k be the accumulated value of the annuity we have

$$C_k = (1+i_k)(1+C_{k-1}) \quad \text{for } k = 2, \cdots, n.$$

Zaks uses straight forward reasoning to find $E(C_k)$ to be

$$\mu_k = \mu(1 + \mu_{k-1}).$$

And,

$$m_k = m(1 + 2\mu_{k-1} + m_{k-1}).$$

Next, by applying $\ddot{s}_{\overline{k}|j} = (1+j)+\cdots+(1+j) = (1+j)(1+\ddot{s}_{\overline{k-1}|j})$ to $\mu_k = \mu(1+\mu_{k-1})$ Zaks obtains the following result[6].

Theorem 2.2 *If C_k denotes the future value after k years of an annuity-due of k yearly payments of 1 and if the yearly rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $Var(i_k) = s^2$, and $i_1, ..., i_n$ are independent variables, then $\mu_k = E(C_k) = \ddot{s}_{\overline{k}|j}$; in general, $\mu_n = E(C_n) = \ddot{s}_{\overline{n}|j}$.*

We now wish to derive the value of $Var(C_k)$. We have shown this value to be $Var(C_k) = m_k - \mu_k^2$. Thus we need a closed value of $E(C_k^2) = m_k$. Zaks uses induction to show

$$m_k = (m + \cdots + m^k) + 2(m\ddot{s}_{\overline{k-1}|j} + m^2\ddot{s}_{\overline{k-2}|j} + \cdots + m^{k-1}\ddot{s}_{\overline{1}|j}.$$

For simplicity we will let

$$M_{1k} = m + \cdots + m^k,$$

$$M_{2k} = m\ddot{s}_{\overline{k-1}|j} + \cdots + m^{k-1}\ddot{s}_{\overline{1}|j}.$$

Recall, we defined m to be $m = 1 + f$. Thus it follows that

$$M_{1k} = \ddot{s}_{\overline{k}|f}.$$

Further,

$$M_{2k} = (1+f)\frac{(1+\mathrm{j})^{\mathrm{k}-1}-1}{\mathrm{d}} + \cdots + (1+f)^{k-1}\frac{(1+\mathrm{j})-1}{\mathrm{d}}.$$

Next we will define the r to be the solution of

$$1+r = \frac{1+\mathrm{f}}{1+\mathrm{j}}.$$

By applying this and using the equivalent rate of interest j for d, we easily derive.

$$M_{2k} = \frac{(1+\mathrm{j})^{\mathrm{k}+1}}{\mathrm{j}}\left[\frac{(1+\mathrm{f})}{(1+\mathrm{j})} + \cdots + \frac{(1+\mathrm{f})}{(1+\mathrm{j})}^{k-1}\right] - \frac{1+\mathrm{j}}{\mathrm{j}}[(1+f) + \cdots + (1+f)^{k-1}].$$

Hence we have

$$M_{2k} = \frac{(1+\mathrm{j})^{\mathrm{k}+1}\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{r}} - (1+\mathrm{j})\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{j}}.$$

By substitution,

$$m_k = M_{1k} + 2M_{2k},$$

is equivalent to

$$m_k = \ddot{s}_{\overline{k}|f} + 2\left[\frac{(1+\mathrm{j})^{\mathrm{k}+1}\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{r}} - (1+\mathrm{j})\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{j}}\right]$$

Applying basic algebra gives

$$m_k = \frac{2(1+\mathrm{j})^{\mathrm{k}+1}\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}}}{\mathrm{j}} - \frac{2\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{r}} - \mathrm{j}\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{j}}.$$

Thus,

$$m_k = \frac{2(1+\mathrm{j})^{\mathrm{k}+1}\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{r}}}{\mathrm{j}} - \frac{(2-\mathrm{j})\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{j}}.$$

We have thus established a closed formula for m_k. [6]

We have now reached an equation for $E(C_k^2)$. However, we still require a formula for $[E(C_k)]^2 = \mu_k^2 = (\ddot{s}_{\overline{k}|j})^2$. Zaks obtains the following lemma [6].

Lemma 2.3

$$(\ddot{s}_{\overline{k}|j})^2 = \frac{\ddot{\mathrm{s}}_{\overline{2\mathrm{k}}|\mathrm{j}} - 2\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}}}{\mathrm{d}}.$$

Proof. Using straight forward calculations we find

$$(\ddot{s}_{\overline{k}|j})^2 = \frac{[(1+\mathrm{j})^2 - 1]^2}{\mathrm{d}^2} = \frac{(1+\mathrm{j})^{2\mathrm{k}} - 2(1+\mathrm{j})^{\mathrm{k}} + 1}{\mathrm{d}^2}.$$

We manipulate the equation to obtain

$$(\ddot{s}_{\overline{k}|j})^2 = \frac{[(1+\mathrm{j})^{2\mathrm{k}} - 1] - 2[(1+\mathrm{j})^{\mathrm{k}} - 1]}{\mathrm{d}^2}.$$

Which is equivalent to

$$(\ddot{s}_{\overline{k}|j})^2 = \frac{\frac{(1+\mathrm{j})^{2\mathrm{k}}-1}{\mathrm{d}} - 2\left(\frac{(1+\mathrm{j})^{\mathrm{k}}-1}{\mathrm{d}}\right)}{\mathrm{d}}.$$

And thus we have

$$(\ddot{s}_{\overline{k}|j})^2 = \frac{\ddot{\mathrm{s}}_{\overline{2\mathrm{k}}|\mathrm{j}} - 2\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}}}{\mathrm{d}}.$$

This concludes the proof. QED

Hence we can now establish a formula for $Var(C_k)$ in terms of the future values of annuities due for periods of k and $2k$ and in terms of interest j, f and r. Thus we reach the following theorem[6].

Theorem 2.4 *If C_k denotes the future value after k years of an annuity-due of k yearly payments of 1 and if the yearly rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $Var(i_k) = s^2$, and $i_1, \cdots, i_n$ are independent variables, then*

$$E(C_k) = \ddot{s}_{\overline{k}|j}$$

and

$$Var(C_k) = \frac{2(1+\mathrm{j})^{\mathrm{k}+1}\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{r}} - (2+\mathrm{j})\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{f}} - (1+\mathrm{j})\ddot{\mathrm{s}}_{\overline{2\mathrm{k}}|\mathrm{j}} + 2(1+\mathrm{j})\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}}}{\mathrm{j}}.$$

2.2 Increasing Payments

In the previous section we studied fixed annuities-due. Let us now consider the case corresponding to the increasing annuity-due. That is the case in which $c_k = k$ for $k = 1, 2, \cdots, n$. Thus we have n payments of $1, 2, \cdots, n$. Therefore the accumulated value at time k can be given as

$$C_k = (1 + i_k)(C_{k-1} + k).$$

From this Zaks uses straight forward reasoning to derive [6],

$$\mu_k = \mu(\mu_{k-1} + k) \quad \text{for } k = 2, \cdots, n,$$

$$m_k = m(m_{k-1} + 2k\mu_{k-1} + k^2) \quad \text{for } k = 2, \cdots, n.$$

Recall, $\mu = 1 + j$, which is equivalent to $(I\ddot{s})_{\overline{1}|j}$. Hence, we have the following theorem [6].

Theorem 2.5 *If C_k denotes the future value after k years of an increasing annuity-due of k yearly payments of $1, \cdots, k$, and if the yearly rate of interest during the kth year is a random variable, i_k, so that $E(1 + i_k) = 1 + j$ and $Var(i_k) = s^2$, and so $i_1, \cdots, i_n$ are independent variables, then*

$$\mu_k = E(C_k) = (I\ddot{s})_{\overline{k}|j} \quad \textit{for } k = 1, \cdots, n$$

and in general, $\mu_n = E(C_n) = (I\ddot{s})_{\overline{n}|j}$.

Next, recall we can denote $m_k = m(1 + 2\mu_{k-1} + m_{k-1})$ This leads to the following lemma [6]

Lemma 2.6 *Under the assumptions of Theorem 2.5, we have*

$$m_k = (m^k + 2^2 m^{k-1} + \cdots + k^2 m) + 2(2m^{k-1}(I\ddot{s})_{\overline{1}|j} + \cdots + km(I\ddot{s})_{\overline{k-1}|j}).$$

Proof. Let

$$M_{1k} = m^k + 2^2 m^{k-1} + \cdots + k^2 m,$$

$$M_{2k} = 2m^{k-1}(I^2\ddot{s})_{\overline{1}|j} + \cdots + km(I^2\ddot{s})_{\overline{k-1}|j}.$$

We must show that $m_k = M_{1k} + 2M_{2k}$. We proceed by induction. The result holds for $k = 2$ since $\mu = (I\ddot{s})_{\overline{1}|j}$ and $m_1 = m$. So

$$m_2 = m(m_1 + 4\mu + 4) = m^2 + 4(I\ddot{s})_{\overline{1}|j} + 4$$

Now assuming our result is true for $2 \leq k \leq n - 1$, it follows from $m_k = m(1 + 2\mu_{k-1} + m_{k-1}$ that it is also true for $k + 1$. QED

Since $m = 1 + f$, we find

$$M_{1k} = (I\ddot{s})_{\overline{k}|f.}$$

Next, recall we have defined $(I\ddot{s})_{\overline{k}|j} = \frac{\mathrm{\ddot{s}_{\overline{k}|j} - j}}{\mathrm{d}}$ and $1 + r = \frac{\mathrm{1+f}}{\mathrm{1+j}}$. Hence we can deduce

$$M_{2k} = \frac{\mathrm{2(1+f)^{k-1}[\ddot{s}_{\overline{1}|j}] + \cdots + k(1+f)[\ddot{s}_{\overline{k-1}|j} - (k-1)]}}{\mathrm{d}}$$

$$= \left(\frac{\mathrm{2(1+f)^{k-1}\left[\frac{(1+j)-1}{d}\right] + \cdots + k(1+f)\left[\frac{(1+j)^{k-1}-1}{d}\right]}}{\mathrm{d}}\right) - \left(\frac{\mathrm{2(1+f)^{k-1} + \cdots + k(k-1)(1+f)}}{\mathrm{d}}\right)$$

$$= \left(\frac{(1+\mathrm{j})^{\mathrm{k}} 2(1+\mathrm{r})^{\mathrm{k}-1} + \cdots + \mathrm{k}(\mathrm{k}-1)(1+\mathrm{f})}{\mathrm{d}^2} \right) - \left(\frac{2(1+\mathrm{f})^{\mathrm{k}-1} + \cdots + \mathrm{k}(1+\mathrm{f})}{\mathrm{d}^2} \right)$$

$$- \left(\frac{2^2(1+\mathrm{f})^{\mathrm{k}-1} + \cdots + \mathrm{k}^2(1+\mathrm{f})}{\mathrm{d}} \right) + \left(\frac{2(1+\mathrm{f})^{\mathrm{k}-1} + \cdots + \mathrm{k}(1+\mathrm{f})}{\mathrm{d}} \right)$$

$$= \frac{(1+\mathrm{j})^{\mathrm{k}}[(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{r}} - (1+\mathrm{r})^{\mathrm{k}} - (\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}} - (1+\mathrm{f})^{\mathrm{k}}}{\mathrm{d}^2} - \frac{(\mathrm{I}^2\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}} - (1+\mathrm{f})^{\mathrm{k}} + (\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}} - (1+\mathrm{f})^{\mathrm{k}}}{\mathrm{d}}$$

$$= \frac{(1+\mathrm{j})^{\mathrm{k}}(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{r}}}{\mathrm{d}^2} - \frac{(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{d}^2} - \frac{(\mathrm{I}^2\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{d}} + \frac{(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{d}}$$

$$= \frac{(1+\mathrm{j})^{\mathrm{k}+2}(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{r}}}{\mathrm{j}^2} - \frac{(1+\mathrm{j})(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{j}^2} - \frac{(1+\mathrm{j})(\mathrm{I}^2\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{j}}.$$

Which gives us the following lemma [6].

Lemma 2.7 *Under the assumptions of Theorem 2.5, we have*

$$M_{2k} = \frac{(1+\mathrm{j})^{\mathrm{k}=2}(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{r}} - (2+2\mathrm{k}-\mathrm{j}^2)(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}} - \mathrm{j}(1+\mathrm{j})(\mathrm{I}^2\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{j}^2}.$$

Now we obtain the following theorem.

Theorem 2.8 *Under the assumptions of Theorem 2.5, we have*

$$m_k = \frac{2(1+\mathrm{j})^{\mathrm{k}+2}(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{r}} - (2+2\mathrm{k}-\mathrm{j}^2)(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}} - (\mathrm{j}+\mathrm{j})^2(\mathrm{I}^2\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{f}}}{\mathrm{j}^2}.$$

We now only need $E(C_k)^2 = \mu_k^2 = [(I\ddot{s})_{\overline{k}|j}]^2$ in order to evaluate $Var(C_k)$. So,

$$[(I\ddot{s})_{\overline{k}|j}]^2 = \frac{(\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}} - \mathrm{k})^2}{\mathrm{d}^2} = \frac{\ddot{\mathrm{s}}^2_{\overline{\mathrm{k}}|\mathrm{j}} - 2\mathrm{k}\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}} + \mathrm{k}^2}{\mathrm{d}^2} = \frac{\frac{\ddot{\mathrm{s}}_{\overline{2\mathrm{k}}|\mathrm{j}} - 2\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}}}{\mathrm{d}} - 2\mathrm{k}\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}} + \mathrm{k}^2}{\mathrm{d}^2}.$$

Notice that $\ddot{s}_{\overline{2k}|j} - 2\ddot{s}_{\overline{k}|j} = (\ddot{s}_{\overline{2k}|j} - 2k) - 2(\ddot{s}_{\overline{k}|j} - k)$. Hence we have

$$[(I\ddot{s})_{\overline{k}|j}]^2 = \frac{(\mathrm{I}\ddot{\mathrm{s}})_{\overline{2\mathrm{k}}|\mathrm{j}} - 2(\mathrm{I}\ddot{\mathrm{s}})_{\overline{\mathrm{k}}|\mathrm{j}} - 2\mathrm{k}(\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}} - \mathrm{k}) - \mathrm{k}^2}{\mathrm{d}^2}.$$

Thus,

$$[(I\ddot{s})^2_{\overline{k}|j}]^2 = \frac{\mathrm{(I\ddot{s})_{\overline{2k}|j} - 2(1+kd)(I\ddot{s})_{\overline{k}|j} - k^2}}{\mathrm{d^2}}.$$

Which leads us to the following theorem [6].

Theorem 2.9 *If C_k denotes the future value after k years of an increasing annuity-due of k yearly payments of $1, \cdots, k$, and if the yearly rate of interest during the kth year is a random variable, i_k, so that $E(1+i_k) = 1+j$ and $Var(i_k) = s^2$, and so that $i_1, \cdots, i_n$ are independent variables, then*

$$E(C_k) = (I\ddot{s})_{\overline{k}|j}$$

and,

$$\begin{aligned} Var(C_k) = & \frac{\mathrm{2(1+j)^{k+2}(I\ddot{s})_{\overline{k}|r} - (2+2j-j^2)(I\ddot{s})_{\overline{k}|f} - (j+j^2)(I^2\ddot{s})_{\overline{k}|f}}}{\mathrm{j^2}} \\ & - \frac{\mathrm{(I\ddot{s})_{\overline{2k}|j} - 2(1+kd)(I\ddot{s})_{\overline{k}|j} - k^2}}{\mathrm{d^2}}. \end{aligned}$$

Hence we have established formulae to evaluate $E(C_k)$ and $\mathrm{Var}(C_k)$ in terms of future values of increasing annuities-due for periods of k to $2k$, and in terms of rates of interest j, f, and, r.

CHAPTER 3

LIFE ANNUITIES

Thus far we have focused our discussion on annuities-certain. That is, the payments have extended over a fixed term. However not all annuities are annuities-certain. A *contingent annuity* is one whose payments extend over a period of time whose length cannot be accurately foretold [2]. A common type of contingent annuity is a *life annuity.* A life annuity is one in which payments are made only if a person is alive. An example would be monthly retirement benefits from a pension plan, which continue for the life of a retiree. Thus it is a model, often utilized for insurance systems, designed to manage random losses where the randomness is related to how long an individual will survive. The *time-until-death* random variable, $T(x)$, is the basic building block.

As in previous chapters we will focus our discussion on annuities-due as they have a more prominent role in actuarial applications. In particular we will consider an annuity which pays a unit amount at the beginning of each year that the annuitant (x) survives. We will see that life annuity theory is comparable to annuities-certain, but brings in survival as a condition for payment.

3.1 Preliminaries

In this section, we introduce some basic actuarial terminologies. Let X denote the random variable of age-at-death and (x) the life of age x. We use $T(x)$ to denote the random variable of the future life time. Thus, $T(x) = X - x$. The survival function $s(x)$ of (x) is the probability function $Pr[X > x]$. Also, we let ${}_t q_x$ denote the probability that (x) will die within t years. Therefore, ${}_t p_x = 1 -_t q_x$ is the probability that (x) will attain age $x + t$, or the survival function of (x). Note that $T(0) = X$ and ${}_x p_0 = s(x)$ for $x \geq 0$.

If $t = 1$, we use q_x and p_x to replace ${}_1 q_x$ and ${}_1 p_x$, respectively.

Or in terms of the survival function, we have

$$ {}_t p_x = \frac{s(x+t)}{s(x)}, $$

and

$$ {}_t q_x = \frac{s(x) - s(x+t)}{s(x)}. $$

A discrete random variable associated with the future life time is the number of future years completed by (x) prior to death. It is called *curtate-future-lifetime* of (x), denoted by $K(x)$. Thus, $K(x) = \lfloor T(x) \rfloor$ and

$$ Pr[K(x) = k] = {}_k p_x - {}_{k+1} p_x. $$

3.2 Life Annuities With Constant Interest Rates

In this section we assume a constant effective annual rate of interest. Let Y be the present value of a life annuity-due with yearly payment of 1. We note that Y is a random variable for such an annuity and is know to be [1]

$$ Y = \ddot{a}_{\overline{K+1}|i}. $$

The probability associated with the value $\ddot{a}_{\overline{K+1}|i}$ is $Pr[K = k]$. The possible values of K are discrete, and its probability function is,

$$ Pr[K = k] = Pr[k \leq T(x) \leq k + 1]. $$

Recall that

$$ {}_tq_x = Pr[T(x) \leq t] \quad \text{for } t \geq 0 $$

$$ {}_tp_x = 1 - {}_tq_x = Pr[T(x) > t] \quad \text{for } t \geq 0. $$

So we can write, [1]

$$ Pr[K = k] = {}_kp_x \; - \; {}_{k+1}p_x = {}_kp_x \, q_{x+k}. $$

We will now consider the actuarial present value of the life annuities (see [1]). Life annuities play a major role in life insurance operations. A whole life annuity is a series of payments made continuously or at equal intervals while a given life survives.

The actuarial present value of a discrete type of whole life annuity-due payable annually at the rate of 1 per year at the beginning is denoted, $\ddot{a}_x$, which is the expected value of the annuity.

The present value random variable of payments is

$$ Y = \ddot{a}_{\overline{K+1}}. $$

Therefore,

$$ \ddot{a}_x = E[\ddot{a}_{\overline{K+1}|i}] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|i} \, {}_kp_x \, q_{x+k}. $$

Or we may rewrite the actuarial present as

$$ \ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x, $$

by using summation-by-parts, where ${}_kp_x$ is the probability of a payment of size 1 being made at time k (that is, the probability that (x) attains the age $x + k$).

The following recursion formula holds

$$ \ddot{a}_x = 1 + v \, p_x \, \ddot{a}_{x+1}. $$

For whole life annuity-immediate, we have

$$a_x = E[a_{\overline{K}|i}] = \sum_{k=1}^{\infty} a_{\overline{k}|i}\, {}_k p_x\, q_{x+k} = a_x = \sum_{k=1}^{\infty} v^k\, {}_k p_x.$$

CHAPTER 4

LIFE ANNUITIES UNDER RANDOM RATES OF INTEREST

4.1 Present Value

In Chapter 2 we found the expected value and the variance of the accumulated value of some annuities. In this section we use the same methods to examine the present value of the expected accumulated value of annuities-certain. We examine the present value of a single investment and the present value of an annuity-due with yearly payments of 1. Once these are obtained, we extend our results to apply to life annuities. That is we are able to evaluate an annuity with both time and rate of interest being random variables.

As in Chapter 2, we let i_k denote the rate of interest for year k, that is for the interval $k-1$ to k for $k = 1, 2, \cdots, n$. And we assume $i_1, \cdots, i_n$ are independent random variables. Now we first consider the case of a single payment at the start of the term We let PV_k denote the present value of the expected accumulated amount. Recall we can take the present value of an accumulation X_t at time t by $\frac{1}{\mathrm{X_t}}$. We found $E(1+i_k) = 1+j$ which is the expected accumulation at time 1. So the present value of this expected amount is

$$[E(1+i_k)]^{-1} = \frac{1}{\mathrm{E[(1+i_k)]}} = \frac{1}{1+\mathrm{j}} =: \mu*$$

Thus

$$PK_k = \frac{1}{1+\mathrm{j}} \cdots \frac{1}{(1+\mathrm{j})^{\mathrm{k}}}$$

Which leads to the following theorem.

Theorem 4.10 *If PV_n denotes the expected present value of a single investment after n years. And, if the yearly rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $Var(i_k) = s^2$, and $i_1, \cdots, i_n$ are independent variables, then*

$$PV_n = \left(\frac{1}{1+\mathrm{j}}\right)^n$$

Next we will consider the case of an annuity-due with k yearly payments of 1. Letting PV_k be the expected present value of the k year annuity we have

$$PV_k = \mu_k^* = \frac{1}{\ddot{\mathrm{s}}_{\overline{\mathrm{k}}|\mathrm{j}}}$$

which is equivalent to

$$\mu_k^* = \frac{(1+\mathrm{j})^{\mathrm{k}} - 1}{\mathrm{d}(1+\mathrm{j})^{\mathrm{k}}} = \frac{1}{\mathrm{d}}\left(1 - \frac{1}{(1+\mathrm{j})^{\mathrm{k}}}\right)$$

Giving us the following theorem.

Theorem 4.11 *If PV_k denotes the expected present value after k years of an annuity-due of k yearly payments of 1. And if the yearly rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $Var(i_k) = s^2$, and $i_1, \cdots, i_n$ are independent variables, then*

$$PV_k = \ddot{a}_{\overline{k}|j}$$

4.2 Life Annuities With Random Rates of Interest

The equations for life annuities in chapter 3 assume that time until death is a random variable and the distribution is known. The interest rates in the models were assumed to be constant. Yet examination of observed interest rates confirms this assumption can be unrealistic. Thus in this section we will again consider the actuarial present value of life annuities, but we will now assume both time until death and the rate of interest are random variables.

We will assume there to be n possible values for our interest rate. We will denote each of these as i_l where $l \leq n$. As in previous work, we will assume that each individual interest rate i_l has $E(i_l) = j$ and $Var(i_l) = s^2$. And, $\frac{1}{\mathrm{E(1+i_l)}} = \frac{1}{\mathrm{1+j}}$. Now recall the actuarial present value of a life annuity with constant interest rates was defined to be

$$\ddot{a}_x = E[\ddot{a}_{\overline{K+1}|}] = \sum_{k=0}^{\infty} v^k \,{}_kp_x.$$

Where K is the random variable defined as the number of complete future life years of a life at age x. Now let ${}_*\ddot{a}_x$ denote the actuarial present value of a life annuity with random interest rates. Thus we must now consider the expectation with respect to both K, and the interest rate i_l. We will assume that K and the $i_l's$ are independent. In addition we will introduce the actuarial notions E_K to be the expectation with respect to K, and $E_{i_l|K}$ to be the expectation with respect to i_l given K. Thus we can write

$${}_*\ddot{a}_x = E_K \, E_{i_l|K}[{}_{i_l}\ddot{a}_{\overline{K+1}|}].$$

Applying theorem 4.12 we obtain

$$_*\ddot{a}_x = E_K\,[\ddot{a}_{\overline{K+1}|j}].$$

Hence we have

$$_*\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|j}\ {}_kp_x\ q_{x+k}$$

or equivalently we can write

$$_*\ddot{a}_x = \sum_{k=0}^{\infty} \left(\frac{1}{1+\mathrm{j}}\right)^k\ {}_kp_x.$$

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